Difference → Repetition → Identity

A Minimal, Auditable Landing with the Difference Triad

Abstract

We formalize the chain *Difference* → *Repetition* → *Identity* via a minimal structure, the *Difference Triad* $(\mathcal{V}, G, \mathrm{Disc})$ at scale κ for a composite projection $\Phi_\kappa : X \to I_\kappa$. Here $\mathcal{V} = \ker d\Phi_\kappa$ captures invisible displacement, $G \leq \mathrm{Aut}_{\Phi_\kappa}$ captures symmetry (disguise), and $\mathrm{Disc}(\Phi_\kappa)$ is the discriminant (folds and jumps). Difference is absence of connectivity by legal chains generated by $(\mathcal{V}, G)$ without crossing Disc; repetition is the generated invariance; identity is the quotient visible in I_κ. The fold map $(x, y) \mapsto (x, y^2)$ serves as the canonical local model for boundary formation. The presentation is self-contained and ready for audit.

1 Setup and Notation

Standing Assumptions 1.1 (Spaces, map, and discriminant)**.** Let X and I_κ be smooth manifolds (or tame stratified spaces). Fix a composite projection

$$\Phi_\kappa \;=\; q_\kappa \circ \rho \circ \mathrm{Obs}_\star \;:\; X \longrightarrow I_\kappa.$$

Define the *discriminant* of Φ_κ by

$$\mathrm{Disc}(\Phi_\kappa) \;=\; \underbrace{\mathrm{Crit}(\Phi_\kappa)}_{\text{rank drop}} \cup \underbrace{\Phi_\kappa^{-1}(\Sigma)}_{\text{fold locus in the image}} \cup \underbrace{J(\rho \circ \mathrm{Obs}_\star)}_{\text{jump set of representation}},$$

where $\mathrm{Crit}(\Phi_\kappa) = \{x \in X : \mathrm{rank}(d\Phi_\kappa)_x < r\}$ for the constant regular rank r on X^{reg}, $\Sigma \subset I_\kappa$ is the set of image singularities (for instance, folds), and $J(\rho \circ \mathrm{Obs}_\star)$ is the discontinuity set of $\rho \circ \mathrm{Obs}_\star$. Assume $\mathrm{Disc}(\Phi_\kappa)$ is closed and $X^{\mathrm{reg}} := X \setminus \mathrm{Disc}(\Phi_\kappa)$ is open and dense. On X^{reg} the differential is of constant rank r, hence there is a short exact sequence

$$0 \longrightarrow \ker d\Phi_\kappa \longrightarrow TX \xrightarrow{d\Phi_\kappa} \Phi_\kappa^*(TI_\kappa) \longrightarrow 0. \tag{1.1}$$

Remark 1.2 (Invisible motions and symmetries)**.** On X^{reg} the distribution $\mathcal{V} := \ker d\Phi_\kappa \subset TX$ consists of Φ_κ-invisible directions. The group of Φ_κ-symmetries is $\mathrm{Aut}_{\Phi_\kappa} := \{g : X \to X \mid \Phi_\kappa \circ g = \Phi_\kappa\}$.

2 The Difference Triad at scale κ

Definition 2.1 (κ-Difference Structure)**.** A *κ-difference structure* is a tuple

$$\mathbb{D}_\kappa \;=\; (X, \Phi_\kappa;\, \mathcal{V}, G, \mathrm{Disc}),$$

where: (i) $\mathrm{Disc} \subset X$ is closed, $X^{\mathrm{reg}} := X \setminus \mathrm{Disc}$ is open and dense, and $\mathrm{rank}(d\Phi_\kappa)$ is constant on X^{reg}; (ii) $\mathcal{V} := \ker d\Phi_\kappa \subset TX|_{X^{\mathrm{reg}}}$ is integrable (displacement flows); (iii) $G \leq \mathrm{Aut}_{\Phi_\kappa}$ acts smoothly on X^{reg} (disguise).

Definition 2.2 (Legal chains, Identity, Difference)**.** A *legal chain* is a finite composition of: (a) flows along $\mathcal{V}$ in X^{reg}, and (b) actions by G, whose image stays in X and never meets $\mathrm{Disc}(\Phi_\kappa)$. Define $x \sim_\kappa y$ iff there exists a legal chain from x to y. The *identity at scale κ* is the class $[x]_\kappa$; *difference at scale κ* means $x \not\sim_\kappa y$.

