Mathe 2 ab 21.03.19

Moodle-Passwort: 20Frühling19

(8) Hospital-Regel:

$$\underbrace{\lim_{x\to a} \frac{f(x)}{g(x)}}_{\text{falls dies ein „unbestimmter Ausdruck" der Art } \frac{0}{0},\ \frac{\pm\infty}{\pm\infty} \text{ ist}} = \underbrace{\lim_{x\to a} \frac{f'(x)}{g'(x)}}_{\text{und falls dieser Grenzwert existiert}}$$

Bsp.:

$$\lim_{x\to\infty} \frac{3x^5 - 3x^2 + 1}{2x^5 + 3x}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{15x^4 - 6x}{10x^4 + 3}$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{60x^3 - 6}{40x^3} \overset{H}{=} \ldots$$

$$\overset{H}{=} \lim_{x\to\infty} \frac{360x}{240x} = \frac{360}{240} = \frac{3}{2} = 1{,}5$$

alternativ:

$$\lim_{x\to\infty} \frac{3x^5 - 3x^2 + 1}{2x^5 + 3x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^5}\left(3 + \underbrace{\frac{-3}{x^3}}_{\to 0} + \underbrace{\frac{1}{x^5}}_{\to 0}\right)}{\cancel{x^5}\left(2 + \underbrace{\frac{3}{x^4}}_{\to 0}\right)} \longrightarrow \frac{3}{2} = 1{,}5$$

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$$\left(1 + \frac{1}{n}\right)^n \xrightarrow[(n\to\infty)]{} e = 2{,}718\ldots$$

$$\left(1 + \frac{2}{n}\right)^n \longrightarrow e^2 \qquad (n\to\infty)$$

$$\left(1 + \frac{x}{n}\right)^n \longrightarrow e^x \qquad (n\to\infty)$$

Beh.: $\lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

Bew.: zu zeigen: $\left(1 + \frac{x}{n}\right)^n \longrightarrow e^x \quad (n\to\infty)$

$$\Longleftrightarrow \underbrace{\ln\left(\left(1 + \frac{x}{n}\right)^n\right)} \longrightarrow \ln\left(e^x\right) = x$$

$$= n \cdot \ln\left(1 + \frac{x}{n}\right) \qquad \text{„}\infty \cdot 0\text{"}$$

$$= \frac{\ln\left(1 + \frac{x}{n}\right)}{\frac{1}{n}} \qquad \text{„}\frac{0}{0}\text{"}$$

de l'Hôpital:

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{\frac{1}{n}} \overset{H}{=} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{x}{n}}\cdot\left(-\frac{x}{n^2}\right)}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{x}{1+\frac{x}{n}} = x$$

Bsp.: $\lim_{x\to 1} \frac{x^3+x^2-x-1}{x^2-1}$

$$\overset{H}{=} \lim_{x\to 1} \frac{3x^2+2x-1}{2x} \qquad \text{“}\frac{4}{2}\text{”}$$

$$\left[\overset{H}{=} \lim_{x\to 1} \frac{6x+2}{2} = \frac{6\cdot 1+2}{2} = 4 \right]$$

$$= \frac{3\cdot 1^2+2\cdot 1-1}{2\cdot 1} = \underline{\underline{2}}$$

$\boxed{1}$ f)***

$$\lim_{t\to\infty} \frac{2(t+\ln(t))}{\sqrt{1+2t^2}}$$

$$\overset{H}{=} \lim_{t\to\infty} \frac{\cancel{2}\left(1+\frac{1}{t}\right)}{\cancel{\frac{1}{2}}\left(1+2t^2\right)^{-1/2}\cdot \cancel{4}t}$$

$$= \lim_{t\to\infty} \frac{\left(1+\frac{1}{t}\right)\sqrt{1+2t^2}}{t}$$

$$= \lim_{t\to\infty} \left(1+\frac{1}{t}\right)\cdot \frac{\sqrt{t^2\cdot\left(\frac{1}{t^2}+2\right)}}{t}$$

$$= \lim_{t\to\infty} \underbrace{\left(1+\frac{1}{t}\right)}_{\to 1}\cdot \frac{\cancel{\sqrt{t^2}}\cdot\sqrt{2+\frac{1}{t^2}}}{\cancel{t}} \qquad \frac{1}{t^2}\to 0$$

