A Formal Mathematical Structure of Inner and Outer Folds

A Certifiable Framework for Context Alignment

Abstract

We present a formal structure for Deleuze-inspired *inner* and *outer* folds and their transformations. The framework is phrased in category theory and differential geometry, with triadic invariants $\mathcal{V}_\kappa = \ker d\Phi_\kappa$ (kernel-flow distribution), $\mathcal{G}_{\Phi_\kappa}$ (the *étale* groupoid of local Φ_κ-preserving diffeomorphisms, i.e. masquerade arrows), and a discriminant split $\mathrm{Disc}(\Phi_\kappa) = \mathrm{Crit}(\Phi_\kappa) \cup \Phi_\kappa^{-1}(\Sigma) \cup \mathrm{J}(\rho \circ \mathrm{Obs}_*)$. Core results include a pullback criterion presented as a 2-pullback (iso-comma) test for alignment, an equivalence theorem under explicit regularity and generation hypotheses, and a scale monotonicity principle. The analytic versus smooth jet clauses are separated to keep claims auditable.

1 Basic Data and Axioms

Cross-level descriptions are often verbally interchangeable while remaining structurally incompatible. This section lands the minimal data $(X_\kappa, Y_\kappa, \Phi_\kappa)$ and the triad $(\mathcal{V}_\kappa, \mathcal{G}_{\Phi_\kappa}, \mathrm{Disc})$, separating inner flows from outer divergences and isolating a regular window where checks can be carried out.

1.1 Context-alignment triple and triad

Definition 1 (Context-alignment data)**.** *Fix a scale $\kappa \in \mathcal{K}$. A* context-alignment datum *is a triple*

$$\mathbf{A}_\kappa = (X_\kappa, Y_\kappa, \Phi_\kappa),$$

with $\Phi_\kappa : X_\kappa \to Y_\kappa$ a C^1 map (or a regular morphism). Define:

$$\mathcal{V}_\kappa(x) := \mathrm{Ker}(d\Phi_\kappa)_x, \qquad \mathcal{G}_{\Phi_\kappa} := \{\, g : U \to V \ \textit{local } C^1\textit{-diffeo} \ \mid \ \Phi_\kappa \circ g = \Phi_\kappa \ \textit{on } U \,\},$$

interpreted as an étale groupoid *of local masquerade arrows. The discriminant is split as*

$$\mathrm{Disc}(\Phi_\kappa) = \underbrace{\mathrm{Crit}(\Phi_\kappa)}_{\textit{mechanism, rank drop}} \cup \underbrace{\Phi_\kappa^{-1}(\Sigma)}_{\textit{appearance, image singular set}} \cup \underbrace{\mathrm{J}(\rho \circ \mathrm{Obs}_*)}_{\textit{semantic/policy jump}}.$$

Axiom 1 (Regular window)**.** *There exists an open set $H_{\mathrm{reg}} \subset X_\kappa$ on which Φ_κ has constant rank and each fiber is an immersed submanifold tangent to $\mathcal{V}_\kappa$. Put $X_\kappa^\circ := H_{\mathrm{reg}}$ and assume $X_\kappa^\circ \cap \mathrm{Crit}(\Phi_\kappa) = \varnothing$.*

Axiom 2 (Scale process)**.** *$\mathcal{K}$ is a directed poset. If $\kappa' \succeq \kappa$, there exists $q_{\kappa\kappa'} : Y_{\kappa'} \to Y_\kappa$ with $\Phi_\kappa = q_{\kappa\kappa'} \circ \Phi_{\kappa'}$.*

Axiom 3 (Fold normal form)**.** *For each $x \in \mathrm{Disc}(\Phi_\kappa)$ there are local coordinates (u, v) with $\Phi_\kappa(u, v) = (u, v^2)$, modeling a fold singularity.*

1.2 Inner and outer folds

Definition 2 (Inner-fold class and outer-fold separation)**.** *On X_κ°, define $x \sim_{\text{in}} y$ if there exists a piecewise C^1 path $\gamma \subset X_\kappa^\circ$ from x to y with $\dot{\gamma}(t) \in \mathcal{V}_\kappa(\gamma(t))$ almost everywhere; write $[x]_{\text{in}}$ for the class.*

Two points $x, y \in X_\kappa$ are outer-fold separable *if every continuous path from x to y intersects* $\mathrm{Disc}(\Phi_\kappa)$.