Standing Assumptions 2.3 (Triad Axioms: Soundness, Completeness, Boundary Adequacy)**. (S) Soundness.** Every primitive move (flow in $\mathcal{V}$ or $g \in G$) preserves Φ_κ on X^{reg}.
(C) Local Completeness. Any local variation that preserves Φ_κ on X^{reg} is generated by flows in $\mathcal{V}$ and actions of G.
(B) Boundary Adequacy. Any path crossing $\mathrm{Disc}(\Phi_\kappa)$ transversely changes $\sim_\kappa$-class.

Theorem 2.4 (Triad Completeness)**.** *Under Assumption 2.3:*

(i) $\sim_\kappa$ is an equivalence relation on each path component of X^{reg}, and Φ_κ is constant on $\sim_\kappa$-classes.

(ii) For $x, y \in X^{\mathrm{reg}}$, $x \sim_\kappa y$ if and only if they lie in the same path component of the regular fiber $\Phi_\kappa^{-1}(\Phi_\kappa(x)) \cap X^{\mathrm{reg}}$.

(iii) The quotient $X/\!\sim_\kappa \xrightarrow{\bar{\Phi}_\kappa} I_\kappa$ identifies the regular image bijectively. The residual difference resides in the fibers of Φ_κ separated by $\mathrm{Disc}(\Phi_\kappa)$.

3 Fold–Difference Correspondence (Minimal)

Definition 3.1 (Fold locus)**.** Let $\Phi_\kappa : X \to I_\kappa$ be C^2. A point $x_0 \in X$ is a *fold point* if $\mathrm{corank}(d\Phi_\kappa)_{x_0} = 1$ and, after choosing local coordinates (u, v) at x_0 and (u, w) at $\Phi_\kappa(x_0)$, one has the normal form

$$\Phi_\kappa(u, v) = (u,\, v^2) \quad \text{near } x_0.$$

The *fold locus* $\mathrm{Fold}(\Phi_\kappa)$ is the set of fold points; its image $\Sigma = \Phi_\kappa(\mathrm{Fold}(\Phi_\kappa))$ is a smooth hypersurface in I_κ.

Definition 3.2 (Public difference classes)**.** Fix the Difference Triad $(\mathcal{V}, G, \mathrm{Disc})$ with G replaced by the identity component G^0. Let $X^{\mathrm{reg}} := X \setminus \mathrm{Disc}(\Phi_\kappa)$. Define $x \sim_\kappa y$ if a finite composition of flows along $\mathcal{V}$ and actions of G^0 connects x to y without meeting $\mathrm{Disc}(\Phi_\kappa)$. Write $[x]_\kappa$ for the class of x.

Theorem 3.3 (Fold–Difference Correspondence)**.** *Assume the above and let $x_0 \in \mathrm{Fold}(\Phi_\kappa)$. Then there exists a neighborhood U of x_0 such that:*

(i) *For any regular value* (u_0, w_0) *with* $w_0 > 0$ *close to* $\Phi_\kappa(x_0)$*, the* regular fiber

$$F_{u_0,w_0} \;:=\; \Phi_\kappa^{-1}(u_0, w_0) \;\cap\; X^{\mathrm{reg}} \;\cap\; U$$

has exactly two connected components F^+ *and* F^-*.*

(ii) *Points in* F^+ *(resp.* F^-*) lie in the same* $\sim_\kappa$*-class within* U*, but any* $x^+ \in F^+$ *and* $x^- \in F^-$ *satisfy* $x^+ \not\sim_\kappa x^-$*.*

(iii) *Any continuous path in* U *that moves from* F^+ *to* F^- *intersects* $\mathrm{Disc}(\Phi_\kappa)$ *exactly on the local fold sheet* $\{v = 0\}$*.*

Consequently, the fold locus is the local public boundary *between* κ*-difference classes: it is where classes meet but cannot be crossed without leaving the class.*

Proof sketch. Normal form gives $\Phi_\kappa(u, v) = (u, v^2)$. For $w_0 > 0$, the preimage is $\{(u_0, \pm\sqrt{w_0})\}$, two components split by $v = 0$, which equals the local discriminant. With G^0 and no $\mathcal{V}$-flow across $v = 0$ in X^{reg}, components cannot be joined without meeting Disc. Any path switching sign of v must cross $v = 0$, proving (ii)–(iii). The within-component connectivity follows by path-connectedness and closedness of Disc. □

Corollary 3.4 (Boundary law)**.** *In a neighborhood of a fold, the map*

$$\mathrm{sgn} \circ \psi \circ \Phi_\kappa \;:\; U \setminus \mathrm{Disc}(\Phi_\kappa) \to \{+1, -1\},$$

for any local defining function ψ *of* Σ*, is constant on each* $\sim_\kappa$*-class and flips value iff a path meets* $\mathrm{Disc}(\Phi_\kappa)$*. Thus, the fold induces a* $\mathbb{Z}_2$*-labeling of local difference classes.*

4 Repetition as Generated Invariance

Definition 4.1 (Repetition)**.** A *repetition* at scale κ is any variation of $x \in X^{\mathrm{reg}}$ staying within the class $[x]_\kappa$, hence generated by a legal chain (flows in $\mathcal{V}$ and actions of G) without crossing $\mathrm{Disc}(\Phi_\kappa)$.

