

Linear Grammars

Grammars with
at most one variable at the right side
of a production

Examples:

$S \rightarrow aSb$	$S \rightarrow Ab$
$S \rightarrow \lambda$	$A \rightarrow aAb$
	$A \rightarrow \lambda$

A Non-Linear Grammar

Grammar G :

$$S \rightarrow SS$$
$$S \rightarrow \lambda$$
$$S \rightarrow aSb$$
$$S \rightarrow bSa$$

$$L(G) = \{w : n_a(w) = n_b(w)\}$$



Number of a in string w

Another Linear Grammar

Grammar G :

$$S \rightarrow A$$
$$A \rightarrow aB \mid \lambda$$
$$B \rightarrow Ab$$

$$L(G) = \{a^n b^n : n \geq 0\}$$

Right-Linear Grammars

All productions have form: $A \rightarrow xB$

or

$$A \rightarrow x$$



Example: $S \rightarrow abS$

$$S \rightarrow a$$

string of
terminals

Left-Linear Grammars

All productions have form: $A \rightarrow Bx$

or

$$A \rightarrow x$$



Example: $S \rightarrow Aab$

$$A \rightarrow Aab \mid B$$

$$B \rightarrow a$$

string of
terminals

Regular Grammars

Regular Grammars

A regular grammar is any right-linear or left-linear grammar

Examples:

G_1

$$S \rightarrow abS$$

$$S \rightarrow a$$

G_2

$$S \rightarrow Aab$$

$$A \rightarrow Aab \mid B$$

$$B \rightarrow a$$

Observation

Regular grammars generate regular languages

Examples:

G_1

$$S \rightarrow abS$$

$$S \rightarrow a$$

$$L(G_1) = (ab)^* a$$

G_2

$$S \rightarrow Aab$$

$$A \rightarrow Aab \mid B$$

$$B \rightarrow a$$

$$L(G_2) = aab(ab)^*$$

Regular Grammars Generate Regular Languages

Theorem

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} = \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Theorem - Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular grammar generates
a regular language

Theorem - Part 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular language is generated
by a regular grammar

Proof - Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

The language $L(G)$ generated by
any regular grammar G is regular

The case of Right-Linear Grammars

Let G be a right-linear grammar

We will prove: $L(G)$ is regular

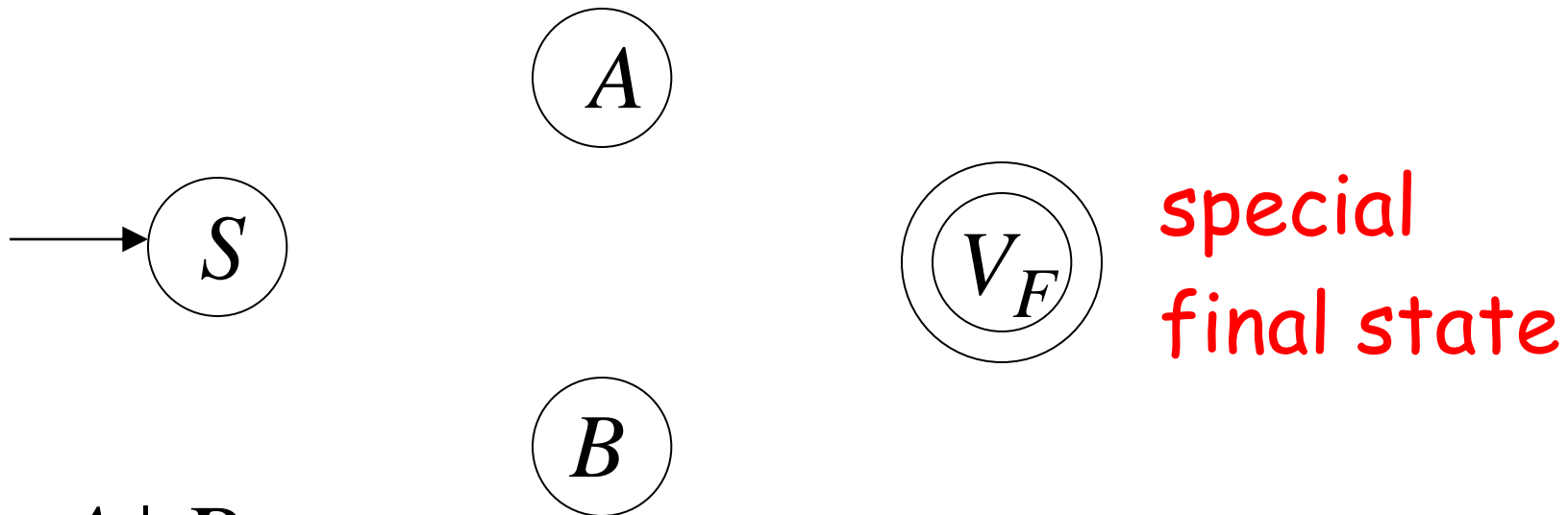
Proof idea: We will construct NFA M
with $L(M) = L(G)$

Grammar G is right-linear

Example: $S \rightarrow aA \mid B$

$$A \rightarrow aa B$$
$$B \rightarrow b B \mid a$$

Construct NFA M such that every state is a grammar variable:

