Inner and Outer Folds: A Minimal Structural Account

Abstract

Fix an observation map $\pi : X \to B$ at scale κ. Inner folds are fiberwise deformations generated by π-preserving flows and actions that avoid the critical set. Outer folds arise along the critical locus where regular fibers split into multiple connected components. Camouflage is characterized as observational coincidence induced by group orbits and, near folds, as a boundary symptom. We state a local normal form, three short propositions, and a scale monotonicity corollary.

1 Introduction

We work in a minimal geometric setting that separates internal generative variation from public boundary phenomena. The account is structural and observation-driven: equivalence and difference are defined relative to an observation map $\pi : X \to B$. Inner folds capture π-invariant deformations within regular fibers. Outer folds capture branching at fold singularities. Camouflage explains when different points appear identical to a fixed observation.

2 Setup and Definitions

Standing Assumptions 2.1. X and B are smooth manifolds. The observation map $\pi : X \to B$ is a submersion on the regular locus $X^\circ := \mathrm{Reg}(\pi)$ and has generic fold singularities on a closed set $\mathrm{Disc} \subset X$ where the rank of $d\pi$ drops by one. All statements are local and apply in a neighborhood avoiding higher singularities.

Definition 2.2 (Flows and π-preserving actions)**.** Let V be a family of smooth vector fields on X tangent to the fibers of π on X°; their time-t flows preserve π. Let $G \le \mathrm{Aut}_\pi(X)$ be a group of diffeomorphisms with $\pi \circ g = \pi$ for all $g \in G$.

Definition 2.3 (Inner fold domain and equivalence)**.** The inner fold domain is $X^\circ = X \setminus \mathrm{Disc}$. Define an equivalence relation $\sim_{\mathrm{in}}$ on X° as the smallest relation containing: (i) points connected by a V-flow trajectory within a single fiber, and (ii) points on the same G-orbit within that fiber. Classes of $\sim_{\mathrm{in}}$ are *inner-fold classes*.

Definition 2.4 (Outer fold phenomenon)**.** Along Disc, regular fibers of π admit a local decomposition into multiple connected components. The resulting branching and the necessity to intersect Disc when passing between components constitute the *outer fold*.

Definition 2.5 (Camouflage)**.** Let $U : X \to \mathcal{O}$ be an observation that factors through π, namely $U = Q \circ \pi$ for some map $Q : B \to \mathcal{O}$. Distinct points $x \neq y$ with $U(x) = U(y)$ are in *camouflage* with respect to U. It is *generative camouflage* if $y = g \cdot x$ for some $g \in G$ and the connecting path avoids Disc.

3 Local fold model

Lemma 3.1 (Fold normal form)**.** *For each $p \in \mathrm{Disc}$ there exist local coordinates $(u, v) \in \mathbb{R}^{m-1} \times \mathbb{R}$ on X and $u \in \mathbb{R}^{m-1}$ on B such that*

$$\pi(u, v) = (u, v^2) \quad \textit{up to diffeomorphism on source and target.}$$

In these coordinates, $\mathrm{Disc} = \{v = 0\}$ and for each fixed u_0 near $u = 0$ the regular fiber $\pi^{-1}(u_0)$ decomposes as $F^+(u_0) \sqcup F^-(u_0)$ given by $v > 0$ and $v < 0$ respectively.

Proof. Standard fold normal form by the Morse lemma with parameters; see any reference on stable map singularities. The rank drop is one and the Hessian is nondegenerate in the normal direction, which yields $(u, v) \mapsto (u, v^2)$ locally. □

4 Main propositions

Proposition 4.1 (Flow generates inner folds)**.** *Let $\gamma : [0, 1] \to X^\circ$ be a concatenation of V-flow segments contained in a single fiber $\pi^{-1}(b)$. Then $\gamma(0) \sim_{\mathrm{in}} \gamma(1)$. In particular, G-equivalent points in the same fiber are also $\sim_{\mathrm{in}}$-equivalent.*

