3–Manifolds Algorithmically Bound 4–Manifolds

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Abstract

Contents

Chapter 1

Introduction

Survey of results: 3–manifolds bounding 4–manifolds.

1. Result using characteristic classes [8].

2. Result by embedding in $\mathbb{R}^5$ [4] [10] [14]

3. Result using Heegard decompositions [12]

Historical arguments prove 3–manifolds bound 4–manifolds, but we normally describe our 3– and 4–manifolds using triangulations. We would like a process that takes us from a triangulated 3–manifold to a triangulated 4–manifold. The work done by Turaev in [13] paved the way for an argument in [1] that includes a process, initially used to estimate a bound on the number of 4–dimensional simplices needed to triangulate a 4–manifold whose boundary is an input 3–manifold, which is made algorithmic in this thesis.

Chapter 2

Manifolds

Our first task is to build the machinery necessary to describe manifolds. We begin with a quick, formal definition, and build some basic properties. Next, we talk about a method of building and augmenting manifolds using handles. We end by defining triangulations: a combinatorial description of a manifold that allows us to cleanly describe our algorithms. This chapter is intended as a review of common tools in geometric topology, so in most places full proof will be omitted. For full details, refer to the texts [3], [5], [6], or [7].

2.1 Fundamentals

At its simplest, most colloquial definition, an n–dimensional manifold is a space that locally looks like an n–dimensional real or half space. We use *charts* and *atlases* to make explicit what is meant by "looks like."

Definition 2.1.1 (Coordinates)**.** Let $\mathbb{H}^n \subset \mathbb{R}^n$ denote the closed real half space under the subspace topology, defined as

$$\mathbb{H}^n = \{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n : x_0 \geq 0\}.$$

We use the notations $\texttt{int}(\mathbb{H}^n)$ and $\partial\mathbb{H}^n$ to denote the topological interior and boundary of $\mathbb{H}^n$ as subsets of $\mathbb{R}^n$ which, when $n > 0$, are

$$\begin{aligned} \texttt{int}(\mathbb{H}^n) &= \{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n : x_0 > 0\}, \\ \partial\mathbb{H}^n &= \{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n : x_0 = 0\} \approx \mathbb{R}^{n-1}. \end{aligned}$$

When $n = 0$, $\mathbb{H}^0 = \mathbb{R}^0 = \{0\}$, so $\mathtt{int}(\mathbb{H}^0) = \mathbb{R}^0$ and $\partial\mathbb{H}^0 = \emptyset$.

Let M be a second-countable Hausdorff space. The pair (U, f) where U is an open subset of M and f is a homeomorphism from U onto an open set of either $\mathbb{R}^n$ or $\mathbb{H}^n$ is called a *chart* of M. The map f is a *coordinate system* on U and its inverse f^{-1} is a *parameterization* of U. Writing f as

$$f(u) = (\xi_0(u), \dots, \xi_{n-1}(u)),$$

the functions ξ_i are *coordinate functions*

Definition 2.1.2 (Atlas)**.** Fix a nonnegative integer n. Let $\mathcal{A} = \{(U_\alpha, f_\alpha) : \alpha \in A\}$ be a collection of charts of M such that the codomain for each chart is of the fixed dimension n. If $\bigcup_A U_\alpha$ contains M, then $\mathcal{A}$ is an *atlas* for M. The homeomorphisms $f_\alpha \circ f_\beta^{-1} : f_\beta(U_\alpha \cap U_\beta) \to f_\alpha(U_\alpha \cap U_\beta)$ are *transition maps* of $\mathcal{A}$. We say that (U_α, f_α) and (U_β, f_β) are *smoothly compatible* if either $U_\alpha \cap U_\beta$ is empty or the transition maps $f_\alpha \circ f_\beta^{-1}$ and $f_\beta \circ f_\alpha^{-1}$ are smooth as maps $\mathbb{R}^n \to \mathbb{R}^n$, i.e. they have continuous partial derivatives of all orders. If every pair of charts in an atlas is smoothly compatible then that atlas is a *smooth atlas*. Two smooth atlases are equivalent if their union is a smooth atlas.

Definition 2.1.3 (Manifold)**.** Let M be a second-countable Hausdorff topological space, $\mathcal{A}$ an atlas for M, and n the dimension of the codomain of each chart in $\mathcal{A}$. If $\mathcal{A}$ is a smooth atlas, then the pair $(M, \mathcal{A})$ is a *smooth n–manifold*, *n–manifold*, or just *manifold*. If $\mathcal{A}$ is a maximal smooth atlas then we call it a *smooth structure* on M.

It eases the notational burden to assume that our manifolds are always equipped with a smooth structure. This assumption allows us to omit writing the an atlas when talking about a manifold.

Definition 2.1.4 (Boundary)**.** Let M be a smooth n–manifold. A chart (U, f) is called an *interior chart* is $f(U)$ is an open subset of $\mathbb{R}^n$ and is a *boundary chart* if $f(U)$ is an open subset of $\mathbb{H}^n$ with $f(U) \cap \partial\mathbb{H}^n \neq \emptyset$. Let $p \in M$. We say that p is an *interior point* if it is in the domain of an interior chart, and is a *boundary point* if it is in the domain of a boundary chart (U, f) so that $f(p) \in \partial\mathbb{H}^n$. The set of all boundary points of M is called the *boundary of* M and is denoted by ∂M. Similarly, the set of all interior points of M is called the *interior* of M and is denoted by $\mathtt{int}(M)$. If M is compact with empty boundary then M is *closed* as a manifold.

There is a potential conflict of this definition with the concept of topological closure. In the rare case that we want to say that a manifold is topologically closed, we will specify that the type of closure is topological. Otherwise, the statement of a "closed manifold" will refer to a manifold with empty boundary.

Example 2.1.5. The n–sphere, denoted S^n and defined as a subset of $\mathbb{R}^{n+1}$, is

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

The $(n+1)$–dimensional topologically open ball, or $(n+1)$–ball, defined as a subset of $\mathbb{R}^{n+1}$, is

$$B^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| < 1\}.$$

We use $\|\cdot\|$ to denote the euclidean norm defined for $x \in \mathbb{R}^{n+1}$ by

$$\|x\| = \left(\sum x_i^2\right)^{1/2}.$$

The space D^{n+1}, the topological closure of B^{n+1}, is the closed $(n+1)$–disc. Both B^{n+1} and D^{n+1} are $(n+1)$–manifolds, and D^{n+1} is a manifold with boundary the n–sphere, a closed n–manifold.

It is often more efficient to think of S^1 and D^2 as submanifolds of the complex plane $\mathbb{C}$.

2.2 Smooth Maps

Definition 2.2.1 (Smooth Map)**.** Let $(X, \{U_\alpha, f_\alpha\})$ and $(M, \{V_\beta, g_\beta\})$ be smooth n– and k–manifolds respectively and let $\varphi : X \to M$ be a map between them. If, for any α and β, the composition $g_\beta \circ \varphi \circ f_\alpha^{-1}$ is smooth as a map $\mathbb{R}^n \to \mathbb{R}^k$, then we say φ is *smooth* as a map between manifolds. If φ is smooth and a well-defined φ^{-1} exists and is smooth, then φ is called a *diffeomorphism* between manifolds. We say that manifolds are *diffeomorphic* if there exists a diffeomorphism between them.

Proposition 2.2.2. Diffeomorphism is an equivalence relation on the space of smooth manifolds.

Manifolds are defined by their local homogeneity, and a tangent space is a precise description of that homogeneity near a point.

Definition 2.2.3 (Tangent Space)**.** Let M be an n–manifold and p be a point in M. A smooth map $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$ is called a *curve* in M through p. Note that this definition is independent of the chart used, as all of our transition maps are smooth.

For p in the boundary of M, we allow an additional type of curve in our definition. Let γ be a smooth map $(-\varepsilon, 0\,] \to M$ or $[\,0, \varepsilon) \to M$ each with $\gamma(0) = p$. A map of the first type is called an *outward curve* in M through p and a map of the second type is called an *inward curve* in M through p.

Let C_pM be the space of smooth curves in M through p and γ_1, γ_2 elements of C_pM. We can define an equivalence relation $\sim$ on $C_p(M)$ by saying that $\gamma_1 \sim \gamma_2$ if

$$\frac{d}{dt}(f \circ \gamma_1)(0) = \frac{d}{dt}(f \circ \gamma_2)(0).$$

An equivalence class of the curve γ in C_pM is a *tangent vector* at p and is written as $\gamma'(0)$. The quotient $C_pM/\sim$ is the *tangent space* at p, denoted T_pM. A vector $\gamma'(0)$ of T_pM is called an *inward vector* if the equivalence class $[\gamma]$ contains only inward curves, and an *outward vector* if $[\gamma]$ contains only outward curves.

Proposition 2.2.4. Let M be an n–manifold and p a point in M. Then T_pM is a vector space isomorphic to $\mathbb{R}^n$.

Proof. Let (U, f) be a chart containing p with $f(p) = \vec{0}$. It follows from our definition of tangent space that the map defined by

$$\begin{array}{rccl} df: & T_pM & \to & \mathbb{R}^n \\ & \gamma'(0) & \mapsto & \frac{d}{dt}(f \circ \gamma)(0). \end{array}$$

is a bijection, so we also have a well defined inverse $(df)^{-1}$. We define operations on

T_pM so that F is strengthened to a vector space isomorphism:

$$\begin{aligned} \gamma_1'(0) + \gamma_2'(0) &= (df)^{-1}(df(\gamma_1'(0)) + df(\gamma_2'(0))), \\ t\gamma_1'(0) &= (df)^{-1}(t\, df(\gamma_1'(0))). \end{aligned}$$

The vector space structure of T_pM is independent of the choice of chart. To see this, let L_v be the parameterized straight line through $\vec{0}$ whose velocity is $v \in \mathbb{R}^n$. Precisely, $L_v(t) = tv$. For any chart (U, f) with $f(p) = 0$, the curve $f^{-1} \circ L_v$ is a curve through p for which $df(f^{-1} \circ L_v) = v$. Because an equivalence class contains such a curve for any applicable chart, the association of a tangent vector with a vector in $\mathbb{R}^n$ does not depend on the chart used. $\square$

In normal calculus, the derivative is a linearization of a function. We use tangent spaces to define the derivative of a smooth map between manifolds because tangent spaces are used as pointwise linearizations of manifolds.

Definition 2.2.5 (Differential)**.** Let $\varphi : X \to M$ be a smooth map between manifolds. Let $p \in X$ and $\varphi(p) = q \in M$. The *differential* of φ at p is a linear map defined as

$$\begin{aligned} d\varphi_p : \quad T_pX &\to T_qM \\ \gamma'(0) &\mapsto (\varphi \circ \gamma)'(0), \end{aligned}$$

and the element $(\varphi \circ \gamma)'(0)$ of T_qM is the *pushforward* of the tangent vector $\gamma'(0)$.

We also classify smooth maps between manifolds. This classification helps us define a submanifold and provides some of the groundwork for defining handles in the next chapter.

Definition 2.2.6 (Embedding)**.** Let $\varphi : X \to M$ be a smooth map between manifolds. The *rank* of φ at the point $p \in X$ is the rank of the differential $d\varphi$. This is computed as the rank of the Jacobian matrix of φ under a coordinate system or as the dimension of $\varphi(T_pX) \subset T_qM$. If φ has rank k for every p in X, then φ is of *constant rank* and we say $\texttt{rank}(\varphi) = k$.

If $d\varphi$ is injective at p, then φ is *immersive* at p. If φ is an everywhere immersive, then it is an *immersion*. This is equivalent to $\texttt{rank}(\varphi) = \texttt{dim}(X)$. If $d\varphi$ is surjective at p, then φ is *submersive* at p, and p is a *regular point* of φ. If φ is everywhere submersive, then it is a *submersion*. This is equivalent to $\texttt{rank}(\varphi) = \texttt{dim}(M)$. If

$\texttt{rank}(d)\varphi_p$ is less than maximal, then p is a *critical point* of φ. In this case, $\varphi(p) = q$ is a critical value. If p is a critical point and $d^2\varphi_p$ is of less than full rank, then p is a *degenerate critical point.* A smooth map with no degenerate critical points is called *generic.* A point $q \in M$ whose preimage $\varphi^{-1}(q)$ consists entirely of regular points is called a *regular value.*

If φ is an injective immersion that is a homeomorphism onto its image $\varphi(X) \subset M$ under the subspace topology, then it is a *smooth embedding* or just *embedding.*

Let $X \subset M$ where M is a manifold, and let $i : X \hookrightarrow M$ be inclusion. If i is an embedding, then we call X an *embedded submanifold* or *submanifold* of M. One can consider X to be a manifold with smooth structure induced by the embedding.

To clarify what it means for a smooth function to be generic, consider a smooth function with degenerate critical points. Through small perturbations, these degenerate critical points can be turned into non-degenerate critical points. Because such a function could be easily 'fixed,' we say that a generic smooth map is one whose critical points are all non-degenerate.

Theorem 2.2.7 (Regular Value Theorem)**.** Let $\varphi : X \to M$ be a smooth map between manifolds. Let $q \in M$ be a regular value. The preimage $\varphi^{-1}(q)$ is either empty or a $(\texttt{dim}(X) - \texttt{dim}(M))$–submanifold of X.

Proposition 2.2.8. Let M be an $(n+1)$–manifold. The boundary ∂M is an n–dimensional closed submanifold of M.

The main result of this work is applicable to orientable 3–manifolds, so we will review what is meant by a space being orientable. Essentially, orientability is a guarantee that when you go for a walk your right and left sides haven't switched places by the time you get home.

Definition 2.2.9. Let V^n be an n–dimensional real vector space, $\mathcal{B}(V^n)$ the set of ordered bases for V^n, and $\mathrm{GL}_n(\mathbb{R})$ the general linear group, i.e. the space of $n \times n$ invertible matrices with entries in $\mathbb{R}$. Let b_1 and b_2 be any two ordered bases from $\mathcal{B}(V^n)$. It is a standard result of linear algebra that there is a unique element A of $\mathrm{GL}_n(\mathbb{R})$ that transforms b_1 into b_2. If the determinant $\det A$ is positive, then b_1 and b_2 are *positively oriented* with respect to each other. If $\det A$ is negative, then b_1

and b_2 are *negatively oriented.* We can define an equivalence relation $\sim$ on $\mathcal{B}(V^n)$ by saying that $b_1 \sim b_2$ if they are positively oriented. An *orientation* of V^n is a choice of one of the two equivalence classes of $\mathcal{B}(V^n)$. This also allows us to classify linear transformations of $\mathrm{GL}_n(\mathbb{R})$ by their action on the quotient space $\mathcal{B}(V^n)/\sim$ by saying they are *orientation preserving* if they have positive determinant and *orientation reversing* if they have negative determinant.

Orientations of $\mathbb{R}^n$ let us define orientations of manifolds.

Definition 2.2.10. Suppose we have fixed an orientation of $\mathbb{R}^n$. Let M be an n–manifold. We say that an *orientation* of M is a consistent choice of orientation of the tangent space at every point of M. A consistent choice of orientation means that for every chart (U, f) with $f : U \to \mathbb{R}^n$, the vector space isomorphism $df : T_pM \to \mathbb{R}^n$ is orientation preserving at every point in U. If M admits an orientation, then it is *orientable.* If M does not admit an orientation, then it is *non-orientable.*

Orientations of manifolds let us decide whether a map is orientation preserving or reversing.

