The real numbers a, b, c, d are such that $a \geq b \geq c \geq d > 0$, $a+b+c+d=1$

Prove that

$$(a+2b+3c+4d)a^a b^b c^c d^d < 1$$

Let's observe $a^a b^b c^c d^d$.

$$a^a b^b c^c d^d = e^{\ln(a^a b^b c^c d^d)} =$$

$$= e^{\ln a^a + \ln b^b + \ln c^c + \ln d^d} = e^{a\ln a + b\ln b + c\ln c + d\ln d}$$

Now, observe $a\ln a + b\ln b + c\ln c + d\ln d$.

We know that $\ln$ is a concave down function and $a+b+c+d=1$, so we can use Jensen's inequality and write

$$a\ln a + b\ln b + c\ln c + d\ln d \leq \ln(a\cdot a + b\cdot b + c\cdot c + d\cdot d) =$$

$$= \ln(a^2+b^2+c^2+d^2)$$

This implies that

$$e^{a\ln a + b\ln b + c\ln c + d\ln d} \leq e^{\ln(a^2+b^2+c^2+d^2)} =$$

$$= a^2+b^2+d^2+c^2$$

So, we can write

$$(a+2b+3c+4d)\,a^{a}b^{b}c^{c}d^{d} \le (a+2b+3c+4d)(a^2+b^2+c^2+d^2) =$$

$$= a^2(a+2b+3c+4d) + b^2(a+2b+3c+4d) +$$
$$+ c^2(a+2b+3c+4d) + d^2(a+2b+3c+4d) \le)$$

Now, note that since $a \ge b \ge c \ge d > 0$,

$(a+2b+3c+4d) \le$

$(a+3b+3c+3d) \le$ (I)

$(3a+b+3c+3d) \le$ (II)

$(3a+3b+c+3d) \le$ (III)

$(3a+3b+3c+d)$ (IV)

$(\le a^2 \cdot I + b^2 \cdot II + c^2 \cdot III + d^2 \cdot IV \Rightarrow)$

Let's denote $a=\alpha_1$ $b=\alpha_2$ $c=\alpha_3$ $d=\alpha_4$

$\sigma(j)$ is denoting the random permutation of first j numbers - $1, 2, \dots, j$

$$(=\sum \alpha^3_{\sigma(j)} + \sum 3\cdot\alpha^2_{\sigma(j)}\cdot\alpha_{\sigma(j)} <$$

$$< \sum \alpha^3_{\sigma(j)} + \sum 3\cdot\alpha^2_{\sigma(j)}\cdot\alpha'_{\sigma(j)} + \sum 6\cdot\alpha_{\sigma(j)}\cdot\alpha'_{\sigma(j)}\cdot\alpha''_{\sigma(j)}$$

Where $\alpha_{\sigma(j)} \neq \alpha'_{\sigma(j)} \neq \alpha''_{\sigma(j)} \neq \alpha_{\sigma(j)}$

Then notice that

$$\sum \alpha^3_{\sigma(j)} + \sum 3 \cdot \alpha^2_{\sigma(j)} \cdot \alpha'_{\sigma(j)} + \sum 6 \cdot \alpha_{\sigma(j)} \cdot \alpha'_{\sigma(j)} \cdot \alpha''_{\sigma(j)} =$$

$$= (a+b+c+d)^3 = 1$$

Proven!