Theorem 4.2 (Generated invariance)**.** *On* X^{reg}*, every local variation that preserves* Φ_κ *is generated by the primitive moves: displacement along* $\mathcal{V}$ *and disguise by* G*.*

Proof sketch. By (1.1), any tangent variation decomposes into an invisible component in $\ker d\Phi_\kappa$ and a visible component tangent to fibers of Φ_κ. The former integrates to displacement flows. The latter corresponds to reparametrization within the Φ_κ-fiber and is realized by an element of G locally. Local integration yields the claim. □

5 Identity as Quotient and Public Test

Proposition 5.1 (Quotient identification)**.** *There is a canonical map* $\pi : X \to X/{\sim_\kappa}$ *such that the induced map* $\overline{\Phi_\kappa} : X/{\sim_\kappa} \to I_\kappa$ *satisfies* $\Phi_\kappa = \overline{\Phi_\kappa} \circ \pi$*. On the regular image* $\Phi_\kappa(X^{\mathrm{reg}})$*, the map* $\overline{\Phi_\kappa}$ *is a bijection onto its image.*

Proof sketch. By Theorem 2.4(i), Φ_κ is constant on $\sim_\kappa$-classes, hence factors through the quotient. Injectivity on the regular image follows from maximality of legal chains within a regular fiber and Boundary Adequacy. $\square$

Corollary 5.2 (Public test for difference)**.** *Points $x, y \in X$ are κ-different if and only if every continuous path from x to y intersects* $\mathrm{Disc}(\Phi_\kappa)$*. Equivalently, x and y project to distinct points of I_κ not joined by a legal chain.*

6 Canonical Local Model: The Fold

Example 6.1 (Fold singularity)**.** Let $X = \mathbb{R}^2$, $I_\kappa = \mathbb{R}^2$, and $\Phi(x, y) = (u, v) = (x, y^2)$. Then

$$\mathrm{Crit}(\Phi) = \{(x, 0) : x \in \mathbb{R}\}, \qquad \Sigma = \{(u, 0) : u \in \mathbb{R}\}, \qquad \mathrm{Disc}(\Phi) = \mathrm{Crit}(\Phi).$$

Distinct points (x, y) and $(x, -y)$ with $y \neq 0$ are $\sim$-equivalent without crossing Disc; the line $y = 0$ is the fold locus where classes meet, in agreement with Theorem 3.3. Any attempt to pass from $y > 0$ to $y < 0$ must cross $\mathrm{Disc}(\Phi)$. Thus the fold converts hidden variation into a visible boundary in the image.

7 Optional Gate to Alignment

Lemma 7.1 (Pullback gate)**.** *Let $\Phi_\kappa : X \to I_\kappa$ and $\Phi'_\kappa : X' \to I_\kappa$ be two projections landing in the same I_κ. If there exists a context L and maps $f : X \to L$, $f' : X' \to L$ with a map $p : L \to I_\kappa$ such that $p \circ f = \Phi_\kappa$ and $p \circ f' = \Phi'_\kappa$, then the fiber product*

$$X \times_{I_\kappa} X' \;\cong\; \{(x, x') \in X \times X' : \Phi_\kappa(x) = \Phi'_\kappa(x')\}$$

collects paired classes $[x]_\kappa = [x']_\kappa$. Alignment holds on the regular part if the pullback is nonempty and intersects neither discriminant.

Proof. Standard property of pullbacks. The discriminant avoidance ensures that identification respects the equivalence classes generated by invisible motions and symmetries. $\square$

8 Summary

At a fixed scale κ, difference is the necessity to cross the discriminant, repetition is the generated invariance under Φ_κ-invisible motions and symmetries, and identity is the quotient visible in I_κ. The Difference Triad $(\mathcal{V}, G, \mathrm{Disc})$ provides a minimal and auditable structure that integrates dynamics (displacement), symmetry (disguise), and boundary (discriminant).