One-loop counterterms for the dimensional regularization of arbitrary Lagrangians.

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Abstract

We present master formulas for the divergent part of the one-loop effective action for an arbitrary (both minimal and nonminimal) operators of any order in the 4-dimensional curved space. They can be considered as computer algorithms, because the one-loop calculations are then reduced to the simplest algebraic operations. Some test applications are considered by REDUCE analitical calculation system.

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1 Introduction.

Progress in the quantum field theory and quantum gravity in particular depends much on the development of methods for the calculation of the effective action. For a lot of problems the analyses can be confined to the one-loop approximation. In this case

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$$\Gamma[\varphi] = S[\varphi] + \frac{i}{2}\,\hbar\ \mathrm{tr}\ \ln\ D + O(\hbar^2), \quad \text{where} \quad D_i{}^j \equiv \frac{\delta^2 S}{\delta\varphi^i\ \delta\varphi_j}. \tag{1}$$

where latin letters denote the whole set of φ indexes.

Finding its divergent part is sometimes a rather complicated technical problem, especially in the curved space. Unfortunately, the usual diagram technique is not manifestly covariant. A very good tool to make calculations in the curved space-time is the Schwinger-DeWitt proper time method [1, 2]. It allows to make manifestly covariant calculations of Feynman graphs if the propagator depends on the background metric [3]. This approach was successfully applied to obtain one-loop counterterms in theories with the simplest second and forth order operators $D_i{}^j$. We should also mention other covariant methods, that in principle allow to get the divergent part of effective action [4, 5, 6].

A different approach was proposed by t'Hooft and Veltman [7]. Instead of calculating Feynman graphs for each new theory, they made it only once for a rather general case. Nevertheless, if $D_i{}^j$ in (1) is not a minimal second order oparetor, their results can not be used directly.

In this paper we extend t'Hooft and Veltman approach to the most general case. We construct the explicit expression for the divergent part of the one-loop effective action without any restrictions to the form and the order of the operator $D_i{}^j$ in the 4-dimentional space-time. Then the divergent part of the one-loop effective action can be found only by making the simplest algebraic operations, for example, by computers.

Our paper is organized as follows.

In Sec. 2 we briefly remind t'Hooft-Veltman diagram technique and introduce some notations and definitions.

In Sec. 3 we consider a theory with an arbitrary minimal operator in the curved space-time. The main result here is an explicit expression for the one-loop contribution to the divergent part of the effective action.

In Sec. 4 we describe a method for the derivation of a master formula for an arbitrary nonminimal operator on the curved background and present the result.

Sec. 5 is devoted to the consideration of some particular cases. We show the agreement of our results with the earlier known ones. Here we proove the correctness of the method and that is why we do not consider here new applications.

In Sec. 6 we give a summary of our results and discuss prospects of using computers for the automatization of calculations.

In Appendix A we describe in details the derivation of the one-loop counterterms for an arbitrary minimal operator.

In Appendix B we illustrate the general method presented in Sec. 4 by calculating the simplest Feynman graphs for an arbitrary nonminimal operator.

Appendix C is devoted to the derivation of some useful identities.

2 Diagramic approach in the background field method

We will calculate the divergent part of the one-loop effective action for a general theory by the diagram tecknique. First we note, that

$$D_i{}^j = \frac{\delta^2 S}{\delta\varphi^i \delta\varphi_j} \tag{2}$$

is a differential operator depending on the background field φ. Its most general form is

$$\begin{aligned} D_i{}^j &= K^{\mu_1\mu_2\ldots\mu_L}{}_i{}^j\ \nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_L} +\ S^{\mu_1\mu_2\ldots\mu_{L-1}}{}_i{}^j\ \nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{L-1}} \\ &+\ W^{\mu_1\mu_2\ldots\mu_{L-2}}{}_i{}^j\ \nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{L-2}} +\ N^{\mu_1\mu_2\ldots\mu_{L-3}}{}_i{}^j\ \nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{L-3}} \\ &+\ M^{\mu_1\mu_2\ldots\mu_{L-4}}{}_i{}^j\ \nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{L-4}} + \ldots, \end{aligned} \tag{3}$$

where ∇_μ is a covariant derivative:

$$\begin{aligned} &\nabla_\alpha T^\beta{}_i{}^j = \partial_\alpha T^\beta{}_i{}^j + \Gamma^\beta_{\alpha\gamma} T^\gamma{}_i{}^j + \omega_{\alpha i}{}^k T^\beta{}_k{}^j - T^\beta{}_i{}^k \omega_{\alpha k}{}^j; \\ &\nabla_\mu \Phi_i = \partial_\mu \Phi_i + \omega_{\mu i}{}^j \Phi_j. \end{aligned} \tag{4}$$

Here $\Gamma^{\alpha}_{\mu\nu}$ is the Cristoffel symbol

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}) \tag{5}$$

and ${\omega_{\mu i}}^j$ is a connection on the principle bundle.

Commuting covariant derivatives we can always make $K, S, W, N, M, \ldots$ symmetric in the greek indexes. This condition is very convenient for the calculations, so we will assume it to be satisfied.

The operator is called minimal if $L = 2l$ and ${K^{\mu\nu\ldots\alpha}}_i{}^j = {K_0}^{\mu\nu\ldots\alpha}\delta_i{}^j$, where K_0 is a totally symmetric tensor, built by $g_{\mu\nu}$:

$${K_0}^{\mu\nu\alpha\beta\ldots} = \frac{1}{(2l-1)!!}(g_{\mu\nu}g_{\alpha\beta}\ldots + \text{permutations of} \quad (\mu\nu\alpha\beta\ldots)). \tag{6}$$

If an operator can not be reduced to this form, we will call it nonminimal one.

Commuting covariant derivatives we can rewrite a minimal operator in the form:

$$\begin{aligned} D_i{}^j \; &= \; \delta_i{}^j \; \Box^l + \; S^{\mu_1\mu_2\ldots\mu_{2l-1}}{}_i{}^j \; \nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{2l-1}} + \; W^{\mu_1\mu_2\ldots\mu_{2l-2}}{}_i{}^j \\ &\times \nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{2l-2}} + \; N^{\mu_1\mu_2\ldots\mu_{2l-3}}{}_i{}^j\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{2l-3}} \\ &+ \; M^{\mu_1\mu_2\ldots\mu_{2l-4}}{}_i{}^j\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{2l-4}} + \ldots, \end{aligned} \tag{7}$$

where $\Box \equiv \nabla_\mu\nabla^\mu$.

The one-loop effective action (1) can be presented as a sum of one-loop diagrams, say, for a minimal operator, as follows:

$$\begin{aligned} \Gamma^{(1)} &= \frac{i}{2}\mathrm{tr}\ \ln\ D_i{}^j = \frac{i}{2}\mathrm{tr}\ \ln\ \Big(\partial^{2l} + V\Big) = \frac{i}{2}\mathrm{tr}\ \ln\ \partial^{2l} \\ &+ \frac{i}{2}\mathrm{tr}\ \ln\ \Big(1 + \frac{1}{\partial^{2l}}V\Big) = \frac{i}{2}\mathrm{tr}\sum_{k=1}^{\infty}\frac{1}{k}\Big(-\frac{1}{\partial^{2l}}V\Big)^k. \end{aligned} \tag{8}$$

where

$$V = S^{\mu_1 \dots \mu_{L-1}} \partial_{\mu_1} \dots \partial_{\mu_{L-1}} + \ W^{\mu_1 \dots \mu_{L-2}} \partial_{\mu_1} \dots \partial_{\mu_{L-2}} + \ N^{\mu_1 \dots \mu_{L-3}}$$

$$\times \partial_{\mu_1} \dots \partial_{\mu_{L-3}} + \ M^{\mu_1 \dots \mu_{L-4}} \partial_{\mu_1} \dots \partial_{\mu_{L-4}} + \dots + O(h,\omega). \tag{9}$$

and we omit an infinite numerical constant.

Terms $O(h,\omega)$ can be found by a series expansion of the operator in powers of $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ and ${\omega_{\mu i}}^j$.

In the momentum space the propagator is δ_i^j / k^{2l}. The form of the vertexes is rather evident, for example, the vertex with the external line of S type can be written as

$$S^{\mu_1 \mu_2 \dots \mu_{L-1}} k_{\mu_1} k_{\mu_2} \dots k_{\mu_{L-1}} \equiv (Sk). \tag{10}$$

Similar notations we will use for other expressions, for example,

$$\begin{aligned} (W(k+p))^\alpha &\equiv W^{\mu\nu\dots\beta\alpha}(k+p)_\mu (k+p)_\nu \dots (k+p)_\beta, \\ (Sk)^{\alpha\beta} &\equiv S^{\mu\nu\dots\gamma\alpha\beta} k_\mu k_\nu \dots k_\gamma. \end{aligned} \tag{11}$$

Numerical factors for the Feynman graphs can be easily found by (8).

The number of diagrams in (8) is infinite, but most of them are convergent. Really, it is easy to see that the degree of divergence of a one-loop graph with s legs of S type, w legs of W type, n legs of N type, m legs of M type and so on ($k = s + w + n + m + \dots$) in the flat space ($h_{\mu\nu} = 0$, ${\omega_{\mu i}}^j = 0$) is

$$I = 4 - s - 2w - 3n - 4m - \dots. \tag{12}$$

Therefore, there are only a finite number of the divergent diagrams. They are presented at the Fig. C. (We excluded divergent graphs, that give zero contribution to the effective action, for example, some tadpole ones.)

The extension of this results to a nonminimal operator will be made below.

3 Effective action for the theory with minimal operator.

Now we should calculate the divergent part of diagrams presented at the Fig. C. We will do it using dimensional regularization. So, in order to find the divergent part of an integral

$$\int d^dk\ f(k,p) \tag{13}$$

it is necessary to expand the function f into series, retain only logarithmically divergent terms and perform the integration according to the following equations [8]

$$\begin{aligned}
&\int d^dk\ \frac{1}{k^{2m+5}} k_{\mu_1} k_{\mu_2} \ldots k_{\mu_{2m+1}} = 0,\\
&\int d^dk\ \frac{1}{k^{2m+4}} k_{\mu_1} k_{\mu_2} \ldots k_{\mu_{2m}} = -\frac{2i\pi^2}{(d-4)} < n_{\mu_1} n_{\mu_2} \ldots n_{\mu_{2m}} >,
\end{aligned} \tag{14}$$

where n_μ is a unit vector ($n_\mu n^\mu = 1$) and

$$\begin{aligned}
&< n_{\mu_1} n_{\mu_2} \ldots n_{\mu_{2m}} > \equiv \frac{1}{2^m (m+1)!}\\
&\times \Big(g_{\mu_1\mu_2} g_{\mu_3\mu_4} \ldots g_{\mu_{2m-1}\mu_{2m}} + \text{permutaions of } (\mu_1 \ldots \mu_{2m}) \Big)
\end{aligned} \tag{15}$$

Using this prescription one can easily find the divergent part of the diagrams at the Fig C. The calculations are presented in details in the appendix A.

Collecting the results for all graphs, we obtain the divergent part of the one-loop effective action for the minimal operator (7) in the flat space:

$$\begin{aligned}
\left(\Gamma^{(1)}_\infty\right)^{flat} &= \frac{1}{16\pi^2(d-4)}\ \mathrm{tr} \int d^4x < (l-1)\partial_\mu \hat{S}\ \hat{W}^\mu - \frac{(2l-1)}{2} \partial_\mu \hat{S}^\mu \hat{W}\\
&+ \frac{l^2}{2(2l+1)} \partial_\mu \hat{S}\ \partial^\mu \hat{S} - \frac{(2l-1)(l^2-1)}{2(2l+1)} \partial_\mu \hat{S}^{\mu\nu} \partial_\nu \hat{S} + \frac{(2l-1)^2 l}{4(2l+1)} \partial_\mu \hat{S}^\mu \partial_\nu \hat{S}^\nu
\end{aligned}$$

$$+\frac{(2l-1)}{3}\partial_\mu \hat{S}^\mu \hat{S}\ \hat{S}-\frac{(2l-1)}{3}\partial_\mu \hat{S}\ \hat{S}\ \hat{S}^\mu+\hat{S}\ \hat{N}+\frac{1}{2}\hat{W}^2+\frac{1}{4}\hat{S}^4-\hat{W}\ \hat{S}^2$$
$$-\hat{M}> . \tag{16}$$

where

$$\hat{N}=(Nn)=N^{\mu\nu\ldots\alpha}n_\mu n_\nu\ldots n_\alpha;$$
$$\hat{S}^\mu=(Sn)^\mu=S^{\mu\nu\ldots\alpha}n_\nu\ldots n_\alpha \quad \text{and so on.} \tag{17}$$

In order to extend this result to the curved space-time we first consider a minimal operator in the form (7).

In this case we can not calculate all divergent graphs, because their number is infinite. (The matter is that the degree of divergence does not depend on the number of $h_{\mu\nu}$ vertexes and there are infinite number of such vertexes too). Nevertheless if we note that the answer should be invariant under the general coordinate transformations, the result can be found by calculating only a finite number of graphs. Really, we should replace derivatives in (16) by the covariant ones and add expressions, containing curvature tensors $R^\sigma{}_{\alpha\mu\nu}$ and $F_{\mu\nu}$. The most general form of additional terms is

$$\frac{1}{16\pi^2(d-4)}\ \mathrm{tr}\int d^4x\ \sqrt{-g}< a_1R^2+a_2R_{\mu\nu}R^{\mu\nu}+a_3\hat{W}^{\alpha\beta}R_{\alpha\beta}+a_4\hat{W}R$$
$$+a_5\nabla_\mu\hat{S}^{\mu\alpha\beta}R_{\alpha\beta}+a_6\nabla_\mu\hat{S}^\mu R+a_7\hat{S}^2R+a_8R_{\alpha\beta}\hat{S}^\alpha\hat{S}^\beta+a_9R_{\alpha\beta}\hat{S}^{\alpha\beta}\hat{S}$$
$$+a_{10}R_{\mu\nu\alpha\beta}\hat{S}^{\mu\alpha}\hat{S}^{\nu\beta}+a_{11}F_{\mu\nu}F^{\mu\nu}+a_{12}F_{\mu\nu}\hat{S}^\mu\hat{S}^\nu+a_{13}F_{\mu\nu}\nabla^\mu\hat{S}^\nu> . \tag{18}$$

where

$$R^\alpha{}_{\beta\mu\nu}=\partial_\mu\Gamma^\alpha_{\nu\beta}-\partial_\nu\Gamma^\alpha_{\mu\beta}+\Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\nu\beta}-\Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\mu\beta},$$
$$F_{\mu\nu i}{}^j=\partial_\mu\omega_{\nu i}{}^j-\partial_\nu\omega_{\mu i}{}^j+\omega_{\mu i}{}^k\omega_{\nu k}{}^j-\omega_{\nu i}{}^k\omega_{\mu k}{}^j,$$
$$R_{\mu\nu}=R^\alpha{}_{\mu\alpha\nu},\qquad R=g^{\mu\nu}R_{\mu\nu}. \tag{19}$$

(We take into account that the expression $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ is a total derivative and may be omitted).