$$= \sqrt{2}$$

Bsp.: $\lim_{x\to\infty} \frac{\sinh(x)}{\cosh(x)} \overset{H}{=} \lim_{x\to\infty} \frac{\cosh(x)}{\sinh(x)}$

$$\overset{H}{=} \lim_{x\to\infty} \frac{\sinh(x)}{\cosh(x)}$$

☹

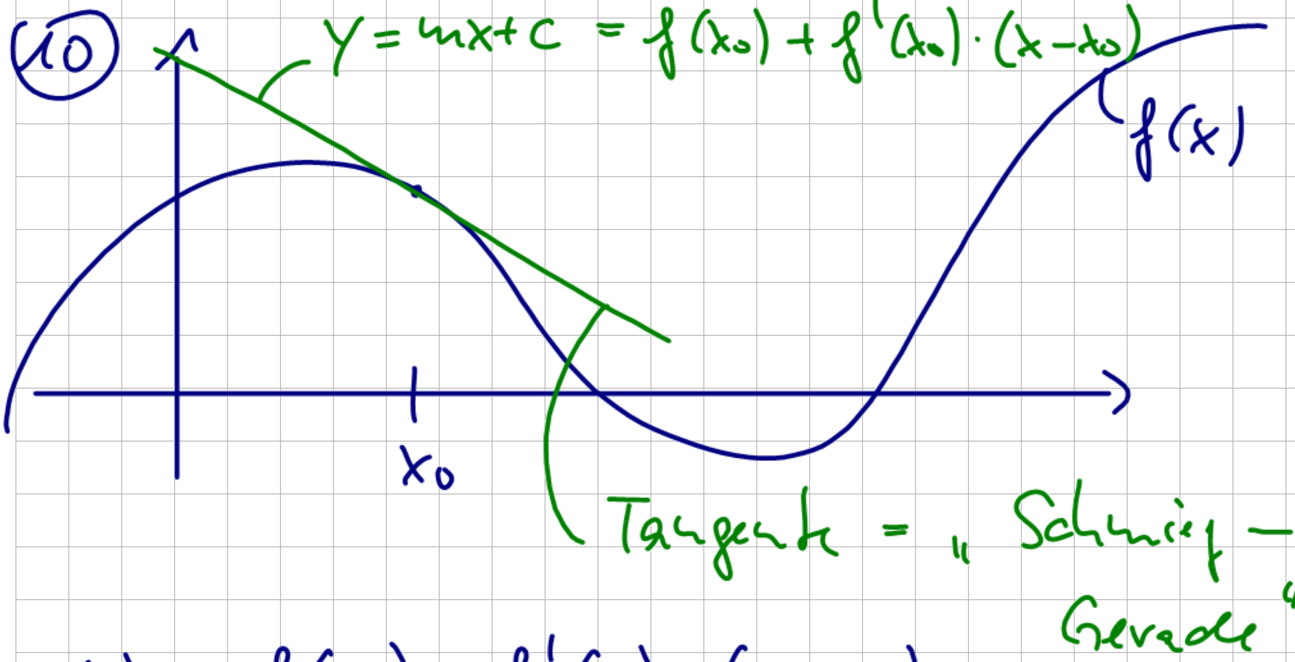
besser:

$$\lim_{x\to\infty} \frac{\sinh(x)}{\cosh(x)} = \lim_{x\to\infty} \frac{\frac{1}{2}\left(e^x-e^{-x}\right)}{\frac{1}{2}\left(e^x+e^{-x}\right)}$$

$$= \lim_{x\to\infty} \frac{\cancel{e^x}\left(1-e^{-2x}\right)}{\cancel{e^x}\left(1+e^{-2x}\right)} = 1 \qquad e^{-2x}\to 0$$