Definition 2.2.11. Let $\phi : X \to M$ be a smooth map between oriented n–manifolds. If the differential $df_p : T_pX \to T_{\phi(p)}M$ is orientation preserving (resp. orientation reversing) as a map between vector spaces for every p in X, then ϕ is *orientation preserving* (resp. *orientation reversing*) as a map.

2.3 Bundles

A common tool in both the construction and description of a manifold is a *bundle.* The ultimate construction in this section is the tubular neighbourhood of a submanifold, which is unique up to a fibre-preserving isotopy.

Definition 2.3.1 (Bundle)**.** A real *vector bundle* is a tuple $\beta = (E, B, \pi : E \to B)$ where B is called the *base space*, E the *total space*, and π the *projection.* The projection is a continuous map for which the subspaces $\pi^{-1}(b) = V_b$ all have the structure of a k–dimensional vector space. The space V_b is called the *fibre* over b.

A bundle is *locally trivial*. That is, for every point $b \in B$ there exists an open neighbourhood $U \subset B$ containing b and a homeomorphism

$$\varphi : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

such that the map $v \mapsto \varphi(b, v)$ is a vector space isomorphism $\mathbb{R}^n \mapsto V_b$ for every $b \in U$. Such a homeomorphism is called a *local trivialization*, and such a pair is a *local coordinate system*. Any pair of local trivializations

$$\begin{array}{rrcl} \varphi_U : & U \times \mathbb{R}^n & \to & \pi^{-1}(U), \\ \varphi_V : & V \times \mathbb{R}^n & \to & \pi^{-1}(V) \end{array}$$

must be compatible in the sense that the composition

$$\varphi_V^{-1} \circ \varphi_U : (U \cap V) \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n,$$

is well defined on $U \cap V$ and satisfies

$$\varphi_V^{-1} \circ \varphi_U(b, v) = (b, A_{UV}(b)(v))$$

for every b in $U \cap V$, where A_{UV} is a function

$$A_{UV} : U \cap V \to \mathrm{GL}_n(\mathbb{R})$$

that assigns a linear transformation from $\mathrm{GL}_n(\mathbb{R})$ to every point b of $U \cap V$. If it is possible to take U to be all of B, then β is a *trivial bundle* and the map $U \times \mathbb{R}^n \to E$ is a *trivialization* of the bundle.

When E and B are smooth manifolds, π is smooth, and the local trivializations are diffeomorphisms, β is a *smooth vector bundle*.

A continuous map $s : B \hookrightarrow E$ that is a right inverse of π is called a *section*. The *zero section* of β is used to refer to both the map

$$\begin{array}{rrcl} z : & X & \to & E \\ & x & \mapsto & (x, 0), \end{array}$$

and its image $z(B)$ in E. Note that the compatibility condition on local trivialization guarantees that the zero section is well defined.

For β a k–vector bundle, we can form a new object called a *k–disc bundle* by restricting the fibres of β to the set of vectors of length at most 1. The machinery we build for vector bundles is applicable to disc bundles via restriction.

Definition 2.3.2 (Bundle Isomorphism)**.** Let $\Phi : \beta_0 \to \beta_1$ be a map between vector bundles. We call Φ a *fibre map* if $\Phi : E_0 \to E_1$ covers a map $\varphi : B_0 \to B_1$. For Φ to cover φ, that means the following diagram commutes:

$$\begin{array}{ccc} E_0 & \xrightarrow{\Phi} & E_1 \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ B_0 & \xrightarrow{\varphi} & B_1 \end{array}$$

This means that if $b \in B_0$ and $\varphi(b) = c$, then Φ maps V_b to V_c by a map we will denote Φ_b.

If Φ_b is a linear map for every $b \in B_0$, then we call Φ a *bundle morphism.* If Φ is a bundle morphism, $B_0 = B_1 = B$, Φ_b bijective for each $b \in B$, and φ is id_B, then Φ is a *bundle isomorphism.* A bundle isomorphism $\Phi : E \to E$ is a *bundle automorphism.*

Definition 2.3.3 (Tangent Bundle)**.** Let M be an m–manifold. We define the total space TM of a smooth vector bundle with base space M and fibres $\mathbb{R}^n \approx T_pM$ at any $p \in M$ by taking the disjoint union of all tangent spaces:

$$TM = \bigsqcup_{p \in M} T_pM.$$

The projection $\pi : TM \to M$ is defined by $\pi(q) = p$ for every $q \in T_pM$. We call TM the *tangent bundle* over M. A section $s : M \to TM$ is called a *vector field* on M.

Definition 2.3.4 (Normal Bundle)**.** Let X be a k–dimensional submanifold of the $(n + k)$–manifold M. At a point $p \in X \subset M$, the tangent space T_pX is a subspace of the tangent space T_pM. Denote the orthogonal complement to T_pX in T_pM by N_pX and call it the normal space at p in X. That is, $T_pX \oplus N_pX = T_pM$. From linear algebra, N_pX is an n–dimensional vector space. We define the total space

$N_M X$ of a vector bundle with base space X and fibres $\mathbb{R}^n \approx N_p X$ at any $p \in X$ by taking the disjoint union of all normal spaces:

$$N_M X = \bigsqcup_{p \in X} N_p X.$$

The projection $\pi : N_M X \to X$ is defined by $\pi(q) = p$ for every $q \in N_p X$. We call $N_M X$ the *normal bundle* over X in N.

Definition 2.3.5 (Tubular Neighbourhood)**.** Let X be a closed submanifold of the closed manifold M. A smooth embedding $f : N_M X \to M$ with $f(x, 0) = x$ and $f(N_M X)$ an open neighbourhood of X in M is called a *tubular neighbourhood* of X in M. We often denote the pair (X, f) by $\nu_M X$.

Similar to the tubular neighbourhood is the *closed tubular neighbourhood*, which has instead the structure of a normal disc bundle over X. We use $D_M X \subset N_M X$ to denote the disc bundle over X, and $\overline{\nu}_M X$ to denote a closed tubular neighbourhood. The notion of "closed" used in this naming convention is that of topological closure.

Theorem 2.3.6 (Existence of Tubular Neighbourhoods, Theorem 4.5.2 in [5])**.** Let X be a closed submanifold of the closed manifold M. Then X has a tubular neighbourhood in M.

Similar to the tubular neighbourhood is the collar of a manifold's boundary.

Proposition 2.3.7 (Collar)**.** Let M be a manifold with nonempty boundary $X = \partial M$. There exists an open neighbourhood U of X in M and a diffeomorphism

$$f : U \to X \times \mathbb{H}^1$$

where $f(X) = X \times \{0\}$. The pair (X, f) is called a *collar neighbourhood* or *collar* of X.

Note that the collar is actually a fibre bundle with base ∂M, fibre $\mathbb{H}^1$, and trivial structure. This allows us to relate our future results on tubular neighbourhoods to include collars. In particular, tubular neighbourhoods are unique up to a fibre-preserving isotopy. Let's define precisely what that means.

Definition 2.3.8 (Homotopy)**.** Let $f, g : X \to M$ be smooth maps between smooth manifolds. Denote the closed unit interval $[0, 1]$ by $\mathbb{I}$. A function

$$\begin{array}{rrcl} H: & X \times \mathbb{I} & \to & M \\ & (x,t) & \mapsto & H_t(x) \end{array}$$

with $H_0(x) = f(x)$ and $H_1(x) = g(x)$ is a *homotopy* between f and g. If a homotopy exists, then f and g are *homotopic*. Less formally, f and g being homotopic means that one can be continuously deformed into the other. The topological spaces X and M are *homotopy equivalent* if there exist continuous maps $f : X \to M$ and $g : M \to X$ for which $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_M.

Because we are primarily interested in smooth functions to build our machinery, we extend our definition of homotopy to a smooth version. With the notation above, a smooth map $H : X \times [0, 1] \to M$ with $H_0(x) = f(x)$ and $H_1(x) = g(x)$ is a *smooth homotopy* between f and g. If a smooth homotopy exists, then f and g are *smoothly homotopic*.

A homotopy through embeddings is called an isotopy. This is useful in general for comparing embedded submanifolds.

Definition 2.3.9 (Isotopy)**.** Let X be a smoothly embedded submanifold of M. An *isotopy* of X in M is a smooth homotopy

$$\begin{array}{rrcl} F: & X \times \mathbb{I} & \to & M \\ & F(x,t) & = & F_t(x) \end{array}$$

such that the related map

$$\begin{array}{rrcl} \hat{F}: & X \times \mathbb{I} & \to & M \times \mathbb{I} \\ & (x,t) & \mapsto & (F_t(x), t) \end{array}$$

is an embedding. The submanifolds $F_0(X)$ and $F_1(X)$ are *isotopic*. When $X = M$ and F_t is a diffeomorphism for each t, F is a *diffeotopy* of M.

Let X be a smoothly embedded closed submanifold of M and consider a pair of tubular neighbourhoods $f, g : N_M X \to M$ of X in M. An isotopy

$$\begin{aligned} F: \quad N_M X \times \mathbb{I} &\to M \\ F(x,t) &= F_t(x) \end{aligned}$$

satisfying the following properties:

1. $F_0 = f$ and $F_1 = g$,
2. $F_0(N_M X) = F_1(N_M X)$,
3. $F_1^{-1} \circ F_0$ is a vector bundle isomorphism $N_M X \to N_M X$,

is an *isotopy of tubular neighbourhoods*, and the tubular neighbourhoods (X, f) and (X, g) are *isotopic*.

This leads into our uniqueness result for tubular neighbourhoods.

Theorem 2.3.10 (Uniqueness of Tubular Neighbourhoods, Theorem 3.3.1 of [6])**.** Let X be a closed submanifold of M. Any pair of tubular neighbourhoods of X in M are isotopic.

A similar theorem also applies to collar neighbourhoods.

Theorem 2.3.11. Let M be a manifold with boundary $X = \partial M$. Let (X, f) and (X, g) be collars of X. Then $f(X \times \mathbb{H}^1)$ and $g(X \times \mathbb{H}^1)$ are isotopic through an isotopy $F : (X \times \mathbb{H}^1) \times \mathbb{I} \to M$ with $F_0 = f$, $F_1 = g$, and $F_t(x) = x$ for every $x \in X$.

There is a stronger uniqueness theorem for tubular neighbourhoods that uses a tighter definition. It essentially says that two of these neighbourhoods are isotopic through a diffeotopy that is stationary outside of a small neighbourhood of the tube.

Theorem 2.3.12 (Isotopy Extension)**.** Let X be a smooth compact submanifold of the smooth closed manifold M, and let

$$\begin{aligned} F: \quad X \times \mathbb{I} &\to M \\ F(x,t) &= F_t(x) \end{aligned}$$

be an isotopy of X in M. Let L be the subset of M equal to the union of the images of F_t for each t. More precisely,

$$L = \bigcup_{t\in[0,1]} F_t(X).$$

There exists a diffeotopy

$$\begin{aligned} G: \quad M \times \mathbb{I} &\rightarrow M \\ G(y,t) &= G_t(y) \end{aligned}$$

with $G_0 = \mathrm{id}_M$, G_1 equal to F_1 on $X \subset M$, and F_t the identity on M outside of an arbitrarily small neighbourhood of L for all t.

Definition 2.3.13 (Ambient Isotopy)**.** Let X, M, F be as above. The isotopy $G : M \times \mathbb{I} \rightarrow M$ guaranteed by Theorem 2.3.12 is called an *ambient isotopy*. The images $F_0(X)$ and $F_1(X)$ are *ambiently isotopic* as submanifolds of M.

Notice that, when the neighbourhood is closed, isotopy extension can strengthen the uniqueness of tubular neighbourhoods theorem to one that is unique through an ambient isotopy. We can perform a similar strengthening on open tubular neighbourhoods as long as we restrict the definition slightly.

Definition 2.3.14. A tubular neighbourhood that is obtained by an arbitrarily small shrinking of another tubular neighbourhood is called *proper*.

Examples of improper tubular neighbourhoods would be $\mathbb{R}^n$ as a neighbourhood of the origin in $\mathbb{R}^n$, or the strip $\{x \in \mathbb{R}^2 : \|x\| < \frac{\pi}{2}\}$ as a neighbourhood of the line $x = 0$ in $\mathbb{R}^2$ whose fibres are the curves $y = \tan x + c$. An example of a proper tubular neighbourhood would be the same strip, but with fibres $y = c$. It should be clear that the interior of a closed tubular neighbourhood is proper, and every proper tubular neighbourhood is the interior of a closed tubular neighbourhood.

Theorem 2.3.15 (Uniqueness of Proper Tubular Neighbourhoods, Theorem 3.3.5 of [6])**.** Let X be a closed submanifold of M. Any two proper tubular neighbourhoods of X are isotopic through an isotopy that can be extended to an ambient isotopy.

For X a closed submanifold of ∂M, our definitions guarantee a tubular neighbourhood of X in ∂M, but not of X in M. Letting $X = \partial M$, notice that ∂M has a trivial normal bundle $\partial M \times \mathbb{R}$ in M, and a collar that is an embedding of $\partial M \times \mathbb{H}^1$ in M. It is clear that an appropriate analogue of the tubular neighbourhood to submanifolds of the boundary would be a generalization of the collar to a sort of "half-tubular neighbourhood."

Definition 2.3.16. Let W be a manifold with boundary and let X be a submanifold of $M = \partial W$. The normal bundle of X in W is the fibre-wise direct sum of $N_M X$ with $(N_W M)|_X$. Because $N_W M$ is a trivial bundle, the normal bundle of X in W is just $N_M X \times \mathbb{R}$. We can only embed one half of $N_M X \times \mathbb{R}$ in W, so we extend f to an embedding $F : N_M X \times \mathbb{H}^1 \to W$ such that F restricted to $z(N_M X) \times \mathbb{H}^1$ coincides with a collar of M on X. Such a neighbourhood is called a *regular neighbourhood.*

Our theorems regarding the existence and uniqueness of tubular neighbourhoods extend with the definition. Proper tubular neighbourhoods are defined in exactly the same way, and uniqueness up to isotopy is now uniqueness up to composition of isotopies of $\nu_M X$ and isotopies of the collar neighbourhood.

2.4 Handles

The main tool used to construct 4–manifolds in later chapters is handle attachment. We focus mainly on the definitions and results needed to meaningfully attach 1– and 2–handles to a 4–manifold.

Remark 2.4.1. There is more than one way to define handle attachment, and we choose to do so in a way that feels more combinatorial in nature. The main concern with this approach is that the object resulting from handle attachment is a "manifold with corners" rather than a smooth manifold. There are arguments that the corners can be smoothed away in a canonical way such that a manifold obtained via handle attachment is smooth and unique up to diffeomorphism, but delving into such an argument at this point would be a distraction. {NOTE: perhaps include an argument as an appendix?} A construction that does not require the smoothing of corners can be found in [6], but the machinery makes explicit handle attachment unnecessarily complicated.

Throughout this section, we will have $n = \lambda + \mu$, M an n–manifold with boundary, and $H^\lambda = D^\lambda \times D^\mu$. Attaching an n–dimensional λ–handle to M is the process of joining M to H^λ along an embedding of $\partial D^\lambda \times D^\mu$ in ∂M.