Then the coefficients a_1 - a_{13} can be found by calculating the diagrams presented at the Fig. C. They conform to the first nontrivial approximation in the counterterm expansion in powers of weak fields $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ and ${\omega_{\mu i}}^j$.

In the appendix A we present detailed calculation of the coefficients a_3 and a_4. The other ones were found in the same way. After rather cumbersome calculations we obtain the following formula for the divergent part of the one-loop effective action for a minimal operator (7) in the curved space:

$$\begin{aligned}
\Gamma^{(1)}_\infty = {} & \frac{1}{16\pi^2(d-4)}\,\mathrm{tr}\int d^4x\,\sqrt{-g} < (l-1)\,\nabla_\mu\hat{S}\;\hat{W}^\mu - \frac{(2l-1)}{2}\nabla_\mu\hat{S}^\mu\hat{W} \\
& -\frac{(2l-1)(l^2-1)}{2(2l+1)}\nabla_\mu\hat{S}^{\mu\nu}\nabla_\nu\hat{S} + \frac{l^2}{2(2l+1)}\nabla_\mu\hat{S}\;\nabla^\mu\hat{S} + \frac{(2l-1)^2 l}{4(2l+1)}\nabla_\mu\hat{S}^\mu \\
& \times\nabla_\nu\hat{S}^\nu + \frac{(2l-1)}{3}\nabla_\mu\hat{S}^\mu\hat{S}\;\hat{S} - \frac{(2l-1)}{3}\nabla_\mu\hat{S}\;\hat{S}\;\hat{S}^\mu + \hat{S}\;\hat{N} + \frac{1}{2}\hat{W}^2 + \frac{1}{4}\hat{S}^4 \\
& -\hat{W}\;\hat{S}^2 - \hat{M} - \frac{(2l-3)(l-1)}{6}\;\hat{W}^{\alpha\beta}R_{\alpha\beta} + \frac{l}{6}\hat{W}R - \frac{(2l-1)l}{12}\;\nabla_\mu\hat{S}^\mu R \\
& +\frac{(2l-1)(2l-3)(l-1)}{12}\;\nabla_\mu\hat{S}^{\mu\alpha\beta}R_{\alpha\beta} - \frac{l}{6}\hat{S}^2 R + \frac{(2l-1)^2(l+2)}{24(2l+1)}\;R_{\alpha\beta} \\
& \hat{S}^\alpha\hat{S}^\beta + \frac{(2l-1)(l-1)}{6}R_{\alpha\beta}\hat{S}^{\alpha\beta}\hat{S} - \frac{(2l-1)(l-1)^2(l+2)}{12(2l+1)}R_{\mu\nu\alpha\beta}\hat{S}^{\mu\alpha}\hat{S}^{\nu\beta} \\
& +l\left(\frac{1}{120}R^2 + \frac{1}{60}R_{\mu\nu}R^{\mu\nu}\right) + \frac{l(2l-1)}{6}\nabla_\mu\hat{S}_\nu F^{\mu\nu} + \frac{(2l-1)^2(l+1)}{4(2l+1)}F_{\mu\nu}\hat{S}^\mu\hat{S}^\nu \\
& +\frac{l}{12}F_{\mu\nu}F^{\mu\nu} > .
\end{aligned} \tag{20}$$

It is more convenient sometimes to use a different form of the operator:

$$\begin{aligned}
D_i{}^j = {} & \delta_i{}^j {K_0}^{\mu_1\mu_2\ldots\mu_{2l}}\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{2l}} + \; S^{\mu_1\mu_2\ldots\mu_{2l-1}}{}_i{}^j\;\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{2l-1}} \\
& + \; W^{\mu_1\mu_2\ldots\mu_{2l-2}}{}_i{}^j\;\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{2l-2}} + \; N^{\mu_1\mu_2\ldots\mu_{2l-3}}{}_i{}^j\;\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{2l-3}} \\
& + \; M^{\mu_1\mu_2\ldots\mu_{2l-4}}{}_i{}^j\;\nabla_{\mu_1}\nabla_{\mu_2}\ldots\nabla_{\mu_{2l-4}} + \ldots,
\end{aligned} \tag{21}$$

where K_0 was defined in (6).

By the help of the same method we found the following answer for the divergent part of the one-loop effective action:

$$\Gamma^{(1)}_{\infty} = \frac{1}{16\pi^2(d-4)}\ \mathrm{tr} \int d^4x\ \sqrt{-g} < (l-1)\ \nabla_\mu \hat{S}\ \hat{W}^\mu - \frac{(2l-1)}{2}\nabla_\mu \hat{S}^\mu \hat{W}$$
$$+\frac{l^2}{2(2l+1)}\nabla_\mu \hat{S}\ \nabla^\mu \hat{S} + \frac{(2l-1)^2 l}{4(2l+1)}\nabla_\mu \hat{S}^\mu \nabla_\nu \hat{S}^\nu - \frac{(2l-1)(l^2-1)}{2(2l+1)}\nabla_\mu \hat{S}^{\mu\nu}$$
$$\times\nabla_\nu \hat{S} + \frac{(2l-1)}{3}\nabla_\mu \hat{S}^\mu \hat{S}\ \hat{S} - \frac{(2l-1)}{3}\nabla_\mu \hat{S}\ \hat{S}\ \hat{S}^\mu + \hat{S}\ \hat{N} + \frac{1}{2}\hat{W}^2 + \frac{1}{4}\hat{S}^4$$
$$-\hat{W}\ \hat{S}^2 - \hat{M} - \frac{l^2(2l-1)}{6(l+1)}\nabla_\mu \hat{S}^\mu R + \frac{(2l-1)(2l-3)(l-1)}{6(l+1)}\nabla_\mu \hat{S}^{\mu\alpha\beta} R_{\alpha\beta}$$
$$+\frac{l^2(2l-1)}{3(l+1)}F^{\mu\nu}\nabla_\mu \hat{S}_\nu + \frac{l^2}{3(l+1)}\hat{W}\ R - \frac{(2l-3)(l-1)}{3(l+1)}\hat{W}^{\mu\nu} R_{\mu\nu}$$
$$+\frac{(2l-1)(l-1)(l+2)}{6(2l+1)}R_{\mu\nu}\hat{S}^{\mu\nu}\hat{S} - \frac{(2l-1)(l-1)^2(l+2)}{12(2l+1)}R_{\mu\nu\alpha\beta}\hat{S}^{\mu\alpha}\hat{S}^{\nu\beta}$$
$$-\frac{(2l-1)^2(l-4)}{24(2l+1)}R_{\mu\nu}\hat{S}^\mu \hat{S}^\nu - \frac{l^2}{2(2l+1)}\hat{S}^2 R + \frac{(l+1)(2l-1)^2}{4(2l+1)}\hat{S}^\mu \hat{S}^\nu F_{\mu\nu}$$
$$+\frac{l^2(-7l^2+4l+12)}{540}R_{\mu\nu}R^{\mu\nu} + \frac{l^2(3l^2+4l+2)}{1080}R^2 + \frac{l^3}{12}F^{\mu\nu}F_{\mu\nu} > . \quad (22)$$

4 Effective action for theory with nonminimal operator.

Let us consider a theory with an arbitrary nonminimal operator (3) first in the flat space ($g_{\mu\nu} = \eta_{\mu\nu}$, ${\omega_{\mu i}}^j = 0$). In this case

$$\Gamma^{(1)} = \frac{i}{2}\ \mathrm{tr}\ \ln\ D_i{}^j = \frac{i}{2}\ \mathrm{tr}\ \ln\ ((K\partial)+V) = \frac{i}{2}\ \mathrm{tr}\sum_{k=1}^{\infty}\frac{1}{k}\left(-\ \frac{1}{\partial^{2l}}V\right)^k, \quad (23)$$

where V is given in (9). There are the following differences from the minimal operator:

1. For the nonminimal operator the propagator in the momentum space is $(Kk)^{-1}{}_i{}^j$, where

$$
\begin{aligned}
&(Kk)_i{}^j \equiv K^{\mu\nu\ldots\alpha}{}_i{}^j\; k_\mu k_\nu \ldots k_\alpha;\\
&(Kk)^{-1}{}_i{}^m\; (Kk)_m{}^j = \delta_i{}^j.
\end{aligned}
\tag{24}
$$

(We assume, that $(Kk)^{-1}{}_i{}^j$ exists. Usually it can be made by adding gauge fixing terms to the action).

2. For a nonminimal operator V depends on some additional fields ϕ^i besides $h_{\mu\nu}$ and $\omega_{\mu i}{}^j$. It will be considered in details below.

The divergent graphs are the same as in the case of a minimal operator (see Fig. C), but because of the difference of the propagators the calculations are also different. They are considered in the appendix B. The result is

$$
\begin{aligned}
&\left(\Gamma^{(1)}_\infty\right)^{flat} = \frac{1}{16\pi^2(d-4)}\mathrm{tr}\int d^4x < \frac{1}{4}\hat{S}^4 - \hat{W}\;\hat{S}^2 + \frac{1}{2}\hat{W}^2 + \hat{S}\;\hat{N} - \hat{M}\\
&+\frac{1}{3}\left((L-1)\partial_\mu\hat{S}^\mu\hat{S}^2 - L\partial_\mu\hat{S}\;\hat{K}^\mu\hat{S}^2 - (L-1)\partial_\mu\hat{S}\;\hat{S}\;\hat{S}^\mu + L\partial_\mu\hat{S}\;\hat{S}^2\hat{K}^\mu\right)\\
&-\frac{1}{2}\partial_\mu\hat{S}\;\partial_\nu\hat{S}\left(-\frac{1}{2}L(L-1)\hat{K}^{\mu\nu} + L^2\hat{K}^\mu\hat{K}^\nu\right) + \frac{1}{2}L(L-1)\partial_\mu\hat{S}\;\partial_\nu\hat{S}^\nu\hat{K}^\mu\\
&-\frac{1}{4}(L-1)(L-2)\partial_\mu\hat{S}\;\partial_\nu\hat{S}^{\mu\nu} - L\partial_\mu\hat{S}\;\hat{W}\hat{K}^\mu + (L-2)\partial_\mu\hat{S}\;\hat{W}^\mu > .
\end{aligned}
\tag{25}
$$

where we use the following notations (compare with (17))

$$
\begin{aligned}
&\hat{W} \equiv (Kn)^{-1}(Wn); && \hat{K}^\alpha \equiv (Kn)^{-1}(Kn)^\alpha;\\
&(Kn) \equiv K^{\mu\nu\ldots\beta}n_\mu n_\nu\ldots n_\beta; && (Kn)^{-1}{}_i{}^m\;(Kn)_m{}^j = \delta_i{}^j
\end{aligned}
\tag{26}
$$

etc.

The generalization of this result to the curved space-time can also be made in the frames of the weak field approximation. We should substitute derivatives in (25) by the covariant ones and add some terms, containing curvature tensors. The additional terms can be found by calculating Feynman diagrams, that conform to the first terms of their expansion over weak fields. Nevertheless, in this case there are some difficulties, for example, now we do not know $K^{\mu\nu\ldots\alpha}{}_i{}^j$ and, therefore, how it depends on $h_{\mu\nu}$. Thus, we do not know expressions for vertexes in the weak field limit.

In order to overcome this difficulty we will use the following trick. Suppose, that $K^{\mu\nu\ldots\sigma}{}_i{}^j$ does not depend on $g_{\mu\nu}$, but depend on some external fields ϕ^b. Also, we impose a condition

$$
\begin{aligned}
0 = \nabla_\alpha K^{\mu\nu\ldots\sigma}{}_i{}^j = \frac{\partial K^{\mu\nu\ldots\sigma}{}_i{}^j}{\partial\phi^b}\partial_\alpha\phi^b + \Gamma^\mu_{\alpha\beta}K^{\beta\nu\ldots\sigma}{}_i{}^j + \ldots \\
+ \Gamma^\sigma_{\alpha\beta}K^{\mu\nu\ldots\beta}{}_i{}^j + \omega_{\alpha i}{}^k K^{\mu\nu\ldots\sigma}{}_k{}^j - K^{\mu\nu\ldots\sigma}{}_i{}^k\omega_{\alpha k}{}^j,
\end{aligned}
\qquad (27)
$$

that is, of cause, satisfied if $K^{\mu\nu\ldots\sigma}{}_i{}^j$ depends only on $g_{\mu\nu}$. From (27) we conclude that $\omega_{\alpha i}{}^j$ should be considered as a weak field.

Thus, unlike the minimal operator, besides the diagrams with external h and ω lines, presented at the Fig C we should consider also graphs with external ϕ lines, presented at the Fig C.

Computing this diagrams we obtain expressions, containing $\partial_\alpha K^{\mu\nu\ldots\sigma}{}_i{}^j$. Substituting it by (27), we found the result depending on $h_{\mu\nu}$ and $\omega_{\mu i}{}^j$. The calculation of graphs, presented at the Fig C gives results, that can not be written as a weak field limit of a covariant expression. Nevertheless, the covariant result should be found by adding a contribution of diagrams with external ϕ-lines.

Therefore, for a nonminimal operator we can not calculate terms, containing $R^\alpha{}_{\beta\mu\nu}$ and $F_{\mu\nu}$ separately.

We illustrate the above discussion in the appendix B by calculating the simplest group of diagrams.

