Exercises on Propositional Logic[1]

Exercises KR 1

[1]Thanks to Fabio Previtali

Example - Entailment

Let α, β, γ be three propositional predicates, tell whether or not:

$$\phi(\alpha, \beta, \gamma) = [(\alpha \wedge \beta) \Rightarrow \gamma \models (\alpha \Rightarrow \gamma) \vee (\beta \Rightarrow \gamma)]$$

Let

$\phi_1(\alpha, \beta, \gamma) = (\alpha \wedge \beta) \Rightarrow \gamma$

$\phi_2(\alpha, \beta, \gamma) = (\alpha \Rightarrow \gamma) \vee (\beta \Rightarrow \gamma)$

Solution:

Example - Solution

The entailment is true. The truth table is as follows:

α	β	γ	$\phi_1(\alpha, \beta, \gamma)$	$\phi_2(\alpha, \beta, \gamma)$	$\phi(\alpha, \beta, \gamma)$
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	1	1
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	1
1	1	1	1	1	1

Example - Tautology

Tell whether the following propositional formula is valid:

$$\phi(A, B) = (A \wedge B) \vee (\neg A \wedge \neg B)$$

Solution:

Example - Solution

FALSE. The truth table is as follows:

A	**B**	$\phi(\mathbf{A}, \mathbf{B})$
0	0	1
0	1	0
1	0	0
1	1	1

Exercise - Propositional Logic Representation

Tell which one among the following formulae is a good representation of the sentence.

If John studies and his father works, then his grandfather is happy.

(1) $(Study \land Work) \Rightarrow Happy$
(2) $Study \land Work \land Happy$
(3) $\neg Study \lor \neg Work \lor Happy$
(4) $(Study \lor Work) \Rightarrow Happy$

Solution:

Exercise - Solution

(1) $(Study \land Work) \Rightarrow Happy$

correct

(2) $Study \land Work \land Happy$

incorrect

(3) $\neg Study \lor \neg Work \lor Happy$

correct, logically equivalent to 1. Why?

(4) $(Study \lor Work) \Rightarrow Happy$

incorrect

Example - Modus Ponens

Consider the following knowledge base:

$$\neg A \Rightarrow B$$
$$B \Rightarrow A$$
$$A \Rightarrow (C \wedge D)$$

Prove the proposition $A \wedge C \wedge D$ using Modus Ponens only. Or else explain why this is not possible.

Solution:

Example - Solution

Modus Ponens:

$$\frac{\alpha \Rightarrow \beta, \alpha}{\beta}$$

It is not possible to prove $A \wedge C \wedge D$ using Modus Ponens only.

In fact: Modus Ponens is not applicable to any pair of formulae in the knowledge base.

Example - Propositional Knowledge Base

Consider a knowledge base consisting of the conjunction of the following propositions:

$$\neg A \Rightarrow B$$
$$B \Rightarrow A$$
$$A \Rightarrow (C \wedge D)$$

1 Tell whether the knowledge base is consistent. In the positive case provide a model

2 Transform the above propositions into a new knowledge base written in conjunctive normal form

3 Which of the clauses in your new knowledge base - if any - are not Horn clauses? Justify your answer

Solution:

Example - Solution 1

Recall: A knowledge base is consistent if it admits at least one model.

The knowledge base is **consistent** because there are two models:

$$\{A, B, C, D\} \text{ and } \{A, C, D\}$$

Example - Solution 2

The new knowledge base written in **CNF** is as follows:

$$A \vee B$$
$$\neg B \vee A$$
$$\neg A \vee C$$
$$\neg A \vee D$$

$A \vee B$ is **NOT** a Horn clause, because it has more than one positive literal.

Example - Solution

Derive $A \wedge C \wedge D$ using Resolution (not proven by MP).

Clausal form including the negated thesis:

$$\{A \vee B\}_1, \{\neg B \vee A\}_2, \{\neg A \vee C\}_3, \{\neg A \vee D\}_4, \{\neg A \vee \neg C \vee \neg D\}_5$$

Proof by **resolution**

From (1) and (2) $\Rightarrow \{A\}_6$
From (3) and (6) $\Rightarrow \{C\}_7$
From (4) and (6) $\Rightarrow \{D\}_8$
From (5) and (6) $\Rightarrow \{\neg C \vee \neg D\}_9$
From (7) and (9) $\Rightarrow \{\neg D\}_{10}$
From (8) and (10) $\Rightarrow \{\}$

Pierino

Given the following propositional symbols:
ST to indicate that Pierino studies;
SY to indicate that Pierino is silly;
LU to indicate that Pierino is lucky;
PS to indicate that Pierino passes Artificial Intelligence.

If Pierino studies and is not silly, he passes Artificial Intelligence.
If Pierino is not lucky and is silly, he does not pass Artificial Intelligence.

Pierino

1. Represent the above sentence in the propositional calculus.
2. Tell which one, among the following sets, is a model, and which one is not a model, for the above formulae.
   $\{ST, SY, PS\}$; $\{LU, PS\}$; $\{\}$; $\{LU, SY, PS\}$.
3. Specify which formulae, different from PS, need to be added to derive that Pierino passes Artificial Intelligence.
4. Show how the above conclusion can be derived by resolution.