Remark 1. If $g \in \mathcal{G}_{\Phi_\kappa}$ with $x \in \mathrm{dom}(g)$, then $x \sim_{\text{in}} g(x)$. If a fiber $\Phi_\kappa^{-1}(y) \cap X_\kappa^\circ$ is connected, it forms one inner-fold class.

The remaining friction is that the agreement described so far is still asserted in words. The next section introduces exo and endo observations together with a 2-pullback witness that arbitrates alignment at the level of structure.

2 Transformers and the Pullback Criterion

External and internal observations often annotate one another without a principled arbiter. Here natural transformations and an iso-comma (2-pullback) witness provide a transparent alignment check and say when the two observations refer to the same discernible content.

2.1 Exo/Endo observations and transformers

Definition 3 (Exo/Endo observations and transformers)**.** *Let*

$$U_{\text{exo}} : \mathcal{C} \to \mathcal{A}_{\text{exo}}, \qquad U_{\text{endo}} : \mathcal{C} \to \mathcal{A}_{\text{endo}}$$

be observation functors. Suppose natural transformations

$$\eta : \text{env} \circ U_{\text{exo}} \Rightarrow U_{\text{endo}}, \qquad \epsilon : \text{dev} \circ U_{\text{endo}} \Rightarrow U_{\text{exo}}.$$

Definition 4 (Pullback criterion (2-pullback / iso-comma witness))**.** *Given $U_{\text{exo}}(x)$ and $U_{\text{endo}}(x)$, a* pullback witness *is an object L with maps*

$$\begin{array}{ccc} L & \xrightarrow{\lambda_{\text{endo}}} & U_{\text{endo}}(x) \\ {\scriptstyle\lambda_{\text{exo}}}\downarrow & & \downarrow{\scriptstyle t_{\text{endo}}} \\ U_{\text{exo}}(x) & \xrightarrow[t_{\text{exo}}]{} & \textit{Target} \end{array}$$

such that the square is a 2-pullback in the ambient 2-category, and the induced invariants $(\mathcal{V}_\kappa, \mathcal{G}_{\Phi_\kappa}, \mathrm{Disc}(\Phi_\kappa))$ agree when restricted to L. Alignment holds at x if such L exists.

An alignment witness closes the verbal gap. The next section moves to the regular window and establishes the basic invariances that turn inner folds and outer folds into operational tests.

3 Main properties on the regular window

Within the regular window, two principles make differences traceable: kernel flows preserve the observation value, while the discriminant enforces observable branching. This separates being co-descriptive from being the same object under the given observation.

Proposition 1 (Inner-fold invariance). *If $\gamma \subset X_\kappa^\circ$ with $\dot\gamma(t) \in \mathcal{V}_\kappa(\gamma(t))$, then $\Phi_\kappa(\gamma(t))$ is constant. Hence $x \sim_{\text{in}} y \Rightarrow \Phi_\kappa(x) = \Phi_\kappa(y)$.*

Proposition 2 (Outer fold forces branching). *If x, y are outer-fold separable, any path γ from x to y meets* $\mathrm{Disc}(\Phi_\kappa)$*. In the fold normal form, crossing $v = 0$ yields a publicly observable branching.*

Theorem 1 (Pullback equivalence under naturality). *Assume the pullback criterion holds at x and η, ϵ are natural. Then*

$$\Phi_\kappa^{\text{endo}}(\text{env} \circ U_{\text{exo}}(x)) \cong \Phi_\kappa^{\text{endo}}(U_{\text{endo}}(x)), \qquad \Phi_\kappa^{\text{exo}}(\text{dev} \circ U_{\text{endo}}(x)) \cong \Phi_\kappa^{\text{exo}}(U_{\text{exo}}(x)).$$

Thus exo and endo transformations change only the reference chart, not the κ-level discernible content.