Summing up the results for all graphs, we find the divergent part of the one-loop effective action for an arbitrary nonminimal operator

$$
\begin{aligned}
\Gamma^{(1)}_\infty = \frac{1}{16\pi^2(d-4)}\,\mathrm{tr}\int d^4x\sqrt{-g} < Flat + WR + \\
+ SR + SSR + FF + FR + RR >,
\end{aligned}
\qquad (28)
$$

where

$$Flat = \frac{1}{4}\hat{S}^4 - \hat{W}\hat{S}^2 + \frac{1}{2}\hat{W}^2 + \hat{S}\hat{N} - \hat{M} - \frac{1}{4}(L-1)(L-2)\nabla_\mu\hat{S}\ \nabla_\nu\hat{S}^{\mu\nu}$$
$$+\frac{1}{2}L(L-1)\nabla_\mu\hat{S}\ \nabla_\nu\hat{S}^\nu\hat{K}^\mu - \frac{1}{2}\nabla_\mu\hat{S}\ \nabla_\nu\hat{S}\Delta^{\mu\nu} + \frac{1}{3}\Big((L-1)\nabla_\mu\hat{S}^\mu\hat{S}^2$$
$$- L\nabla_\mu\hat{S}\ \hat{K}^\mu\hat{S}^2 - (L-1)\nabla_\mu\hat{S}\ \hat{S}\ \hat{S}^\mu + L\ \nabla_\mu\hat{S}\ \hat{S}^2\hat{K}^\mu\Big) - L\nabla_\mu\hat{S}\hat{W}\hat{K}^\mu$$
$$+(L-2)\nabla_\mu\hat{S}\hat{W}^\mu; \tag{29}$$

$$WR = -\ \frac{1}{2}L^2\hat{W}\ \hat{F}_{\mu\nu}(Kn)^\mu\hat{K}^\nu + \frac{1}{3}L\ \hat{W}\ \hat{K}^\alpha\Delta^{\mu\nu}n_\sigma R^\sigma{}_{\mu\alpha\nu} + \frac{1}{3}L^2(L-1)$$
$$\times\hat{W}\ \hat{K}^{\mu\nu}\hat{K}^\alpha n_\sigma R^\sigma{}_{\mu\alpha\nu} - \frac{1}{6}(L-2)(L-3)\hat{W}^{\mu\nu}R_{\mu\nu}; \tag{30}$$

$$SR = -\ \frac{1}{6}L^2(L-1)\hat{S}\ \nabla_\mu\hat{F}_{\alpha\nu}(Kn)^{\mu\nu}\hat{K}^\alpha + \frac{2}{3}L\hat{S}\ \nabla_\mu\hat{F}_{\nu\alpha}(Kn)^\alpha\Delta^{\mu\nu}$$
$$-\frac{1}{12}(L-1)(L-2)(L-3)\hat{S}^{\alpha\mu\nu}\nabla_\alpha R_{\mu\nu} - \frac{1}{12}L^2(L-1)(L-2)\hat{S}\hat{K}^{\mu\nu\alpha}\hat{K}^\beta$$
$$\times n_\sigma\nabla_\alpha R^\sigma{}_{\mu\beta\nu} + L(L-1)\hat{S}\ \hat{K}^{\mu\nu}\Delta^{\alpha\beta}n_\sigma\left(\frac{5}{12}\nabla_\alpha R^\sigma{}_{\nu\beta\mu} - \frac{1}{12}\nabla_\mu R^\sigma{}_{\alpha\nu\beta}\right)$$
$$-\frac{1}{2}L\ \hat{S}\ \hat{K}^\beta\Delta^{\mu\nu\alpha}n_\sigma\nabla_\alpha R^\sigma{}_{\mu\beta\nu}; \tag{31}$$

$$SSR = -\frac{1}{2}L(L-1)\hat{S}\hat{S}^\mu\hat{F}_{\mu\nu}\hat{K}^\nu + \frac{1}{2}L^2\hat{S}^2\hat{F}_{\mu\nu}(Kn)^\mu\hat{K}^\nu + \frac{1}{6}\hat{S}^2\Delta^{\mu\nu}R_{\mu\nu}$$
$$+\frac{1}{3}L(L-1)\hat{S}\hat{S}^\mu\hat{K}^\nu R_{\mu\nu} + \frac{1}{12}(L-1)(L-2)\hat{S}\hat{S}^{\mu\nu}R_{\mu\nu} - \frac{1}{3}L^2(L-1)\hat{S}^2$$
$$\times\hat{K}^{\mu\nu}\hat{K}^\alpha n_\sigma R^\sigma{}_{\mu\alpha\nu} - \frac{1}{6}L(L-1)(L-2)\hat{S}\hat{S}^{\mu\nu}\hat{K}^\alpha n_\sigma R^\sigma{}_{\mu\alpha\nu} + \frac{1}{3}(L-1)\hat{S}$$
$$\times\hat{S}^\alpha\Delta^{\mu\nu}n_\sigma R^\sigma{}_{\mu\alpha\nu} - \frac{1}{3}L\hat{S}^2\hat{K}^\alpha\Delta^{\mu\nu}n_\sigma R^\sigma{}_{\mu\alpha\nu}; \tag{32}$$

$$FF = -\ \frac{1}{24}L^2(L-1)^2\hat{K}^{\mu\nu}F_{\mu\alpha}\hat{K}^{\alpha\beta}F_{\nu\beta} + \frac{1}{24}L^2\hat{K}^\mu F_{\beta\nu}\Delta^{\alpha\beta}\hat{K}^\nu F_{\alpha\mu} - \frac{5}{24}L^2$$
$$\times\hat{K}^\mu F_{\beta\mu}\Delta^{\alpha\beta}\hat{K}^\nu F_{\alpha\nu} - \frac{1}{48}L^2(L-1)\hat{K}^\mu F_{\beta\nu}\Delta^\nu\hat{K}^{\alpha\beta}F_{\alpha\nu} - \frac{1}{48}L^2(L-1)\hat{K}^\mu$$
$$\times F_{\beta\mu}\Delta^\nu\hat{K}^{\alpha\beta}F_{\alpha\nu}; \tag{33}$$

$$FR = \frac{1}{40}L^2(L-1)(L-2)\Delta^\mu \hat{K}^\nu \hat{K}^{\alpha\beta\gamma} F_{\mu\alpha} n_\sigma R^\sigma{}_{\gamma\beta\nu} - L^2(L-1)(L-2)$$
$$\times \Delta^\nu \hat{K}^{\alpha\beta\gamma} \hat{K}^\mu n_\sigma \left(\frac{1}{60} R^\sigma{}_{\beta\gamma\mu} F_{\alpha\nu} + \frac{1}{12} R^\sigma{}_{\beta\gamma\nu} F_{\alpha\mu} \right) + L^2(L-1)^2 \Delta^\alpha \hat{K}^{\beta\gamma} \hat{K}^{\mu\nu}$$
$$\times n_\sigma \left(\frac{1}{60} R^\sigma{}_{\beta\mu\gamma} F_{\alpha\nu} + \frac{1}{20} R^\sigma{}_{\alpha\mu\gamma} F_{\nu\beta} + \frac{1}{15} R^\sigma{}_{\gamma\mu\alpha} F_{\nu\beta} + \frac{1}{60} R^\sigma{}_{\mu\nu\gamma} F_{\alpha\beta} \right) + L^2$$
$$\times (L-1) \Delta^{\alpha\beta} \hat{K}^{\gamma\delta} \hat{K}^\mu n_\sigma \left(\frac{4}{15} R^\sigma{}_{\delta\beta\gamma} F_{\alpha\mu} - \frac{1}{30} R^\sigma{}_{\beta\delta\alpha} F_{\gamma\mu} - \frac{1}{15} R^\sigma{}_{\alpha\gamma\mu} F_{\beta\delta} \right.$$
$$\left. - \frac{1}{30} R^\sigma{}_{\gamma\alpha\mu} F_{\beta\delta} \right) + L^2(L-1) \Delta^{\alpha\beta} \hat{K}^\gamma \hat{K}^{\mu\nu} n_\sigma \left(\frac{7}{60} R^\sigma{}_{\alpha\beta\mu} F_{\gamma\nu} - \frac{11}{60} R^\sigma{}_{\beta\mu\gamma} \right.$$
$$\left. \times F_{\alpha\nu} + \frac{1}{5} R^\sigma{}_{\mu\alpha\gamma} F_{\beta\nu} + \frac{1}{60} R^\sigma{}_{\mu\alpha\nu} F_{\gamma\beta} \right) + L^2 \Delta^{\mu\alpha\beta} \hat{K}^\gamma \hat{K}^\nu n_\sigma \left(\frac{7}{20} R^\sigma{}_{\alpha\gamma\beta} F_{\nu\mu} \right.$$
$$\left. + \frac{1}{10} R^\sigma{}_{\alpha\beta\nu} F_{\gamma\mu} \right); \tag{34}$$

$$RR = \frac{1}{10} L^2 \hat{K}^\delta \Delta^{\mu\nu\alpha\beta} \hat{K}^\gamma n_\sigma n_\rho R^\sigma{}_{\alpha\beta\gamma} R^\rho{}_{\mu\nu\delta} + L^2(L-1)^2(L-2) \hat{K}^{\beta\gamma\delta} \Delta^\alpha$$
$$\times \hat{K}^{\mu\nu} n_\sigma n_\rho \left(\frac{2}{45} R^\rho{}_{\alpha\delta\nu} R^\sigma{}_{\beta\mu\gamma} - \frac{1}{120} R^\rho{}_{\delta\alpha\nu} R^\sigma{}_{\beta\mu\gamma} \right) + L^2(L-1) \hat{K}^\delta \Delta^{\alpha\beta\gamma} \hat{K}^{\mu\nu}$$
$$\times n_\sigma n_\rho \left(-\frac{1}{10} R^\rho{}_{\mu\gamma\nu} R^\sigma{}_{\alpha\delta\beta} + \frac{1}{15} R^\rho{}_{\delta\alpha\nu} R^\sigma{}_{\beta\mu\gamma} + \frac{1}{60} R^\rho{}_{\beta\delta\nu} R^\sigma{}_{\gamma\mu\alpha} \right) + L^2$$
$$\times (L-1)^2 \hat{K}^{\gamma\delta} \Delta^{\alpha\beta} \hat{K}^{\mu\nu} n_\sigma n_\rho \left(-\frac{1}{20} R^\rho{}_{\mu\beta\nu} R^\sigma{}_{\delta\alpha\gamma} + \frac{1}{180} R^\rho{}_{\alpha\nu\beta} R^\sigma{}_{\gamma\delta\mu} - \frac{7}{360} \right.$$
$$\left. \times R^\rho{}_{\mu\gamma\nu} R^\sigma{}_{\alpha\delta\beta} - \frac{1}{240} R^\rho{}_{\delta\beta\nu} R^\sigma{}_{\gamma\alpha\mu} - \frac{1}{120} R^\rho{}_{\beta\gamma\nu} R^\sigma{}_{\alpha\delta\mu} - \frac{1}{30} R^\rho{}_{\delta\beta\nu} R^\sigma{}_{\alpha\gamma\mu} \right)$$
$$+ L^2(L-1)(L-2)\ \hat{K}^\delta \Delta^{\mu\nu} \hat{K}^{\alpha\beta\gamma} n_\sigma n_\rho \left(-\frac{1}{30} R^\rho{}_{\gamma\nu\beta} R^\sigma{}_{\alpha\delta\mu} - \frac{1}{180} R^\rho{}_{\mu\gamma\nu} \right.$$
$$\left. \times R^\sigma{}_{\alpha\beta\delta} + \frac{1}{180} R^\rho{}_{\mu\gamma\delta} R^\sigma{}_{\alpha\beta\nu} \right) + L^2(L-1)^2(L-2)\ \hat{K}^{\mu\nu} \Delta^\delta \hat{K}^{\alpha\beta\gamma} n_\sigma n_\rho$$
$$\times \left(\frac{1}{45} R^\rho{}_{\mu\gamma\nu} R^\sigma{}_{\alpha\beta\delta} - \frac{1}{80} R^\rho{}_{\beta\nu\gamma} R^\sigma{}_{\mu\alpha\delta} + \frac{1}{90} R^\rho{}_{\beta\nu\gamma} R^\sigma{}_{\delta\alpha\mu} \right) + L^2(L-1) \hat{K}^{\mu\nu}$$
$$\times \Delta^{\alpha\beta\gamma} \hat{K}^\delta n_\sigma n_\rho \left(\frac{7}{120} R^\rho{}_{\beta\gamma\nu} R^\sigma{}_{\mu\alpha\delta} - \frac{3}{40} R^\rho{}_{\beta\gamma\delta} R^\sigma{}_{\mu\alpha\nu} + \frac{1}{120} R^\rho{}_{\delta\gamma\nu} R^\sigma{}_{\alpha\beta\mu} \right)$$
$$+ L^2(L-1)(L-2) \hat{K}^{\alpha\beta\gamma} \Delta^{\mu\nu} \hat{K}^\delta n_\sigma n_\rho \left(-\frac{1}{24} R^\rho{}_{\mu\gamma\nu} R^\sigma{}_{\alpha\beta\delta} - \frac{1}{180} R^\rho{}_{\nu\gamma\delta} \right.$$