Solution:

Solution-1

1. Represent the sentences in the propositional calculus.

   If Pierino studies and is not silly, he passes Artificial Intelligence.

   $ST \wedge \neg SY \Rightarrow PS$

   If Pierino is not lucky and is silly, he does not pass Artificial Intelligence.

   $\neg LU \wedge SY \Rightarrow \neg PS$

Solution-2

2. Tell which one, among the following sets, is a model, and which one is not a model, for the above formulae.

$\{ST, SY, PS\}$ no
$\{LU, PS\}$ yes
$\{\}$ yes
$\{LU, SY, PS\}$ yes.

Solution-3

3. Specify which formulae, different from PS, need to be added to derive that Pierino passes Artificial Intelligence.

   $\neg SY \wedge ST$

4. Show how the above conclusion can be derived by resolution.

   (1) $\neg ST, SY, PS$
   (2) ST
   (3) $\neg SY$
   (4) $\neg PS$
   From (1) and (4) $\Rightarrow \{\neg ST, SY\}_5$
   From (5) and (2) $\Rightarrow \{SY\}_6$
   From (3) and (6) $\Rightarrow \{\}$

DA QUI

Example - Resolution

Using **resolution**, tell whether the following formula can be proven:

$$\{A \Leftrightarrow B, A \vee B\} \vdash_R (A \wedge B)$$

Solution: $\underset{1}{\{\neg A \vee B\}}, \underset{2}{\{\neg B \vee A\}}, \underset{3}{\{A \vee B\}}, \underset{4}{\{\neg A \vee \neg B\}}$

$2 \text{ e } 3 = \{A\}_5$

$1 \text{ e } 4 = \{\neg A\}_6$

$6 \text{ e } 5 = \{\ \}$

NON collegato

Collegato

$2 \text{ e } 3 = \{A\}_5$

$5 \text{ e } 1 = \{B\}_6$

$6 \text{ e } 4 = \{\neg A\}_7$

$7 \text{ e } 5 = \{\ \}$

Example - Solution

First step is to negate the thesis and then transform the given formula in clausal form:

$$\{A \Leftrightarrow B, A \vee B, \neg(A \wedge B)\}$$

$$\{A \Rightarrow B, B \Rightarrow A, A \vee B, \neg(A \wedge B)\}$$

$$\{\neg A, B\}_1, \{\neg B, A\}_2, \{A, B\}_3, \{\neg A, \neg B\}_4$$

From (1) and (3) $\Rightarrow \{B\}_5$
From (2) and (3) $\Rightarrow \{A\}_6$
From (4) and (5) $\Rightarrow \{\neg A\}_7$
From (6) and (7) $\Rightarrow \{\}$

Example - Resolution

If I leave and go on vacation, then I am happy
If I leave then I go on vacation
I leave

Question: Can I derive, *I go on vacation and I am happy?*

Solution: $V \wedge H$

$$\{\{(L \wedge V) \Rightarrow H\}, \{L \Rightarrow V\}, \{L\}, \{\neg(V \wedge H)\}\}$$

$$\overset{1}{\{\neg L \vee \neg V \vee H\}} \quad \overset{2}{\{\neg L \vee V\}} \quad \overset{3}{\{L\}} \quad \overset{4}{\{\neg V \vee \neg H\}}$$

$1 \, e \, 2 = (\neg L \vee H)_5 \quad 5 \, e \, 3 = (H)_6 \quad 6 \, e \, 4 = (\neg V)_7 \quad 7 \, e \, 2 = \neg L_8 \quad 8 \, e \, 3 = \{\}$

Example - Resolution - Representation

$$\Gamma = \{\{(L \wedge V) \Rightarrow H\}, \{L \Rightarrow V\}, \{L\}\} \vdash_R (V \wedge H)$$

Negate the thesis:

$$\{\{(L \wedge V) \Rightarrow H\}, \{L \Rightarrow V\}, \{L\}, \{\neg(V \wedge H)\}\}$$

Example - Solution

Transform into clausal form:

$$\{\neg L, \neg V, H\}_1, \{\neg L, V\}_2, \{L\}_3, \{\neg V, \neg H\}_4$$

From (1) and (2) $\Rightarrow \{\neg L, H\}_5$
From (3) and (5) $\Rightarrow \{H\}_6$
From (4) and (6) $\Rightarrow \{\neg V\}_7$
From (2) and (7) $\Rightarrow \{\neg L\}_8$
From (3) and (8) $\Rightarrow \{\}$

Exercise

Let A, B, C be propositional symbols. Given

$$KB = \{A \Rightarrow C, B \Rightarrow C, A \vee B\}$$

tell whether the formula C can be derived from KB in each of the following cases:

Non si può fare perchè non c'è un punto di partenza ergo A oppure B

1 using **Modus Ponens**

2 using **Resolution**

Both for (1) and (2), in case of positive answer show the derivation, in case of negative answer explain why.