Definition 2.4.2 (Handle)**.** Let $\varphi : \partial D^\lambda \times D^\mu \to \partial M$ be an embedding. A new space is defined through the equivalence $x \sim \varphi(x)$ where x in $\partial D^\lambda \times D^\mu$ by first considering the space $M \cup H^\lambda / \sim$, a "manifold with corners." Through canonical smoothing of the corners mentioned in Remark 2.4.1, we obtain space $M \cup_\varphi H^\lambda$ that is a smooth n–manifold. We call H^λ an n*–dimensional* λ*–handle*, and say that $M \cup_\varphi H^\lambda$ is the result of an n–dimensional λ*–handle attachment*.

Note that the domain of φ has the structure of a trivial μ–disc bundle over $S^{\lambda-1}$ and, using the language of vector bundles, the image of φ has the structure of a closed tubular neighbourhood of $f_0(S^{\lambda-1}) = \varphi \circ z(S^{\lambda-1})$. Our uniqueness theorems for tubular neighbourhoods tell us that the closed tubular neighbourhood

$$f : D_{\partial M} f_0(S^{\lambda-1}) \to \overline{\nu}_{\partial M}\, f_0(S^{\lambda-1})$$

of $f_0(S^{\lambda-1})$ is unique up to ambient isotopy of the embedding f_0. Thus, if we have an embedding $f_0 : S^{\lambda-1} \to \partial M$ (i.e. a knot) with trivial disc bundle, then a handle attachment can be defined using f_0 and an explicit embedding

$$f : S^{\lambda-1} \times D^\mu \to D_{\partial M}\, f_0(S^{\lambda-1})$$

for a closed tubular neighbourhood $\overline{\nu}_{\partial M}\, f_0(S^{\lambda-1})$.

In other words, the characteristics of handle attachment that fully describe the smooth n–manifold $M \cup_\varphi H^\lambda$ up to diffeomorphism are:

1. The isotopy class of an embedding $f_0 : S^{\lambda-1} \to \partial M$ with trivial normal disc bundle, and

2. the isotopy class of an identification of $S^{\lambda-1} \times D^\mu$ with $\overline{\nu}_{\partial M}\, f_0(S^{\lambda-1})$.

There is some specific language that is useful to the description of handle attachment.

Definition 2.4.3. Let $M \cup_\varphi H^\lambda$ be an n–manifold with a λ–handle attached. The embedding $f_0 : S^{\lambda-1} \to \partial M$ is called the *attaching map*, its image $f_0(S^{\lambda-1})$ is the *attaching sphere*, and its tubular neighbourhood is the *attaching neighbourhood.* The embedding $f : S^{\lambda-1} \times D^{\mu} \to \overline{\nu}_{\partial M}\, f_0(S^{\lambda-1})$ is called a *normal framing* or just *framing.* Inside of the λ–handle $H^{\lambda} = D^{\lambda} \times D^{\mu}$, the disc $D^{\lambda} \times \{\vec{0}\}$ is the *core*, and the disc $\{\vec{0}\} \times D^{\mu}$ is the *cocore.* The boundary circle $\{\vec{0}\} \times S^{\mu-1}$ of the cocore is the *belt sphere.* The integer λ is the *index* of the handle.

There are two special cases of handle attachment to discuss. First, let M be orientable and path–connected with ∂M compact, connected, and nonempty. The attaching sphere of a 1–handle is $S^0 = \partial D^1$, which is a pair of points. There is a unique isotopy class of embeddings $f_0 : S^0 \to \partial M$. This means that $M \cup_{\varphi} H^1$ is determined entirely by the framing f. The normal disc bundle of $f_0(S^0)$ is a bundle over S^0, so it is vacuously trivial. Using the vector bundle structure of the tubular neighbourhood, we write an embedding of $S^0 \times D^{n-1} \to S^0 \times D^{n-1}$ as a pair of length–preserving linear transformations $\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, i.e. elements of $\mathcal{O}_{n-1}(\mathbb{R})$, each restricted to act on one of the connected components of $S^0 \times D^{n-1}$. The determinant of an element of $\mathcal{O}_{n-1}(\mathbb{R})$ is either 1 or -1, and $\mathcal{O}_{n-1}(\mathbb{R})$ has two path–connected components corresponding to these two cases. Every element of $\mathcal{O}_{n-1}(\mathbb{R})$ is an embedding $\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, so any path in $\mathcal{O}_{n-1}(\mathbb{R})$ is an isotopy of its endpoints. It is then easy to see that are four isotopy classes of our trivialization, and those fall into two types. Either both transformations are are orientation preserving (reversing), or one is orientation preserving (reversing) and the other is orientation reversing (preserving). Under the first type of automorphism, $M \cup_{\varphi} H^1$ is a non-orientable manifold. Under the second, $M \cup_{\varphi} H^1$ is orientable.

Next, let W be a 4–manifold with orientable boundary M. To attach a 2–handle to W, we need an embedding $f_0 : S^1 \to M$ and a framing $S^1 \times D^2 \to \overline{\nu}_{\partial M} f_0(S^1)$. The space $S^1 \times D^2$ is what we call a solid torus.

Definition 2.4.4. A space V that is homeomorphic to $S^1 \times D^2$ is called a *solid torus.* A simple closed curve J in ∂V that bounds a 2–disc in V is called a *meridian.* A simple closed curve K in ∂V that intersects a meridian at a single point is called a *longitude.* Note that there are infinitely many isotopy classes of longitudes of V, whereas there's exactly one isotopy class of meridians.

Once an embedding $f_0 : S^1 \to M$ is chosen, $W \cup_{\varphi} H^2$ is determined by the framing $f : S^1 \times D^2 \to \overline{\nu}_{\partial M}\, f(S^1)$. To investigate possible framings, we consider the *mapping class group* of the solid torus. The mapping class group of a space X is the group of isotopy classes of automorphisms X, and it is denoted $mpg(X)$. For V a solid torus, $mpg(V)$ is very closely linked to the mapping class group of the torus $T^2 = S^1 \times S^1 = \partial V$, which is the special linear group $\mathrm{SL}_2(\mathbb{Z})$, the group of 2×2 matrices with integer entries and determinant ± 1.

Lemma 2.4.5. Let V be a solid torus and let f be an automorphism of the torus ∂V. Then h extends to an automorphism of V if and only if f maps a meridian to a meridian.

Lemma 2.4.6. A pair of automorphisms f, g of V that agree on ∂V and map a meridian to a meridian are isotopic.

Theorem 2.4.7. The mapping class group of a solid torus V is the subgroup of the mapping class group of ∂V containing automorphisms that map meridians to meridians. This subgroup is isomorphic to $\mathbb{Z}$.

Proof. In [11], it is shown that the mapping class group of T^2 is $\mathrm{SL}_2(\mathbb{Z})$ by generating the group from a pair of basic automorphisms of T^2 called *twists*, and the swapping map $(z, w) \to (w, z)$. The twists are defined as:

$$\begin{aligned} h_L(e^{i\theta}, e^{i\phi}) &= (e^{i(\theta+\phi)}, e^{i\phi}) \qquad \text{“longitudinal twist”} \\ h_M(e^{i\theta}, e^{i\phi}) &= (e^{i\theta}, e^{i(\theta+\phi)}) \qquad \text{“meridinal twist”} \end{aligned}$$

Fix the meridinal and longitudinal directions of the boundary of a solid torus V to coincide with the meridinal and longitudinal directions of V. This would mean that if we are writing $V = S^1 \times D^2$, $\{1\} \times S^1$ is a meridian and $S^1 \times \{1\}$ is a longitude. Notice that the meridinal twist maps a meridian to a meridian, so it extends to an automorphism of V. Also, neither the longitudinal twist nor the swap map a meridian to a meridian, so neither extend to an automorphism of V.

Let F be an automorphism of V with restriction f to ∂V. Because f is an automorphism of ∂V, it can be written as the product of twists and swaps. Because f preserves meridians, it can be written entirely as a power of the meridinal twist. In the other direction, any automorphism that is the power of a meridinal twist clearly preserves meridians. Because these automorphisms are written as powers of h_M, there is a clear isomorphism from this subgroup to $\mathbb{Z}$. $\square$

The content of these results means that the attachment of a 2–handle to a 4–manifold can be described by an embedding $f_0 : S^1 \to M$ and an integer k called the *framing constant* that determines the framing automorphism as long as we know what a framing constant of 0 means. To see this, first notice that h_M can be extended to

$$\begin{array}{rccc} H_M : & S^1 \times D^2 & \to & S^1 \times D^2 \\ & (e^{i\theta}, re^{i\phi}) & \mapsto & (e^{i\theta}, re^{i(\phi+\theta)}) \end{array}$$

which represents a generator of $mpg(S^1 \times D^2)$. Because $\nu_{\partial M}\, f(S^1)$ is a solid torus, a one-to-one correspondence ψ between $mpg(S^1 \times \mathbb{R}^2)$ and the isotopy classes of maps $S^1 \times D^2 \to \nu_{\partial M}\, f(S^1)$ is determined entirely by $\psi(\mathrm{id})$. The image of $\psi(\mathrm{id})$ is called the class of *preferred framings*. Sometimes, there is a canonical class of preferred framings. One important case is when M is S^3.

Let $f_0 : S^1 \to S^3$ be an embedding. Let V be a solid torus that is a closed tubular neighbourhood of the knot $f_0(S^1)$ in S^3. Then there is exactly one isotopy class $[J]$ of longitudes of V, unique up to ambient isotopy of f_0, such that any curve in $[J]$ bounds a disc in $S^3 \setminus V$. Moreover, for any representative J of $[J]$ there is a framing $f_J : S^1 \times D^2 \to V$ such that $f_J(S^1 \times \{\vec{1}\}) = J$, and the isotopy class of $[f_J]$ is in one–to–one correspondence with $[J]$. Our canonical choice for $\psi(\mathrm{id})$ is $[f_J]$, and this generates the rest of the map, i.e. $\psi([H_M^k]) = [f_J \circ H_M^k]$. The isotopy class of a framing $f : S^1 \times D^2 \to \overline{\nu}_{S^3} f_0(S^1)$ can then be classified entirely by an integer k that describes the number of meridinal twists it takes to get to f from the preferred one. We call k the *framing constant* for f.

2.5 Triangulations

A simplicial manifold is a special description of a piecewise–linear structure on a manifold. We describe it as being built from a patchwork of pieces carved from a finite dimensional Euclidean space which we call simplices.

Let $E = \{e_0, \ldots, e_n\}$ be a set of $n+1$ points in some n–dimensional affine space F such that

$$e_i \neq \sum_{j \neq i} t_j e_j$$

for each i and any collection of nonnegative real t_j with $\sum t_j = 1$. The *standard*

n–*simplex* σ is defined as

$$\sigma = \{p \in F : p = \sum_{i=1}^{n} t_i e_i,\ e_i \in E,\ t_i \geq 0 \text{ for each } i, \text{ and } \sum_{i=1}^{n} t_i = 1\}.$$

The above set is also called the *convex hull* of E.

Let E' be a $k+1$ element subset of E. The convex hull on E' is called a *facet* of σ and is itself a k–simplex. A zero–dimensional facet is a *vertex*, one–dimensional an *edge*, two–dimensional a *triangle*, three–dimensional a *tetrahedron* and four–dimensional a *pentachoron*. These names are also applied to the 0–, 1–, 2–, 3–, and 4–simplices. A facet with dimension $n-1$ is called a *face* of the simplex that contains it. We number the vertices of the n–simplex with the numbers $0, \ldots, n$. Every face of σ contains all but one vertex of σ, and this gives a numbering to the faces of σ. The i^{th} face of σ is found via the *face map* $F^i(\sigma)$. Removing from a simplex all of its proper faces gives us the *interior* of σ.

{FIGURE HERE IS GOOD}

A *simplicial complex* is a locally finite collection Σ of simplices embedded in some affine space satisfying two conditions. First, any facet of a simplex in Σ is also in Σ. Second, the intersection of two simplices in Σ is either empty or a facet of both.

An n–*dimensional gluing complex* is a space obtained from a finite set of n–simplices and a collection of maps between faces $\{F^i(\sigma) \to F^j(\tau)\}$ called *gluings*. We demand that any face appears in exactly one of the gluings and that each gluing is an affine linear map that dictates how the faces $F^i(\sigma)$ and $F^j(\tau)$ are identified. A gluing $F^i(\sigma) \to F^j(\tau)$ is defined completely by a bijection of the vertices of $F^i(\sigma)$ to the vertices of $F^j(\tau)$ and extended linearly in order of ascending dimension over all facets of the faces. The quotient space of the union of our simplices relative to the equivalence relation defined by the gluings is the *gluing conplex*. An n–dimensional gluing complex can be seen as a simplicial complex by embedding it in some high–dimensional Euclidean space. Denote an n–gluing complex by T and define the k–*skeleton* of T to be the union of all facets of T of dimension at most k with $0 \leq k \leq n$. Denote the k–skeleton of T by T^k, and note that $T^n = T$.

Recall from Definition 2.1.3 that an n–dimensional piecewise–linear manifold M is a manifold whose atlas of M has piecewise linear transition maps. If a gluing satisfies this property, we call it a *simplicial manifold.* To build an atlas for an n–gluing T, a point p in $T \setminus T^{n-1}$ has an obvious chart $(\texttt{int}(\sigma), f)$ where σ is the n–simplex whose interior contains p and f is the trivial linear map from $\texttt{int}(\sigma)$ to the interior of the standard n–simplex, an open subset of $\mathbb{R}^n$. Because a chart for every point interior to an n–simplex exists and because no face of this complex is unglued, we may iteratively define piecewise–linear homeomorphisms for points in $T \setminus T^k$ as k descends to zero. This leaves the task of defining charts for the vertices of T. An important notion in constructing these charts is the *link* of a simplex.

Definition 2.5.1. If σ is a k–simplex in an n–gluing T, consider a simplex τ containing σ. Then there is a facet τ' that is "opposite" σ in the sense that $\sigma \cap \tau' = \emptyset$ and τ is the convex hull of $\sigma \cup \tau'$. The gluing made up of the $F^i(\tau)$ corresponding to all τ containing σ is the *link* of σ and is denoted $\texttt{lk}(\sigma)$.

If the link of a vertex v in T has $\texttt{lk}(v) = S^{n-1}$, then the cone on that link is a neighbourhood of v in the gluing and is piecewise-linearly homeomorphic to the n–disc. Because the rest of the atlas is immediate from our definition of gluing, the condition that vertex links are spheres is necessary and sufficient to say that a gluing is a simplicial manifold.

Until now, our definitions only allow for closed simplicial manifolds. If we allow our n–dimensional gluing to have unpaired faces, then we impose the additional restriction that the links of unglued vertices are $(n-1)$–dimensional discs with boundary $(n-2)$–spheres triangulated by facets entirely from unglued faces. If this additional condition is met, then the unglued faces form an $(n-1)$–dimensional simplicial manifold in their own right.

We also consider a similar and more general construction of a piecewise–linear structure on a manifold called a cell decomposition. Their main use in this work is as a combinatorial structure dual to the triangulation.