$$\times R^{\sigma}{}_{\alpha\beta\mu} - \frac{1}{360} R^{\rho}{}_{\delta\gamma\nu} R^{\sigma}{}_{\alpha\beta\mu}\Big) - \frac{1}{120} L^2(L-1)(L-2)(L-3)\hat{K}^{\mu\nu\alpha\beta}\Delta^{\delta}$$
$$\times \hat{K}^{\gamma} n_\sigma n_\rho R^{\rho}{}_{\alpha\beta\gamma} R^{\sigma}{}_{\mu\nu\delta} - \frac{1}{80} L^2(L-1)^2(L-2)(L-3)\hat{K}^{\alpha\beta\gamma\delta}\hat{K}^{\mu\nu} n_\sigma n_\rho$$
$$\times R^{\rho}{}_{\beta\gamma\mu} R^{\sigma}{}_{\alpha\delta\nu} + L^2 \hat{K}^{\mu}\Delta^{\alpha\beta\gamma}\hat{K}^{\nu} n_\rho \Big(-\frac{1}{8} R_{\beta\gamma} R^{\rho}{}_{\nu\alpha\mu} + \frac{3}{20} R_{\beta\gamma} R^{\rho}{}_{\mu\alpha\nu} + \frac{3}{40}$$
$$\times R_{\alpha\mu} R^{\rho}{}_{\beta\gamma\nu} + \frac{1}{40} R^{\sigma}{}_{\beta\gamma\mu} R^{\rho}{}_{\nu\alpha\sigma} - \frac{3}{20} R^{\sigma}{}_{\alpha\beta\mu} R^{\rho}{}_{\gamma\nu\sigma} + \frac{1}{10} R^{\sigma}{}_{\alpha\beta\nu} R^{\rho}{}_{\gamma\mu\sigma}\Big)$$
$$+ L^2(L-1)\hat{K}^{\gamma}\Delta^{\alpha\beta}\hat{K}^{\mu\nu} n_\rho \Big(\frac{1}{20} R_{\alpha\nu} R^{\rho}{}_{\gamma\beta\mu} + \frac{1}{20} R_{\alpha\gamma} R^{\rho}{}_{\mu\beta\nu} + \frac{1}{10} R_{\alpha\beta} R^{\rho}{}_{\mu\gamma\nu}$$
$$+ \frac{1}{20} R^{\sigma}{}_{\alpha\nu\gamma} R^{\rho}{}_{\sigma\beta\mu} - \frac{1}{60} R^{\sigma}{}_{\mu\alpha\nu} R^{\rho}{}_{\beta\sigma\gamma} + \frac{1}{10} R^{\sigma}{}_{\alpha\beta\gamma} R^{\rho}{}_{\mu\sigma\nu} - \frac{1}{12} R^{\sigma}{}_{\alpha\beta\nu} R^{\rho}{}_{\mu\sigma\gamma}\Big)$$
$$+ L^2(L-1)^2\hat{K}^{\alpha\beta}\Delta^{\gamma}\hat{K}^{\mu\nu} n_\rho \Big(\frac{1}{60} R_{\alpha\mu} R^{\rho}{}_{\beta\nu\gamma} - \frac{1}{20} R_{\alpha\mu} R^{\rho}{}_{\gamma\nu\beta} + \frac{1}{120} R_{\alpha\beta}$$
$$\times R^{\rho}{}_{\mu\nu\gamma} + \frac{3}{40} R_{\alpha\gamma} R^{\rho}{}_{\nu\beta\mu} + \frac{1}{20} R^{\sigma}{}_{\gamma\mu\alpha} R^{\rho}{}_{\nu\sigma\beta} + \frac{1}{120} R^{\sigma}{}_{\alpha\mu\gamma} R^{\rho}{}_{\beta\nu\sigma} - \frac{1}{40} R^{\sigma}{}_{\alpha\mu\gamma}$$
$$\times R^{\rho}{}_{\sigma\nu\beta} + \frac{1}{40} R^{\sigma}{}_{\alpha\mu\beta} R^{\rho}{}_{\sigma\nu\gamma} - \frac{1}{20} R^{\sigma}{}_{\alpha\mu\beta} R^{\rho}{}_{\gamma\nu\sigma} - \frac{1}{40} R^{\sigma}{}_{\mu\beta\nu} R^{\rho}{}_{\gamma\sigma\alpha}\Big) + L^2$$
$$\times (L-1)\hat{K}^{\alpha\beta}\Delta^{\mu\nu}\hat{K}^{\gamma} n_\rho \Big(\frac{1}{20} R^{\sigma}{}_{\mu\nu\beta} R^{\rho}{}_{\gamma\sigma\alpha} - \frac{7}{60} R^{\sigma}{}_{\beta\mu\alpha} R^{\rho}{}_{\gamma\nu\sigma} + \frac{1}{20} R^{\sigma}{}_{\beta\mu\alpha}$$
$$\times R^{\rho}{}_{\sigma\nu\gamma} + \frac{1}{10} R^{\sigma}{}_{\mu\beta\gamma} R^{\rho}{}_{\nu\alpha\sigma} + \frac{1}{60} R^{\sigma}{}_{\beta\mu\gamma} R^{\rho}{}_{\alpha\nu\sigma} + \frac{7}{120} R_{\alpha\beta} R^{\rho}{}_{\nu\gamma\mu} + \frac{11}{60} R_{\beta\mu}$$
$$\times R^{\rho}{}_{\nu\alpha\gamma}\Big) + L^2(L-1)(L-2)\hat{K}^{\alpha\beta\gamma}\Delta^{\mu}\hat{K}^{\nu} n_\rho \Big(\frac{7}{240} R_{\alpha\beta} R^{\rho}{}_{\gamma\mu\nu} + \frac{7}{240} R_{\alpha\nu}$$
$$\times R^{\rho}{}_{\beta\gamma\mu} - \frac{1}{60} R_{\alpha\mu} R^{\rho}{}_{\beta\gamma\nu} - \frac{1}{24} R^{\sigma}{}_{\alpha\beta\nu} R^{\rho}{}_{\sigma\gamma\mu} + \frac{1}{15} R^{\sigma}{}_{\alpha\beta\nu} R^{\rho}{}_{\mu\gamma\sigma} + \frac{1}{40} R^{\sigma}{}_{\alpha\beta\mu}$$
$$\times R^{\rho}{}_{\sigma\gamma\nu} + \frac{1}{40} R_{\beta\gamma} R^{\rho}{}_{\nu\mu\alpha} + \frac{1}{48} R^{\sigma}{}_{\beta\gamma\mu} R^{\rho}{}_{\nu\alpha\sigma}\Big) + L^2(L-1)^2(L-2)\hat{K}^{\alpha\beta\gamma}$$
$$\times \hat{K}^{\mu\nu} n_\rho \Big(-\frac{7}{240} R_{\alpha\mu} R^{\rho}{}_{\beta\gamma\nu} + \frac{1}{240} R_{\beta\gamma} R^{\rho}{}_{\mu\alpha\nu} - \frac{1}{40} R^{\sigma}{}_{\alpha\mu\beta} R^{\rho}{}_{\nu\gamma\sigma}\Big) + L(L-1)$$
$$\times (L-2)(L-3)\hat{K}^{\mu\nu\alpha\beta}\Big(\frac{1}{180} R_{\mu\nu} R_{\alpha\beta} + \frac{7}{720} R^{\sigma}{}_{\alpha\beta\rho} R^{\rho}{}_{\mu\nu\sigma}\Big), \quad (35)$$

and an exact form of $\Delta^{\mu}, \Delta^{\mu\nu}, \Delta^{\mu\nu\alpha}$ and $\Delta^{\mu\nu\alpha\beta}$ are given in (119).

We should note, that deriving (28) we omitted total derivatives, and therefore we must make the following substitutions

$$R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = 2 R_{\alpha\beta\mu\nu} R^{\alpha\mu\beta\nu} \to 4 R_{\mu\nu} R^{\mu\nu} - R^2. \quad (36)$$

Although the result is very large, the calculations of the effective action can be easily made by computers. For example, we considered particular cases (see next section) using tensor package [9] for the REDUCE analytical calculation system [10].

5 Applications

The presented algorithm can be applied for calculations of the effective action for theories regularized by higher derivatives [11] and theories in the nonminimal gauges. May be there are some other prospects. Here we consider only the simplest test examples in order to check the correctness of the method. We prove, that our results agree with the earlier known ones and each other.

5.1 Arbitrary minimal operator

An arbitrary minimal operator of order $2l$ can be considered as a particular case of a nonminimal one, if $L = 2l$ and $K^{\mu\nu\ldots\alpha}{}_i{}^j = K_0{}^{\mu\nu\ldots\alpha}\delta_i{}^j$, where K_0 was defined in (6). Substituting it to (28) and taking into account (15), we obtain (22).

So, we proved the agreement of the results for the minimal and nonminimal operators.

5.2 The minimal second order operator

If $l = 1$ the operator (7) takes the form

$$D_i{}^j = \delta_i{}^j\Box + S^\mu{}_i{}^j\nabla_\mu + W_i{}^j. \tag{37}$$

Using the following concequences of (15)

$$< 1 >= 1, \qquad < n_\mu n_\nu >= \frac{1}{4}g_{\mu\nu}, \tag{38}$$

$$< n_\mu n_\nu n_\alpha n_\beta >= \frac{1}{24}(g_{\mu\nu}g_{\alpha\beta} + g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha}),$$

it is easy to see that (20) gives then the following well-known result [7]:

$$\Gamma^{(1)}_\infty = \frac{1}{16\pi^2(d-4)}\mathrm{tr}\int d^4x\sqrt{-g}\left(\frac{1}{12}Y_{\mu\nu}Y^{\mu\nu} + \frac{1}{2}X^2 + \frac{1}{60}R_{\mu\nu}R^{\mu\nu} - \frac{1}{180}R^2\right), \quad (39)$$

where

$$\begin{aligned} Y_{\mu\nu} &= \frac{1}{2}\nabla_\mu S_\nu - \frac{1}{2}\nabla_\mu S_\nu + \frac{1}{4}S_\mu S_\nu - \frac{1}{4}S_\nu S_\mu + F_{\mu\nu}, \\ X &= W - \frac{1}{2}\nabla_\mu S^\mu - \frac{1}{4}S_\mu S^\mu + \frac{1}{6}R. \end{aligned} \quad (40)$$

5.3 The minimal forth order operator

A minimal forth order operator ($l = 2$) has the form

$$D_i{}^j = \delta_i{}^j\Box^2 + S^{\mu\nu\alpha}{}_i{}^j\ \nabla_\mu\nabla_\nu\nabla_\alpha + W^{\mu\nu}{}_i{}^j\ \nabla_\mu\nabla_\nu + N^\mu{}_i{}^j\ \nabla_\mu + M_i{}^j. \quad (41)$$

From (20) we obtained the result, that coincided with the one found in [3] by Barvinsky and Vilkovisky up to the total derivatives. (We do not presented it here because it is too large)

5.4 Vector field

As another example we consider the vector field operator

$$D = \Box\ \delta_\alpha{}^\beta - \lambda\ \nabla_\alpha\nabla^\beta + P_\alpha{}^\beta, \qquad \text{where} \qquad P_{\alpha\beta} = P_{\beta\alpha}. \quad (42)$$

In order to rewrite it in the form (3), we should make the second term symmetric in α and β by the commutation of covariant derivatives. Then we found, that

$$D_\alpha{}^\beta = \left(g^{\mu\nu}\delta_\alpha{}^\beta - \frac{\lambda}{2}\,(g^{\mu\beta}\delta_\alpha{}^\nu + g^{\nu\beta}\delta_\alpha{}^\mu) \right) \nabla_\mu\nabla_\nu + P_\alpha{}^\beta + \frac{\lambda}{2}\,R_\alpha{}^\beta. \quad (43)$$

So, nonzero coefficients are

$$\begin{aligned} K^{\mu\nu}{}_\alpha{}^\beta &= g^{\mu\nu}\delta_\alpha{}^\beta - \frac{\lambda}{2}\,(g^{\mu\beta}\delta_\alpha{}^\nu + g^{\nu\beta}\delta_\alpha{}^\mu); \\ W_\alpha{}^\beta &= P_\alpha{}^\beta + \frac{\lambda}{2}\,R_\alpha{}^\beta \end{aligned} \quad (44)$$

and, therefore,

$$(Kn)_\alpha{}^\beta = \delta_\alpha{}^\beta - \lambda\, n_\alpha n^\beta; \qquad (Kn)^{-1}{}_\alpha{}^\beta = \delta_\alpha{}^\beta + \gamma\, n_\alpha n^\beta, \quad (45)$$

where $\gamma = \dfrac{\lambda}{1-\lambda}$.

After substituting it to (28) we found the following result

$$\begin{aligned} \Gamma^{(1)}_\infty = \frac{1}{16\pi^2(d-4)} \int d^4x\,\sqrt{-g}\,\, & \left(\left(\frac{1}{24}\gamma^2 + \frac{1}{4}\gamma + \frac{1}{2} \right) P_{\mu\nu}P^{\mu\nu} + \frac{1}{48}\gamma^2 P^2 \right. \\ + \left(\frac{1}{12}\gamma^2 + \frac{1}{3}\gamma \right) R_{\mu\nu}P^{\mu\nu} & + \left(\frac{1}{24}\gamma^2 + \frac{1}{12}\gamma + \frac{1}{6} \right) RP + \left(\frac{1}{24}\gamma^2 + \frac{1}{12}\gamma \right. \\ \left. - \frac{4}{15} \right) R_{\mu\nu}R^{\mu\nu} & \left. + \left(\frac{1}{48}\gamma^2 + \frac{1}{12}\gamma + \frac{7}{60} \right) R^2 \right), \end{aligned} \quad (46)$$

that is in agreement with [3] and [12]. (Here $P \equiv P_\alpha{}^\alpha$.)

In order to check the result (28) we calculated one-loop counterterms for the squared operator (42), that is a nonminimal operator of the forth order:

$$\begin{aligned} D^2{}_\alpha{}^\beta = {} & \delta_\alpha{}^\beta\Box^2 - \lambda\nabla_\alpha\nabla^\beta\Box + 2P_\alpha{}^\beta\Box - \lambda\Box\nabla_\alpha\nabla^\beta + \lambda^2\nabla_\alpha\Box\nabla^\beta \\ & -\lambda P_\alpha{}^\mu\nabla_\mu\nabla^\beta - \lambda P_\mu{}^\beta\nabla_\alpha\nabla^\mu + (\Box P_\alpha{}^\beta) + 2(\nabla_\mu P_\alpha{}^\beta)\nabla^\mu - \lambda(\nabla_\alpha\nabla^\mu P_\mu{}^\beta) \\ & -\lambda(\nabla_\alpha P_\mu{}^\beta)\nabla^\mu - \lambda(\nabla_\mu P^{\mu\beta})\nabla^\alpha + P_\alpha{}^\mu P_\mu{}^\beta. \end{aligned} \quad (47)$$

By commuting covariant derivatives it can be rewritten in the form (3), where

$$K^{\mu\nu\gamma\delta}{}_{\alpha}{}^{\beta} = \delta_{\alpha}{}^{\beta}\frac{1}{3}\left(g^{\mu\nu}g^{\gamma\delta} + g^{\mu\gamma}g^{\nu\delta} + g^{\mu\delta}g^{\nu\gamma}\right) + \frac{1}{12}(-2\lambda + \lambda^2)\Big(g^{\mu\nu}\delta_{\alpha}{}^{\gamma}g^{\beta\delta}$$
$$+ g^{\mu\nu}\delta_{\alpha}{}^{\delta}g^{\beta\gamma} + g^{\mu\gamma}\delta_{\alpha}{}^{\nu}g^{\beta\delta} + g^{\mu\gamma}\delta_{\alpha}{}^{\delta}g^{\beta\nu} + g^{\mu\delta}\delta_{\alpha}{}^{\nu}g^{\beta\gamma} + g^{\mu\delta}\delta_{\alpha}{}^{\gamma}g^{\beta\nu} + g^{\nu\gamma}\delta_{\alpha}{}^{\mu}$$
$$\times g^{\beta\delta} + g^{\nu\gamma}\delta_{\alpha}{}^{\delta}g^{\beta\mu} + g^{\nu\delta}\delta_{\alpha}{}^{\mu}g^{\beta\gamma} + g^{\nu\delta}\delta_{\alpha}{}^{\gamma}g^{\beta\mu} + g^{\gamma\delta}\delta_{\alpha}{}^{\mu}g^{\beta\nu} + g^{\gamma\delta}\delta_{\alpha}{}^{\nu}g^{\beta\mu}\Big); \tag{48}$$

$$S^{\mu\nu\gamma}{}_{\alpha}{}^{\beta} = 0; \tag{49}$$

$$W^{\mu\nu}{}_{\alpha}{}^{\beta} = 2P_{\alpha}{}^{\beta}g^{\mu\nu} - \frac{\lambda}{2}P_{\alpha}{}^{\mu}g^{\nu\beta} - \frac{\lambda}{2}P_{\alpha}{}^{\nu}g^{\mu\beta} - \frac{\lambda}{2}P^{\beta\mu}\delta_{\alpha}{}^{\nu} - \frac{\lambda}{2}P^{\beta\nu}\delta_{\alpha}{}^{\mu}$$
$$-\frac{2}{3}R^{\mu\nu}\delta_{\alpha}{}^{\beta}; +\frac{1}{6}(\lambda - 2\lambda^2)\left(R_{\alpha}{}^{\mu}g^{\nu\beta} + R_{\alpha}{}^{\nu}g^{\mu\beta} + R^{\beta\mu}\delta_{\alpha}{}^{\nu} + R^{\beta\nu}\delta_{\alpha}{}^{\mu}\right)$$
$$+\frac{1}{6}(2\lambda - \lambda^2)\left(R_{\alpha}{}^{\mu\beta\nu} + R_{\alpha}{}^{\nu\beta\mu}\right) + \frac{1}{2}(2\lambda - \lambda^2)\ g^{\mu\nu}R_{\alpha}{}^{\beta} \tag{50}$$

$$M_{\alpha}{}^{\beta} = P_{\alpha\mu}P^{\mu\beta} + \frac{\lambda}{2}P_{\alpha\mu}R^{\mu\beta} + \frac{\lambda}{2}P_{\mu\nu}R^{\mu}{}_{\alpha}{}^{\nu\beta} + \frac{1}{4}(2\lambda - \lambda^2)R_{\alpha\mu\nu\gamma}R^{\gamma\mu\nu\beta}$$
$$+\frac{1}{12}(4\lambda + 7\lambda^2)\ R_{\mu\alpha\nu}{}^{\beta}R^{\mu\nu} + \frac{1}{6}(\lambda - 2\lambda^2)R_{\alpha\mu}R^{\mu\beta} - \frac{1}{2}R_{\mu\nu\gamma\alpha}R^{\mu\nu\gamma\beta}. \tag{51}$$

It is easy to see that in this case $(Kn)^{-1}{}_{\alpha}{}^{\beta} = \delta_{\alpha}{}^{\beta} + (2\gamma + \gamma^2)\ n_{\alpha}n^{\beta}$. Substituting this expressions to (28) we obtain by explicit calculation that an identity $\mathrm{tr}\ \ln\ D^2 = 2\ \mathrm{tr}\ \ln\ D$ is satisfied.