Solution:

$$\{\underbrace{\neg A \vee C}_{1}, \underbrace{\neg B \vee C}_{2}, \underbrace{A \vee B}_{3}, \underbrace{\neg C}_{4}\}$$

$1 \text{ e } 3 = \{B\ C\}\ 5$

$5 \text{ e } 2 = \{C\}\ 6 \qquad 6 \text{ e } 4 = \{\}$

Solution

C cannot be derived with Modus Ponens as we only know that $A \vee B$ is true, but we do not know which one of them A or B is true.

Hence, we can neither apply Modus Ponens to:

- A and $A \Rightarrow C$ nor to
- B and $B \Rightarrow C$

Solution 2

$$\{\neg A \vee C\}_1, \{\neg B \vee C\}_2, \{A \vee B\}_3, \{\neg C\}_4$$

From (1) and (3) $\Rightarrow \{B \vee C\}_5$
From (2) and (5) $\Rightarrow \{C\}_6$
From (4) and (6) $\Rightarrow \{\}$

Wet and rain (home)

Consider the following set of sentences:

If it rains then it is wet $R \rightarrow W$

If it is wet then it does not rain $W \rightarrow \neg R$

$\vdash_R$ It rains R

(a) Write the corresponding propositional formulae

(b) Prove, via resolution, that they are inconsistent.

Solution: $\{\neg R \vee W\}_1, \{\neg W \vee \neg R\}_2, \{R\}_3$

Solution

(a) Write the corresponding propositional formulae

If it rains then it is wet
1.$Rains \Rightarrow Wet$

If it is wet then it does not rain
2.$Wet \Rightarrow \neg Rains$

It rains
3. $Rains$

Solution

(b) Prove, via resolution, that they are inconsistent.

1.$\neg Rains \vee Wet$
2. $\neg Wet \vee \neg Rains$
3. $Rains$

4. $\neg Rains$ from 1 and 2
5. $\{\}$ from 3 and 4

Exercise - home

Consider the following propositional formulae:

$$P \Rightarrow (Q \Leftrightarrow R) \quad \neg P \vee (\neg Q \vee R \wedge \neg R \vee Q)$$
$$\neg Q \vee (P \vee R) \quad Q \Rightarrow (P \vee R)$$
$$\neg R \vee Q \wedge \neg P \quad R \Rightarrow (Q \wedge \neg P)$$

- **1** Convert them into Conjunctive Normal Form
- **2** Tell whether or not the resulting set of clauses is Horn
- **3** Tell whether or not the resulting set of clauses is satisfiable, in the positive case show a model

$\{\neg P \vee (\neg Q \vee R\}_1, \{\neg P \vee (\neg R \vee Q)\}_2, \{\neg Q \vee (P \vee R)\}_3, \{\neg R \vee Q\}_4, \{\neg R \vee \neg P\}_5$

NOT HORN

$\{\}, \{P\}$

Exercise - home

$H \iff (L \vee G)$

I'm happy iff I won the lottery or my girlfriend is with me

$R \Rightarrow \neg G$

If it is raining my girlfriend is not with me

It is raining and I am happy

$R \wedge H \quad \vdash_R H \iff L$

Question: Can I derive, *I am happy iff I won the lottery?*

$$\Gamma = \{H \Leftrightarrow (L \vee G), R \Rightarrow \neg G, R \wedge H\} \boxed{\vdash_R (H \Leftrightarrow L)}$$

$\{\neg H \vee L \vee G\}_1, \{\neg L \vee H\}_2, \{\neg G \vee H\}_3, \{\neg R \vee \neg G\}_4, \{R\}_5, \{H\}_6, \{H \vee L\}_7, \{\neg H \vee \neg L\}_8$

DEDUCTION

$\vdash_R (H \iff L)$

$\neg(H \iff L) = \neg((H \to L) \wedge (L \to H))$

$= \neg((\neg H \vee L) \wedge (\neg L \vee H)) = \neg(\neg H \wedge L) \vee \neg(\neg L \wedge H) = (H \vee \neg L) \wedge (L \vee \neg H)$

$1 \text{ e } 8 = \{\neg H \vee G\}_9$

$9 \text{ e } 4 = \{\neg H \vee \neg R\}_{10}$

$10 \text{ e } 5 = \{\neg H\}_{11}$

$11 \text{ e } 6 = \{\}$

$= (H \vee L) \wedge (\cancel{H \vee \neg H}) \wedge (\cancel{\neg L \vee L}) \wedge (\neg L \vee \neg H)$

semplifico

Back to the Wumpus - home

Knowing that there is breeze in [2, 1] and not in [1, 2], infer using resolution that :

- there is no pit in [2, 2]
- there is a pit in [3, 1]

Recall the rules of the environment:

$$
\begin{aligned}
B_{1,2} &\Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{1,3}) \\
B_{2,1} &\Leftrightarrow (P_{1,1} \vee P_{2,2} \vee P_{3,1})
\end{aligned}
$$

Exploring a wumpus world