

More Applications of the Pumping Lemma

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m (critical length)
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x y z$
- with $|x y| \leq m$ and $|y| \geq 1$
- such that: $x y^i z \in L \quad i = 0, 1, 2, \dots$

Non-regular languages

$$L = \{vv^R : v \in \Sigma^*\}$$



Regular languages

Theorem: The language

$$L = \{vv^R : v \in \Sigma^*\} \quad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Let m be the critical length for L

Pick a string w such that: $w \in L$

and length $|w| \geq m$

We pick $w = a^m b^m b^m a^m$

From the Pumping Lemma:

we can write: $w = a^m b^m b^m a^m = x y z$

with lengths: $|x y| \leq m, \quad |y| \geq 1$

$$w = xyz = \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m}$$

$x \quad y \quad z$

Thus: $y = a^k, \quad 1 \leq k \leq m$

$$x y z = a^m b^m b^m a^m \quad y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma: $x y^i z \in L$
 $i = 0, 1, 2, \dots$

Thus: $x y^2 z \in L$

$$x y z = a^m b^m b^m a^m \quad y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma: $x y^2 z \in L$

$$xy^2z = \overbrace{a \dots a}^{m+k} \overbrace{a \dots a}^m \overbrace{a \dots a}^m \overbrace{a \dots a}^m \in L$$

$\underbrace{\hspace{1.5cm}}_x \quad \underbrace{\hspace{1.5cm}}_y \quad \underbrace{\hspace{1.5cm}}_y \quad \underbrace{\hspace{4.5cm}}_z$

Thus: $a^{m+k} b^m b^m a^m \in L$

$$a^{m+k}b^mb^ma^m \in L \quad k \geq 1$$

BUT: $L = \{vv^R : v \in \Sigma^*\}$



$$a^{m+k}b^mb^ma^m \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L
is a regular language is not true

Conclusion: L is not a regular language

END OF PROOF

Non-regular languages

$$L = \{a^n b^l c^{n+l} : n, l \geq 0\}$$



Regular languages

Theorem: The language

$$L = \{a^n b^l c^{n+l} : n, l \geq 0\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : n, l \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : n, l \geq 0\}$$

Let m be the critical length of L

Pick a string w such that: $w \in L$ and
length $|w| \geq m$

We pick $w = a^m b^m c^{2m}$

From the Pumping Lemma:

We can write $w = a^m b^m c^{2m} = x y z$

With lengths $|x y| \leq m, |y| \geq 1$

$$w = xyz = \overbrace{a \dots a}^m \overbrace{a \dots a}^m \overbrace{ab \dots bc \dots cc \dots c}^{2m}$$
$$\underbrace{\hspace{1.5cm}}_x \underbrace{\hspace{1.5cm}}_y \underbrace{\hspace{4.5cm}}_z$$

Thus: $y = a^k, 1 \leq k \leq m$

$$x y z = a^m b^m c^{2m}$$

$$y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma: $x y^i z \in L$
 $i = 0, 1, 2, \dots$

Thus: $x y^0 z = xz \in L$

$$x y z = a^m b^m c^{2m} \qquad y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma: $xz \in L$

$$xz = \overbrace{a \dots a}^{m-k} \overbrace{a \dots a}^m \overbrace{b \dots b}^m \overbrace{c \dots c}^{2m} \in L$$

$$\underbrace{\hspace{1.5cm}}_x \underbrace{\hspace{4.5cm}}_z$$

Thus: $a^{m-k} b^m c^{2m} \in L$

$$a^{m-k}b^m c^{2m} \in L \quad k \geq 1$$

BUT: $L = \{a^n b^l c^{n+l} : n, l \geq 0\}$



$$a^{m-k}b^m c^{2m} \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L
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END OF PROOF

Non-regular languages

$$L = \{a^{n!} : n \geq 0\}$$



Regular languages

Theorem: The language $L = \{a^{n!} : n \geq 0\}$
is not regular

$$n! = 1 \cdot 2 \cdots (n-1) \cdot n$$

Proof: Use the Pumping Lemma

$$L = \{a^{n!} : n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^{n!} : n \geq 0\}$$

Let m be the critical length of L

Pick a string w such that: $w \in L$

$$\text{length } |w| \geq m$$

We pick $w = a^{m!}$

From the Pumping Lemma:

We can write $w = a^{m!} = x y z$

With lengths $|x y| \leq m, |y| \geq 1$

$$w = xyz = a^{m!} = \overbrace{a \dots a}^m \overbrace{a \dots a}^{m! - m}$$

The diagram illustrates the decomposition of the string $w = a^{m!}$ into xyz . The string is represented as a sequence of $m!$ 'a' characters. Braces above the string indicate the lengths of segments: a brace of length m covers the first m 'a's, and a brace of length $m! - m$ covers the remaining $m! - m$ 'a's. Below the string, three braces identify the segments x , y , and z . The brace for x covers the first m 'a's. The brace for y covers a portion of the first m 'a's, with a length of at least 1. The brace for z covers the remaining $m! - m$ 'a's.

Thus: $y = a^k, 1 \leq k \leq m$

$$x y z = a^{m!}$$

$$y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma: $x y^i z \in L$
 $i = 0, 1, 2, \dots$

Thus: $x y^2 z \in L$

$$x y z = a^{m!}$$

$$y = a^k, \quad 1 \leq k \leq m$$

From the Pumping Lemma: $x y^2 z \in L$

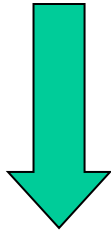
$$xy^2z = \overbrace{a \dots a}^{m+k} \overbrace{a \dots a}^{m!-m} \in L$$

$\underbrace{\hspace{1.5cm}}_x \underbrace{\hspace{1.5cm}}_y \underbrace{\hspace{1.5cm}}_y \underbrace{\hspace{4.5cm}}_z$

Thus: $a^{m!+k} \in L$

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

Since: $L = \{a^{n!} : n \geq 0\}$



There must exist p such that:

$$m!+k = p!$$

However: $m!+k \leq m!+m$ for $m > 1$

$$\leq m!+m!$$

$$< m!m + m!$$

$$= m!(m + 1)$$

$$= (m + 1)!$$



$$m!+k < (m + 1)!$$



$$m!+k \neq p! \quad \text{for any } p$$

$$a^{m!+k} \in L \qquad 1 \leq k \leq m$$

BUT: $L = \{a^{n!} : n \geq 0\}$



$$a^{m!+k} \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L
is a regular language is not true

Conclusion: L is not a regular language

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