Having fixed the invariances, we turn to the main semantic risk: masquerade. It is confined to Φ_κ-preserving local arrows and kernel-flow orbits.

4 Masquerade vs. Kernel Flow

Masquerade is the attempt to let linguistic likeness pass for structural sameness. By restricting it to the *étale* groupoid of Φ_κ-preserving local maps and to kernel flows, any purported identification must be generated by verifiable primitives.

Definition 5 (Kernel flow and masquerade arrow). *Integral curves of $\mathcal{V}_\kappa$ are* kernel flows*. An arrow $g : U \to V$ in $\mathcal{G}_{\Phi_\kappa}$ is a* masquerade *if $\Phi_\kappa \circ g = \Phi_\kappa$ on U.*

Proposition 3 (Inner-class generation by primitives). *On X_κ°, if $x \sim_{\text{in}} y$, then y is reachable from x by a finite concatenation of kernel flows and local arrows in $\mathcal{G}_{\Phi_\kappa}$. A global diffeomorphism need not exist.*

Definition 6 (Masquerade failure). *An exo/endo correspondence is a* masquerade failure *if no pullback witness exists. Then some path must cross* $\mathrm{Disc}(\Phi_\kappa)$*, breaking inner-fold equivalence and producing outer-fold separation.*

Proposition 4 (No masquerade across an outer fold). *If x, y are outer-fold separable, there is no local arrow $g \in \mathcal{G}_{\Phi_\kappa}$ with $g(x) = y$. Any identification of x and y is a masquerade failure.*

The boundary on masquerade is now explicit. The next step uses scale and jet clauses to regulate detectability: coarse differences do not vanish at finer scales, and finite-order breaks surface under refinement.

5 Scale Monotonicity and Jets

Refinement acts as a pressure amplifier: a difference visible at a coarse level remains visible when the observation is refined; if only a finite-order break is present, refinement brings it to the surface. The analytic and smooth cases are separated so that claims track their verification class.

Definition 7 (Scale monotonicity). *If $\kappa' \succeq \kappa$, then $\Phi_\kappa = q_{\kappa\kappa'} \circ \Phi_{\kappa'}$.*

Theorem 2 (Irreversibility of separation). *If $\Phi_\kappa(x) \neq \Phi_\kappa(y)$, then $\Phi_{\kappa'}(x) \neq \Phi_{\kappa'}(y)$ for any $\kappa' \succeq \kappa$. The converse need not hold.*

Definition 8 (Finite-jet indistinguishability)**.** *Let $j^r\Phi_\kappa$ be the r-jet. If $j^r\Phi_\kappa(x) = j^r\Phi_\kappa(y)$, then x, y are indistinguishable up to order r.*

Proposition 5 (Jet clauses: analytic vs. smooth)**.** *(a)* ***Analytic case.*** *If Φ_κ is real-analytic near a path from x to y and all finite jets agree along the path, then there exists a local arrow $g \in \mathcal{G}_{\Phi_\kappa}$ with $g(x) = y$.*

(b) ***Smooth case.*** *In the C^∞ category, if a finite-order discrepancy appears at some point on any path from x to y, then for a refinement $\kappa' \succeq \kappa$ this discrepancy is detected and manifests as outer-fold separation near that point.*

With detectability in place, we add explicit gates that allow an equivalence theorem: stability and completeness of the primitives inside the regular window, and a boundary rule for crossings of the discriminant.

6 Alignment Equivalence with Explicit Gates

The equivalence needs clear gates. Stability ensures primitives preserve the observation, completeness ensures they generate all local preservers, and the boundary rule makes class changes visible at the discriminant.

Standing Hypotheses 1 (Primitive generation and boundary gate on X_κ°)**.** *(i)* ***Stability (S).*** *Each primitive move (kernel flow or $\mathcal{G}_{\Phi_\kappa}$-arrow) preserves Φ_κ.*

(ii) ***Completeness (C).*** *Any local Φ_κ-preserving variation is generated by these primitives.*

(iii) ***Boundary (B).*** *Any transverse crossing of* $\mathrm{Disc}(\Phi_\kappa)$ *changes the inner-fold class.*

Theorem 3 (Alignment equivalence on the regular window)**.** *Let $x_1, x_2 \in X_\kappa^\circ$ and assume* (S)(C)(B)*. The following are equivalent.*

(a) There exists a 2-pullback witness L exhibiting exo/endo agreement with invariant preservation.