27.3.19

(10)



$$g(x) = \underbrace{f(x_0)}_{\text{Zahl}} + \underbrace{f'(x_0)}_{\text{Zahl}} \cdot (\underset{\text{Variable}}{x} - \underset{\text{Zahl}}{x_0})$$

$$g(x_0) = f(x_0) + f'(x_0) \cdot (x_0 - x_0) = f(x_0) \checkmark$$

$$g'(x) = 0 + f'(x_0) \cdot 1 = f'(x_0)$$

$$g'(x_0) = f'(x_0) \checkmark$$

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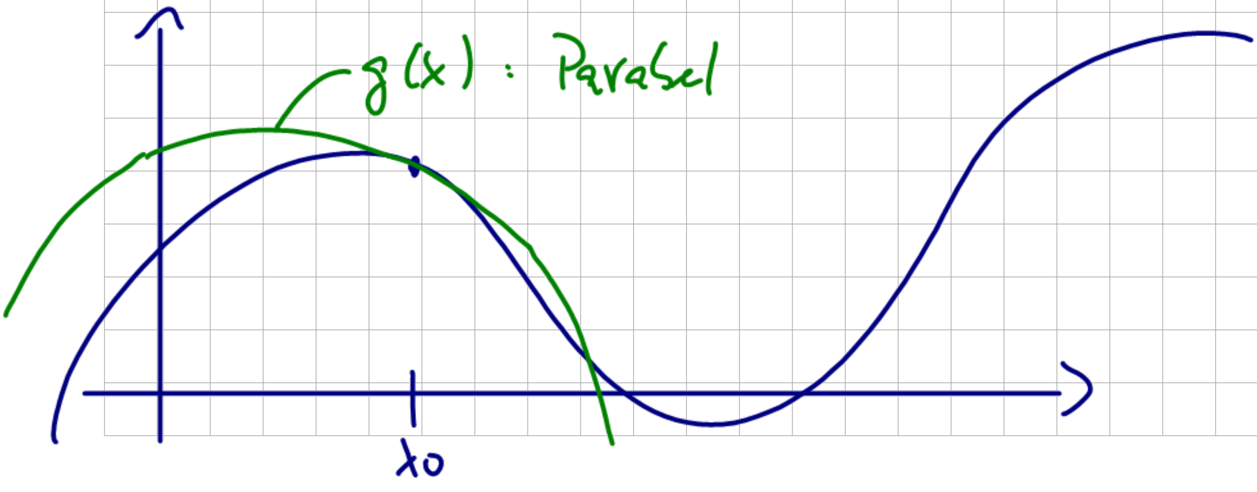
gesucht: $g(x)$ = „Schmieg-Parabel"



$$g(x) = f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2!}\cdot(x-x_0)^2$$

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$$g(x) = f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}\cdot(x-x_0)^3$$

$$= \sum_{k=0}^{3} \frac{f^{(k)}(x_0)}{k!}\cdot(x-x_0)^k$$

$$g(x_0) = f(x_0) + \underbrace{f'(x_0)\cdot(x_0-x_0) + \frac{f''(x_0)}{2!}(x_0-x_0)^2 + \frac{f'''(x_0)}{3!}(x_0-x_0)^3}_{=0}$$

$$= f(x_0) \checkmark$$

$$g'(x) = 0 + f'(x_0)\cdot 1 + \frac{f''(x_0)}{2!}\cdot 2(x-x_0)\cdot 1 + \frac{f'''(x_0)}{3!}\cdot 3(x-x_0)^2\cdot 1$$