5.5 Gravity theory. The λ-family of gauge conditions

The gravitational field is described by the action

$$S = \int d^4x\ \sqrt{-g}\ R, \tag{52}$$

Its second variation can be found by making a substitution $g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}$ and retaining terms quadratic in $h_{\mu\nu}$. Due to the invariance under the general coordinate transformations $x^{\mu} \to x^{\mu} + \xi^{\mu}$ $(Kn)_{\alpha\beta}{}^{\mu\nu}$ has a zero mode

$$h_{\mu\nu} \to h_{\mu\nu} + n_\mu \xi_\nu + n_\nu \xi_\mu, \tag{53}$$

that should be deleted by adding to the action gauge fixing terms

$$S_{gf} = -\frac{1}{2} \int d^4x \sqrt{-g}\, g_{\mu\nu} \chi^\mu \chi^\nu, \tag{54}$$

where

$$\chi^\mu = \frac{1}{\sqrt{1+\lambda}} \left(g^{\mu\alpha} \nabla^\beta h_{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} \nabla^\mu h_{\alpha\beta} \right). \tag{55}$$

Then the second variation of the action takes the form

$$\frac{\delta^2 S}{\delta h_{\alpha\beta}\, \delta h_{\mu\nu}} = \sqrt{-g}\, C^{\alpha\beta,\gamma\delta} \Bigg(\delta^{\mu\nu}_{\gamma\delta} \Box + \frac{\lambda}{2(1+\lambda)} g^{\mu\nu} \left(\nabla_\gamma \nabla_\delta + \nabla_\delta \nabla_\gamma \right)$$
$$- \frac{\lambda}{2(1+\lambda)} \left(\delta^\mu_\gamma \nabla_\delta \nabla^\nu + \delta^\nu_\gamma \nabla_\delta \nabla^\mu + \delta^\mu_\delta \nabla_\gamma \nabla^\nu + \delta^\nu_\delta \nabla_\gamma \nabla^\mu \right) + P_{\gamma\delta}{}^{\mu\nu} \Bigg), \tag{56}$$

where

$$\delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2} \left(\delta^\mu_\alpha \delta^\nu_\beta + \delta^\nu_\alpha \delta^\mu_\beta \right); \qquad C^{\alpha\beta,\gamma\delta} = \frac{1}{4} \left(g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma} - g^{\alpha\beta} g^{\gamma\delta} \right);$$
$$P_{\gamma\delta}{}^{\mu\nu} = R_\gamma{}^\mu{}_\delta{}^\nu + R_\gamma{}^\nu{}_\delta{}^\mu + \frac{1}{2} \left(\delta^\mu_\gamma R_\delta{}^\nu + \delta^\nu_\gamma R_\delta{}^\mu + \delta^\mu_\delta R_\gamma{}^\nu + \delta^\nu_\delta R_\gamma{}^\mu \right) - g^{\mu\nu} R_{\gamma\delta}$$
$$- g_{\gamma\delta} R^{\mu\nu} - \delta^{\mu\nu}_{\gamma\delta} R + \frac{1}{2} g_{\gamma\delta} g^{\mu\nu} R. \tag{57}$$

tr ln $(\sqrt{-g}\, C^{\alpha\beta,\mu\nu})$ gives zero contribution in the dimensional regularization. So, in this case

$$D_{\alpha\beta}{}^{\mu\nu} = \delta^{\mu\nu}_{\gamma\delta} \Box - \frac{\lambda}{2(1+\lambda)} \left(\delta^\mu_\gamma \nabla_\delta \nabla^\nu + \delta^\nu_\gamma \nabla_\delta \nabla^\mu + \delta^\mu_\delta \nabla_\gamma \nabla^\nu + \delta^\nu_\delta \nabla_\gamma \nabla^\mu \right)$$
$$+ \frac{\lambda}{2(1+\lambda)} g^{\mu\nu} \left(\nabla_\gamma \nabla_\delta + \nabla_\delta \nabla_\gamma \right) + P_{\gamma\delta}{}^{\mu\nu}. \tag{58}$$

and

$$K^{\mu\nu}{}_{\alpha\beta}{}^{\gamma\delta} = g^{\mu\nu}\delta_{\alpha\beta}{}^{\gamma\delta} - \frac{\lambda}{4(1+\lambda)}\Big(\delta_\alpha{}^\gamma\delta_\beta{}^\mu g^{\delta\nu} + \delta_\alpha{}^\gamma\delta_\beta{}^\nu g^{\delta\mu} + \delta_\alpha{}^\delta\delta_\beta{}^\mu g^{\gamma\nu}$$

$$+\,\delta_\alpha{}^\delta\delta_\beta{}^\nu g^{\gamma\mu} + \delta_\beta{}^\gamma\delta_\alpha{}^\mu g^{\delta\nu} + \delta_\beta{}^\gamma\delta_\alpha{}^\nu g^{\delta\mu} + \delta_\beta{}^\delta\delta_\alpha{}^\mu g^{\gamma\nu} + \delta_\beta{}^\delta\delta_\alpha{}^\nu g^{\gamma\mu}\Big)$$

$$+\frac{\lambda}{2(1+\lambda)}g^{\gamma\delta}\left(\delta_\alpha{}^\mu\delta_\beta{}^\nu + \delta_\alpha{}^\nu\delta_\beta{}^\mu\right); \tag{59}$$

$$W_{\alpha\beta}{}^{\gamma\delta} = P_{\alpha\beta}{}^{\gamma\delta} - \frac{\lambda}{2(1+\lambda)}\Big(R_\alpha{}^\gamma{}_\beta{}^\delta + R_\alpha{}^\delta{}_\beta{}^\gamma\Big) + \frac{\lambda}{4(1+\lambda)}\Big(\delta_\alpha{}^\gamma R_\beta{}^\delta$$

$$+\delta_\alpha{}^\delta R_\beta{}^\gamma + \delta_\beta{}^\gamma R_\alpha{}^\delta + \delta_\beta{}^\delta R_\alpha{}^\gamma\Big); \tag{60}$$

$$(Kn)^{-1}{}_{\alpha\beta}{}^{\gamma\delta} = \delta^{\gamma\delta}_{\alpha\beta} + \frac{\lambda}{2}\Big(\delta_\alpha{}^\gamma n_\beta n^\delta + \delta_\alpha{}^\delta n_\beta n^\gamma + \delta_\beta{}^\gamma n_\alpha n^\delta + \delta_\beta{}^\delta n_\alpha n^\gamma\Big)$$

$$-\lambda g^{\gamma\delta}n_\alpha n_\beta. \tag{61}$$

The effective action is calculated by (28). For an arbitrary $P_{\mu\nu}{}^{\alpha\beta}$, symmetric in each pair of indexes, we found the following result:

$$\frac{1}{16\pi^2(d-4)}\int d^4x\sqrt{-g}\ \Big(\frac{1}{48}\lambda^2(P^\mu{}_{\mu\nu}{}^\nu)^2 + (-4\lambda^2-12\lambda)\ P_{\mu\nu\alpha}{}^\alpha P_\beta{}^{\beta\mu\nu}$$

$$+\,(4\lambda^2+24\lambda+24)P_{\alpha\beta\mu\nu}P^{\mu\nu\alpha\beta} + 4\lambda^2 P_{\mu\nu\alpha\beta}P^{\nu\beta\mu\alpha} + 4\lambda^2 P_{\mu\alpha\nu}{}^\alpha P_\beta{}^{\nu\beta\mu}$$

$$+\,2\lambda^2 P^\alpha{}_{\alpha\mu\nu}P_\beta{}^{\beta\mu\nu} - 8\lambda^2 P_{\mu\alpha\nu}{}^\alpha P_\beta{}^{\beta\mu\nu} + (4\lambda^2+8\lambda+8)P_{\mu\nu}{}^{\mu\nu}R + (-4\lambda^2$$

$$-\,4\lambda)P^\mu{}_{\mu\nu}{}^\nu R + 4\lambda^2 P^{\mu\nu\alpha}{}_\alpha R_{\mu\nu} + (8\lambda^2+32\lambda)P_\alpha{}^{\mu\alpha\nu}R_{\mu\nu} + (-4\lambda^2+8\lambda)$$

$$\times P_\alpha{}^{\alpha\mu\nu}R_{\mu\nu} + (-8\lambda^2-48\lambda)P_{\mu\nu\alpha\beta}R^{\mu\alpha\nu\beta} + (44\lambda^2+32\lambda-88)\ R_{\mu\nu}R^{\mu\nu}$$

$$+\,(-4\lambda^2+24\lambda+28)\ R^2\Big). \tag{62}$$

Substituting here the expression (57) for $P_{\mu\nu}{}^{\alpha\beta}$ we obtain

$$\frac{1}{16\pi^2(d-4)}\int d^4x\sqrt{-g}\Big(\frac{1}{6}\Big(4\lambda^2+4\lambda+7\Big)R^{\mu\nu}R_{\mu\nu}$$

$$+\frac{1}{12}\left(4\lambda^2+7\right)R^2\Big). \quad (63)$$

Moreover, we should add the contribution of Faddeev-Popov ghosts with the Lagrangian density

$$L_{gh}=\bar{c}^{\alpha}\left(\delta^{\alpha}{}_{\beta}\nabla^{\mu}\nabla_{\mu}+R^{\alpha}{}_{\beta}\right)c^{\beta}. \quad (64)$$

It can be easily found by (22). The result is

$$-2\times\frac{1}{16\pi^2(d-4)}\int d^4x\,\sqrt{-g}\,\left(\frac{7}{30}R_{\mu\nu}R^{\mu\nu}+\frac{17}{60}R^2\right). \quad (65)$$

The sum of (63) and (65) gives the ultimate expression for the divergent part of the one-loop effective action

$$\Gamma^{(1)}_{\infty}=\frac{1}{16\pi^2(d-4)}\int d^4x\,\sqrt{-g}\,\Big(\frac{1}{60}\left(20\lambda^2+1\right)R^2$$
$$+\frac{1}{30}\left(20\lambda^2+20\lambda+21\right)R^{\mu\nu}R_{\mu\nu}\Big), \quad (66)$$

that is in agreement with the results, found in [13] and [3].

6 Conclusion

In this paper we calculate the divergent part of the one-loop effective action for an arbitrary (minimal and nonminimal) operators without any restrictions to their form and order in the curved space-time, using t'Hooft-Veltman diagram technique [7, 14, 15].

Actually, we made some operations, that encounter in calculating Feynman diagrams, namely, integration over a loop momentum, summation of all divergent graphs and obtaining a manifestly covariant result by its weak field limit in the general case. Then, in order to calculate the divergent part of the effective action, one should only substitute the explicit expression for second variation of an action.

Unfortunately, the master formula is very large and can hardly be used for calculations without computers. Nevertheless, on the base of the general algorithm we obtained one-loop counterterms for some examples, namely, an arbitrary minimal operator, the vector field operator and the gravity theory in the λ-gauge. Our results were in agreement with the ones found earlier. The calculations were made by the tensor package for the REDUCE analytical calculation system [9]. For the considered examples the required memory and execution time were the following (we used IBM-486/DX-2/66/8Mb):

1. Calculation of the RR contribution for an arbitrary minimal operator by (35) took 17 seconds. In this case the required memory was about 500 kb.

2. For the vector field operator and its square execution time was 174 and 515 seconds respectively. Required memory was 2 Mb.

3. The calculation of the one-loop counterterms for the gravity theory in the λ-gauge took 170 minutes while the required memory was 8 Mb.

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Appendix

A Divergent diagrams calculation for a minimal operator

Let us consider first logarithmically divergent graphs (1a)-(1e). In order to find the divergent part of the diagram in this case we should retain only terms without external momentums and perform the remaining integration. As an example let us calculate a diagram (1.d).

$$(1.d) = \frac{i}{2(2\pi)^4}\mathrm{tr}\int d^dk\ \frac{(Sk)\ (N(k-p))}{k^{2l}(k-p)^{2l}}. \tag{67}$$

Using the method described above, we can easily conclude that

$$(1.d)_\infty = \frac{i}{2(2\pi)^4}\ \int d^dk\ \frac{1}{k^4}\mathrm{tr} < \hat{S}\ \hat{N} >= \frac{1}{16\pi^2(d-4)}\mathrm{tr} < \hat{S}\ \hat{N} >, \tag{68}$$

where

$$\begin{aligned} \hat{S} &= (Sn) = S^{\mu\nu\ldots\alpha}n_\mu n_\nu \ldots n_\alpha; \\ \hat{N} &= (Nn) = N^{\mu\nu\ldots\alpha}n_\mu n_\nu \ldots n_\alpha. \end{aligned} \tag{69}$$

and $n_\mu = k_\mu/\sqrt{k^\alpha k_\alpha}$ is a unit vector. In a similar fashion we have

$$\begin{aligned} (1.a)_\infty &= \frac{1}{64\pi^2(d-4)}\mathrm{tr} < \hat{S}^4 >; \\ (1.b)_\infty &= -\ \frac{1}{16\pi^2(d-4)}\mathrm{tr} < \hat{W}\ \hat{S}^2 >; \\ (1.c)_\infty &= \frac{1}{32\pi^2(d-4)}\mathrm{tr} < \hat{W}^2 >; \\ (1.e)_\infty &= -\ \frac{1}{16\pi^2(d-4)}\mathrm{tr} < \hat{M} > . \end{aligned} \tag{70}$$

The calculation of linearly divergent graphs is a bit more difficult. For example, in order to find the divergent part of the diagram

$$(1.f) = \frac{i}{2(2\pi)^4}\mathrm{tr}\int d^dk\ \frac{(Sk)(W(k-p))}{k^{2l}(k-p)^{2l}} \tag{71}$$

we should retain only terms linear in external momentum p. Using the rule (109) formulated in the appendix C we obtain

$$\begin{aligned} &\frac{1}{16\pi^2(d-4)}\mathrm{tr} < 2l(pn)\hat{S}\ \hat{W} - (2l-2)p_\mu\hat{W}^\mu\hat{S} > \\ &= \frac{1}{32\pi^2(d-4)}\mathrm{tr} < (2l-1)p_\mu\hat{S}^\mu\hat{W} - (2l-2)\ p_\mu\hat{S}\ \hat{W}^\mu > . \end{aligned} \tag{72}$$