(b) x_2 is obtained from x_1 by a finite concatenation of kernel flows and arrows of $\mathcal{G}_{\Phi_\kappa}$.

(c) x_1, x_2 lie in the same inner-fold class and no path between them crosses $\mathrm{Disc}(\Phi_\kappa)$*.*

Proof sketch. $(a) \Rightarrow (b)$: naturality and invariant preservation give a path inside a connected fiber component; by (C) it is generated by primitives. $(b) \Rightarrow (c)$: primitives preserve Φ_κ by (S) and avoid boundary change by (B). $(c) \Rightarrow (a)$: let L be the connected fiber component through x_1, x_2; it yields the 2-pullback witness with invariant agreement. □

The gate is now explicit and testable. The final technical step gives minimal examples and a counterexample routine so that decisions return either a witness or a smallest failure.

7 Minimal Examples and Counterexample Windows

Operational testing needs two instruments: a minimal visible difference (the fold normal form) and a minimal counterexample routine (push toward the critical region). Together they provide a practical audit: a witness when alignment holds, a counterexample otherwise.

7.1 Fold normal form

For $\Phi(u, v) = (u, v^2)$, the discriminant is $\{v = 0\}$. For $v \neq 0$, each fiber splits into two branches. Any cross-branch path meets Disc, yielding an outer fold. Any motion along a single branch tangent to $\mathcal{V}$ preserves the observation, yielding an inner fold.

7.2 Counterexample window

If alignment is judged only by a partial observation $Q \circ \Phi_\kappa$, there exist x, y with $Q(\Phi_\kappa(x)) = Q(\Phi_\kappa(y))$ but $\Phi_\kappa(x) \neq \Phi_\kappa(y)$. This is *boundary camouflage*. Pushing inputs toward the fold normal form's critical region exposes a minimal counterexample.

With the testing protocol complete, we close the technical development and return to a concise conceptual synthesis.

8 Conclusion

Inner folds are generated by kernel flows and local masquerade arrows; outer folds arise from the discriminant split and produce observable branching. The pullback criterion as a 2-pullback supplies a robust alignment test. Scale monotonicity and the analytic versus smooth jet clauses separate true symmetries from masquerades. All gates are explicit and auditable.

Philosophical Reflection: Re-Embedding into Deleuzian Discourse

Difference. $\mathcal{V}_\kappa$ articulates an intrinsic space of motion within a fixed observation value. Difference is not a nominal opposition but a generative flow that preserves the observable while varying the underlying state. This gives a concrete seat for a production of difference.

Repetition. $\mathcal{G}_{\Phi_\kappa}$ formalizes repetition as the class of Φ_κ-preserving local equivalences. Repetition is not copy but orbit: it is produced by composing kernel flows with local masquerade arrows. The result is a structured equivalence relation rather than a rhetorical slogan.

Identity and its illusions. Without a pullback witness, any apparent sameness is an unlicensed identification. The discriminant split separates mechanism-level rank drops, image-level fold singularities, and semantic jumps induced by the observation policy. Identity becomes a claim with a test: certify by witness or return a counterexample.

Event and fold. The fold normal form presents a minimal model for events. Crossing the critical set creates a visible bifurcation. Events cease to be metaphysical declarations and become the emergence of a new branch under a named discriminant condition.

Operational afterthought. (i) Any philosophical alignment must land in the triad $(\mathcal{V}_\kappa, \mathcal{G}_{\Phi_\kappa}, \mathrm{Disc})$. (ii) Any cross-context equivalence must present a 2-pullback witness or yield a smallest counterexample. (iii) The analytic versus smooth clauses state the verification class and cost. These rules embed the discourse back at the top layer while preserving the generative core: difference as flow, repetition as orbit, identity as a publicly testable claim.