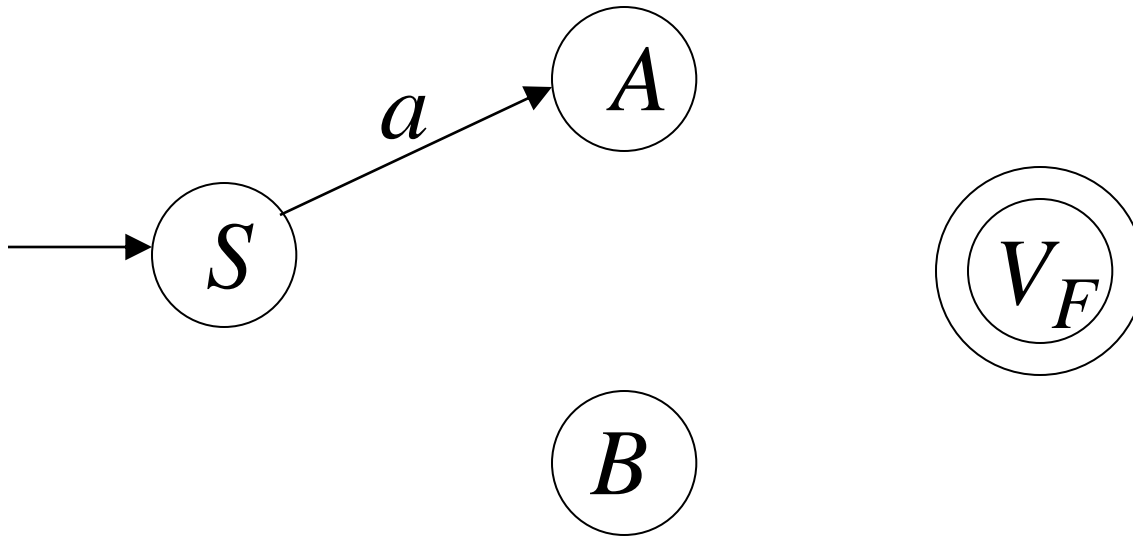


$$S \rightarrow aA \mid B$$

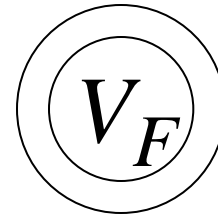
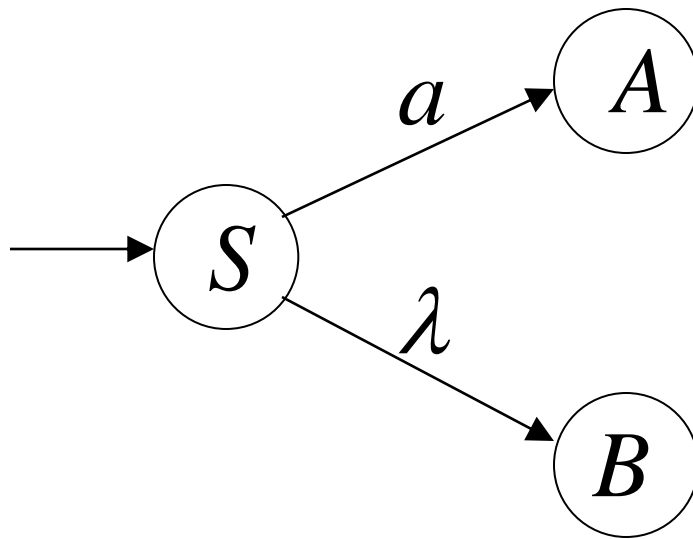
$$A \rightarrow aa B$$

$$B \rightarrow b B \mid a$$

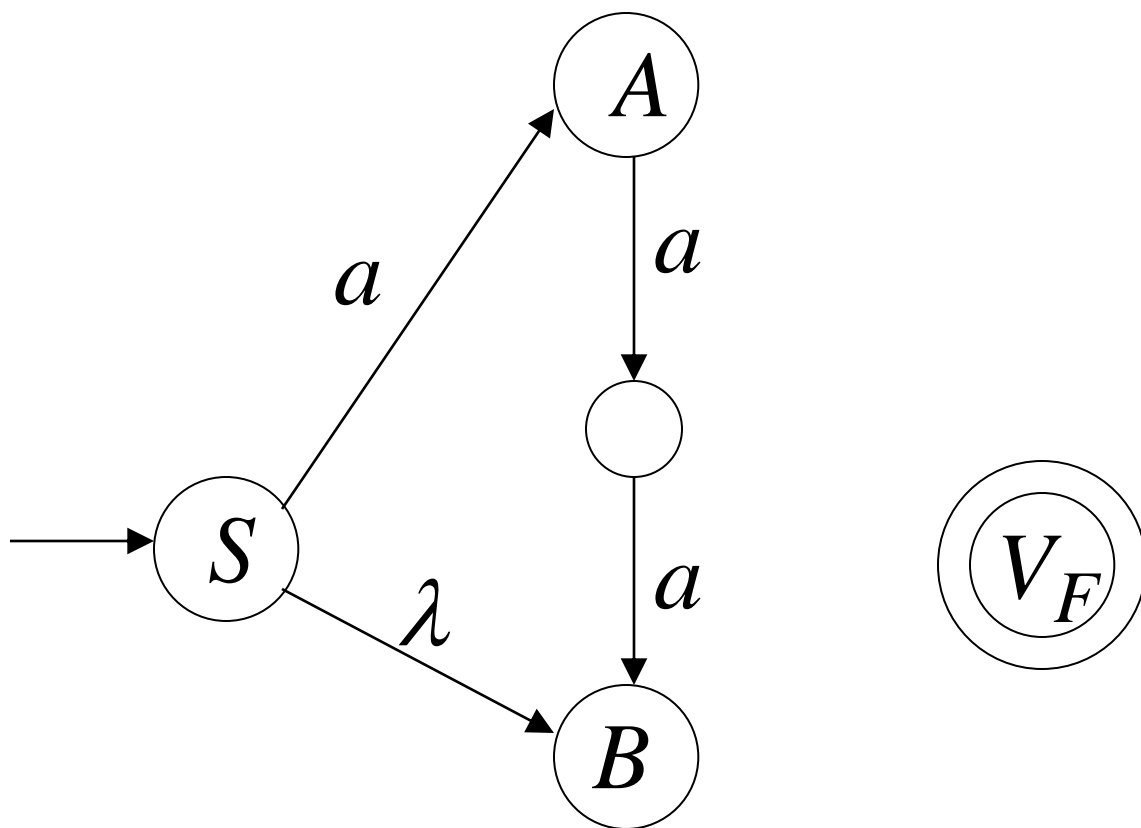
Add edges for each production:



$$S \rightarrow aA$$

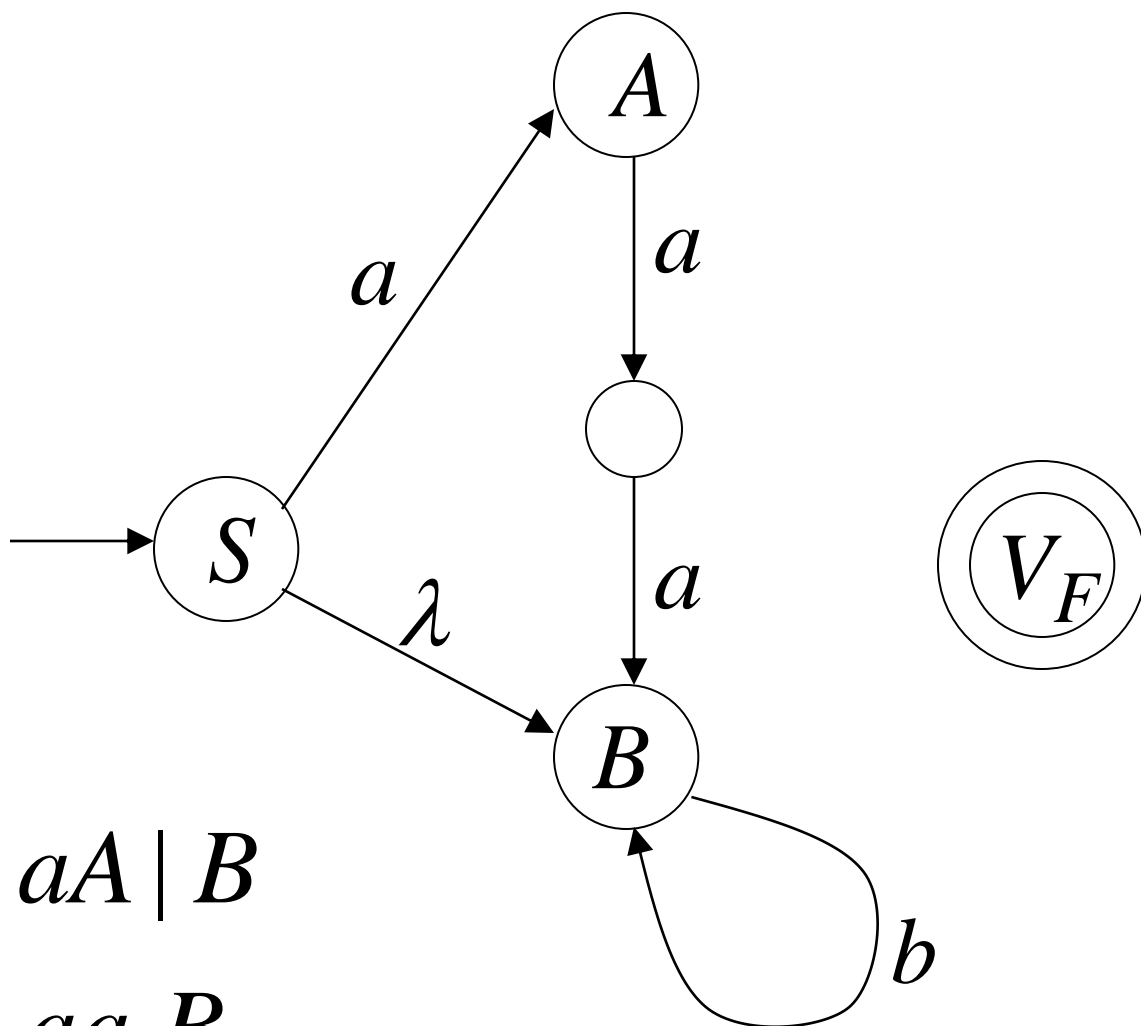


$$S \rightarrow aA \mid B$$



$$S \rightarrow aA \mid B$$

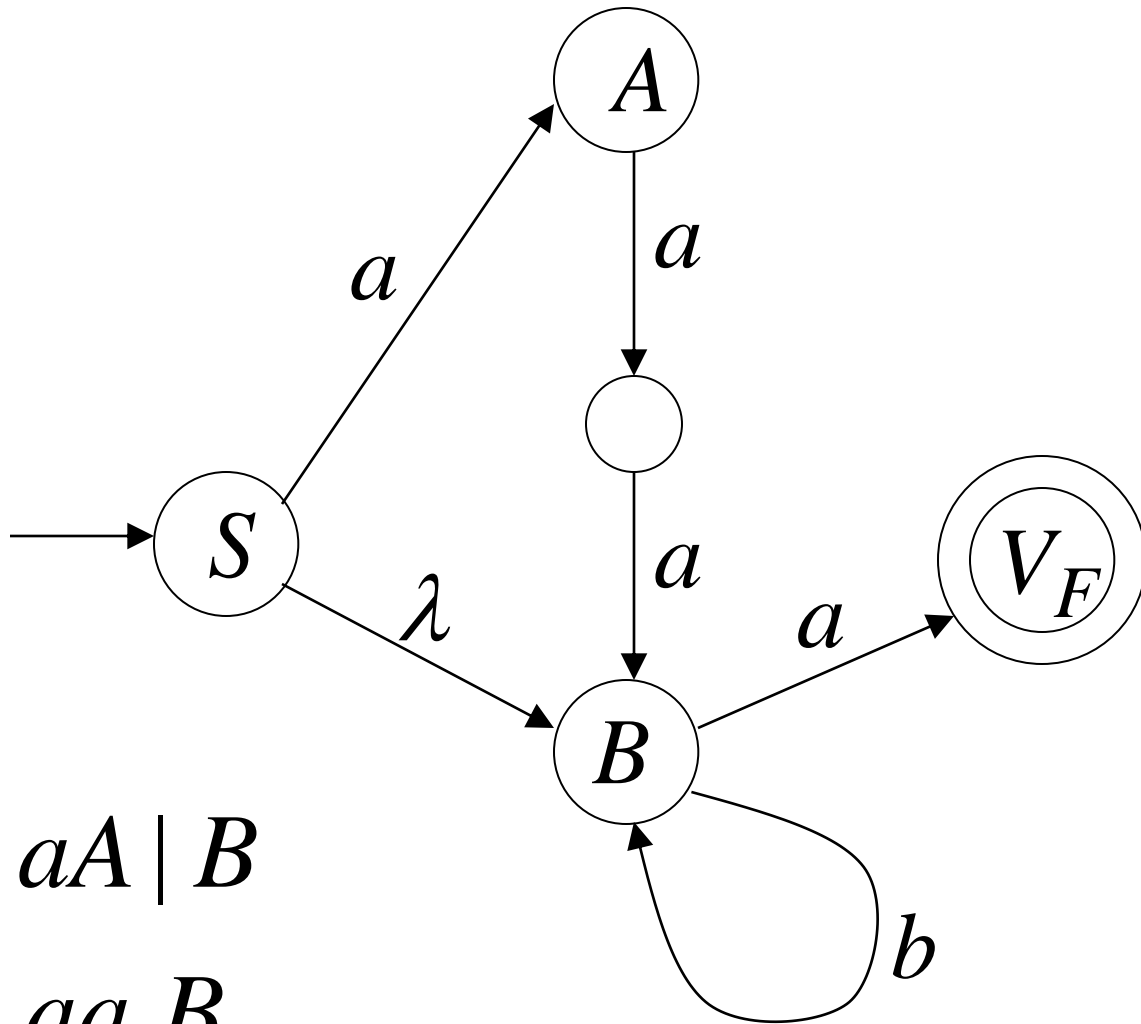
$$A \rightarrow aa B$$



$$S \rightarrow aA \mid B$$

$$A \rightarrow aa B$$

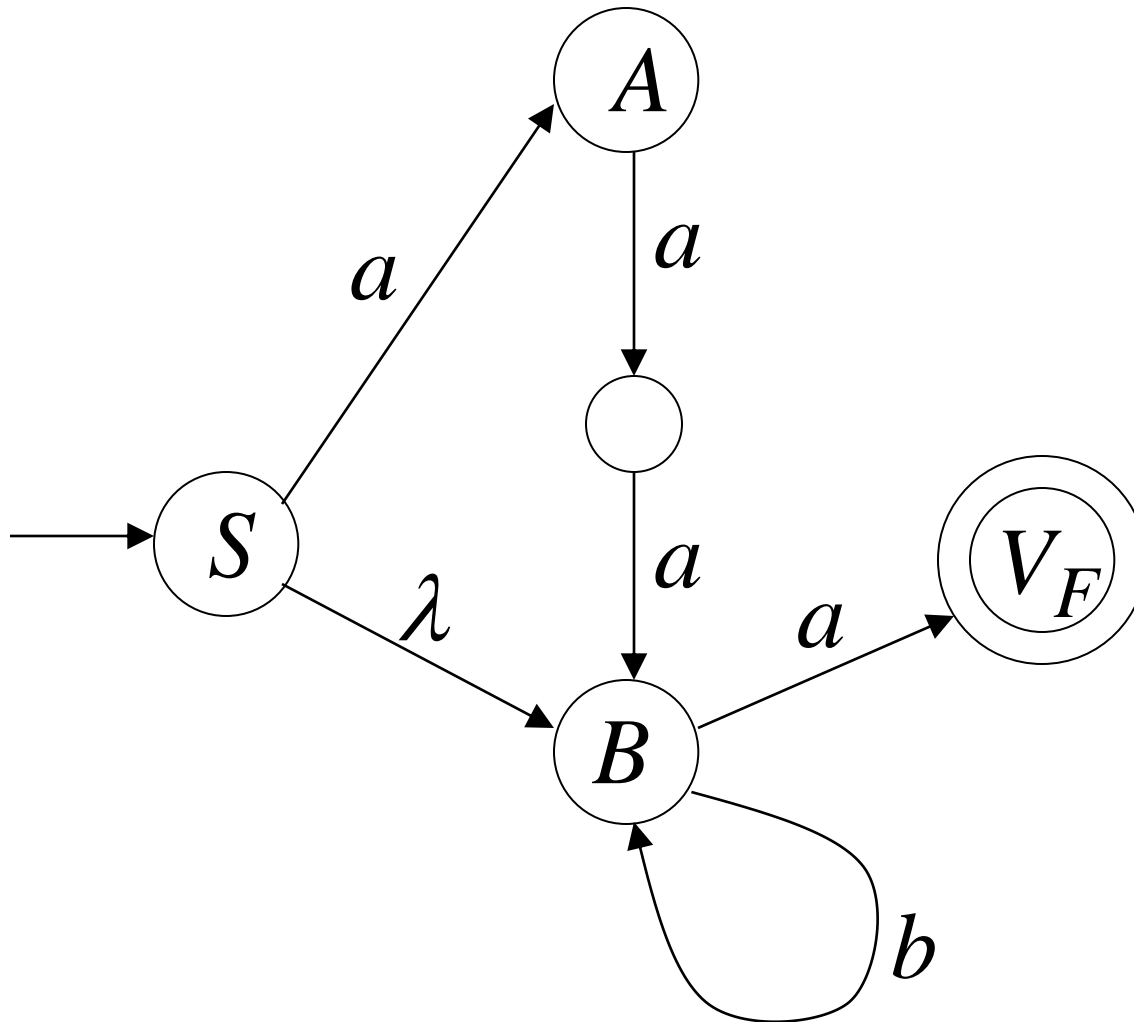
$$B \rightarrow bB$$



$$S \rightarrow aA \mid B$$

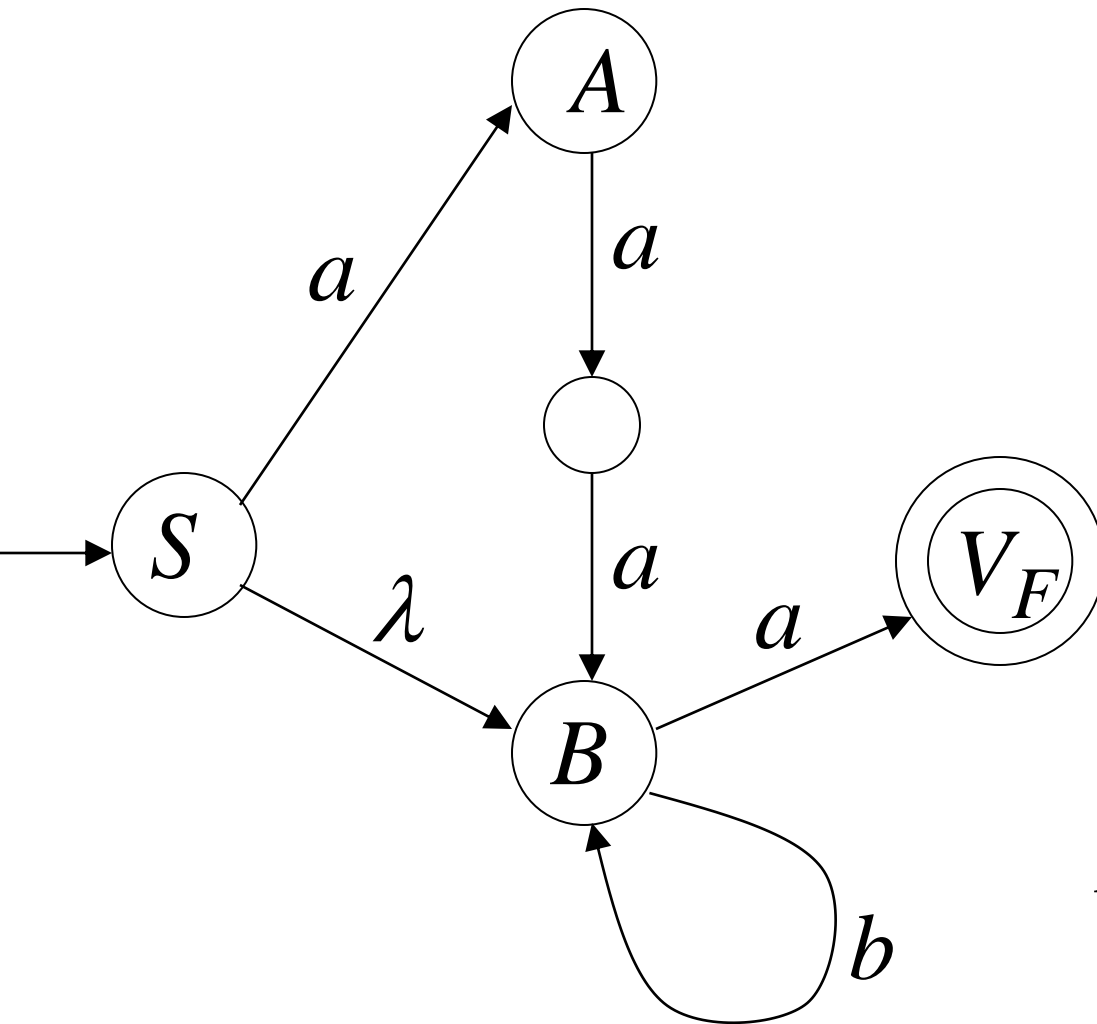