Definition 2.5.2. A space homeomorphic to B^n is called an *n–cell.*

Chapter 3

Background

3.1 3–Manifolds Bound 4–Manifolds

In their paper "3–Manifolds Efficiently Bound 4–Manifolds," Francesco Costantino and Dylan Thurston demonstrated that M, a closed, orientable 3–manifold triangulated with n simplices, bounds a 4–manifold W triangulated using a number of simplices bounded by a quadratic polynomial in n. First, they argue that 3–manifolds bound 4–manifolds. Their argument builds a 4–manifold W from a closed, oriented 3–manifold M by filling in one boundary component of $M \times I$ with 4–dimensional 2–, 3–, and 4– handles. Attachment instructions for these handles are obtained by a Morse 2–function from M to $\mathbb{R}^2$. A Morse 2–function is defined similarly to an ordinary Morse function.

Definition 3.1.1 (Morse 2–function [2])**.** Let M be an n–manifold, Σ a 2–manifold, and $f : M \to \Sigma$ a generic smooth map. We call such a map a *Morse 2–function*.

We now proceed to a central theorem that outlines the course of action to be taken in later chapters.

Theorem 3.1.2. Let M be a closed orientable 3–manifold. There exists a 4–manifold whose boundary is M.

Proof. To begin, let $f : M \to \mathbb{R}^2$ be a Morse 2–function. From the regular value theorem, the preimage of a regular value of f is the disjoint union of oriented circles in M. The image of the singular set of f is a collection of arcs in $\mathbb{R}^2$ that only intersect pairwise and transversally. This leaves us with a classification of critical values of f into codimension 1 singularities, i.e. arcs away from crossings, and codimension 2 singularities, i.e. arc crossings. We obtain a 4–manifold with boundary M by gluing discs to regular preimage circles and extending over each singularity type.

To get an idea of how to extend over these singularities, we examine what happens one dimension down. A closed 2–manifold Σ bounds a 3–manifold if and only if the Euler characteristic $\chi(\Sigma)$ is even. In particular, every orientable surface bounds a 3–manifold. Let Σ be an oriented 2–manifold and f a Morse function $\Sigma \to \mathbb{R}$. Preimages of regular values are again circles, which we fill with discs. We use the language of handles to describe this filling. Let $\Sigma \times I$ be a 3–manifold with oriented boundary components $\Sigma \times \{0\} = \Sigma_-$ and $\Sigma \times \{1\} = \Sigma_+$, where Σ_- has orientation opposite that of Σ and Σ_+. Specify a regular value of f in each connected component of $f(\Sigma_-)$ minus the critical values of f. Each of these regular values has preimage a disjoint collection of circles in Σ_- which we take to be the attaching spheres of 3–dimensional 2–handles. The resulting object is a 3–manifold with one boundary component of Σ_+ and the rest corresponding to singular values of f. To obtain a 3–manifold whose boundary is Σ_+, we need only fill the boundary components corresponding to the singular values of f.

{FIGURE: MORSE FUNCTION AND 2–HANDLE ATTACHMENT}

Because f is a Morse function whose domain is a surface, its critical points are classified. Let p be a critical point of f, and x its critical level. Locally in Σ_-, a critical point of index 0 is a minimum, of index 1 a saddle, and of index 2 a maximum of f. Let $\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$ be an interval containing x and no other critical values of f. When p is of index 0 or 2, we can immediately deduce that the component of $f^{-1}(\varepsilon(x))$ containing p is a disc in Σ_-. To see this, recognize that $f^{-1}(x)$ will consist of a disjoint collection of circles corresponding to the regular points of Σ_- that also map to x through f along with a single point – p.

{FIGURE: NEIGHBOURHOODS OF INDEX 0,1,2 CRIT. PTs}

The case of the saddle needs more care. Suppose that p is of index 1. Critical points of Morse functions are non–degenerate, so $f^{-1}(x)$ forms a cross near p. For $y \in \varepsilon(x)$ not equal to x, y is a regular value whose preimage is a disjoint union of circles. As Σ_- is oriented, pulling back the orientation of $\mathbb{R}$ to Σ_- allows us to coherently orient the preimage circles of the regular values. The circles above and below the saddle singularity are the result of smoothing out the cross into a pair of oriented arcs, done in two possible ways. The orientations of these circles orient the cross, which has two incoming arms and two outgoing arms which appear in alternating order. A Morse function has distinct critical levels, so the cross we know about in the connected component of $f^{-1}(x_0)$ containing p_0 must have its arms connected in Σ_- through nonsingular orientation–preserving arcs. The connected component of $f^{-1}(x_0)$ containing p_0 must then be a figure 8. Thus, the component of $f^{-1}(\varepsilon)$ containing p is a pair of pants in Σ_-.

{FIGURE: SMOOTHING OUT A CROSS AND SADDLE}

We return now to the 3–manifold constructed from $\Sigma \times I$ by gluing 2–handles to the Σ_- boundary component. The boundary components consist of Σ_+ as well as a collection of components corresponding to the critical points of f left over in the Σ_- side of the boundary. If p has index 0 or 2, then the component of $f^{-1}(\varepsilon(x))$ contianing p is a disc. For some ϵ, the boundary of this disc meets the belt sphere of the nearest 2–handle, and the boundary component is seen to be a 2–sphere, which becomes the attaching sphere of a 3–handle. Similarly, if p has index 1 then the pair of pants near the saddle singularity of p is grown until it meets three belt spheres from three separate 2–handle attachments. The boundary component is again seen to be a 2–sphere, so we can attach another 3–handle. It is reasonable to note that, had Σ not been oriented, then the boundary component corresponding to a saddle singularity could have been a copy of $\mathbb{RP}^2$. As $\mathbb{RP}^2$ has Euler characteristic 1, it does not bound any 3–manifold.

{FIGURE: THREE TYPES OF 3–BALL BLOCKS}

Return now to the case of a closed, oriented 3–manifold M and the generic smooth map $f : M \to \mathbb{R}^2$. It is assumed that M is compact, so we will assume that the image $f(M)$ is also compact. To build a 4–manifold W whose boundary is exactly M, we take the 4–manifold with boundary $M \times I$. We again let $M \times \{0\} = M_-$ be the boundary component with orientation opposite that of M, and $M \times \{1\} = M_+$ the boundary component with orientation exactly that of M. For reasons that will become clear later, we fill in the boundary component M_+, setting the domain of f to be M_+. This will result in a 4–manifold whose boundary is M_-. The image of the singular set of f is, as stated before, a set of arcs in the plane. These arcs separate the plane into connected regions of regular values, and we can choose f so that these regions are all homeomorphic to discs. The discs of regular values all pull back through f to open solid tori in M. We fill in M_+ by attaching 2–, 3–, and 4–handles where the attachment instructions are obtained through f.

First, we take care of the regular values of f. Let R_0 be a disc of regular values, and shrink the region away from the critical values of f slightly. This "shrunken" region is topologically closed, and homeomorphic to the closed 2–dimensional disc. Name this "shrunken region" by R, and take $f^{-1}(R)$ to be a collection of attaching neighbourhoods for 4–dimensional 2–handles. Take a pair of arbitrary points p, q interior to R, and let T be an open solid torus that projects over R through f. Let the circle $f^{-1}(p) \cap T$ be an attaching sphere for a 2–handle. The orientation of $\mathbb{R}^2$ and the curve $f^{-1}(q) \cap T$ determines a framing of $f^{-1}(p)$, and this pair completely determines the 2–handle attached over $f^{-1}(R)$.

Next, we extend over the codimension 1 singularities. An arc a_0 of codimension 1 critical values are the image of an arc s_0 of critical points of f in M_+, and the connected component of $f^{-1}(a_0)$ containing s_0 is an interval crossed with the singularities that occur in the 2–dimensional case. The endpoints of the arc a_0 are codimension 2 critical values of f, and the endpoints of the arc s_0 are critical points that map to the endpoints of a_0. Slightly shrink a_0 away from its boundary to the arc a of critical values of f. Take A to be a tubular neighbourhood of a in $\mathbb{R}^2$. Then A is a copy of $I \times I$ so that lines in the x direction of A are parallel to a and lines in the y direction are normal to a. The boundary components $I \times \{0\}$ and $I \times \{1\}$ run parallel to a and sit in the boundary of the shrunken regions to either side of a_0, and the normal boundary components $\{0\} \times I$ and $\{1\} \times I$ normally intersect a_0 at ∂a. For a given normal arc $\{x\} \times I$, our analysis of the singularities of the Morse function one dimension down allow us to say that $f^{-1}(\{x\} \times I)$ is a disjoint collection of cylinders with either a pair of pants or a disc. Each point of $(I \times \{0\}) \cup (I \times \{1\})$ pulls back to a circle that is a boundary component of one of these three shapes. Pulling $I \times \{0\}$ and $I \times \{1\}$ back through f then yields cylinders. The 2–handles attached over the preimages of the shrunken regions to either side of a_0 have filled these cylinders with discs. Boundary circles of the $f^{-1}(\{x\} \times I)$ are filled by discs in $(M \times I) \cup$ (2–handles), so f pulls the normal arcs of A back to spheres in $(M \times I) \cup$ (2–handles). This means that each connected component of $f^{-1}(A)$ is a copy of $S^2 \times D^1$, to which we attach 4–dimensional 3–handles.

Finally, we extend over codimension 2 singularities. The codimension 1 singularities can be classified into definite or indefinite folds. This classification depends on the type of 1–dimensional Morse singularity found by pulling an arc in the plane, normal to the given singular arc, back through f and examining f restricted to the connected component of this preimage containing the critical point that maps through f to the given singular arc. If the component we've found is a pair of pants, then the singular arc is an indefinite fold. If it is a disc, then the arc is a definite fold. Every codimension 2 critical value is either the crossing of a pair of definite folds, a pair of indefinite folds, or of one definite and one indefinite fold. A codimension 2 critical is mapped to by exactly two critical points in M. Put x to be our codimension 2 critical point, and p_1, p_2 to be the critical points of f projecting over x. If x is the crossing of a pair of definite folds then we can analyze p_1, p_2 using Morse theory, so p_1, p_2 are each of index either 0 or 2. We get that the connected components of $f^{-1}(x)$ containing p_1, p_2 are just the points p_1, p_2. If x is the crossing of one definite and one indefinite fold, then, without loss of generality, p_1 is of index 1 and p_2 is index 0 or 2. Then the connected component containing p_1 is the figure–eight graph from above and the component containing p_2 is just p_2. When x is the crossing of two indefinite folds, each of p_1, p_2 is of index 1. Our analysis only guarantees that p_1, p_2 are the centres of crosses with arms alternating in and out of p_i in the connected components of $f^{-1}(x)$ containing p_i. If $p_{1,2}$ are in different connected components, then each of those components is a figure–eight graph. If $p_{1,2}$ are in the same connected component, there are exactly two different directed graphs that contain exactly two vertices, each of degree four, that each have two edges entering and two leaving.

Each crossing is in the boundary of exactly four regions of regular values and in the boundary of exactly four singular arcs. Label the regions $R_1, \ldots, R_4$ and the singular arcs by $a_{1,2}, a_{2,3}, a_{3,4}, a_{4,1}$. The regions are labeled in an anticlockwise order about x, and the arcs are named for the pair of regions they border. Note that every point in $\mathbb{R}^2$ outside of the range of f is vacuously a regular value of f. Let x, $p_{1,2}$

be as they were above. Let D_x be a small disc in the plane containing x. There is an obvious triangulation of D_x with exactly four triangles so that one vertex of each triangle corresponds to x, the shared boundary of a pair of triangles corresponds to an arc $a_{i,i+1}$, and the interior of each triangle corresponds to a region R_i of regular values whose boundary contains x. Let R_i be any of the regions containing x so that $f^{-1}(R_i)$ is nonempty. At this point in our construction, we may form an arc γ_i inside of R_i from $a_{i-1,i}$ to $a_{i,i+1}$ that consists entirely of points in the boundary of the shrunken region pulled back through f to attach 2–handles and points in the boundary of the tubular neighbourhood of the shrunken arc pulled back through f to attach 3–handles. If every R_i is in the image of f, then the union of the γ_i is the boundary of D_x. If some region R_i is outside of the image of f, then the arcs $a_{i-1,i}$ and $a_{i,i+1}$ are each definite folds, and the regions to either side of R_i are in the image of f. This is a special case where x is the crossing of a pair of definite folds, and x sits on the boundary of $f(M) \subset \mathbb{R}^2$, and in this case we do not define a γ_i.

Each D_x pulls back through f to a collection of disjoint 3–handlebodies that, together with the boundaries of handles already attached in $(M \times I)\cup$(2– and 3–handles), form copies of S^3 to which we attach 4–dimensional 4–handles. When all 4–handles are attached, the component M_+ is completely filled, and we have a 4–manifold whose boundary is exactly M. We prove this first for the connected components that consist entirely of regular points, then use similar arguments to prove this for all other cases.

Let U be a component of $f^{-1}(D_x)$ that consists entirely of regular points. Then the connected component of $f^{-1}(x)$ that sits inside of U is a circle, and T is a closed regular neighbourhood of this circle, i.e. a solid torus in M. A meridian of this solid torus maps through f to the boundary of D_x. Through handle attachment we have filled every circle in M_+ that maps to a point in ∂D_x with a disc. Then U shares a boundary with a solid torus V in $(M \times I)\cup$(2– and 3–handles), and the meridian of U is a longitude of V, so $U \cup V \subset (M \times I)\cup$(2– and 3–handles) is a 3–sphere. We use this 3–sphere as an attaching neighbourhood of a 4–handle.

Next, we consider the case where U contains only one of p_1 or p_2, abbreviated to p, which we recall as being a critical point of f so that $f(p) = x$. If p is a definite fold, then a pair of opposite singular arcs, say $a_{4,1}$ and $a_{2,3}$, are also definite folds, and the image of f near the strand of critical points $f^{-1}(a_{4,1} \cup x \cup a_{2,3})$ lies entirely on one side of $a_{4,1} \cup x \cup a_{2,3}$ in the plane, which we will take to be $R_1 \cup a_{1,2} \cup R_2$ without loss of generality. The strand $a_{1,2}$ pulls back through f to only regular points in U, so these are circles. It is easy to see that U is a 3–ball, and $f^{-1}(\gamma_1 \cup \gamma_2)$ is a 2–sphere filled with discs to make another 3–ball. Gluing these 3–balls together results in another 3–sphere over which we may attach a 4–handle.

If p is an indefinite fold, then the component of $f^{-1}(x)$ containing p is an oriented figure–eight graph embedded in U, U is a regular neighbourhood of the figure–eight graph, and f maps ∂U to the boundary of D_x. Again, the boundary of U maps through f to

In each remaining case U shares a boundary with another handlebody of the same genus and the two form a Heegard Diagram of S^3. The proof is covered in detail in Section 4.4 of [1]. At this point, $M \cup$(2–, 3–, and 4–handles) has only one boundary component: M_-. We obtain a 4–manifold whose boundary is M, and whose handle decomposition contains handles of index at most 2, by building the handle decomposition dual to that we've just described. The process can be simplified somewhat. For example, the only 4–handles that can't be dealt with by extending 2– and 3– handles are those that were added when p_1 and p_2 were in the same connected component of $f^{-1}(x)$. The object of study that simplifies our work considerably is called the *Stein complex* of the Morse 2–function $f : M \to \mathbb{R}^2$.

