Part 5 Dot product & Outer product

#vector dot product and vector length

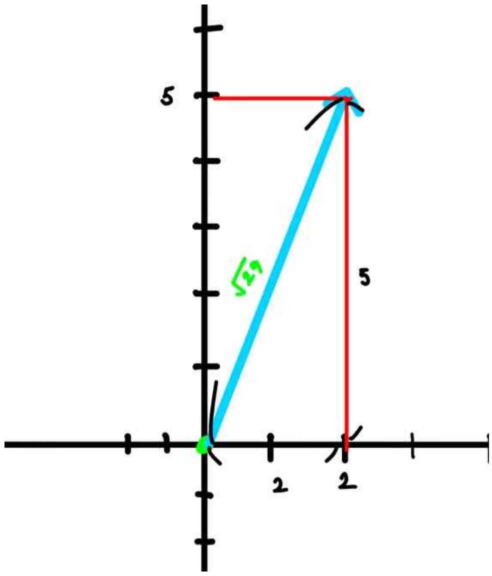
-Dot product

$\vec{a} \bullet \vec{b} = [a_1, a_2 \dots a_n] \bullet [b_1, b_2 \dots b_n] = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

-Length

$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$

Ex) if $\vec{b} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\|\vec{b}\| = \sqrt{2^2 + 5^2} = \sqrt{29}$



So, we can say that

$\vec{a} \bullet \vec{a} = \|\vec{a}\|^2$

-dot product

commutative property O

associative property O

distributive property O

#Proving Cauchy-Schwarz inequality

$$p(t) = \|t\vec{y} - \vec{x}\|^2 \geq 0$$

$$= (t\vec{y} - \vec{x}) \cdot (t\vec{y} - \vec{x})$$

$$= t\vec{y} \cdot t\vec{y} - \vec{x} \cdot t\vec{y} - \vec{x} \cdot t\vec{y} + \vec{x} \cdot \vec{x}$$

$$= t^2 \underbrace{(\vec{y} \cdot \vec{y})}_{a} - \underbrace{2(\vec{x} \cdot \vec{y})}_{b} t + \underbrace{(\vec{x} \cdot \vec{x})}_{c}$$

$$= t^2 a - tb + c \geq 0$$

$$p\left(\frac{b}{2a}\right) = \frac{b^2}{4a} \cdot a - \frac{b}{2a} \cdot b + c$$

$$= \frac{-b^2}{4a} + c \geq 0$$

$$= c \geq \frac{b^2}{4a}$$

$$= 4ac \geq b^2$$

$$= \cancel{4}(\vec{y} \cdot \vec{y})(\vec{x} \cdot \vec{x}) \geq \cancel{4}(\vec{x} \cdot \vec{y})^2$$

$$= \|\vec{y}\|^2 \|\vec{x}\|^2 \geq (\vec{x} \cdot \vec{y})^2$$

$$= \|\vec{y}\| \|\vec{x}\| \geq |\vec{x} \cdot \vec{y}|$$

if $\vec{x} = c\vec{y}$

$$|\vec{x} \cdot \vec{y}|$$

$$= |c\vec{y} \cdot \vec{y}|$$

$$= |c| |\vec{y} \cdot \vec{y}|$$

$$= |c| \|\vec{y}\|^2$$

$$= |c| \|\vec{y}\| \|\vec{y}\|$$

$$= \|c\vec{y}\| \|\vec{y}\|$$

$$= \|\vec{x}\| \|\vec{y}\|$$

#Vector triangle inequality

$\vec{x}\cdot\vec{y} \le |\vec{x}\cdot\vec{y}| \le \|\vec{x}\|\|\vec{y}\|$

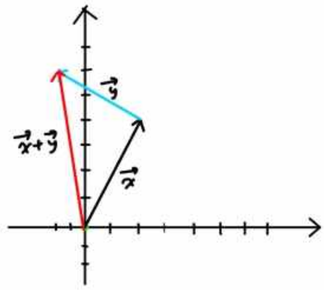
$\|\vec{x}+\vec{y}\|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$

$= \vec{x}\cdot\vec{x} + \vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{x} + \vec{y}\cdot\vec{y}$

$= \|\vec{x}\|^2 + 2(\vec{x}\cdot\vec{y}) + \|\vec{y}\|^2 \le \|\vec{x}\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|y\|^2$