Proof. By definition of $\sim_{\mathrm{in}}$ and the π-preserving property, each segment remains in $\pi^{-1}(b)$. Concatenation stays within the generated equivalence. The G-case is immediate since $g \in G$ preserves π. □

Proposition 4.2 (Disjunctive branching generates outer folds)**.** *In the coordinates of Lemma 3.1, any path $\eta : [0, 1] \to X$ joining a point in $F^+(u_0)$ to a point in $F^-(u_0)$ intersects* Disc*. Hence the components $F^\pm(u_0)$ are separated by* Disc*, and passing between them forces a crossing of the critical set.*

Proof. In the normal form $\pi(u, v) = (u, v^2)$, the sign of v determines the component. Any continuous path from $v > 0$ to $v < 0$ must pass through $v = 0$, which equals Disc. □

Proposition 4.3 (Camouflage: inner illusion and boundary symptom)**.** *Let $U = Q \circ \pi$ be as above.*

(i) If $y = g \cdot x$ for some $g \in G$ and a connecting V-path avoids Disc*, then $U(y) = U(x)$ and $x \sim_{\mathrm{in}} y$. This is generative camouflage.*

(ii) In a fold neighborhood, distinct points $x^\pm \in F^\pm(u_0)$ satisfy $U(x^+) = U(x^-)$ while any path from x^+ to x^- crosses Disc*. Camouflage acts as a boundary symptom of the outer fold.*

Proof. (i) Since $\pi \circ g = \pi$ and $U = Q \circ \pi$, $U(g \cdot x) = U(x)$. Avoiding Disc keeps the motion in X°, hence $x \sim_{\text{in}} y$ by Proposition 4.1. (ii) In the normal form, $\pi(x^+) = \pi(x^-) = u_0$ so U agrees. Proposition 4.2 forces any connecting path to meet Disc. $\square$

Corollary 4.4 (Scale monotonicity). *Let $\kappa' \succeq \kappa$ be a refinement of observation. Suppose $U_\kappa = Q \circ \pi$ and $U_{\kappa'} = R \circ U_\kappa$ for some R. Then the induced equivalence $\sim_{\text{in},\kappa'}$ on X° refines $\sim_{\text{in},\kappa}$.*

Proof. If $x \sim_{\text{in},\kappa'} y$ then $U_{\kappa'}(x) = U_{\kappa'}(y)$. Since $U_{\kappa'} = R \circ U_\kappa$, equality at κ' implies equality at κ. The generators of $\sim_{\text{in},\kappa'}$ are a subset of those for $\sim_{\text{in},\kappa}$, hence refinement. $\square$

5 Minimal example and test rule

Example 5.1 (Canonical fold). Let $X = \mathbb{R}^m$ with coordinates (u, v) and $B = \mathbb{R}^{m-1}$ with coordinate u. Take $\pi(u, v) = (u, v^2)$, $V = \{\partial_v\}$ on $X^\circ = \{v \neq 0\}$, and $G = \{\pm 1\}$ acting by $v \mapsto \pm v$ with u fixed. Then inner-fold classes are the connected components within each fiber away from Disc $= \{v = 0\}$ modulo G-action. Outer fold is the separation of $F^\pm(u_0)$ by Disc. Camouflage at an observation $U = Q \circ \pi$ identifies $F^\pm(u_0)$ while any transition crosses Disc.

Remark 5.2 (Scope and costs). All statements are local near folds and ignore higher codimension singularities. The cost of observation is encoded in the factorization $U = Q \circ \pi$. Stronger results require extra structure such as properness, compactness, or finite presentation of fibers.

6 Conclusions

Inner folds are generated by π-invariant flows and actions on the regular locus. Outer folds are branching phenomena forced at the critical set. Camouflage is observational coincidence produced by inner generative motion or by boundary geometry near folds. The model is minimal and composable with measurement-driven protocols.