$$= f'(x_0) + f''(x_0)\cdot(x-x_0) + \frac{f'''(x_0)}{2}\cdot(x-x_0)^2$$

$$g'(x_0) = f'(x_0) \checkmark$$

$$g''(x) = f''(x_0) + f'''(x_0)\cdot(x-x_0)$$

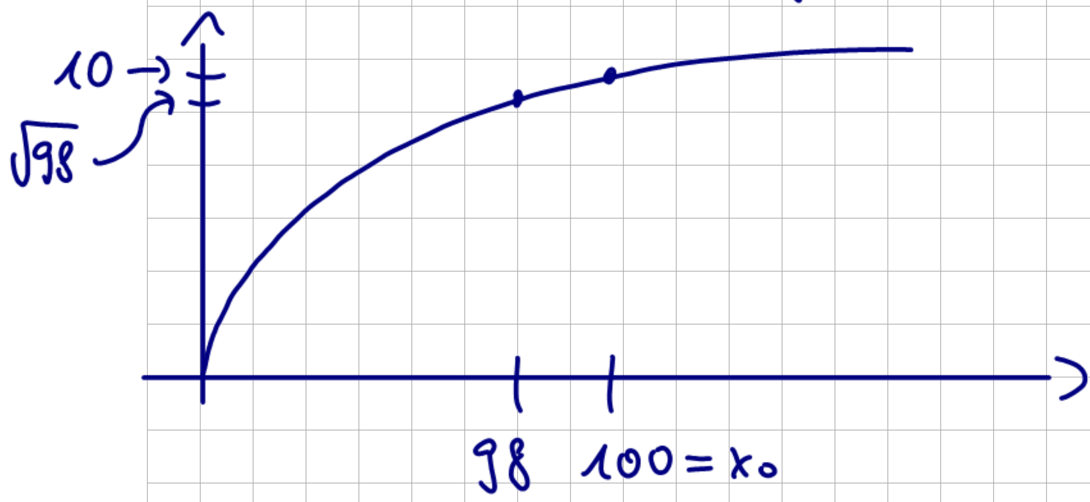
$$g''(x_0) = f''(x_0) \checkmark$$

$g'''(x) = f'''(x_0)$

$g'''(x_0) = f'''(x_0)$ ✓

Bsp. $\boxed{3}$:

Berechne angenähert $\sqrt{98}$ mittels Taylor 3. Ordnung!



$x_0 = 100 \quad , \quad x = 98 \quad , \quad f(x) = \sqrt{x}$

Bestimme $f(x_0)$, $f'(x_0)$, $f''(x_0)$, $f'''(x_0)$:

$f(x_0) = \sqrt{100} = 10$

$f'(x) = \left(x^{1/2}\right)' = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$f'(x_0) = \frac{1}{2 \cdot \sqrt{100}} = \frac{1}{20}$

$f''(x) = \left(\frac{1}{2} x^{-\frac{1}{2}}\right)' = -\frac{1}{4} x^{-\frac{3}{2}} = -\frac{1}{4\sqrt{x}^3}$

$f''(x_0) = -\frac{1}{4\sqrt{100}^3} = -\frac{1}{4000}$

$$f'''(x) = \left(-\frac{1}{4}\, x^{-\frac{3}{2}}\right)' = \frac{3}{8}\, x^{-\frac{5}{2}} = \frac{3}{8\sqrt{x}^{\,5}}$$

$$f'''(x_0) = \frac{3}{8\cdot\sqrt{100}^{\,5}} = \frac{3}{800\,000}$$

Taylor:

$$\begin{aligned} f(98) = \sqrt{98} \approx g(98) &= f(x_0) + f'(x_0)\cdot(x-x_0) \\ &\quad + \frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad + \frac{f'''(x_0)}{3!}(x-x_0)^3 \end{aligned}$$

$$\begin{aligned} &= 10 + \frac{1}{20}\cdot(98-100) + \frac{-\frac{1}{4000}}{2}\cdot(-2)^2 \\ &\quad + \frac{\frac{3}{800\,000}}{6}\cdot(-2)^3 \\ &= 10 - \frac{1}{10} - \frac{1}{2000} - \frac{1}{200\,000} \\ &= 9{,}899495 \end{aligned}$$

Wahrer Wert: $\sqrt{98} = 9{,}899494\underline{9366}\ldots$

(14) Taylor-Entwicklung von $f(x) = \sin(x)$,
Entwicklungsstelle $x_0 = 0$

a) Ableitungen bereitstellen:

$$f(x) = \sin(x)$$
$$f'(x) = \cos(x)$$
$$f''(x) = -\sin(x)$$
$$f'''(x) = -\cos(x)$$
$$f^{(4)}(x) = \sin(x) = f(x)$$

b) Ableitungen auswerten bei x_0:

$$f(0) = \sin(0) = 0$$
$$f'(0) = \cos(0) = 1$$
$$f''(0) = -\sin(0) = 0$$
$$f'''(0) = -\cos(0) = -1$$
$$f^{(4)}(0) = \sin(0) = 0$$

c) Taylor-Polynom bilden:

$$g(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} \underbrace{x^k}_{(x-x_0)^k \text{ mit } x_0=0}$$