$= \|\vec{x}+\vec{y}\|^2 \le (\|\vec{x}\|+\|\vec{y}\|)^2$

$= \|\vec{x}+\vec{y}\| \le \|\vec{x}\|+\|\vec{y}\|$



if) $\vec{x} = c\cdot\vec{y}$

$\vec{x}\cdot\vec{y} = |\vec{x}\cdot\vec{y}|$

$= c\vec{y}\cdot\vec{y}$

$= c\|\vec{y}\|^2$

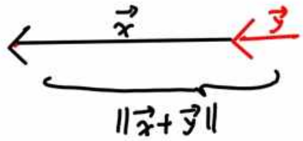
$= c\|\vec{y}\|\|\vec{y}\|$

$= \|c\vec{y}\|\|\vec{y}\|$

$= \|\vec{x}\|\|\vec{y}\|$

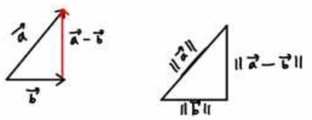
$\therefore \|\vec{x}+\vec{y}\| = \|\vec{x}\|+\|\vec{y}\|$

$\vec{x} = c\vec{y},\ c>0$



#Define angle with Vector

$\vec{a}, \vec{b} \in \mathbb{R}^n$, non-zero



not hold inequality

$\|\vec{b}\| > \|\vec{a}\| + \|\vec{a}-\vec{b}\|$

$\|\vec{a}\| > \|\vec{b}\| + \|\vec{a}-\vec{b}\|$

$\|\vec{a}-\vec{b}\| > \|\vec{a}\| + \|\vec{b}\|$

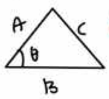
$\|\vec{x}+\vec{y}\| \le \|x\| + \|y\|$

$\|\vec{a}\| \le \|\vec{b}\| + \|\vec{a}-\vec{b}\|$

$\|\vec{b}\| \le \|\vec{a}\| + \|\vec{b}-\vec{a}\|$

$\|\vec{a}-\vec{b}\| \le \|\vec{a}\| + \|-\vec{b}\|$

law of cosine



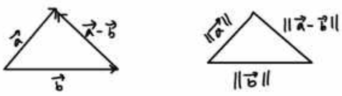
$C^2 = A^2 + B^2 - 2AB\cos\theta$ $\longrightarrow$

$\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$

$(\vec{a}-\vec{b})(\vec{a}-\vec{b}) = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$

$\|\cancel{\vec{a}}\|^2 - \cancel{2}(\vec{a}\cdot\vec{b}) + \|\cancel{\vec{b}}\|^2 = \|\cancel{\vec{a}}\|^2 + \|\cancel{\vec{b}}\|^2 - \cancel{2}\|\vec{a}\|\|\vec{b}\|\cos\theta$

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$



cf) if $\vec{a} = c\vec{b}$ $\quad c>0 \rightarrow \theta=0$

$c<0 \rightarrow \theta = 180$

Orthogonal

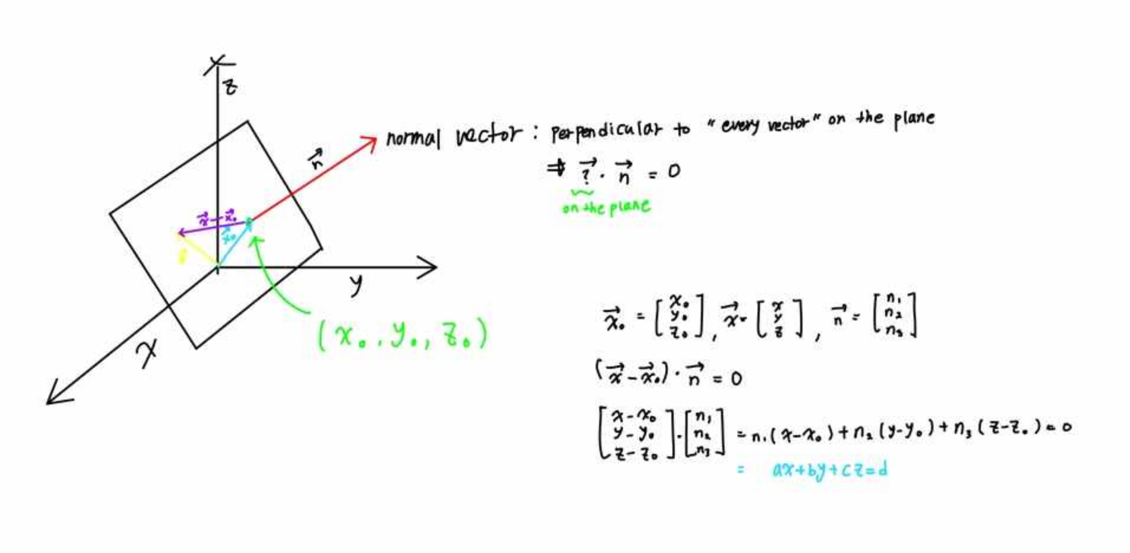
if $\vec{a}, \vec{b}$ perpendicular $\overset{\not\leftarrow}{\longrightarrow}$ $\vec{a}\cdot\vec{b}=0$ why? $\cos 90^\circ = 0$