$$A \rightarrow aa B$$

$$B \rightarrow bB \mid a$$



$S \Rightarrow aA \Rightarrow aaaB \Rightarrow aaabB \Rightarrow aaaba$

NFA M



Grammar

G

$S \rightarrow aA \mid B$

$A \rightarrow aa B$

$B \rightarrow bB \mid a$

$$L(M) = L(G) = aaab^*a + b^*a$$

In General

A right-linear grammar G

has variables: V_0, V_1, V_2, \dots

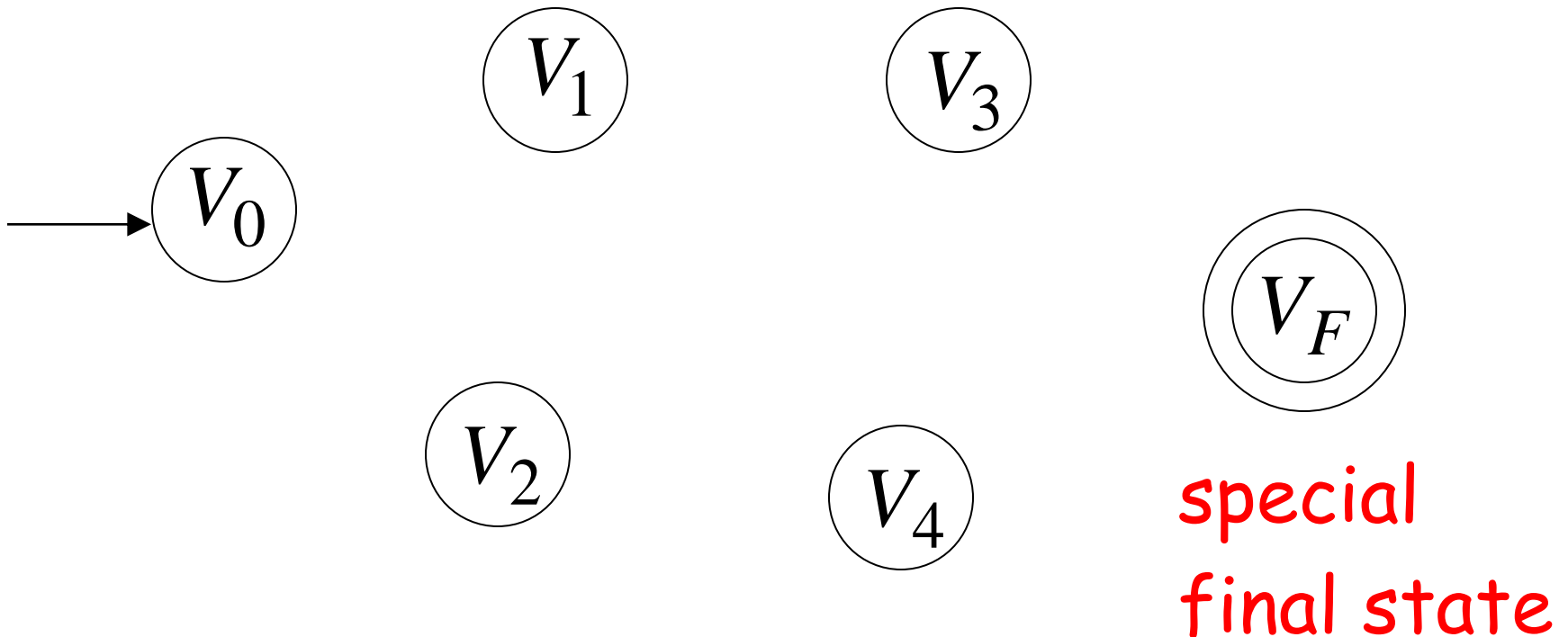
and productions: $V_i \rightarrow a_1 a_2 \cdots a_m V_j$

or

$$V_i \rightarrow a_1 a_2 \cdots a_m$$

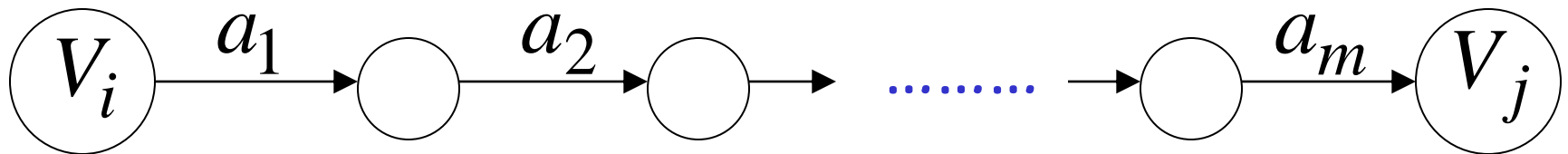
We construct the NFA M such that:

each variable V_i corresponds to a node:



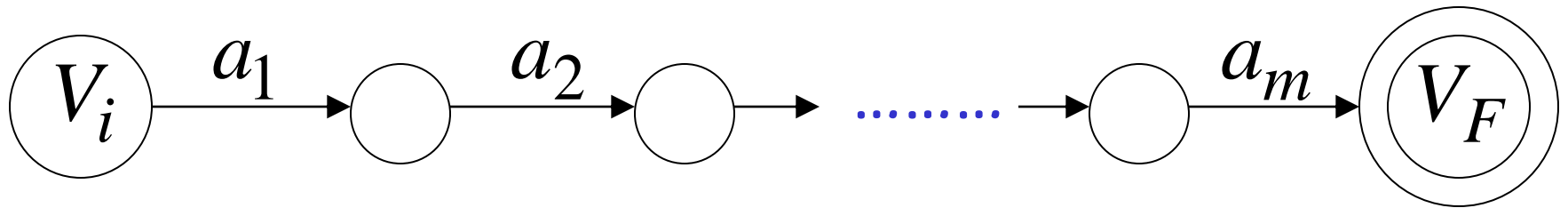
For each production: $V_i \rightarrow a_1 a_2 \cdots a_m V_j$