Let f be a Morse function or Morse 2–function with compact fibers as before. Define the *Stein factorization* of f to be a factorization $f = g \circ h$ so that the following are satisfied:

1. The map h has connected fibers, and
2. the map g is finite–to–one.

We can also see the image of h as the quotient space of the domain of f by the connected fibers of f. The image of h will also be called the *Stein complex* of the map f.

In the case of the oriented surface Σ, the Stein complex of f is a graph whose vertices correspond to the singularities of f. Vertices corresponding to critical points of index 0 or 2 being of degree 1, and vertices corresponding to critical points of index

1 being of degree 3. We form a 3–manifold whose boundary is Σ by starting with a disc bundle over the edges of $h(\Sigma)$, then extending over the vertices by attaching 3–balls as in the analysis above. There is an obvious deformation retraction of the resulting 3–manifold to the Stein complex of f.

In the case of the 3–manifold M, the Stein complex is generically a surface and the possible singularities are more interesting. As in the analysis above, some singularities are just the singularities from the lower dimensional Morse function crossed with an interval, so the corresponding shape in the Stein complex is simply a degree–one or degree–three vertex crossed with an interval. The more interesting parts of the Stein complex are brought forth at the crossing of indefinite folds in the plane. As in the case of Σ, M is a circle bundle over the Stein complex at generic points and the 4–manifold W with $\partial W = M$ is a disc bundle. We extend over singularities by attaching handles and, as before, W retracts onto the Stein complex. The Stein complex, along with some extra information we can obtain from f, provides a set of building instructions for W explored in the next section. $\square$

Corollary 3.1.3. Let M be a closed, orientable 3–manifold. Then there exists a Morse 2–function $f : M \to \mathbb{R}^2$ with Stein factorization $f = g \circ h$ and a Stein complex $S = h(M)$ whose regions can be assigned framing constants such that S provides a full set of instructions for the construction of a 4–manifold W with $\partial W = M$.

3.2 Shadows of 3– and 4–Manifolds

The building instructions mentioned in the previous section were studied by Turaev in [13] and are called *shadows*. Shadows are piecewise–linear 2–dimensional structures that live inside of piecewise–linear, compact, oriented 4–manifolds. The central algorithm of this work uses shadows of 3– and 4– manifolds to construct a triangulation of a 4–manifold with a given 3–manifold boundary.

Definition 3.2.1. Let P to be a compact topological space. If every point p of P has a neighbourhood homeomorphic to an open set in one of the following local models

1. the closed 2–disc D^2,

2. the product of the interval $I = [0, 1]$ with the Y–shaped graph $K_{1,3}$, or

3. the cone on the complete graph K_4,

then P is a *simple polyhedron.* { FIGURE HERE DEPICTING THE LOCAL MODELS} The set of points which do not have a neighbourhood homeomorphic to an open set in D^2 form a 4–valent graph which we call the *singular set* of P and denote by $\mathrm{Sing}(P)$. The vertices and edges of $\mathrm{Sing}(P)$ are called the vertices and edges of P. We call the connected components of $P \setminus \mathrm{Sing}(P)$ the *regions* of P.

The local models each have boundary. Being a surface, the 2–disc has a well defined boundary. In the local model $K_{1,3} \times I$, the boundary consists of the set $(K_{1,3} \times \{0\}) \cup (K_{1,3} \times \{1\}) \cup (V(K_{1,3}) \times I)$, where $V(K_{1,3})$ denotes the vertex set of the graph $K_{1,3}$. The boundary points of the cone on K_4, defined at $K_4 \times I / \sim$, where $\sim$ is defined to be $(x, 1) \sim (y, 1)$ for any x, y, are the points in $K_4 \times \{0\}$.

If a point p of P has a neighbourhood homeomorphic to an open set containing a point in the boundary of one of our three local models, then p is a *boundary point* of P. The set of all boundary points of P is the boundary of P and is denoted by ∂P. A region of P is *internal* if its closure is disjoint from ∂P. If ∂P is empty then P is *closed.*

Simple polyhedra are almost shadows. To define a shadow, we need to consider simple polyhedra embedded in a 4–manifold. First, we introduce a concept from simple–homotopy theory.

Definition 3.2.2. Let K be a simplicial complex, and let τ, σ be simplices so that $\dim \sigma$ is maximal, $\dim \tau = \dim \sigma - 1$, and no simplex of dimension $\dim \sigma$ other than σ contains τ. We obtain the complex L from K by removing σ and τ. That is, $L = K \setminus (\mathtt{int}(\sigma) \cup \tau)$. We say that L is an *elementary collapse* of K.

If a complex L is obtained from K through iterated elementary collapses, then we say that L is a *simplicial collapse* of K. We may also say that K *collapses* onto L.

We are finally ready to define a shadow.

Definition 3.2.3. Let W be a piecewise–linear, compact, oriented 4–manifold. Let $P \subset W$ be a closed simple sub–polyhedron of W such that W collapses onto P and the regions of P are *locally flat* in W. That is to say, if p is in $P \setminus \mathrm{Sing}(P)$ then there is a chart (U, f) of W around p so that $f(P \cap U)$ is contained in $\mathbb{R}^2 \subset \mathbb{R}^4$. We define P to be a *shadow polyhedron* of W.

Remark 3.2.4. There is a notion of shadow equivalence in [13] via basic shadow moves. A shadow polyhedron whose regions are all homeomorphic to discs is called *standard*, and through Turaev's shadow moves any shadow polyhedra can be made standard. Furthermore, the algorithms which make up the bulk of this document produce a standard shadow polyhedron, so from here forward we always consider our polyhedra to be standard.

Not every piecewise–linear, compact, oriented 4–manifold contains a shadow polyhedron. A necessary and sufficient condition for the existence of a shadow polyhedron in such a 4–manifold is the existence of a handle decomposition of W containing no handles of index greater than 2, as shown in [13]. This requirement tells us that W has a connected non-empty 3–manifold boundary M. A shadow is defined for M as well.

Definition 3.2.5. A shadow polyhedron of an oriented, closed 3–manifold M is a shadow polyhedron P of a compact 4–manifold W with $\partial W = M$

This gives us the following theorem for free.

Theorem 3.2.6. Every closed, oriented 3–manifold has a shadow polyhedron.

Proof. To sketch the proof, we use the results of Lickorish and Kirby as summarized in [3]. Any closed, oriented smooth 3–manifold M has a presentation as integral surgery over a link L in S^3. A handle decomposition of a 4–manifold with boundary equal to M can be obtained from the integral surgery diagram of M. We begin with a 0–handle, which is just a copy of B^4 with boundary S^3. We put our surgery presentation in this S^3. Each component of the surgery link in S^3 will be the attaching sphere of a 2–handle, and the integer surgery coeffifient will be the element of $\pi_1(\mathrm{GL}_2(\mathbb{R}))$ that fully describes the framing of this 2–handle. Adding a 2–handle over every component of the link results in a 4–manifold W such that $\partial W = M$.

The link diagram also determines the shadow of W hence of M. Take $\pi : L \to D^2$ to be a regular projection. That is, π is injective everywhere except for at a finite number of points which coincide with the crossings of L. Then the mapping cylinder $(I \times L) \coprod D^2/(0, x) \sim \pi(x)$ with a disc glued to the free link ends at $\{1\} \times L_i$ is, as per our definition, a shadow of W. $\square$

It is natural to wonder how closely related shadows are with their associated 3– and 4–manifolds. Just as simple–homotopy type is not a complete invariant of 4–manifolds, a shadow polyhedron does not uniquely determine a 4–manifold. The following example demonstrates this explicitly and hints at what kind of additional information is needed to uniquely identify a shadow with a 4–manifold.

Example 3.2.7. We examine disc bundles over S^2. Take W_0 to be the trivial disc bundle $D^2 \times S^2$ and W_1 to be the bundle whose 0–section has self–intersection number of 1 in the ambient 4–manifold. Each W_i collapses onto S^2, so each have shadow S^2. Because W_0 is $D^2 \times S^2$, it's boundary is $S^1 \times S^2$. One can see that W_1 has handle decomposition with exactly one 0–handle B^4 and one 2–handle attached over the unknot in $S^3 = \partial B^4$ with framing coefficient 1. This manifold is a punctured $\mathbb{CP}^2$, so has boundary S^3. We've defined a shadow polyhedron of an oriented, closed 3–manifold M to be a shadow polyhedron P of a compact 4–manifold W with $\partial W = M$, so S^3 and $S^2 \times S^1$ each have shadow polyhedron S^2.

We've found a pair of distinct 4–manifolds with distinct boundaries, but the shadow of each is S^2. We want to construct a 4–manifold whose boundary is a given 3–manifold, and this shows that we need more than just a naked shadow polyhedron to do so. The information needed is carried by the internal regions of a polyhedron and are named "gleams" by Turaev. One can intuit from the Example 3.2.7 that a "gleam" might describe the regular neighbourhood of an embedded shadow.

Definition 3.2.8. Let P be a polyhedron embedded in the 4–manifold W in a locally flat way. Then there exists a canonical colouring of the internal regions of P by elements of $\frac{1}{2}\mathbb{Z}$ called *gleams*. A gleam necessarily depends on the embedding of P. We may also discern a canonical colouring of the internal regions of P by elements of $\mathbb{Z}_2$ called the Z_2*–gleam* that depends only on the combinatorial structure of P. The Z_2–gleam of a region of P determines whether the gleam of that region is an integer or half–integer.

Let D be an internal region of P. Because P is assumed to be standard, we know that D is an open disc and the closure of D is a closed disc. The embedding of D in P extends to an embedding $e : \bar{D} \to P$ so that e takes $\partial\bar{D}$ into $\mathrm{Sing}(P)$. Denote by $U(D)$ the simple polyhedron that is a small open regular neighborhood of D in P. We may construct $U(D)$ from $\bar{D}$ by first gluing the core of either an annulus or Möbius strip, to $\partial\bar{D}$. Then, for each point p of $\partial\bar{D}$ so that $e(p)$ is a vertex of P, let A_p be an arc in the band attached to $\partial\bar{D}$ so that A_p intersects the core of the band only at p. Obtain $U(D)$ by gluing half the boundary of a disc to A_p for each p. The map e extends easily to the map $e' : U(D) \to P$. Define the $\mathbb{Z}_2$–gleam of D in P to be equal to 1 if the band attached to $\partial\bar{D}$ was a Möbius strip and 0 if the band was an annulus.

{FIGURE: $U(D)$ FOR Z_2–GLEAM EVEN AND ODD}

Now, suppose that $f : P \to W$ is our locally flat embedding of P and let $D, \bar{D}, e : D \to P, U(D)$ and e' be defined as before. Because e' embeds $U(D)$ in P, we consider $U(D)$ to be a subset of P. A regular neighbourhood of $f(U(D))$ in W collapsing onto $f(U(D))$ is an oriented 4–ball B^4. Let p_0 be a point in $\partial\bar{D}$ and (V, g) a chart of W with V containing p_0 such that the intersection $V_P = V \cap f(U(D))$ is contained in $f(U(D))$. The embedding f is locally flat, so $g(V_P)$ is contained in a 3–dimensional slice B^3 of $g(V)$ and $g(f(\bar{D}) \cap V)$ is contained in a 2–dimensional slice of B^3. If p_0 is not a vertex of P, then there are exactly two other regions D' and D'' of P that meet D at p_0. The direction in B^3 in which D' and D'' separate away from p_0 can be extended to a direction in $g(V)$ where it is an element of the projective line P^1 of lines orthogonal to $f(D)$ sufficiently close to $f(p_0)$. If p_0 is a vertex of P, then ignoring the region of P that meets D only at p_0 leaves two suitable separating regions that meet D at p_0. We form a smooth bundle of directions over $\partial\bar{D}$ which is a section of a P^1 bundle over $\partial\bar{D}$. The obstruction to extend this section to all of D is a class of $H^2(D, \partial D; \pi_1(P^1))$. The ambient space $g(V)$ is oriented so D is oriented. The class of $H^2(D, \partial D; \pi_1(P^1)) = \mathbb{Z}$ is an integer z that corresponds to the number of times a section of the boundary of D loops around a P^1 bundle, which is the number of half loops made around a S^1 bundle. We are interested in the S^1 bundle, so we take the gleam of D to be $z/2$. Note that z modulo 2 is exactly the $\mathbb{Z}_2$–gleam of D.

Theorem 3.2.9. [13] Let S be a polyhedron whose internal regions are equipped with gleams. Then there exists a canonical construction associating to S the a pair (W, S), where W is a piecewise–linear, compact, oriented 4–manifold W containing an embedded copy of S with shadow S can be reconstructed from the combinatorics of S.

We can extend our definition of shadows to shadows of pairs (M, G) where M is a 3–manifold whose boundary is not necessarily empty and G is an embedded framed graph whose vertices have degree either 1 or 3. This extension is useful because it allows us to build a shadow for a closed 3–manifold from a reasonable decomposition into blocks whose shadows are known. In this case, the polyhedron representing our shadow will have boundary and the 1–cells of the boundary will be classified.

Definition 3.2.10. Define a *boundary–decorated* standard polyhedron to be a standard polyhedron P with boundary so that ∂P is a graph whose edges are coloured one of i for internal, e for external, or f for false. The graph ∂P then has three distinct subgraphs $\partial_i P$, $\partial_e P$ and $\partial_f P$ intersecting only at vertices and whose union is ∂P. If $\partial_f P = \emptyset$ then we call P as *proper*.

Boundary decorated polyhedra can be turned into a shadows for a 3–manifolds with boundary and with framed graphs embedded in their interior. The boundary of the 3–manifold is represented by the subgraph $\partial_e P$ and the embedded graph is represented by the subgraph ∂_i.

Definition 3.2.11. Let P be a boundary decorated standard polyhedron properly embedded in a 4–manifold W so that W collapses onto P with a framing on $\partial_i P$. An embedding $f : X \to Y$ is *proper* if $f(\partial X) = f(X) \cap \partial Y$ and $f(X)$ is transverse to ∂Y everywhere in ∂X.

Let M be the complement of an open regular neighbourhood of $\partial_e P$ in ∂W and let G be a framed graph embedded in M whose core is $\partial_i P$. Then P is a *shadow* of the pair (M, G). If the false boundary is empty, then P is a *proper* shadow of (M, G). Gleams are defined on the interior regions of P as before.

Chapter 4

Projections and initial data

A gleamed shadow of a 3-manifold M is a shadow of a 4-manifold bounded by M. The ultimate goal of this thesis is to provide an algorithm that takes as input an edge-distinct triangulation T of the closed orientable 3–manifold M and produces as output a triangulation of a 4–manifold W such that ∂W is equivalent in the sense of triangulations to T. This algorithm is split into three chapters. Chapter 4 builds a piecewise–linear Morse 2–function $T \to \mathbb{R}^2$ and collects all data from that function necessary to build a gleamed shadow shared by T and a 4–manifold bounded by T. Chapter 5 builds a gleamed shadow from the information obtained in 4. Chapter 6 assumes we have a gleamed shadow $(S, \mathtt{gl})$ and constructs a triangulated 4–manifold whose shadow is $(S, \mathtt{gl})$.