After a substitution $p_\mu \hat{S} \to -\partial_\mu \hat{S}$, the result for this diagram takes the form

$$(1.f)_\infty = \frac{1}{32\pi^2(d-4)} \mathrm{tr} < -(2l-1)\partial_\mu \hat{S}^\mu \hat{W} + (2l-2)\partial_\mu \hat{S}\ \hat{W}^\mu > . \tag{73}$$

The second linearly divergent graph

$$(1.g) = -\frac{i}{6(2\pi)^4} \mathrm{tr} \int d^d k\ \frac{\underset{(-p)}{(Sk)}\underset{(-q)}{(S(k+q))}\underset{(p+q)}{(S(k-p))}}{k^{2l}(k-p)^{2l}(k+q)^{2l}} \tag{74}$$

can be calculated in the same way. (Here indexes in the bottom point the argument of S). The result is

$$(1.g)_\infty = \frac{(2l-1)}{48\pi^2(d-4)} \mathrm{tr} < \partial_\mu \hat{S}^\mu \hat{S}\ \hat{S} - \partial_\mu \hat{S}\ \hat{S}\ \hat{S}^\mu > . \tag{75}$$

So, we should consider only the rest quadratically divergent diagram.

$$(1.h) = \frac{i}{4(2\pi)^4} \mathrm{tr} \int d^d k\ \frac{(Sk)(S(k-p))}{k^{2l}(k-p)^{2l}}. \tag{76}$$

Retaining logarithmically divergent terms we can easily find that

$$(1.h)_\infty = \frac{1}{16\pi^2(d-4)} \mathrm{tr} \int d^4x < -\frac{(2l-1)(l^2-1)}{2(2l+1)} \partial_\mu \hat{S}^{\mu\nu} \partial_\nu \hat{S}$$
$$+ \frac{l^2}{2(2l+1)} \partial_\mu \hat{S}\ \partial^\mu \hat{S} + \frac{(2l-1)^2 l}{4(2l+1)} \partial_\mu \hat{S}^\mu \partial_\nu \hat{S}^\nu > . \tag{77}$$

The calculation of graphs in the curved space is much more difficult. Here we consider as an example only the simplest group of diagrams, namely, we find a_3 and a_4 coefficients in the equation (18) by calculating graphs (2.a) and (2.b).

The vertex with $h_{\mu\nu}$ in (2a) should be found according to (9) by series expansion of $\Box^l$ to the first order and has the form

$$\sum_{m=0}^{l-1} k^{2m}(k-p)^{2l-2m-2} \left(-h^{\mu\nu} k_\mu (k-p)_\nu + \frac{1}{2} h^\alpha{}_\alpha p^\mu (k-p)_\mu \right). \tag{78}$$

Then the graph (2.a) can be written as

$$(2.a) = \frac{i}{2(2\pi)^4}\mathrm{tr}\int d^dk\ \frac{1}{k^{2l}(k-p)^{2l}}(Wk)\sum_{m=0}^{l-1} k^{2m}(k-p)^{2l-2m-2}$$
$$\times\left(-h^{\mu\nu}k_\mu(k-p)_\nu + \frac{1}{2}h^{\alpha}{}_{\alpha}p^\mu(k-p)_\mu\right). \qquad (79)$$

It is easy to see that it is quadratically divergent. So, retaining terms quadratic in external momentum p we obtain the divergent part

$$\frac{1}{16\pi^2(d-4)}\mathrm{tr} < \hat{W}\left(-\frac{l}{2}h^{\alpha}{}_{\alpha}p^2 - h^{\mu\nu}n_\mu n_\nu\left(+\frac{2}{3}l(l+1)(l+2)(n^\alpha p_\alpha)^2\right.\right.$$
$$\left.\left. -\frac{1}{2}l(l+1)p^2\right) + l(l+1)(n^\gamma p_\gamma)\left(h^{\mu\nu}n_\mu p_\nu + \frac{1}{2}h^{\alpha}{}_{\alpha}(n^\beta p_\beta)\right)\right) > . \qquad (80)$$

Using rules formulated in the appendix C, it can be written as

$$(2.a)_\infty = \frac{1}{16\pi^2(d-4)}\mathrm{tr} < -\frac{1}{12}(2l-3)(2l-4)(2l-5)\hat{W}^{\mu\nu\alpha\beta}h_{\alpha\beta}p_\mu p_\nu$$
$$-\frac{1}{12}(l-1)(2l-3)\hat{W}^{\mu\nu}(-p^2h_{\mu\nu} - p_\mu p_\nu h^{\alpha}{}_{\alpha} + 2p_\mu p_\alpha h^{\alpha}{}_{\nu}) + \frac{l}{6}\hat{W}(-h^{\alpha}{}_{\alpha}$$
$$\times p^2 + h^{\mu\nu}p_\mu p_\nu) > . \qquad (81)$$

Nevertheless, (81) can not be presented as a weak field limit of a covariant expression. The matter is that the graphs (2.a) and (2.b) can not be considered separately.

The vertex in the tadpole diagram (2.b) can be found by series expansion of $W^{\mu\nu\ldots\alpha}{}_i{}^j\nabla_\mu\nabla_\nu\ldots\nabla_\alpha$ in powers of $h_{\mu\nu}$ to the first order. Then, retaining only nontrivial contributions, we have

$$(2.b) = \frac{i}{2(2\pi)^4}\ \mathrm{tr}\int d^dk\sum_{m=0}^{2l-4}(2l-3-m)(k+p)_{\mu_1}\ldots(k+p)_{\mu_m}\Gamma^{\alpha\ (1)}_{\mu_{m+1}\mu_{m+2}}$$
$$\times k_\alpha k_{\mu_{m+3}}\ldots k_{\mu_{2l-2}}\frac{1}{k^{2l}}W^{\mu_1\mu_2\ldots\mu_{2l-2}}, \qquad (82)$$

where

$$\Gamma^{\alpha\ (1)}_{\beta\gamma} = \frac{1}{2}(p_\beta h_\gamma{}^\alpha + p_\gamma h_\beta{}^\alpha - p^\alpha h_{\beta\gamma}) \tag{83}$$

is the Cristoffel symbol in the weak field limit.

By (107) after simple transformations the divergent part of (2.b) can be written as

$$(2.b)_\infty = \frac{(2l-3)(2l-4)(2l-5)}{192\pi^2(d-4)} \mathrm{tr} < \hat{W}^{\mu\nu\alpha\beta} h_{\alpha\beta} p_\mu p_\nu > . \tag{84}$$

The sum of (81) and (84) unlike each of the graphs considered separately can be presented as a weak field approximation of a covariant expression

$$\frac{1}{16\pi^2(d-4)} \mathrm{tr} \int d^4x \sqrt{-g} < \frac{l}{6}\hat{W}R - \frac{1}{6}(l-1)(2l-3)R_{\mu\nu}\hat{W}^{\mu\nu} >, \tag{85}$$

that is the ultimate answer for the graphs (2.a) and (2.b).

B Divergent graphs calculation for nonminimal operator

The flat space divergent graphs are the same as for a minimal operator. Nevertheless, the calculations are a bit different. As earlier, we begin with the consideration of logarithmically divergent graphs. For example,

$$(1.d)_\infty = \frac{i}{2(2\pi)^4} \ \mathrm{tr} \int d^dk \ (Sk) \ (Kk)^{-1} (N(k-p)) \ (K(k-p))^{-1} \Big|_\infty$$
$$= \frac{1}{16\pi^2(d-4)} \ \mathrm{tr} < \hat{S} \ \hat{N} > . \tag{86}$$

where

$$\hat{S} \equiv (Kn)^{-1}(Sn); \qquad \hat{N} \equiv (Kn)^{-1}(Nn), \qquad \text{an so on} \tag{87}$$

(compare with (69)!)

The other logarithmically divergent graphs can be calculated in the same way. A form of the result coincides with (70). (But the notations differ!)

Now let us consider linearly divergent graphs.

$$(1.f) = \frac{i}{2(2\pi)^4}\mathrm{tr}\int d^dk(Sk)(Kk)^{-1}(W(k-p))(K(k-p))^{-1}. \qquad (88)$$

We should expand this expression into series over external momentum p and retain logarithmically divergent terms (It is easy to see that they are linear in p). The expansion of $(K(k-p))^{-1}$ is found in the appendix C. By eq. (117) and (119) we obtain, that

$$(1.f)_\infty = \frac{1}{16\pi^2(d-4)}\mathrm{tr} < -L\partial_\mu\hat{S}\ \hat{W}\ \hat{K}^\mu + (L-2)\partial_\mu\hat{S}\ \hat{W}^\mu > . \qquad (89)$$

In a similar fashion we find the divergent parts of the diagrams (1.g) and (1.h). The results are presented in (25).

As for a minimal operator we do not consider here all curved space diagrams. In order to illustrate the method we present only the calculation of the WR contribution (or the graphs (2.a), (2.b), (2.o), (2.p) and (3.a)). The other graphs are computed in the same way, but the calculations are much more cumbersome.

If $K^{\mu\nu\ldots\alpha}$ does not depend on metric, the vertexes with $h_{\mu\nu}$ should be obtained by the series expansion of covariant derivatives in the operator (3) to the first order. Then, retaining only logarithmically divergent terms and performing the integration as described in the section 4, we find the divergent part of a diagram.

By this method we obtain the following answer for the quadratically divergent graph (2.a)

$$(2.a)_\infty = \frac{1}{16\pi^2(d-4)}\mathrm{tr} < \frac{1}{3}L(L-1)(L-2)\hat{W}\hat{K}^{\mu\nu\alpha}p_\alpha\Gamma^\sigma_{\mu\nu}n_\sigma$$
$$+\frac{1}{2}L(L-1)\hat{W}\hat{K}^{\mu\nu}\Gamma^\sigma_{\mu\nu}p_\sigma + \frac{1}{2}L(L-1)\hat{W}\hat{K}^{\mu\nu}\Delta^\alpha\Gamma^\sigma_{\mu\nu}n_\sigma p_\alpha >, \qquad (90)$$

where Δ^μ is the first coefficient in the propagator expansion. Its explicit form is found in the appendix C.

Similarly we obtain

$$(2.b)_\infty = \frac{1}{6}(L-2)(L-3)(L-4) < \hat{W}^{\mu\alpha\beta} p_\mu \Gamma^\sigma_{\alpha\beta} n_\sigma > . \qquad (91)$$

Using rules, derived in the appendix, it can be rewritten as

$$\frac{1}{16\pi^2(d-4)}\mathrm{tr} < \frac{1}{3}\left(p_\alpha\Gamma^\sigma_{\mu\nu} + p_\mu\Gamma^\sigma_{\nu\alpha} + p_\nu\Gamma^\sigma_{\mu\alpha}\right)\hat{W} n_\sigma \left(\frac{1}{6}L(L-1)(L-2)\right.$$
$$\left. \times \hat{K}^{\mu\nu\alpha} + \frac{1}{2}L(L-1)\hat{K}^{\mu\nu}\Delta^\alpha \ + L\hat{K}^\alpha\Delta^{\mu\nu}\right) + \frac{1}{3}\left(p_\sigma\Gamma^\sigma_{\mu\nu} + 2p_\mu\Gamma^\sigma_{\nu\sigma}\right)\hat{W}$$
$$\times\left(\frac{1}{2}L(L-1)\ \hat{K}_{\mu\nu} + L\ \hat{K}^\mu\Delta^\nu\right) > . \qquad (92)$$

Summing up the results (90) and (92), we obtain

$$\frac{1}{16\pi^2(d-4)}\mathrm{tr} < \frac{1}{2}L(L-1)(L-2)\hat{W}\hat{K}^{\mu\nu\alpha} p_\alpha\Gamma^\sigma_{\mu\nu} n_\sigma + \frac{1}{3}L(L-1)\hat{W}$$
$$\times\hat{K}^{\mu\nu}\Delta^\alpha n_\sigma\left(2p_\alpha\Gamma^\sigma_{\mu\nu} + p_\mu\Gamma^\sigma_{\nu\alpha}\right) + \frac{1}{3}L\hat{W}\ \hat{K}^\alpha\Delta^{\mu\nu} n_\sigma\left(p_\alpha\Gamma^\sigma_{\mu\nu} + 2p_\mu\Gamma^\sigma_{\nu\alpha}\right)$$
$$+\frac{1}{3}L(L-1)\hat{W}\ \hat{K}^{\mu\nu}\left(2p_\sigma\Gamma^\sigma_{\mu\nu} + p_\mu\Gamma^\sigma_{\nu\sigma}\right) + \frac{1}{3}L\ \hat{W}\ \hat{K}^\mu\Delta^\nu\left(p_\sigma\Gamma^\sigma_{\mu\nu} + 2p_\mu\right.$$
$$\left.\times\Gamma^\sigma_{\nu\sigma}\right) > . \qquad (93)$$

Diagrams, containing the connection ${\omega_{\alpha i}}^j$ are calculated in a similar fashion. (We remind, that $\omega^j_{\alpha i}$ should be considered as a weak field.) We should consider 2 linear divergent graphs (2.o) and (2.p). By the same arguments as above, it is easy to see, that

$$(2.o)_\infty = \frac{1}{16\pi^2(d-4)}\mathrm{tr} < -\frac{1}{2}L(L-1)\hat{W}\hat{K}^{\mu\nu}\omega_\mu p_\nu - L\hat{W}\hat{K}^\mu\omega_\mu\Delta^\nu p_\nu >;$$
$$(2.p)_\infty = -\ \frac{1}{16\pi^2(d-4)}\ \mathrm{tr} < L(L-1)\hat{W}^{\mu\nu}\omega_\mu p_\nu >$$
$$= -\ \frac{1}{16\pi^2(d-4)}\mathrm{tr} < \hat{W}\ p_\nu\omega_\mu\Delta^{\mu\nu} > . \qquad (94)$$

So, the whole contribution of this diagrams is

$$\frac{1}{16\pi^2(d-4)}\mathrm{tr} < -\frac{1}{2}L(L-1)\hat{W}\hat{K}^{\mu\nu}p_\nu\omega_\mu - L\ \hat{W}\hat{K}^{\mu}p_\nu\omega_\mu\Delta^\nu$$
$$-\hat{W}p_\nu\omega_\mu\Delta^{\mu\nu} > . \quad (95)$$

As we mention above, (93) and (95) can not be presented as a weak field limit of covariant expressions. The matter is that we should take into account a contribution of fields ϕ^b. For this purpose we consider a quadratically divergent diagram (3.a). It is written as

$$(3.a) = \frac{1}{16\pi^2(d-4)}\mathrm{tr}\int d^dk (Wk)(Kk)^{-1}\frac{\partial(K(k-p))}{\partial\phi^b}\phi^b(K(k-p))^{-1}. \quad (96)$$