we add transitions and intermediate nodes

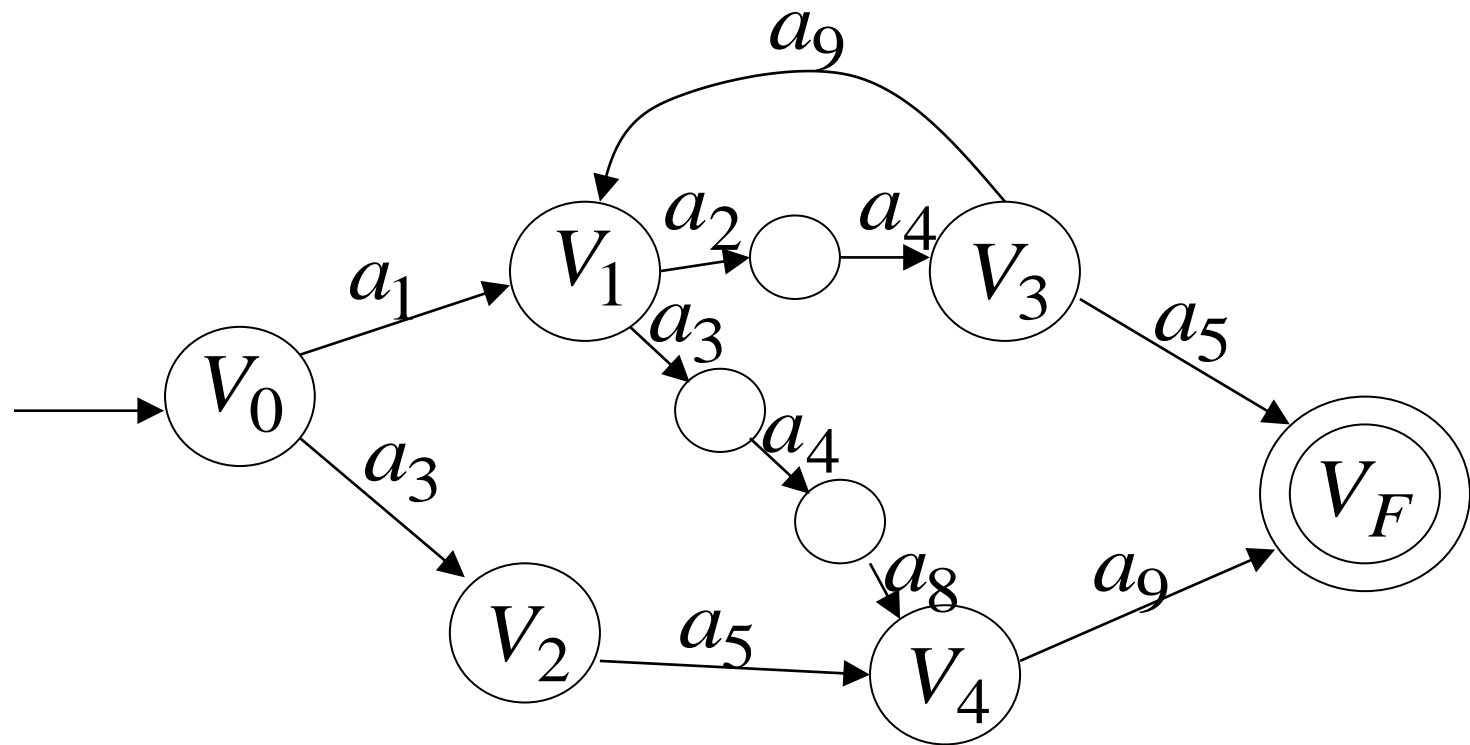


For each production: $V_i \rightarrow a_1 a_2 \cdots a_m$

we add transitions and intermediate nodes



Resulting NFA M looks like this:



It holds that: $L(G) = L(M)$

The case of Left-Linear Grammars

Let G be a left-linear grammar

We will prove: $L(G)$ is regular

Proof idea:

We will construct a right-linear grammar G' with $L(G) = L(G')^R$

Since G is left-linear grammar
the productions look like:

$$A \rightarrow Ba_1a_2 \cdots a_k$$

$$A \rightarrow a_1a_2 \cdots a_k$$

Construct right-linear grammar G'

Left
linear

G

$$A \rightarrow Ba_1a_2 \cdots a_k$$

$$A \rightarrow Bv$$



Right
linear

G'

$$A \rightarrow a_k \cdots a_2a_1B$$

$$A \rightarrow v^R B$$

Construct right-linear grammar G'

Left
linear

G

$$A \rightarrow a_1 a_2 \cdots a_k$$

$$A \rightarrow v$$



Right
linear

G'

$$A \rightarrow a_k \cdots a_2 a_1$$

$$A \rightarrow v^R$$

It is easy to see that: $L(G) = L(G')^R$

Since G' is right-linear, we have:



Proof - Part 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular language L is generated
by some regular grammar G

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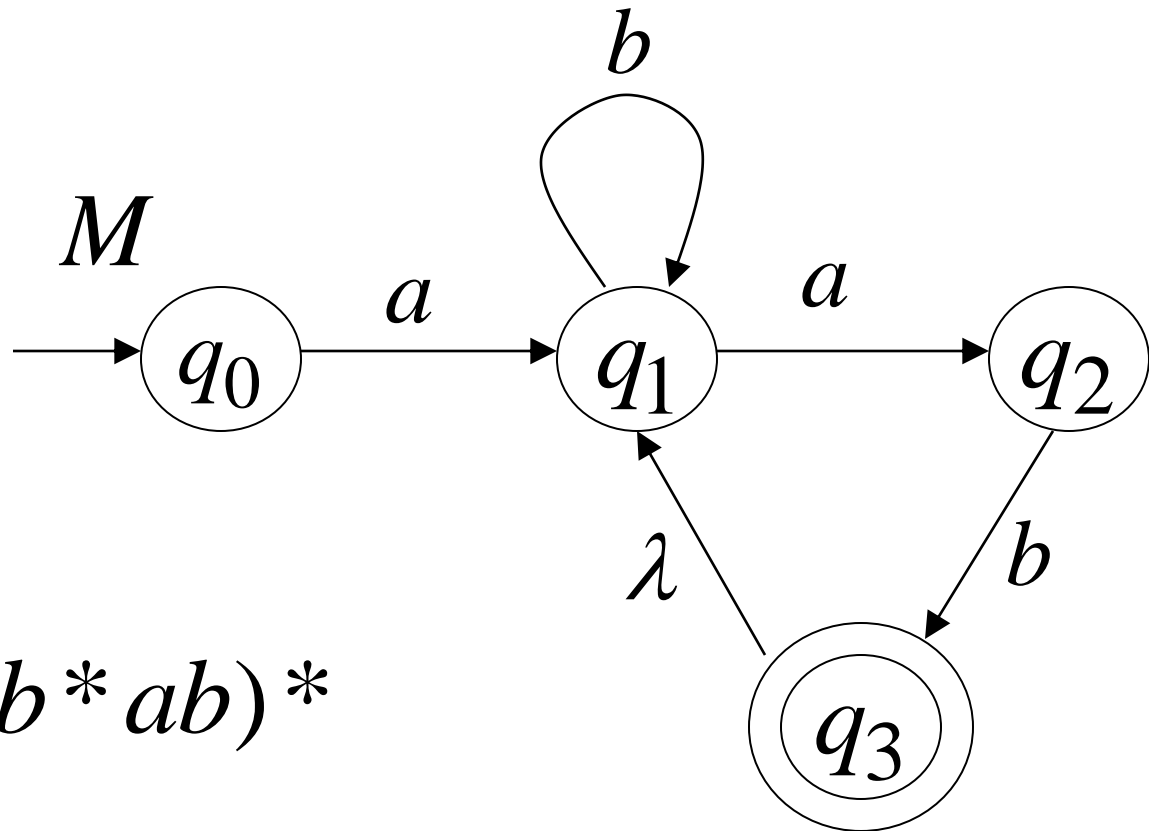
Proof idea:

Let M be the NFA with $L = L(M)$.