The data we are trying to collect and retain with the algorithms of this chapter are as follows:

1. A list of polygons which we will call regions.

2. A colouring of every edge of every region using one of the three colours i, f, h.

3. A wedge number of 0, 1, or 2 for any edge coloured i.

4. A list of adjacencies between regions. An item in this list consists of a pair of regions and an i–coloured boundary edge of each region.

5. A list of piecewise–linear circles associated to each region. An item in this list is an ordered list of polygonal 2–cells which live in a closed polyhedral 3–complex so that any two consecutive 2–cells lie on the boundary of the same

3–cell, yet any 3–cell contributes either zero or two 2–cells to any given list. This list is indexed by the list of regions in item 1.

This data is sufficient to build a shadow, as seen in Chapter 5.

4.1 Preliminaries

Before beginning the algorithm, we should establish some properties of piecewise–linear maps from polyhedra to $\mathbb{R}^2$. These properties are called upon to support some algorithms in this chapter. We also review the basics of planar graphs, as these are the objects that encode the information we wish to carry to Chapter 5.

Definition 4.1.1. Let G be a connected graph such that an embedding of G in $\mathbb{R}^2$ exists. Fix $p : G \to \mathbb{R}^2$ to be such an embedding. Then the pair (G, p) is called a *planar graph*. The image $p(G)$ separates $\mathbb{R}^2$ into path connected components called the *regions* of (G, p). There is exactly one region of (G, p) which is unbounded in the plane, and whose boundary is a cycle of G. This region, its boundary cycle and all edges and vertices in that boundary cycle are said to be *outer*. All other other regions, vertices and edges are *inner*. Regardless of the choice of embedding p, every chordless cycle of G is the boundary of exactly one inner region of (G, p), and every inner region of (G, p) is bounded by exactly one chordless cycle of G, so the inner regions and chordless cycles of (G, p) are in 1–1 correspondence.

Definition 4.1.2. Let T be a tetrahedron with the four vertices u, v, w, x, six edges uv, uw, ux, vw, vx, wx, and faces $\hat{u}$, $\hat{v}$, $\hat{w}$, $\hat{x}$, where a face is named by the vertex of T it does not contain. Define a projection $\pi : T \to D^2$ by first choosing a map from the vertices of T to distinct points in S^1. Each point p of T is decribed by the convex combination

$$p = t_u u + t_v v + t_w w + t_x x$$

with the t_* nonnegative and summing to 1. We can define π at p by

$$\begin{aligned} \pi(p) &= \pi(t_u u + t_v v + t_w w + t_x x) \\ &= t_u \pi(u) + t_v \pi(v) + t_w \pi(w) + t_x \pi(x) \end{aligned} \tag{4.1}$$

Without loss of generality, we assume that the points $\pi(u), \pi(v), \pi(w), \pi(x)$ are ordered in a clockwise orientation about S^1. We call $\pi : T \to D^2$ a *linear tetrahedral projection*.

Definition 4.1.3. A point of D^2 in the image of π is one of five types:

1. The four points of type 1 are the images of the vertices of T under π.
2. The single point of type 2 is the intersection $\pi(uw) \cap \pi(vx)$.
3. All points in $\pi(uv) \cup \pi(vw) \cup \pi(wx) \cup \pi(xu)$, excluding the points of type 1, are of type 3.
4. All points in $\pi(uw) \cup \pi(vx)$ excluding the points of type 1 or 2 are of type 4.
5. The points of type 5 are the points outside of the image of any vertex or edge of T.

The image of the 1–skeleton of T forms a planar graph G inside of D^2 whose vertex set consists of the points of types 1 and 2 and whose edges consist of points of type 3 and 4. The planar graph embedding cuts the plane into five connected regions: four inner regions that contain all points of type 5, and and one outer region. This graph has four outer edges, consisting of the points of type 3, and four inner edges, consisting of the points of type 4.

By definition, the preimage of a point of type 1 is a vertex of T. The preimage of the single point of type 2 is a line segment between the edges uw and vx interior to T. A point of type 3 is the image of exactly one point in an edge of T. A point of type 4 is in the image of exactly one edge and one face, so the preimage of one of these points is a line segment interior to T between those two facets. Finally, points of type 5 are in the image of exactly two faces of T, and pull back to line segments interior to T between those two faces.

4.2 Build a projection

Our algorithm begins by defining a projection $\pi : T \to \mathbb{R}^2$ satisfying some properties that are very similar to the properties of a smooth Morse 2–function. We demand first that if we restrict π to any tetrahedron Δ^3 of T, then $\pi|_{\Delta^3}$ is an affine–linear tetrahedral projection. Next, let E, E' be edges of T and $\pi(E), \pi(E')$ their images. From the first condition it is guaranteed that if $\pi(E) \cap \pi(E')$ is nonempty then it consists of the single point z. We demand that if z is interior to D^2 then it is in the image of no other edge of T through π.

The first requirement can be met by choosing arbitrary images for the vertices of T, then extending to a linear tetrahedral projection on each tetrahedron as in Equation 4.1. An arbitrary placement of the m vertices of T does not, however, guarantee the second condition. Choosing as images for the vertices odd n^{th} complex roots of unity (with $n \geq m$) does guarantee this condition by a theorem in [9]. Line segments between vertices of a regular polygon in the plane are called *diagonals*. The theorem places a bound on the number of diagonals that can intersect at a point. If the polygon has an odd number of vertices, then this bound is two.

To see how we use this theorem, let T be our input triangulation, and let $|T^0| = m$. Put an arbitrary ordering on the elements of T^0. Let n be the least odd number greater than or equal to m. Define $\pi(v_k) = e^{2\pi ik/n}$ for every v_k in T^0. Extend π over all of T. Then $\pi : T \to \mathbb{R}^2$ satisfies our desired properties.

4.3 Obtain a planar graph

We construct here a planar graph (G, p) from the projection $\pi : T \to \mathbb{R}^2$ that will be used throughout this chapter. The projection $\pi : T \to \mathbb{R}^2$ defined in the previous section maps vertices of T to the n^{th} roots of unity, where n is odd. The construction of (G, p) here is essentially an extension of the construction seen in Definition 4.1.3 to the whole of T.

Every vertex v of T has a distinct image $\pi(v)$ in S^1, so our graph G begins with these points as its vertex set V. The vertex of G associated to v in T will be named $G(v)$. To fill the rest of the vertex set $V(G)$, we examine the images of each pair of non–adjacent edges of T for interior intersections. Here, edges in T are adjacent if they share a vertex. For a pair of non–adjacent edges E, F of T with boundary vertices $\partial E = v_E \cup w_E$ and $\partial F = v_F \cup w_F$, we may determine whether the line segments $\pi(E)$ and $\pi(F)$ in the plane intersect by checking the order in which the points $\pi(v_E)$, $\pi(w_E)$, $\pi(v_F)$, and $\pi(w_F)$ occur around S^1. Each of these points is at an n^{th} root of unity, so there is an obvious ordering that assigns each of these points to an integer. If $\pi(v_E)$ and $\pi(w_E)$ or $\pi(v_F)$ and $\pi(w_F)$ are adjacacent in this ordering, then $\pi(E)$ and $\pi(F)$ do not intersect. Otherwise, $\pi(E)$ and $\pi(F)$ intersect. Each intersection of this type adds a vertex to $V(G)$ which we will name $G(E,F)$ and these are all of the vertices we get.

To make G a planar graph, we need to fix an embedding $p : G \to \mathbb{R}^2$. For a vertex $G(v)$ directly associated with the vertex v of T, we define $p(G(v)) = \pi(v)$. For a vertex $G(E,F)$ associated to the intersection of $\pi(E)$ and $\pi(F)$ in the plane, we define $p(G(E,F)) = \pi(E) \cap \pi(F)$. This embedding is chosen so that an edge $e = uv$ of G, all of which will be added next, can be embedded in the plane as the line segment connecting $p(u)$ and $p(v)$.

Every edge of G comes from an edge of T, and every edge of T produces at least one edge of G. If an edge E of T has image $\pi(E)$ that intersects no other image $\pi(F)$ with E, F non–adjacent in T, then E adds exactly one edge to $E(G)$ whose vertices are $G(v_E)$ and $G(w_E)$. Otherwise, $\pi(E)$ intersects the line segments $\pi(E_j)$ for each of the edges in $\{E_j\}_{j=1}^m$ with E_j not adjacent to E in T. In this case, we have a vertex $G(E,E_j)$ in $V(G)$ for each j, and $p(G(E,E_j))$ is the point of intersection between $\pi(E)$ and $\pi(E_j)$. The edges we add to G from E form a path P in G with tails $G(v_E)$ and $G(w_E)$ that passes through every $G(E,E_j)$, and all we need to know is the order in which the vertices occur in P. When this is known, we populate P with edges.

To determine the order of the vertices of P, we assume that $\pi(E)$ is a vertical line segment, which is done without loss of generality. For this discussion, E, $\{E_j\}_{j=1}^m$ are as above. Then

$$\pi(v_E) = e^{2\pi ik/n} \text{ and } \pi(w_E) = e^{2\pi ik/n}.$$

and

$$\pi(v_{E_j}) = e^{2\pi ip/n} \text{ and } \pi(w_{E_j}) = e^{2\pi iq/n}.$$

Then the x–co-ordinate of the intersection $\pi(E) \cap \pi(E_j)$ is fixed at $\cos(2\pi k/n)$, and the y–co-ordinate y_j can be found to be

$$y_j = \frac{\sin(2\pi i \frac{q-p}{n}) + (\sin(2\pi i \frac{p}{n}) - \sin(2\pi \frac{q}{n}))\cos(2\pi \frac{k}{n})}{\cos(2\pi i \frac{p}{n}) - \cos(2\pi i \frac{q}{n})}.$$

We may compute and order the real numbers y_j. This orders the v_j in the path P from w_E to v_E. This ordering is given by the permutation $\sigma \in \Sigma_m$, so we populate P with the edges $(w_E, \sigma^{-1}(1))$, $(\sigma^{-1}(i), \sigma^{-1}(i+1))$ for $i \in \{1, \ldots, m-1\}$, and the edge $(\sigma^{-1}(m), v_E)$. We perform this process for each edge of T. As stated above, p is defined on an edge $e = uv$ of G as the line segment between $p(u)$ and $p(v)$. We then have our planar graph (G, p).

A particularly nice property of a planar graph is that it has well–defined regions, and those regions are in one–to–one correspondence with the chordless cycles of the graph. Further, there exist standard algorithms to find all chordless cycles in a graph that we may use to find all regions of our planar graph. We do not include the details of such algorithms here.

{A FIGURE WOULD BE GREAT HERE}

4.4 Find all mapping triangles

This algorithm takes as input a region R of G, represented by chordless cycle C in G, and gives as output a list of piecewise–linear circles in the triangulation T that map to a generic point interior to the region R bounded by C. A piecewise–linear circle is unambiguously described as an ordered list of triangles $\{t_i\}$ in T so that every consecutive pair of triangles are from the same Δ^3, and so that every Δ^3 of T represented by a triangle in this list is represented by exactly two triangles in this list. The piecewise–linear circle is then formed by connecting the centres of consecutive triangles with line segments.

The projection π maps the triangle t of T to the triangle t' in D^2 so that the vertices u', v', w' of t' are all in S^1, and the edges of t' are straight line segments between the vertices of t'. Now t' splits D^2 into four path-connected regions: one region which is the image of t and three regions which are not. Intuitively, take a point p in the disc and a point q in S^1 with $q \neq u', v', w'$. A straight line from p to q crosses 0, 1, or 2 edges of t'. In the case of 0 or 2 crossings, p is not in the image of t. If there is exactly 1 crossing, then p is in the image of t.

To implement this into a graph algorithm, recognize that the edges e_i of t map through π to paths P_i in G. To check if a given region, represented by the chordless region C, is in the image of t we first check whether every vertex of C is in a path P_i. If this is true then we are done — R is in the image of t. Otherwise, there is a vertex v of C which is not in any P_i. If v is an outer vertex of G, then R is not in the image of t. Otherwise, v is in the image of exactly two edges of T, and we choose one of those edges arbitrarily. That edge maps through π to the path P whose tail vertices are each outer. At least one tail vertex, call it x, is not a tail of any P_i as otherwise P would be some P_i, and this contradicts the statement that v is not in any P_i.

Now we have a graph representation of our intuitive understanding of the situation. The path from v to x represents a straight line segment connecting a point in the disc to a point on the boundary different from a vertex of our triangle. This is evident by our construction of G. We need only count the number of vertices that P shares with the paths P_i. If exactly 1 vertex is shared, then the region R is in the image of t. Otherwise, R is not in the image of t.

4.5 Find all mapping circles

This algorithm takes as input a region of R of G and gives as output a set of lists of triangles in T. Each list unambiguously defines a piecewise-linear circle in T built from line segments between the centres of consecutive triangles.

Take as input a region R of G and build from R a list L of triangles using the above algorithm: 4.4. We partition L into sublists corresponding to the connected components of the preimage of a generic point of R. To separate a connected component, we examine a sublist generated from any given triangle of L. Begin by adding an arbitrary triangle t to a sublist l of L. The triangle t is inside exactly two tetrahedra of T, so we choose one of these tetrahedra, S, arbitrarily. Now S contains exactly one other t' of L, which itself is inside of another tetrahedron S' of T. We add t' to l and continue this process with S' until we find ourselves back at t. Reduce L to $L' = L \setminus l$, and continue until $L^{(n)}$ is empty. Each sublist l defines a piecewise–linear circle in T. {FIGURE: A PL CIRCLE}

4.6 Count wedge numbers

In the previous section, we built lists of circles that map through π over the regions of (G, p). What we do not yet have is a way of describing the fibers of π that map over the edges of (G, p). For an edge e of (G, p) with $p(e)$ in the image $\pi(E)$ for E an edge of T, the shape of the fibers that map over $p(e)$ is entirely classified by a positive integer associated to E that we call the wedge number. We define the wedge number and provide a method of computation.

Definition 4.6.1. Let e be an edge of G, x a point in $p(e)$, and E the unique edge in T so that $p(e)$ is entirely contained in $\pi(T)$. The preimage $\pi^{-1}(x)$ in T consists of the disjoint union of n circles with a wedge of m circles, for some nonnegative integers n, m. More precisely,

$$\pi^{-1}(x) = \Big(\coprod_{i=1}^{n} S^1\Big) \coprod \Big(\bigvee_{i=1}^{m} S^1\Big).$$

We define the *wedge number* of E to be exactly m.

Remark 4.6.2. A justification that $\pi^{-1}(x)$ has exactly the structure described in Definition 4.6.1 is easier to articulate once we have a method of computation. Furthermore, it does double duty as a justification of the correctness of our computation method.

Before we compute the wedge number of an edge E of T, we find an easier number to get our hands on. We call this object the *wedge sum* of E, which ends up being exactly twice the wedge number we will assign to E. Begin with the link $\texttt{lk}(E)$ in T. Our triangulation is of a closed manifold, so $\texttt{lk}(E)$ is a triangulated S^1, and each edge of $\texttt{lk}(E)$ is opposite E in some tetrahedron Δ^3. If F is an edge of $\texttt{lk}(E)$ then we associate to F a crossing coefficient with respect to E. Because T is an edge–distinct triangulation, E and F are non–adjacent. Using the method discussed in Section 4.3, we check whether $\pi(E)$ and $\pi(F)$ intersect. The crossing coefficient of F with respect to E is 0 if $\pi(E)$ and $\pi(F)$ do not intersect P_E, and +1 otherwise. Because $\texttt{lk}(E) \cong S^1$ is closed, the sum of the crossing coefficients with respect to e is even. The sum of the crossing coefficients with respect to E is the wedge sum of E, and our definition will allow us to assign the wedge number of E to be half the wedge sum of E.

