\section{LECTURE 9}

- Readings: Sections 3.4-3.5

\section{Outline}

- PDF review

- Multiple random variables

- conditioning

- independence

- Examples

\section{Summary of concepts}

$\begin{array}{rcl}p_{X}(x) & & f_{X}(x) \\ & F_{X}(x) & \\ \sum_{x} x p_{X}(x) & \mathbf{E}[X] & \int x f_{X}(x) d x \\ & \operatorname{var}(X) & \\ p_{X, Y}(x, y) & & f_{X, Y}(x, y) \\ p_{X \mid A}(x) & & f_{X \mid A}(x) \\ p_{X \mid Y}(x \mid y) & & f_{X \mid Y}(x \mid y)\end{array}$

Joint PDF $f_{X, Y}(x, y)$

$\mathbf{P}((X, Y) \in S)=\iint_{S} f_{X, Y}(x, y) d x d y$

- Interpretation:

$\mathrm{P}(x \leq X \leq x+\delta, y \leq Y \leq y+\delta) \approx f_{X, Y}(x, y) \cdot \delta^{2}$

- Expectations:

$\mathrm{E}[g(X, Y)]=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X, Y}(x, y) d x d y$

- From the joint to the marginal:

$$
f_{X}(x) \cdot \delta \approx \mathbf{P}(x \leq X \leq x+\delta)=
$$

- X and Y are called independent if

$$
f_{X, Y}(x, y)=f_{X}(x) f_{Y}(y), \quad \text { for all } x, y
$$

\section{Continuous r.v.'s and pdf's}

`![](https://cdn.mathpix.com/cropped/2023_04_06_60ab0a2faf50c68aa93cg-1.jpg?height=228&width=680&top_left_y=268&top_left_x=1189)`

$$
\mathbf{P}(a \leq X \leq b)=\int_{a}^{b} f_{X}(x) d x
$$

- $\mathbf{P}(x \leq X \leq x+\delta) \approx f_{X}(x) \cdot \delta$

- $\mathbf{E}[g(X)]=\int_{-\infty}^{\infty} g(x) f_{X}(x) d x$

\section{Buffon's needle}

- Parallel lines at distance d Needle of length ℓ (assume $\ell<d$)

- Find P (needle intersects one of the lines)

`![](https://cdn.mathpix.com/cropped/2023_04_06_60ab0a2faf50c68aa93cg-1.jpg?height=171&width=231&top_left_y=1671&top_left_x=1405)`

- $X \in[0, d / 2]$: distance of needle midpoint to nearest line

- Model: X, Θ uniform, independent $f_{X, \Theta}(x, \theta)=\quad 0 \leq x \leq d / 2,0 \leq \theta \leq \pi / 2$

- Intersect if $X \leq \frac{\ell}{2} \sin \Theta$

$$
\begin{aligned}
\mathbf{P}\left(X \leq \frac{\ell}{2} \sin \Theta\right) & =\iint_{x \leq \frac{\ell}{2} \sin \theta} f_{X}(x) f_{\Theta}(\theta) d x d \theta \\
& =\frac{4}{\pi d} \int_{0}^{\pi / 2} \int_{0}^{(\ell / 2) \sin \theta} d x d \theta \\
& =\frac{4}{\pi d} \int_{0}^{\pi / 2} \frac{\ell}{2} \sin \theta d \theta=\frac{2 \ell}{\pi d}
\end{aligned}
$$

Conditioning

- Recall

$$
\mathbf{P}(x \leq X \leq x+\delta) \approx f_{X}(x) \cdot \delta
$$

- By analogy, would like:

$\mathbf{P}(x \leq X \leq x+\delta \mid Y \approx y) \approx f_{X \mid Y}(x \mid y) \cdot \delta$

- This leads us to the definition:

$f_{X \mid Y}(x \mid y)=\frac{f_{X, Y}(x, y)}{f_{Y}(y)} \quad$ if $f_{Y}(y)>0$

- For given y, conditional PDF is a

(normalized) "section" of the joint PDF

- If independent, $f_{X, Y}=f_{X} f_{Y}$, we obtain

$$

f_{X \mid Y}(x \mid y)=f_{X}(x)
$$

\section{Stick-breaking example}

- Break a stick of length ℓ twice: break at X : uniform in $[0,1]$; break again at Y, uniform in $[0, X]$

$$
f_{X, Y}(x, y)=f_{X}(x) f_{Y \mid X}(y \mid x)=
$$

on the set:

$\mathbf{E}[Y \mid X=x]=\int y f_{Y \mid X}(y \mid X=x) d y=$

Joint, Marginal and Conditional Densities

Image by MIT OpenCourseWare, adapted from Probability, by J. Pittman, 1999.

$$
f_{X, Y}(x, y)=\frac{1}{\ell x}, \quad 0 \leq y \leq x \leq \ell
$$

$$
\begin{aligned}

& f_{Y}(y)=\int f_{X, Y}(x, y) d x \\
&=\int_{y}^{\ell} \frac{1}{\ell x} d x \\
&=\frac{1}{\ell} \log \frac{\ell}{y}, \quad 0 \leq y \leq \ell \\
& \mathrm{E}[Y]=\int_{0}^{\ell} y f_{Y}(y) d y=\int_{0}^{\ell} y \frac{1}{\ell} \log \frac{\ell}{y} d y=\frac{\ell}{4}
\end{aligned}
$$

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$$\mathbf{E}[Y \mid X=x]=\int y f_{Y \mid X}(y \mid X=x) d y=f_{X, Y}(x, y)=\frac{1}{\ell x}, \quad 0 \leq y \leq x \leq \ell$$