+ ← non-zero vector

so if $\vec{a},\vec{b}$ is zero vector, we can say only orthogonal not perpendicular.

if $\vec{a},\vec{b}$ is non-zero vector, we can say either orthogonal and perpendicular.

#Defining a plane in R3

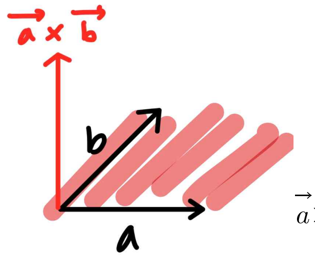


#Cross product

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \vec{a} \times \vec{b} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}$$

Ex)

$$\vec{a} = \begin{bmatrix} 1 \\ -7 \\ 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix} \quad \vec{a} \times \vec{b} = \begin{bmatrix} -7 \cdot 4 & - & 1 \cdot 2 \\ 1 \cdot 5 & - & 1 \cdot 4 \\ 1 \cdot 2 & - & (-7) \cdot 5 \end{bmatrix}$$



$\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and $\vec{b}$

$\vec{a} \times \vec{b}$ is orthogonal means that $\vec{a} \cdot \vec{b} = 0$

prove)

$(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$

$$\begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

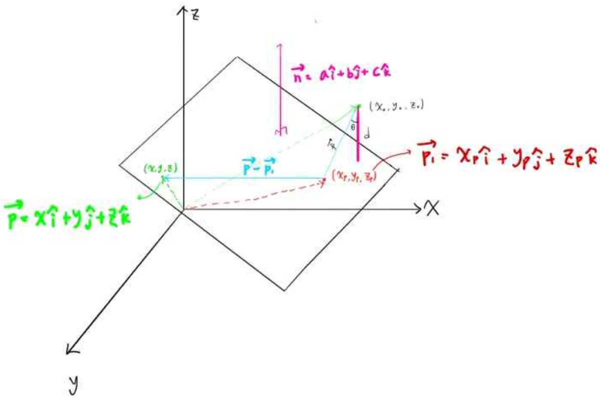
$a_1a_2a_3 - a_1a_3b_2 + a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1$=0

0=0

$(\vec{a}\times\vec{b}) \cdot \vec{b} = 0$

prove)
same as $(\vec{a}\times\vec{b}) \cdot \vec{a} = 0$

#Normal vector from plane equation & Point distance to plane



- $\vec{P} - \vec{P_1} = (x - x_p)\hat{i} + (y - y_p)\hat{j} + (z - z_p)\hat{k}$

  $\vec{n} \cdot (\vec{P} - \vec{P_1}) = 0 = ax - ax_p + by - by_p + cz - cz_p$

  $\Rightarrow ax_p + by_p + cz_p = ax + by + cz$

  $\downarrow \quad \downarrow \quad \downarrow$

  $AX + BY + CZ = D$ (surface equation)

  $\therefore \vec{n} = A\hat{i} + B\hat{j} + C\hat{k}$

- $\vec{f} = (x_0 - x_p)\hat{i} + (y_0 - y_p)\hat{j} + (z_0 - z_p)\hat{k}$

  $\Rightarrow \cos\theta = \dfrac{d}{\|\vec{f}\|}$

  $\Rightarrow \|\vec{f}\| \cos\theta = d$

  $\Rightarrow \dfrac{|\vec{n}||\vec{f}| \cos\theta}{|\vec{n}|} = d$

  $\Rightarrow \dfrac{\vec{n}\cdot\vec{f}}{|\vec{n}|} = d$

  $\vec{n}\cdot\vec{f} \rightarrow Ax_0 - Ax_p + By_0 - By_p + Cz_0 - Cz_p$

  $|\vec{n}| \rightarrow \sqrt{A^2 + B^2 + C^2}$

  $\Rightarrow \dfrac{Ax_0 + By_0 + Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}} = d$