With the notation as in Definition 4.6.1, we justify that $\pi^{-1}(x)$ has the structure as the disjoint union of n circles with the wedge of m circles. Consider the tetrahedra Δ_i^3 that contain E. Following the convention in Definition 4.1.3, x is a point of either type 3 or type 4 in each Δ_i^3. In a tetrahedron Δ^3 not containing E, x will either be a point of type 5 or will not be in the image of Δ^3. We can conclude from this that connected components of $\pi^{-1}(x)$ missing the edges of T will be circles, and exactly one component of $\pi^{-1}(x)$ will not miss every edge of T, and this is the component that hits E in exactly one point. This shows that all but one connected component of $\pi^{-1}(x)$ is a circle, and all that is left is to show that the last component is the wedge of some number of circles.

The component of $\pi^{-1}(x)$ that hits E will miss every other edge of T outside of the Δ_i^3, and outside of the Δ_i^3 it will consist of arcs in T whose endpoints are contained in the triangles of the Δ_i^3 opposite to E. What about the portion of $\pi^{-1}(x)$ inside of the Δ_i^3? We have x as either type 3 or type 4 in each of the Δ_i^3, and x is of type 3 in exactly the Δ_i^3 contributing 0 to the wedge sum and of type 4 in exactly the Δ_i^3 contributing $+1$ to the wedge sum. This means that the portion of $\pi^{-1}(x)$ inside of the union of the tetrahedra containing E is a star with a number of leaves equal to the wedge sum. Pairs of leaves of this star are connected by the arcs we found previously, and the object we end up with is a wedge of circles, the number of which coincides exactly with half the wedge sum – i.e. the wedge number of e.

Remark 4.6.3. We may expand our interpretation of wedge numbers from the structure of $\pi^{-1}(x)$, notation as in Defintion 4.6.1, to the structure of a surface in T containing $\pi^{-1}(x)$. Let P be the path in G which is induced by an edge E of T, as in Section 4.3. Let e_i be an edge of P, and let's further assume that P is a path whose edges are all interior to G. Then e_i is the border of a pair of regions R_y and R_z. Let $y \in R_y$ and $z \in R_z$. We examine the preimage of y, z, and simple arc $\gamma_{y,z}$ that connects the two, intersecting e_i in exactly one point and intersecting no other edges of G. We know that the preimages of y and z through π are the disjoint unions of circles

$$\pi^{-1}(y) = \coprod_{i=1}^{n} S^1_{y,i} \text{ and } \pi^{-1}(z) = \coprod_{i=1}^{m} S^1_{z,i}.$$

So $\pi^{-1}(y)$ and $\pi^{-1}(z)$ are cobordant through a surface Σ, and Σ lives in T as $\pi^{-1}(\gamma_{y,z})$. A connected component of Σ is either a cylinder, corresponding to one of the n circles found in the preimage through π of a point in $p(e_i)$, or a copy of S^2 punctured k–times, corresponding to the wedge of $k-1$ circles found in the preimage of a point in an edge of G. The wedge number of an edge is interpreted as the number of circles in a bouqet of circles that projects to a generic point of that edge through π – that is, $k-1$.

4.7 Truncate tetrahedra

In order to carry out edge blowups and reduce wedge numbers, we need to replace all of the tetrahedra in T with truncated tetrahedra. Essentially, we are removing a small ball around every vertex of T and leaving an S^2 boundary for every removed vertex. This is identical to replacing every tetrahedron of T with a truncated tetrahedron. At this point in our algorithm, the only information we need from T is adjacency between tetrahedra. Truncating tetrahedra does not change this information, so we truncate every tetrahedron of T and obtain T'. We also have π', defined as π restricted to $T' \subset T$. The objects that make up the truncated complex T' are solid polyhedra that are glued together over hexagonal faces. We'll call this complex a *solid polyhedral gluing*. We previously built lists of triangles in T that formed PL–circles in T. Truncating the tetrahedra of T alters these lists only by replacing the triangles in these lists with the associated truncated triangles, which are hexagonal 2–cells.

In Section 4.3 we outlined how to obtain a planar graph (G, f) associated to T from the projection π. To obtain a planar graph associated to T' and π', we begin with the graph (G, f) and examine the difference between the images $\pi'(T')$ and $f(G)$ in the plane. This picture suggests that we should "cut away" the parts of (G, f) that are outside of $\pi'(T')$.

Consider a vertex v of G that is associated to a vertex of T. Then $f(v)$ lies on the circle in the plane. Recall that the vertices of T are mapped under π to the

n^{th} roots of unity in $\mathbb{C}$, where n is an odd number at least as large as the number of vertices of T, so we'll assume without loss of generality that $f(v)$ is at $1 \in \mathbb{C}$. A chord in the circle with endpoints $e^{\pm\pi/2n}$ intersects $\pi(T)$ in a line segment near 1. The preimage of this segment is an embedded copy of $\texttt{lk}(v)$ that separates v from the rest of T^0. To truncate the tetrahedra of T containing v, we remove the interior of this link. Truncating the tetrahedra around every vertex of T^0 yields T' and π', where π' is just π restricted to T'. We obtain a new planar graph (G', f') by first subdividing every edge of $f(G)$ intersecting $\pi'(\partial T')$ into a pair of edges. Each subdivision deletes a single edges and introduces a pair of new edges and a single vertex. Connect sequential vertices on the line segments $\pi'(\partial T')$ with new edges. Finally, remove the vertices of G that are associated to vertices of T as well as all edges adjacent to these vertices. The result is a graph G'.

To make G' planar, we need to fix an embedding. We obtained G' from G by subdividing some edges of G, adding new edges between the new subdivision vertices, and deleting edges and vertices outside of the desired image. We can then define an embedding f' on G' by first making $f'(e) = f(e)$ and $f'(v) = f(v)$ for any edges and vertices of G' that are untouched by our creation of G'. Let v in G' be a vertex added when subdividing the edge e of G. We added v because $f(e)$ intersected $\pi'(\partial T')$, and we define $f'(v) = f(e) \cap \pi'(\partial T')$. We've defined f' on all vertices of G'. By design, we can define f' on an edge of G' to be the line segment in the plane connecting the vertices adjacent to that edge.

{FIGURE: GREAT PLACE FOR A FIGURE}.

The faces of (G', f') are in 1–1 correspondence with the faces of (G, f). The gluing T' has none of the vertices of T, but retains all of the edges of T which we call "old" edges and we have some new edges that are projected through π' to outer edges of (G', f'), hence these edges are not asked for in any future algorithms.

4.8 Blow up edges

This algorithm takes as input an edge e of the solid polyhedral gluing T' with wedge number $w(e)$ at least three, and returns an altered solid polyhedral gluing T'' and an altered graph G'', where e is replaced by a set of edges e_i, $i = 1, \ldots, w(e) - 1$, where each edge has wedge number exactly 2. We blow up edges to reduce wedge numbers and we want to reduce all wedge numbers to at most 2.

To blow up the edge $e = xy$, we need to choose a pair of 3–cells over which e will be blown up. We need to slightly modify our definition of the link of an edge because we've truncated all of our tetrahedra into 3–cells, and some edges are going to be blown up. Each tetrahedra has six edges and we can easily keep track of the associated edges in a truncated tetrahedron as well as a set of blown up edges in a blown up truncated tetrahedron. An edge or set of edge copies in a tetrahedron or blown up tetrahedron can then be said to have an associated "old" edge, as it would appear in T. We take $\texttt{lk}(e)$ during this algorithm to be a triangulated circle L whose edges are exactly the edges of the "old" link in T. The main information contained in an "old" link is a string of crossing coefficients with respect to the edge e, and this information is not altered by the following algorithm. The structure of a tetrahedron and the requirement that our initial triangulation T was edge distinct means that a pair of "old" edges from T can be opposite edges of at most one tetrahedron of T, hence a pair of such "old" edges uniquely determine a 3–cell of T'.

Because the wedge number of e is at least 3, there are at least 6 edges of $\texttt{lk}(e)$ with $+1$ crossing coefficients. We choose a connected path $Q = q_0 \dots q_n$ in $\texttt{lk}(e)$ whose crossing coefficients sum to exactly 4, and whose tail edges, q_0q_1 and $q_{n-1}q_n$, each have $+1$ crossing coefficient. The tail edges of this path along with the edge e determine two 3–cells t_0,t_1 over which we blow e up. Two of the vertices of the t_0 are x and y.

The image of e under π is the path $P = v^1 \dots v^k$. We choose to map x to v^1 and y to v^k. The vertices of G on the circle which are not v^1 or v^k are partitioned into two sets which we arbitrarily designate as "upper" and "lower" with respect to P. Subscripts containing l's and u's refer to lower and upper designations of the objects in question. Without loss of generality, we'll say that the edge opposite e in t_0 is q_0q_1 and the edge opposite e in t_1 is $q_{n-1}q_n$. Either way, the edge opposite e in $t_{0,1}$ has crossing coefficient $+1$ with respect to e. Because Q has crossing sum exactly 4, the vertices q_0, q_n are both either upper or lower. We'll say that they're both lower, which forces q_1, q_{n-1} to each be upper. Blowing up the edge of a pair of solid polyhedra adds a face to each of the two solid polyhedron over which the two become glued. The edge is called e and has vertices x, y. One of the edges that e splits into forms an edge of two 2–cells, one containing q_0 and one containing q_n. We call this edge e_l, and say that e_l has vertices x_l, y_l. The other edge that e splits into forms the edge of two 2–cells, one containing q_1 and one containing q_{n-1}. This edge is called e_u and has vertices x_u, y_u.

Every vertex v^i of P is adjacent to exactly two other vertices v^i_{ll} and v^i_{uu} of G. When v^i is internal, each of v^i, v^i_{ll}, and v^i_{uu} is on the path P^i, which is the image of an edge e_i of T through π. Then v^i_{ll} is the vertex on the path P^i from v^i to a lower vertex and v^i_{uu} is the vertex on the path P^i from v^i to an upper vertex. When v^i is external, v^i_{ll} and v^i_{uu} are already upper and lower vertices, so v^i_{uu} is the upper vertex and v^i_{ll} is the lower vertex.

{FIGURE: SAMPLE P^i AND VERTICES AS DEFINED}

{FIGURE: BLOWUP ILLUSTRATED}.

The blowup of P is the graph $P \times I$ which gives us a copy of P at $\{0\}$ and at $\{1\}$, called P_l and P_u respectively. The new vertices are similarly subscripted, and the graph $P \times I$ also contains the edges $v^i_l v^i_u$ for every $i = 1, \ldots, k$. Blowing up P inside of G means that we remove P and all edges incident to P from G. We then add to G the graph $P \times I$, and add the edges $v^i_{ll} v^i_l$ and $v^i_{ll} v^i_u$ for each i. The embedding of G incorporates $P \times I$ in the way depicted in the figure.

{FIGURE: EMBEDDING OF G}

The regions of G incident to P are pushed away from each other and new regions are created. These new regions are bounded by the edges $v^i_u v^i_l$, $v^i_l v^{i+1}_l$, $v^{i+1}_l v^{i+1}_u$, $v^{i+1}_u v^i_u$ for $i = 1, \ldots, k-1$ and are the image of the new face f of T. The paths P_l and P_u and the edges $v^1_u v^1_l$ and $v^k_u v^k_l$ are the image of the boundary edges of f. Recall that these edges are e_u, e_l, $x_u x_l$ and $y_l y_u$. We get that $x_l x_u$ and $y_l y_u$ map to $v^1_l v^1_u$ and $v^k_l v^k_u$ so that lower and upper subscripts are mapped to each other. This also makes e_u map to P_u and e_l map to P_l.

The wedge numbers of e_u and e_l are exactly 2 and $w(e) - 1$ respectively. We can see this by saying that an edge of $\mathtt{lk}(e)$ contributes to the wedge number of e_l, (resp. e_u) if and only if that edge is adjacent to a 2–cell of T' containing e_l, (resp. e_u). If $w(e_l)$ is still greater than 2, we proceed by choosing another pair of 3–cells to blow e_l up over. This pair is found by extending Q in both directions to $q_{-k}q_{-k+1}\dots q_0\dots q_n\dots q_{n+k'}q_{n+k'+1}$ where $q_{-k}q_{-k+1}$ and $q_n+k'q_{n+k'+1}$ each have crossing coefficient $+1$ with respect to e_l, and there are no other edges between $q_{-k}q_{-k+1}$ and q_0q_1 and between $q_{n-1}q_n$ and $q_n+k'q_{n+k'+1}$ with crossing coefficient $+1$ with respect to e_l. The residual triangulation structure leaves us with exactly two 3–cells, one containing e_l and $q_n+k'q_{n+k'+1}$ and one containing e_l and $q_{-k}q_{-k+1}$, over which to blow up e_l. The algorithm for this blow up works for blowing up e_l into e_lu and e_ll, where e_lu has wedge number 2 and e_ll has wedge number $w(e_l)-1$. Iterate until e has been blown up into exactly $w(e)-1$ edges whose wedge numbers are exactly 2.

There are a few new regions in G, and we need to know how generic points in those regions are pulled back into T'' by π''. Fortunately, we are able to derive the mapping circles of a region of G'' entirely from the mapping circles of an adjacent region. Let R be a new region of G'' whose mapping circles are unknown, and let R' be a region of G'', adjacent to R over the edge e of G'', whose mapping circles are known and given as the list $L=\{l_i\}$. We know R' is adjacent to R over the edge e of G'' and e is in the image of the edge E in T''. Pull from L the circles $l_i\in L$ which have in their description at least one 2–cell containing E. Deconstruct the circles l_i into a single unordered list K of 2–cells. Call the list of 2–cells in T containing E by $\sigma(E)$. Note that in $\sigma(E)$ is a 2–cell created by an edge blowup that is projected over R but not R', and that 2–cell is absent from K. The list of 2–cells that project over R includes all of the piecewise linear circles in $L\setminus K$, as well as the symmetric difference of K and $\sigma(E)$. Recreating the circles in $L\setminus K$ is unnecessary as they are unchanged by passing over e. We've gone through great pains in our explicit description of the dge blow up in order to ensure that the piecewise–linear circles in $K\triangle L$ are realized using the exact algorithm in Section 4.4. More precisely, the chasing of 2–cells over 3–cells is unambiguous. We now have a list of piecewise–linear circles in T'' for any region of G''.

4.9 Final Modifications

Once all offensive edges have been blown up, we are left with a solid polyhedral gluing T'', a map $\pi'':T''\to D^2$ and a planar graph (G'',p''). To most efficiently assemble a shadow we perform one last set of modifications to (G'', p'') colour all of the edges and regions of (G'', p'') for easy identification. The wedge numbers of our edges are all at most 2, so the singularities of π'' are well understood. We remove the more complicated singularities that occur at the interior vertices of (G'', p'').