Expanding the integrant into series over external momentum p and retaining only terms, quadratic in it (they are logarithmically divergent), we obtain

$$(3.a)_\infty = \frac{1}{16\pi^2(d-4)}\mathrm{tr} < \frac{1}{2}L(L-1)\hat{W}\ (Kn)^{-1}\frac{\partial(Kn)^{\mu\nu}}{\partial\phi^b}\phi^b p_\mu p_\nu + L\ \hat{W}$$
$$\times\ (Kn)^{-1}\frac{\partial(Kn)^{\mu}}{\partial\phi^b}\phi^b\Delta^\nu p_\mu p_\nu + \hat{W}\ (Kn)^{-1}\frac{\partial(Kn)}{\partial\phi^b}\phi^b\Delta^{\mu\nu}p_\mu p_\nu >, \quad (97)$$

that can be written as

$$\frac{1}{16\pi^2(d-4)}\mathrm{tr} < \frac{1}{2}L(L-1)\hat{W}\ (Kn)^{-1}\partial_\nu\partial_\mu(Kn)^{\mu\nu} + L\ \hat{W}\ (Kn)^{-1}$$
$$\times\ \partial_\nu\partial_\mu(Kn)^\mu\Delta^\nu + \hat{W}\ (Kn)^{-1}\partial_\nu\partial_\mu(Kn)\Delta^{\mu\nu} > . \quad (98)$$

Taking into account (27) and using rules, formulated in the appendix C, we find additional terms

$$(3.a)_\infty = -\ \frac{1}{16\pi^2(d-4)}\mathrm{tr} < \frac{1}{2}L(L-1)p_\nu\hat{W}\Big((L-2)\Gamma^\sigma_{\mu\alpha}\hat{K}^{\mu\nu\alpha}n_\sigma + \Gamma^\mu_{\mu\alpha}$$
$$\times\hat{K}^{\nu\alpha} + \Gamma^\nu_{\mu\alpha}\hat{K}^{\mu\alpha} + (Kn)^{-1}\omega_\mu(Kn)^{\mu\nu} - \hat{K}^{\mu\nu}\omega_\mu\Big) + Lp_\nu\hat{W}\Big((L-1)\Gamma^\sigma_{\mu\alpha}$$

$$\times \hat{K}^{\mu\alpha} n_\sigma + \Gamma^\mu_{\mu\alpha}\hat{K}^\alpha + (Kn)^{-1}\omega_\mu (Kn)^\mu - \hat{K}^\mu\omega_\mu\Big)\,\Delta^\nu + p_\nu \hat{W}\left(L\Gamma^\sigma_{\mu\alpha}\hat{K}^\alpha n_\sigma\right.$$

$$\left.+(Kn)^{-1}\omega_\mu (Kn) - \hat{K}\ \omega_\mu\right)\Delta^{\mu\nu} >=$$

$$= \frac{1}{16\pi^2(d-4)}\mathrm{tr} < \frac{1}{2}L(L-1)p_\nu\hat{W}\Big((L-2)\Gamma^\sigma_{\mu\alpha}\hat{K}^{\mu\nu\alpha}n_\sigma + \Gamma^\mu_{\mu\alpha}\hat{K}^{\nu\alpha}$$
$$+\Gamma^\nu_{\mu\alpha}\hat{K}^{\mu\alpha} - \hat{K}^{\mu\nu}\omega_\mu\Big) + Lp_\nu\hat{W}\Big((L-1)\ \Gamma^\sigma_{\mu\alpha}\hat{K}^{\mu\alpha}n_\sigma + \Gamma^\mu_{\mu\alpha}\hat{K}^\alpha - \hat{K}^\mu\omega_\mu\Big)$$
$$\times\ \Delta^\nu + p_\nu\hat{W}\left(L\Gamma^\sigma_{\mu\alpha}\hat{K}^\alpha n_\sigma - \omega_\mu\right)\Delta^{\mu\nu} > . \tag{99}$$

Adding (99) to (93) and (95), we obtain the following result in the weak field limit

$$\frac{1}{16\pi^2(d-4)}\mathrm{tr} < \frac{1}{3}L\ \hat{W}\hat{K}^\alpha\Delta^{\mu\nu}n_\sigma\left(p_\alpha\Gamma^\sigma_{\mu\nu} - p_\nu\Gamma^\sigma_{\mu\alpha}\right) + \frac{1}{6}L(L-1)\hat{W}\hat{K}^{\mu\nu}$$
$$\times\Delta^\alpha n_\sigma\left(p_\alpha\Gamma^\sigma_{\mu\nu} - p_\nu\Gamma^\sigma_{\mu\alpha}\right) + \frac{1}{6}L(L-1)\hat{W}\hat{K}^{\mu\nu}\left(p_\sigma\Gamma^\sigma_{\mu\nu} - p_\mu\Gamma^\sigma_{\nu\sigma}\right) - \frac{1}{6}L\ \hat{W}$$
$$\times\hat{K}^\mu\Delta^\nu\left(p_\sigma\Gamma^\sigma_{\mu\nu} - p_\mu\Gamma^\sigma_{\nu\sigma}\right) - \frac{1}{2}L^2\hat{W}\left(p_\mu\omega_\nu - p_\nu\omega_\mu\right)\hat{K}^\mu\hat{K}^\nu >, \tag{100}$$

that can be (unlike (93) and (95) !) presented as a first term in the expansion of

$$\frac{1}{16\pi^2(d-4)}\mathrm{tr} < -\frac{1}{2}L^2\hat{W}F_{\mu\nu}\hat{K}^\mu\hat{K}^\nu + \frac{1}{6}L(L-1)\hat{W}\hat{K}^{\mu\nu}R_{\mu\nu} + \frac{1}{6}L^2\hat{W}$$
$$\times\hat{K}^\mu\hat{K}^\nu R_{\mu\nu} + \frac{1}{3}L\ \hat{W}\hat{K}^\alpha\Delta^{\mu\nu}n_\sigma R^\sigma{}_{\mu\alpha\nu} + \frac{1}{6}L(L-1)\hat{W}\hat{K}^{\mu\nu}\Delta^\alpha n_\sigma R^\sigma{}_{\mu\alpha\nu} > \tag{101}$$

in powers of $h_{\mu\nu}$ and $\omega_{\mu i}{}^j$.

This expression is a final result for the considered group of diagrams. Now we try to present it in the most compact form. Using rules, formulated in the appendix, we find, that

$$-\ \frac{1}{2}L(L-1)\hat{W}\ \hat{K}^{\mu\nu}R_{\mu\nu} + L^2\hat{W}\ \hat{K}^\mu\hat{K}^\nu R_{\mu\nu}$$
$$= \hat{W}\Delta^{\mu\nu}R_{\mu\nu} = \frac{1}{2}(L-2)(L-3)\hat{W}^{\mu\nu}R_{\mu\nu} \tag{102}$$

Then, taking into account the following simple identities

$$
\begin{aligned}
&(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)(Kn) = Ln_\rho R^\rho{}_{\alpha\mu\nu}(Kn)^\alpha + F_{\mu\nu}(Kn) - (Kn)F_{\mu\nu};\\
&(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)(Kn)^\beta = (L-1)n_\rho R^\rho{}_{\alpha\mu\nu}(Kn)^{\alpha\beta} + R^\beta{}_{\alpha\mu\nu}(Kn)^\alpha\\
&+ F_{\mu\nu}(Kn)^\beta - (Kn)^\beta F_{\mu\nu};\\
&(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)(Kn)^{\beta\gamma} = (L-2)\ n_\rho R^\rho{}_{\alpha\mu\nu}(Kn)^{\alpha\beta\gamma} + R^\beta{}_{\alpha\mu\nu}(Kn)^{\alpha\gamma}\\
&+ R^\gamma{}_{\alpha\mu\nu}(Kn)^{\alpha\beta} + F_{\mu\nu}(Kn)^{\beta\gamma} - (Kn)^{\beta\gamma}F_{\mu\nu},
\end{aligned}
\tag{103}
$$

we obtain the final result

$$
\begin{aligned}
&\frac{1}{16\pi^2(d-4)}\mathrm{tr} < -\ \frac{1}{2}L^2\hat{W}\ \hat{F}_{\mu\nu}(Kn)^\mu\hat{K}^\nu + \frac{1}{3}L\ \hat{W}\ \hat{K}^\alpha\Delta^{\mu\nu}n_\sigma R^\sigma{}_{\mu\alpha\nu}\\
&+\frac{1}{3}L^2(L-1)\hat{W}\ \hat{K}^{\mu\nu}\hat{K}^\alpha n_\sigma R^\sigma{}_{\mu\alpha\nu} - \frac{1}{6}(L-2)(L-3)\hat{W}^{\mu\nu}R_{\mu\nu} > .
\end{aligned}
\tag{104}
$$

The other diagrams are considered in the same way.

C Integration over angles

Let us first us start with (15):

$$
\begin{aligned}
&< n_{\mu_1}n_{\mu_2}\ldots n_{\mu_{2m}} > \equiv \frac{1}{2^m(m+1)!}\\
&\times \Big(g_{\mu_1\mu_2}g_{\mu_3\mu_4}\ldots g_{\mu_{2m-1}\mu_{2m}} + \text{permutations of } (\mu_1\ldots\mu_{2m})\Big)
\end{aligned}
\tag{105}
$$

It can be interpreted in the following way: In order to obtain the result of the angle integration we should make pairs of n_α by all possible ways and add a numerical constant. Each pair of n_α and n_β should be substituted by $g_{\alpha\beta}$.

The sum contains $(2m-1)!!$ terms. Hence, if we contract (105) with a totally symmetric tensor $A^{\mu_1\mu_2\dots\mu_{2m}} \equiv A_{(2m)}$ (here the bottom index points the tensor rank) the result will be

$$< (A_{(2m)}n) >= \frac{(2m-1)!!}{2^m(m+1)!} A^{\mu_1\dots\mu_m}{}_{\mu_1\dots\mu_m}, \tag{106}$$

Similarly one can find that for a symmetric tensor $A_{(2m-1)}$ with $2m-1$ indexes the following equation takes place:

$$\begin{aligned} < n_\alpha(A_{(2m-1)}n) >= &\frac{(2m-1)!!}{2^m(m+1)!} A^{\mu_1\dots\mu_{m-1}}{}_{\mu_1\dots\mu_{m-1}\alpha} \\ &= \frac{2m-1}{2(m+1)} < (A_{(2m-1)}n)_\alpha >, \end{aligned} \tag{107}$$

In the more general case we will use the following consequence of (15):

$$\begin{aligned} < n_{\mu_1}n_{\mu_2}\dots n_{\mu_{2m}} >= &\frac{1}{2(m+1)} \Big(g_{\mu_1\mu_2} < n_{\mu_3}n_{\mu_4}\dots n_{\mu_{2m}} > \\ &+ g_{\mu_1\mu_3} < n_{\mu_2}n_{\mu_4}\dots n_{\mu_{2m}} > + \dots + g_{\mu_1\mu_{2m}} < n_{\mu_2}n_{\mu_3}\dots n_{\mu_{2m-1}} > \Big) . \end{aligned} \tag{108}$$

Making contraction with 2 symmetric tensors we find that

$$\begin{aligned} < n_\alpha(A_{(2m)}n)(B_{(2p-1)}n) >= &\frac{1}{2(m+p+1)} < \Big(2m(A_{(2m)}n)_\alpha(B_{(2p-1)}n) \\ &+(2p-1)(A_{(2m)}n)(B_{(2p-1)}n)_\alpha \Big) > . \end{aligned} \tag{109}$$

This equation can be easily generalized to a greater number of symmetric tensors.

The rules (106), (107) and (109) are used in calculating Feynman graphs for a minimal operator. For a nonminimal operator we should formulate different identities.

Let us consider an integral

$$\int d^dk\ f(k) \tag{110}$$

and assume, that all terms in it are more than logarithmically divergent. Therefore in the dimensional regularization it is equal to 0.

A substitution $k_\mu \to k_\mu + p_\mu$ do not change the divergent part of the integral. Nevertheless, on the other hand, we can calculate it explicitly, using the method described above. For example, let us consider that $f(k)$ transforms as

1. $f(k) \to \alpha^{-3} f(k)$ if $k_\mu \to \alpha k_\mu$. Then, retaining only logarithmically divergent terms, we obtain

$$0 = \left(\int d^d k \ f(k+p) \right)_\infty = - \frac{2i\pi^2}{d-4} < \delta_\mu f(n) > p^\mu, \tag{111}$$

if

$$f(k+p) \equiv f(k) + \delta_\mu f(k) p^\mu + \delta_{\mu\nu} f(k) p^\mu p^\nu + \ldots. \tag{112}$$

($\delta_{\mu\nu\ldots}$ must be considered as symmetric in its indexes)

From (111) we have

$$< \delta_\mu f(n) >= 0. \tag{113}$$

2. If $f(k)\ g(k) \to \alpha^{-3} f(k)\ g(k)$ we find

$$\left(\int d^d k \ g(k)\ f(k+p) \right)_\infty = \left(\int d^d k \ g(k-p)\ f(k) \right)_\infty \tag{114}$$

and therefore

$$< g(n)\ \delta_\mu f(n) > = - \ < \delta_\mu g(n)\ f(n) > . \tag{115}$$

3. If $f(k) g(k) \to \alpha^{-2} f(k) g(k)$, using an identity

$$\left(\int d^d k \ g(k) f(k+p) \right)_\infty = \left(\int d^d k \ g(k-p) f(k) \right)_\infty$$

it is easy to see, that

$$< g(n) \delta_{\mu\nu} f(n) >=< \delta_{\mu\nu} g(n) f(n) > . \tag{116}$$

Other cases are considered in a similar fashion.