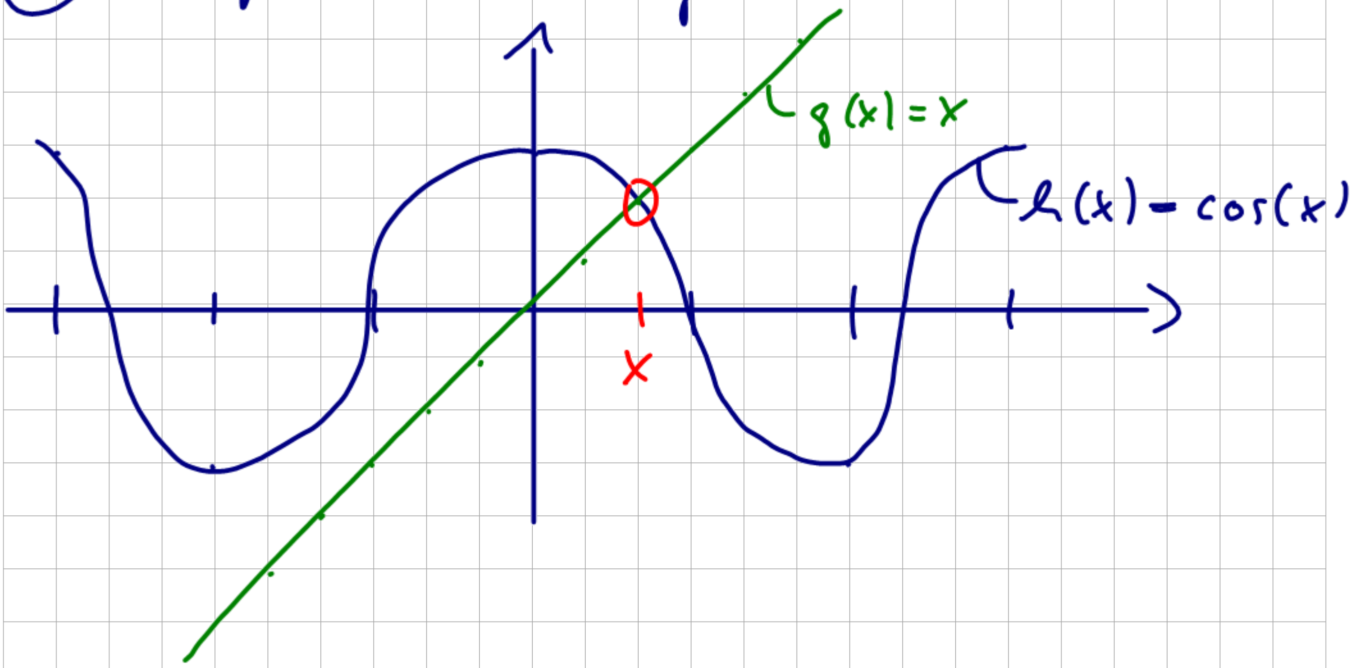
$$= f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(4)}(0)}{4!}x^4$$

$$= 0 + 1\cdot x + 0\cdot x^2 - \frac{1}{3!}x^3 - 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7 - 0 + \ldots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$$

28.03.19

(15) Bsp.: Gleichung $\cos(x) = x$



Berechnung der Lösung x:

Idee: Formuliere Gl. $\cos(x) = x$ als Nullstellenproblem:

$$f(x) := \cos(x) - x$$

Es gilt: $f(x) = 0 \iff \cos(x) = x$

Berechne Nullstelle x von $f(x)$ näherungsweise:

1. Bisektion (15)
2. Newton-Verfahren (16)

(16) Newton-Verfahren:

x_n : Näherungswert für NSt. von $f(x)$

Tangentengleichung zur Stelle x_n:

$$y = f(x_n) + f'(x_n) \cdot (x - x_n)$$

Setze $y = 0$, löse auf nach x:

$$0 = f(x_n) + f'(x_n) \cdot (x - x_n)$$

$$\Leftrightarrow \quad -\frac{f(x_n)}{f'(x_n)} = x - x_n$$

$$\Leftrightarrow \quad x = x_n - \frac{f(x_n)}{f'(x_n)}$$

nächster Näherungswert, $x_{n+1} := x$

Also:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Bsp.: $\cos(x) = x$

Fkt.: $f(x) = \cos(x) - x$

$f'(x) = -\sin(x) - 1$

Iteration: $x_{n+1} = x_n - \frac{\cos(x_n) - x_n}{-\sin(x_n) - 1}$

Startwert: $x_0 = 1$ (z. B.)

$x_1 = 1 - \frac{\cos(1) - 1}{-\sin(1) - 1} \approx \underline{0{,}7}50363867$

$x_2 \approx \underline{0{,}739}11289$

$x_3 \approx \underline{0{,}739085133}$

$x_4 \approx \underline{\underline{0{,}739085133}}$