Construct from M a regular grammar G
such that $L(M) = L(G)$

Since L is regular
there is an NFA M such that $L = L(M)$

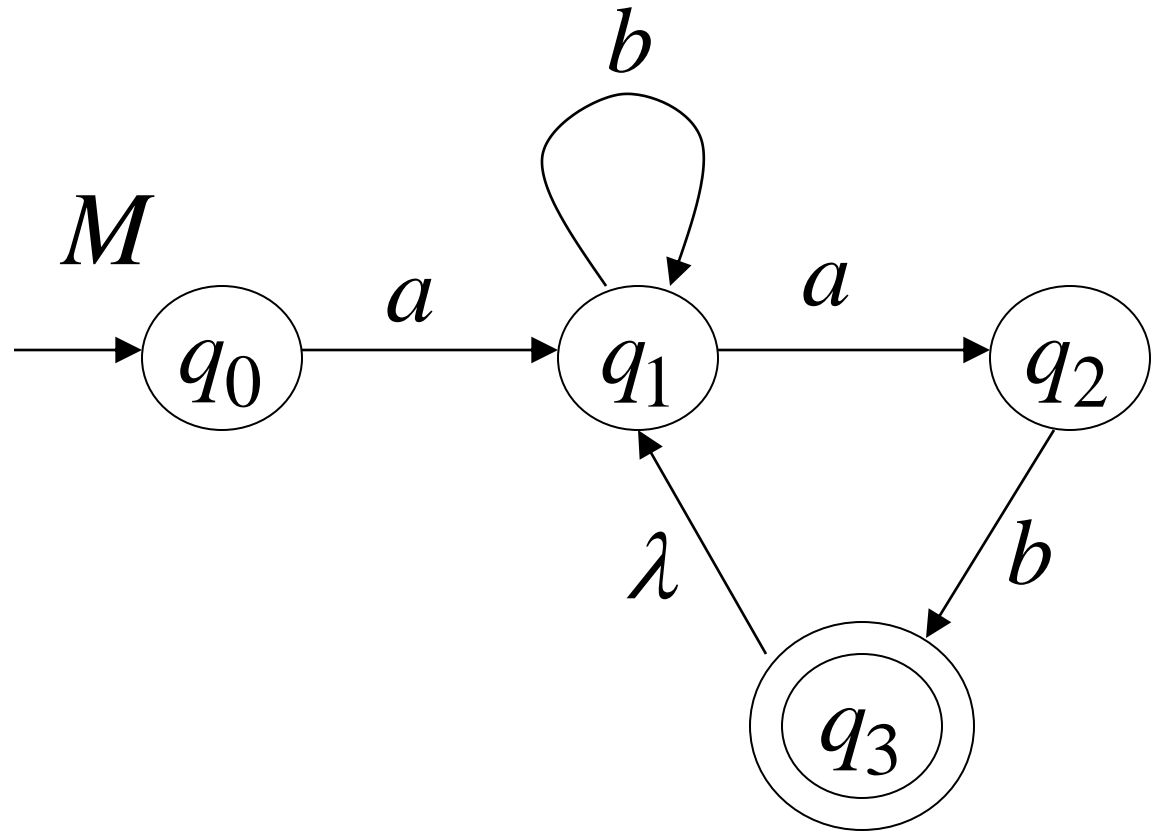
Example:



$$L = ab^*ab(b^*ab)^*$$

$$L = L(M)$$

Convert M to a right-linear grammar

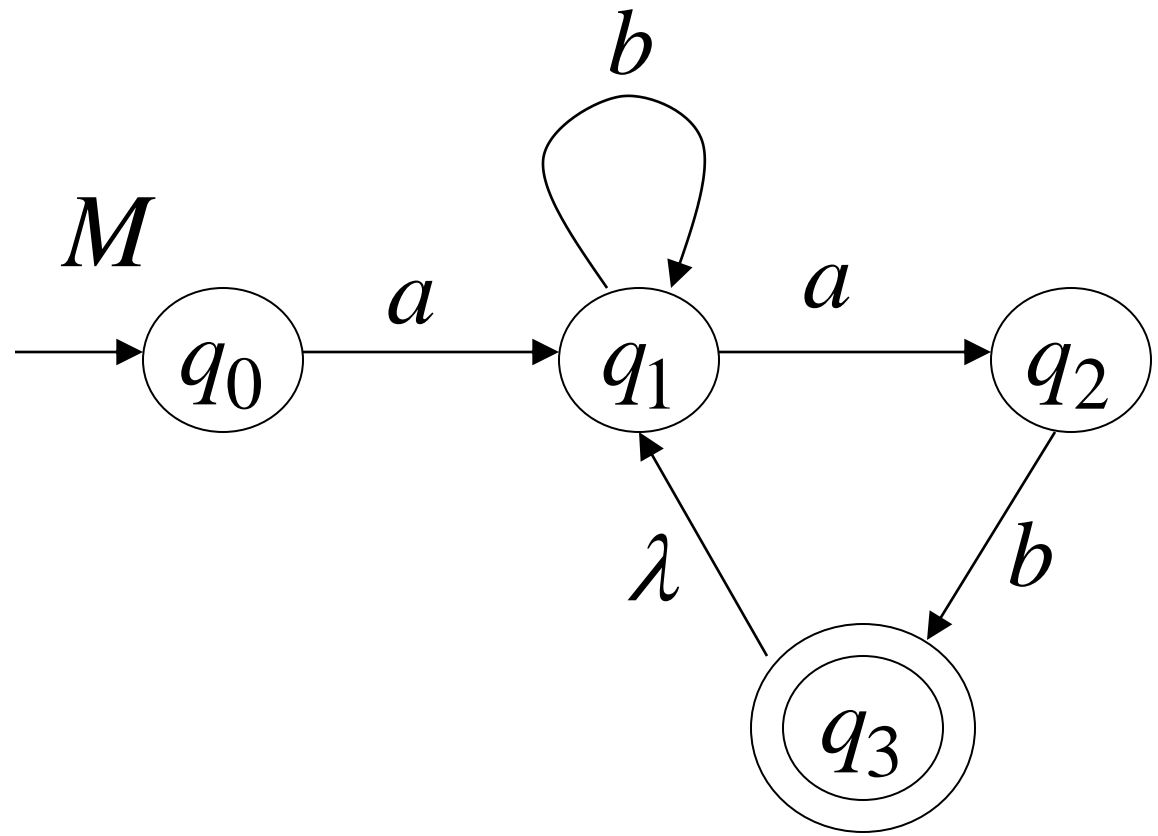


$$q_0 \rightarrow aq_1$$

$q_0 \rightarrow aq_1$

$q_1 \rightarrow bq_1$

$q_1 \rightarrow aq_2$

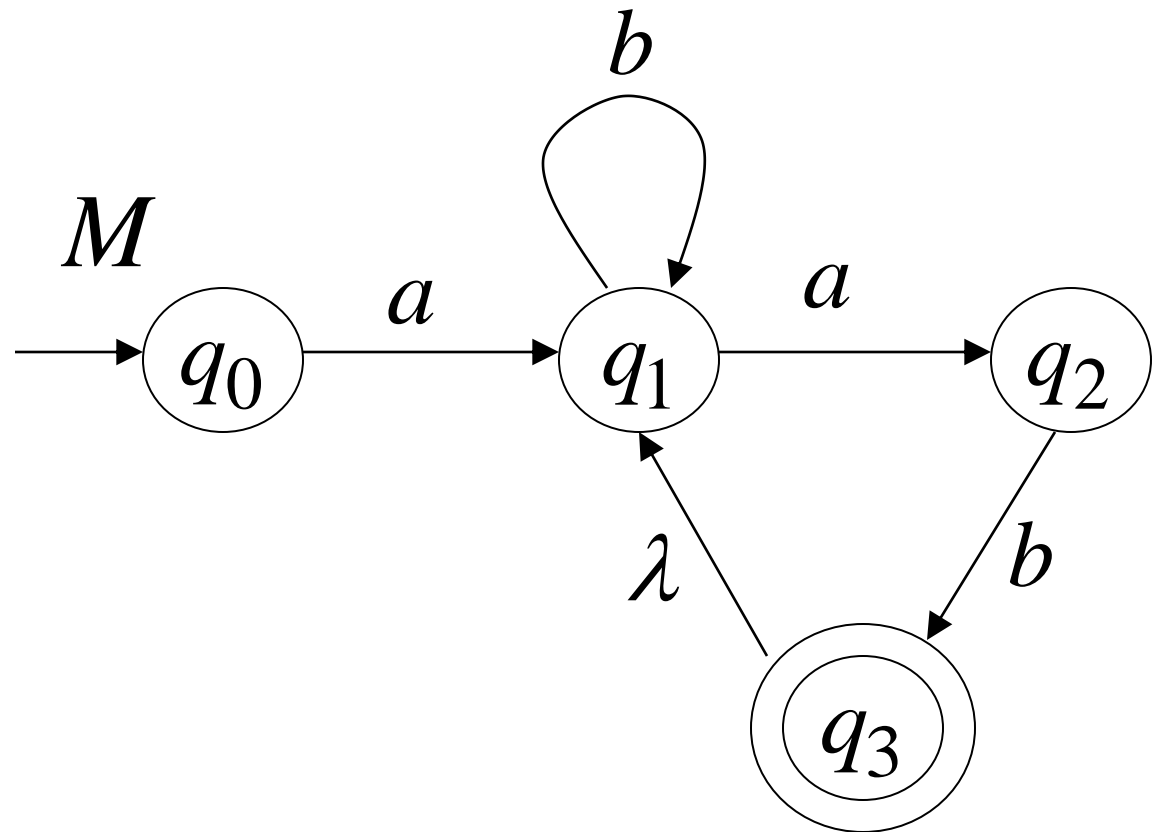


$q_0 \rightarrow aq_1$

$q_1 \rightarrow bq_1$

$q_1 \rightarrow aq_2$

$q_2 \rightarrow bq_3$



$$L(G) = L(M) = L$$

G

$q_0 \rightarrow aq_1$

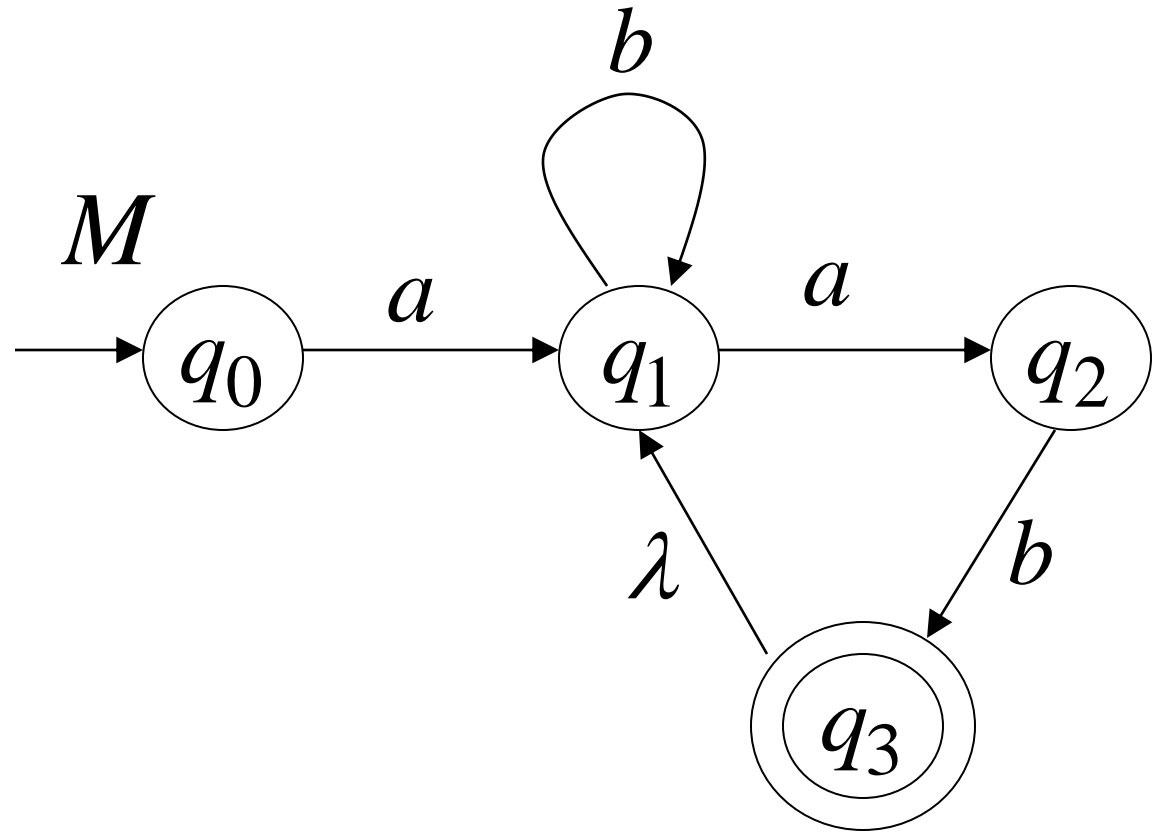
$q_1 \rightarrow bq_1$

$q_1 \rightarrow aq_2$

$q_2 \rightarrow bq_3$

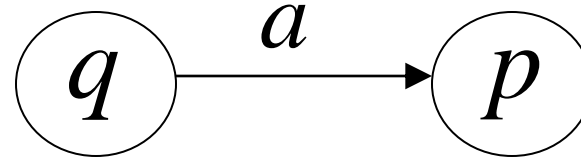
$q_3 \rightarrow q_1$

$q_3 \rightarrow \lambda$

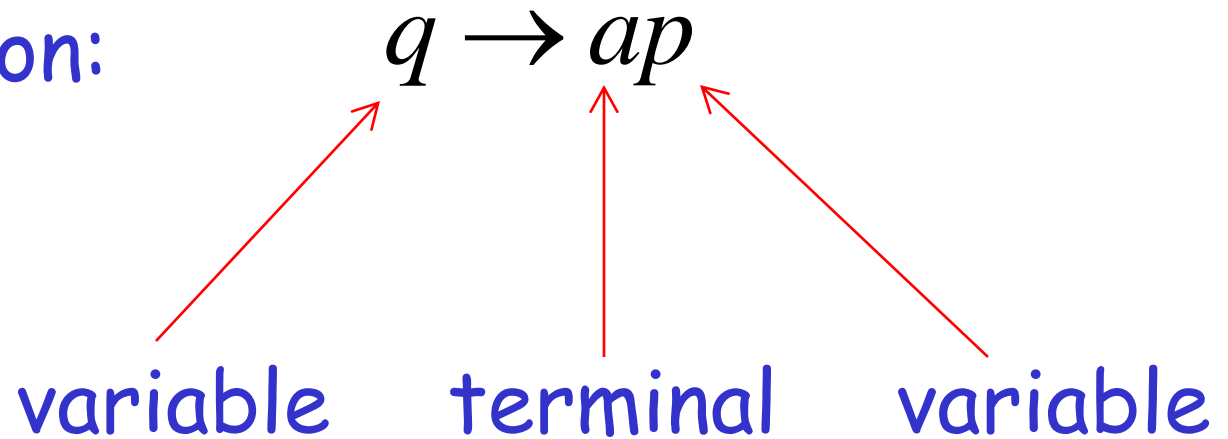


In General

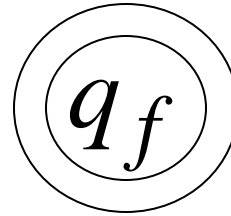
For any transition:



Add production:



For any final state:



Add production:

$$q_f \rightarrow \lambda$$

Since G is right-linear grammar

G is also a regular grammar

with $L(G) = L(M) = L$