In order to apply this rules for calculating Feynman graphs, we should know coefficients of the propagator expansion in powers of the external momentum

$$(K(k+p))^{-1} = (Kk)^{-1} + \delta^{\mu}(Kk)^{-1}p_{\mu} + \delta^{\mu\nu}(Kk)^{-1}p_{\mu}p_{\nu} + \delta^{\mu\nu\alpha}(Kk)^{-1}$$
$$\times p_{\mu}p_{\nu}p_{\alpha} + \ldots \equiv \Big(1 + \frac{1}{k}\Delta^{\mu}p_{\mu} + \frac{1}{k^2}\Delta^{\mu\nu}p_{\mu}p_{\nu} + \frac{1}{k^3}\Delta^{\mu\nu\alpha}p_{\mu}p_{\nu}p_{\alpha} + \frac{1}{k^4}$$
$$\times\, \Delta^{\mu\nu\alpha\beta}p_{\mu}p_{\nu}p_{\alpha}p_{\beta} + \ldots\Big)(Kk)^{-1}. \qquad (117)$$

The coefficient Δ^{μ} – $\Delta^{\mu\nu\alpha\beta}$ can be found by expanding the identity

$$1_i{}^j = (K(k+p))_i{}^m (K(k+p))^{-1}{}_m{}^j \qquad (118)$$

in powers of p_{μ}. The result is

$$\Delta^{\mu} = -L\hat{K}^{\mu};$$
$$\Delta^{\mu\nu} \equiv -\frac{1}{2}L(L-1)\hat{K}^{\mu\nu} + L^2\hat{K}^{(\mu}\hat{K}^{\nu)};$$
$$\Delta^{\mu\nu\alpha} = -\frac{1}{6}L(L-1)(L-2)\hat{K}^{\mu\nu\alpha} + \frac{1}{2}L^2(L-1)\hat{K}^{(\mu\nu}\hat{K}^{\alpha)} + \frac{1}{2}L^2(L-1)$$
$$\times\, \hat{K}^{(\alpha)}\hat{K}^{\mu\nu)} - L^3\hat{K}^{(\mu}\hat{K}^{\nu}\hat{K}^{\alpha)};$$
$$\Delta^{\mu\nu\alpha\beta} = -\frac{1}{24}L(L-1)(L-2)(L-3)\hat{K}^{\mu\nu\alpha\beta} + \frac{1}{6}L^2(L-1)(L-2)\hat{K}^{(\mu\nu\alpha}$$
$$\times \hat{K}^{\beta)} + \frac{1}{6}L^2(L-1)(L-2)\hat{K}^{(\beta}\hat{K}^{\mu\nu\alpha)} + \frac{1}{4}L^2(L-1)^2\hat{K}^{(\mu\nu}\hat{K}^{\alpha\beta)}$$
$$-\frac{1}{2}L^3(L-1)\hat{K}^{(\mu\nu}\hat{K}^{\alpha}\hat{K}^{\beta)} - \frac{1}{2}L^3(L-1)\hat{K}^{(\alpha}\hat{K}^{\mu\nu}\hat{K}^{\beta)} - \frac{1}{2}L^3(L-1)$$
$$\times \hat{K}^{(\alpha}\hat{K}^{\beta}\hat{K}^{\mu\nu)} + L^4\hat{K}^{(\mu}\hat{K}^{\nu}\hat{K}^{\alpha}\hat{K}^{\beta)}, \qquad (119)$$

where

$$A^{(i_1 i_2 \ldots i_n)} \equiv \frac{1}{n!}\left(A^{i_1 i_2 \ldots i_n} + A^{i_2 i_1 \ldots i_n} + \ldots + A^{i_n i_{n-1} \ldots i_1}\right). \qquad (120)$$

(Other coefficients is not needed for the calculation of the divergent part of the effective action).

The coefficient Δ^{μ} – $\Delta^{\mu\nu\alpha\beta}$ satisfy the following useful identities

$$
\begin{aligned}
&\Delta^{\mu\nu} + \frac{1}{2}L(L-1)\hat{K}^{\mu\nu} + L\hat{K}^{(\mu}\Delta^{\nu)} = 0;\\
&\Delta^{\mu\nu\alpha} + \frac{1}{6}L(L-1)(L-2)\hat{K}^{\mu\nu\alpha} + \frac{1}{2}L(L-1)\hat{K}^{(\mu\nu}\Delta^{\alpha)} + L\hat{K}^{(\mu}\Delta^{\nu\alpha)} = 0;\\
&\Delta^{\mu\nu\alpha\beta} + \frac{1}{24}L(L-1)(L-2)(L-3)\hat{K}^{\mu\nu\alpha\beta} + \frac{1}{6}L(L-1)(L-2)\hat{K}^{(\mu\nu\alpha}\Delta^{\beta)}\\
&+\frac{1}{2}L(L-1)\hat{K}^{(\mu\nu}\Delta^{\alpha\beta)} + L\hat{K}^{(\mu}\Delta^{\nu\alpha\beta)} = 0
\end{aligned}
\tag{121}
$$

and so on.

Now we illustrate formulated rules by the simplest example. Let us consider

$$
< (L-3)(Nn)^{\alpha}(Kn)^{-1} >,
\tag{122}
$$

where $N^{\mu\nu\ldots\alpha}$ is a totally symmetric tensor with $L-3$ indexes. $(L-3)(Nn)^{\alpha}$ can be presented as $\delta^{\alpha}(Nn)$. Using (115), we obtain

$$
< (L-3)(Nn)^{\alpha}(Kn)^{-1} >= - < (Nn)\ \delta^{\alpha}(Kn)^{-1} > .
\tag{123}
$$

Substituting here Δ^{α} we find an identity

$$
< (L-3)\hat{N}^{\alpha} >= L < \hat{N}\hat{K}^{\alpha} > .
\tag{124}
$$

Other cases can be considered in the same way.

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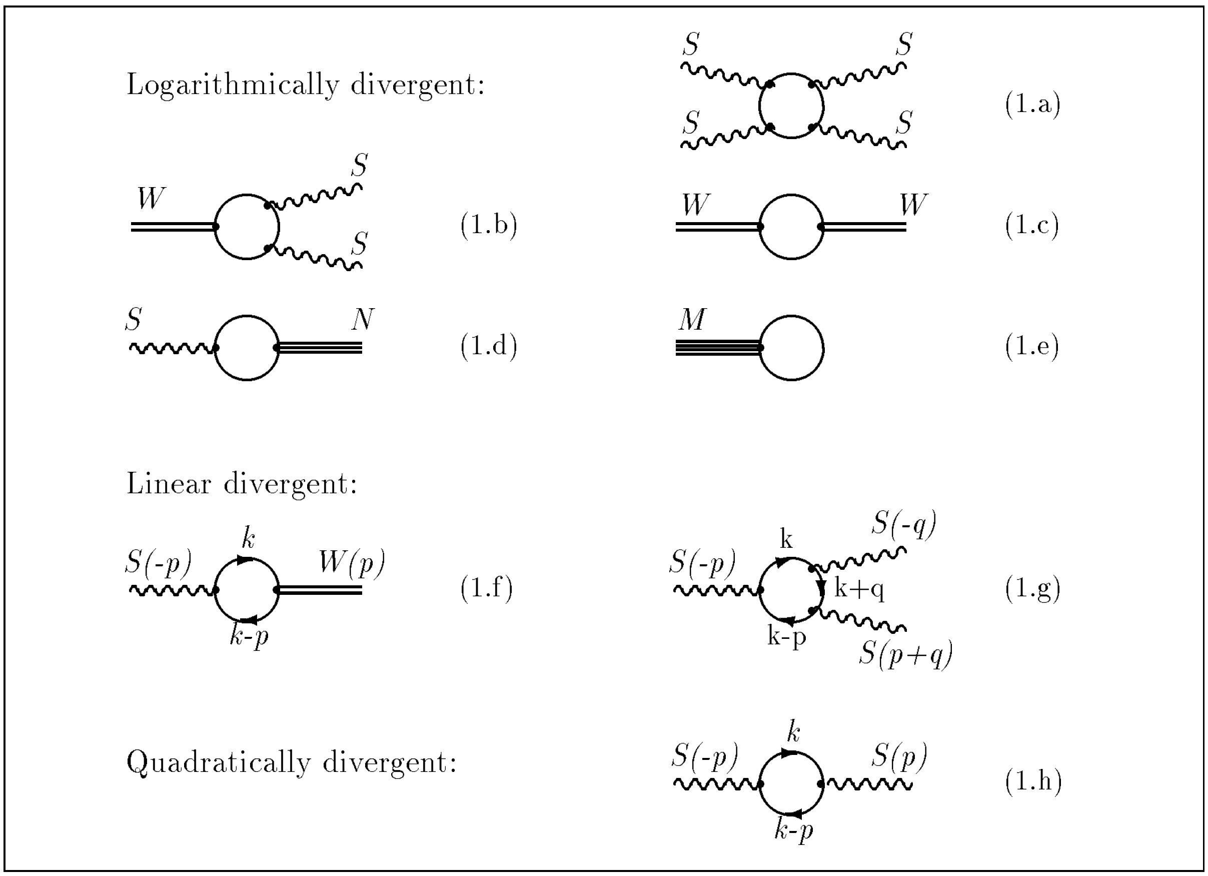

Figure 1: Divergent graphs in the flat space.

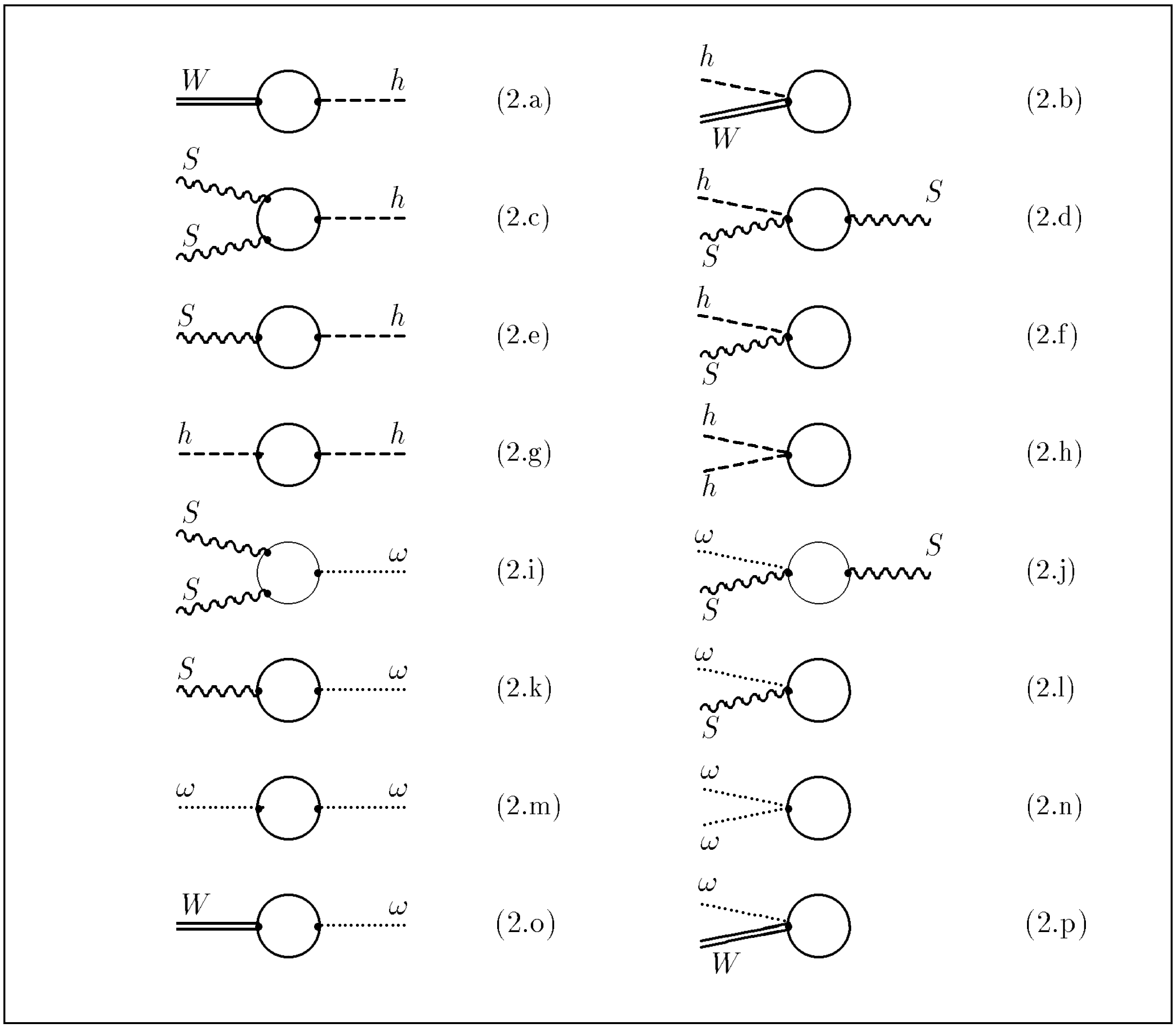

Figure 2: Graphs for the effective action calculation in the curved space.

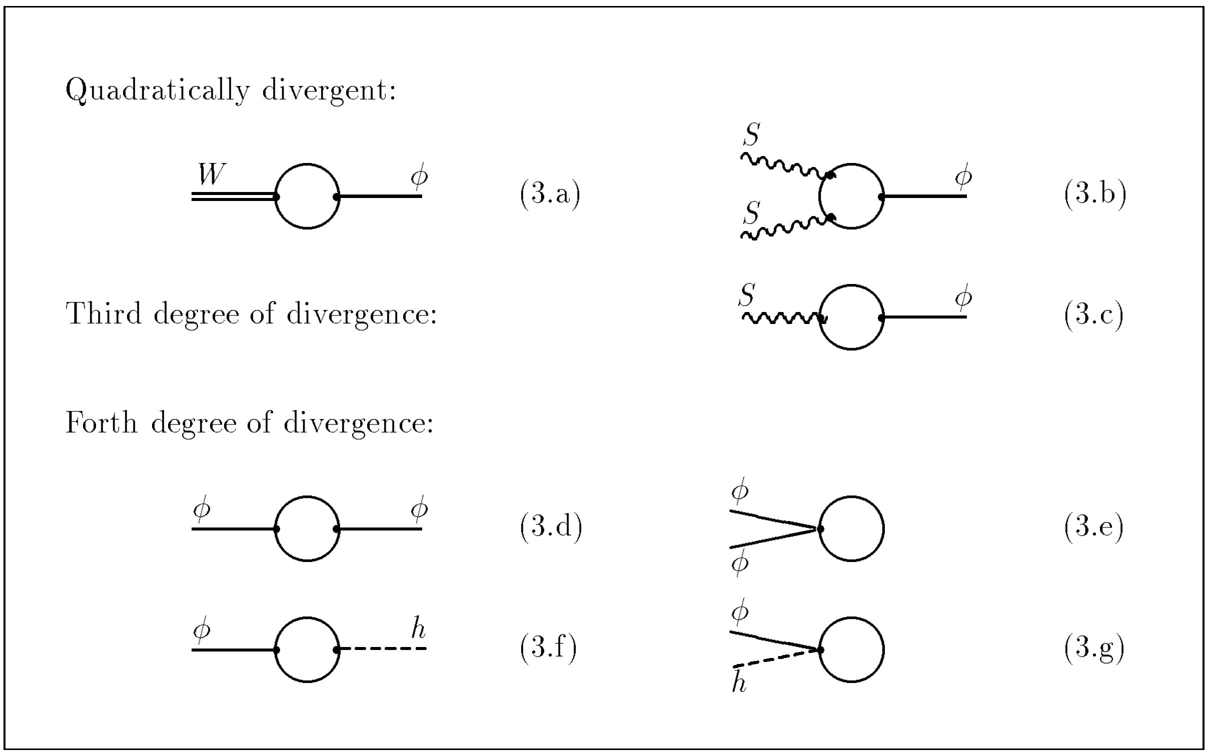

Figure 3: Diagrams for the calculation of the ϕ^b contribution in the first nontrivial approximation