Let v be an interior vertex of (G'', p''). By design, v has degree 4 and we may order the edges containing v in clockwise order, following the standard orientation of D^2. These edges will be e_0, e_1, e_2, e_3. Because v is interior, the e_i are also interior. The e_i have boundary vertices v and v_i in G''. Delete v and the edges e_i. Add to the vertex set the vertices v_i', and to the edge set the edges e_i', $i = 0, \ldots, 4$, where $e_i' = v_i v_i'$ and directed edges $v_i' \to v_{i+1}'$, subscript addition modulo 4. When all interior vertices are replaced, we have the graph G'''.

Recall that our planar embedding p'' embedded edges of G'' to straight line segments in the plane. That is, we have $p''(e_i) = tp''(v) + (1-t)p''(v_i)$. To define a planar embedding on G''', we again define the embedding of the vertices of G''' and embed the edges of G''' as straight line segments connecting these vertices. The intended effect is to "carve away" small discs around each embedded interior vertex of G. For a new vertex v_i' with the notation above, defining p''' as

$$p'''(v_i') = \frac{1}{5}p''(v) + \frac{4}{5}p''(v_i)$$

will suffice. Every exterior vertex of G'' has an associated vertex in G''', and p''' will agree with p'' on these vertices. Finally, the edges of G''' are embedded as straight line segments in the plane. The result is a planar graph (G''', p''').

The difference in regions between (G''', p''') and (G'', p'') amounts to a new region for each interior vertex of (G'', p''). These regions represent the "carved away" portions of the plane near the critical values of π'' that may have the form of the crossing of a pair of indefinite folds (see Section 3.1). We colour these regions H for *hole*. All other interior regions of (G''', p''') are coloured I for *interior*. Using the notation above, the new edges e_i' are each the shared boundary of a pair of interior regions, hence will be coloured I. Directed edges $v_i' \to v_{i+1}'$ and the vertices v_i' are on the boundary of a region coloured H, so are also coloured H. The exterior region of (G''', p''') is ignored. These final modifications are mostly an attempt to simplify the language and algorithms of Chapter 5. We don't bother modifying T'' or π'' to reflect our changing of G'' into G'''.

The algorithms of Chapter 4 have provided us with two objects and the data connecting them. Namely, we have a solid polyhedral gluing T'' that is equivalent to the 3–manifold triangulation with some 3–balls carved away. The edges of T'' inherited from T and produced as copies of edges inherited from T via blow ups are also equipped with wedge numbers. We also have a planar graph (G''', p''') which is connected to T'' by a projection $\pi'' : T'' \to D^2$ and a list of piecewise–linear circles associated to each interior region of G'''.

Chapter 5

Shadows from 3–Manifolds

The previous chapter produced a map $\pi'' : T'' \to \mathbb{R}^2$, where T'' is a 3–manifold with only spherical boundary components. In Section 3.1 we discussed the construction of a Stein complex from a Morse 2–function f from a 3–manifold to $\mathbb{R}^2$. The only two properties we needed from f were that f had compact fibers and that f was generic. Genericity was only used to determine how the fibers of f joined over singularities, and the piecewise–linear analogue to the critical values of the smooth Morse 2–function f are the vertices and edges of (G'', p''). Our map π'' has compact fibers, and the wedge numbers of Section 4.6, each reduced to at most two in Section 4.8, provide the information needed to determine how the fibers of π'' behave near the edges and vertices of (G'', p''). This means we may construct a Stein complex for T'' from the data produced in Chapter 4. The data used to construct this complex is a planar graph (G''', p'''), lists of circles projecting over the regions of (G''', p'''), and a colouring and wedge number for each edge of G'''. The Stein complex constructed this way is a shadow polyhedron of T'', as defined in Section 3.2. By Lemma 4.4 of [1], the polyhedron constructed is standard, i.e. each region is a disc. To obtain a shadow of T'', we equip each region not touching the boundary of the polyhedron with a gleam

5.1 Construct and connect regions

This algorithm concerns itself with producing a shadow of T'' from π''. Because we do not refer to the inital triangulation or its projection in this algorithm or the next, we will abbreviate T'' by T, π'' by π, and G'' by G to ease notation. We begin by constructing an empty shadow S and populating it with regions. Each region R of G is a polygon, i.e. a topological disc, whose boundary is a triangulated S^1. The edges of this S^1 are all associated with edges of G, and each edge of G is coloured I if the edge might produce an internal edge of our shadow, F if the edge will produce a boundary edge of our shadow, or H if the edge was introduced by removing an internal vertex of G. Similarly, the regions of G are coloured I for "internal" and H for "hole."

Let p be a generic point in an internal region R. The preimage of the point p is the disjoint union $l_1 \cup \cdots \cup l_m$ where each l_i is a piecewise–linear circle in T. To split π into $g \circ h$ with g finite–to–one, we just say that p pulls back through g to exactly m points. The points in the regions of our shadow represent circles in T, the points in the interior edges of our shadow represent two circles joining together to become a single circle, the points in the boundary edges of our shadow represent circles collapsing to points, and the vertices of our shadow are interpreted using two local models.

We restrict our consideration to the internal regions of G. The region R pulls back through π to a disjoint collection of m open solid tori in T, and any torus N in this collection can be seen as a circle bundle $N \xrightarrow{\pi} R$. We pull R back through g to m disjoint copies of itself. We let a *copy of* R be the pair $(P(R), l_i)$ where P is a polygon identical to R whose edges are coloured identically to the edges of R and l_i is one of the piecewise–linear circles of T in the preimage of p. For every internal region R of G and every piecewise–linear circle l_i of R, we add $(P(R), l_i)$ to S.

We analyzed how the edges of G are pulled back to T through π in Section 4.6 and this analysis can be extended to see how edges of G are pulled back to our shadow through g. Let e be an internal edge of G and let A, B be the internal regions of G which are adjacent over e. There is an associated edge E of T so that E projects over e through π. Let a in A and b in B be points near e and let $\gamma_t = ta + (1-t)b$ be the line segment in the plane that connects a and b, and let $t'a + (1 - t'b)$ be the point on this line segment that lies in e. We know that $\pi^{-1}(a)$ consists of exactly m piecewise–linear circles $k_1 \cup \cdots \cup k_m$ and $\pi^{-1}(b)$ of exactly n piecewise–linear circles $l_1 \cup \cdots \cup l_n$ in T. The wedge number of E is at most 2, so the numbers m, n differ by at most 1. Recall that piecewise–linear circles in T are represented by ordered lists of 2–cells in T. If a circle k_i does not contain E in any of its 2–cells, then the circle k_i is unchanged as we pass over e. There is a circle l_j that is represented by the same ordered list of 2–cells (up to a cyclic permutation) as k_i. We glue the region copies $(P(A), k_i)$ and $(P(B), l_j)$ over the edge of each corresponding to e. The result is a copy of $A \cup_e B$, and the polygon $P(A \cup_e B)$ is identical, in terms of edge identifications, to the polygon obtained by gluing A to B over the edge e in an orientation preserving way. The piecewise–linear circles associated to the region copies $(P(A), k_i)$ and $(P(B), l_j)$ are equivalent, so we may take either circle as the piecewise–linear circle associated to $P(A \cup_e B)$. All copies of A and B whose associated piecewise–linear circles not containing E are dealt with in this manner. We reduced all wedge numbers to at most 2 via edge blowups, so there are exactly three cases to consider when dealing with the copies of A and B containing E.

If the wedge number of E is 0, then there is exactly one region copy (P, l) among the copies of A, B where l contains E. The edge copy in P corresponding to e is coloured F for *false*. This edge corresponds to a shrinking singularity of π, where a circle in T collapses to a point.

If the wedge number of E is 1, then there are exactly two region copies $(P(A), k)$ and $(P(B), l)$ among the copies of A and B where k and l each contain E. We combine $P(A)$ and $P(B)$ as before into $P(A \cup_e B)$, but the piecewise–linear circles k and l are distinct in a way that has been hinted at before. The intersection of k and l as a list of 2–cells consists of exactly the 2–cells in either circle that do not contain E, and the symmetric difference of k and l is exactly the set of 2–cells in T containing E. We are left with the region copy pair $(P(A \cup_e B), (k, l))$. If the region copy $P(A \cup_e B)$ is considered in the future, it is with respect to some edge f of $A \cup_e B$ and f is exclusively an edge of either A or B. The representative circle for $P(A \cup_e B)$ is then taken to be k if f is from A, and l if f is from B. In this case, the edge of G did not correspond to a singularity of π at all.

If the wedge number of E is 2, then there are exactly three region copies. Without loss of generality, two of these are copies of A and the last is a copy of B and we call these copies $(P(A), k_1)$, $(P(A), k_2)$ and $(P(B), l)$. In this case we have encountered the boundary of all three region copies along an internal edge of S, so we glue each region copy along their edge e. This gluing should be orientation reversing when restricted to $(P(A), k_1)$, $(P(A), k_2)$ and orientation preserving when restricted to $(P(A), k_1)$, $(P(B), l)$ and to $(P(A), k_2)$, $(P(B), l)$. Colour the shared edge e by i for *internal*. This edge corresponds to a simple saddle singularity of π, where a single circle splits into two distinct circles.

Iterating over all adjacent internal region pairs produces a connected simple polyhedron S with boundary a graph whose connected components are coloured either F or H. The false boundary is not considered until the next chapter, but we fill in each boundary component coloured H with a simple block.

5.2 Extend over vertices

There are some holes in our shadow represented by digraphs whose edges are coloured H. Whenever we work with one of these graphs, it will be named X. The projection of any X through π is one of the copies of the directed cycle on 4 vertices introduced during interior vertex deletion in Section 4.9. Each of the four vertices of $\pi(X)$ is adjacent to a single edge, and those edges are projected onto by a finite number of strands in S by $g : S \to D^2$. At most one of these projecting strands per edge corresponds to a strand of singular points through π. These singular strands are determined by π and extend along the entirety of the edge of T in which they lie, corresponding to edges or shrinking singularites in S. This means if one of our graphs X is adjacent to a strand of singular points, then it is adjacent to another strand from the same edge of T, hence these singular strands occur in pairs.

In every case, X is one of five possible graphs which we classify by their number of simple directed cycles. The paper on which this thesis is based contains a detailed justification of the form of polyhedron we glue to S in order to fill these holes. The five possible graphs are classified by the number of singular and nonsingular strands adjacent to X.

{ FIGURE: THE FIVE POSSIBLE GRAPHS }

{ FIGURE: THE FOUR FILLING POLYHEDRA }

The first case occurs when X is adjacent to exactly two strands of shrinking singularities and one nonsingular strand. Here, we colour the edges of X by F and let X join the rest of the false boundary.

The second case occurs when X is adjacent to no singular strands. We fill in the hole with a disc whose boundary has triangulation X.

The third case occurs when X is adjacent to exactly two strands of non–shrinking singularities and three nonsingular strands. We fill in the hole with the model shadow edge.

The fourth case occurs when X is adjacent to exactly four strands of non–shrinking singularites and four nonsingular strands. We fill this hole with the model shadow vertex.

The fifth case occurs when X is adjacent to exactly four strands of non–shrinking singularites and no nonsingular strands. We will this hole with a shadow polyhedron that has exactly two vertices and adjust the gleams of the regions adjacent to where we glue this polyhedron slightly. The polyhedron in this case is the shadow of X as it sits in S^3.

All boundary components of S are filled this way. At the end of this algorithm we have a shadow of T. Once the internal regions of S are assigned gleams, we will be able to construct a triangulated 4–manifold.

5.3 Compute Framing Constants

Data types used:

- A cell complex T_2, modified by the actions in Sections 4.7 and 4.8;
- A cell complex $^*T_2$, obtained as the dual complex to T_0 and modified by the actions in Sections 4.7 and 4.8;
- A simple polyhedron S that is a Stein complex for T_2;
- The regions of D_i of S which are homeomorphic to open 2–discs per Lemma 4.4 of [1] and were built in Section 5.1 from a set of subregions $\{B_{i,k}\}$;
- The edges e_i of S, of which some are the false boundary edges coloured F in Section 5.1.

Before delving into the details of framing constant computation, we will briefly discuss the theory of the method outlined in Section 4.6 of [1]. Because S is a Stein complex for T_2, we can describe π_2 as a Stein factorization $\pi_2 = f \circ g$, where f is a map $T_2 \to S$ and g is a map $S \to \mathbb{R}^2$. Let $\overline{D}$ denote the abstract compactification of D in S, i.e. $\overline{D}$ is a 2–complex whose 2–cell is D and whose 1– and 0–cells are the edges and vertices of S that are incident to D. Suppose that D has not been assigned a framing constant and $\overline{D}$ has no false edges. The preimage of $\overline{D}$ through f is a closed solid torus in T_2. Recall from Corollary 3.1.3 that As demonstrated in the proof of Theorem 3.1.2, the framing constant that will be used to attach a 4–dimensional 2–handle

Chapter 6

4–Manifolds from Shadows

Section 3.2 discussed constructing a 4–manifold W with a given a shadow $S = (P, \texttt{gl})$ out of handles with index at most two. For every vertex of P we attach a 0–handle, every edge a 1–handle, and every region a 2–handle. The 0– and 1–handles are attached according to the combinatorics of P, but the 2–handles needed more information. In particular, P alone didn't tell us how to frame the attaching sphere of any particular 2–handle.

Our goal is to take a disc bundle over a polyhedron P, which is standard hence has a canonical cell decomposition as a 2–complex. If we define the bundle over the 0–, 1–, then 2–skeleton of P, there is an obstruction to extending this bundle over the 2–cells of P. This obstruction only exists on 2–cells whose closures are disjoint from the boundary of P. We modify P by removing a small neighbourhood from each region whose closure does not touch the boundary of P, and put a disc bundle on the remaining space. A disc bundle near a vertex of P is a 4–dimensional 0–handle and near an edge is a 4–dimensional 1–handle. Extending the disc bundle over the removed regions then amounts to attaching 2–handles whose cores are the removed regions.

This is where we argue that ignoring false regions/edges during reconstruction produces an equivalent 4–manifold.

A trivalent boundary graph is made of edges coloured f. We coloured certain edges f because those edges were introduced to G via shrinking singularities of our map. Examine the preimage of a generic point in a sufficiently small neighbourhood of a point p in a false edge. If q is near p in the region whose boundary contains p, then the preimage of q is a single circle in T. Every false edge of our shadow comes directly from an edge of T, so the preimage of p is a point. If we walk a path from q to p in our neighbourhood of p, then this path pulls back to a sequence of circles in T which collapse to the point $\pi^{-1}(p)$ in T. The intuition we're bringing to this method is that every preimage circle of π is filled with a disc in order to construct a bounded 4–manifold. We see that filling in circles near a false edge amounts to attaching a 4–ball over a pair of boundary 3–balls. This attachment has no effect on the 4–manifold we're constructing, so we neglect to fill in discs near any false edge. Extending this to any region containing a false edge is easily justified, as a disc bundle over a disc is contractible.

Furthermore, ignoring the false edges and regions entirely rather than collapsing them removes any dependence on shadow moves and equivalences to reconnect a disconnected shadow, get rid of embedded graphs or make a nonstantard polyhedron standard again.

6.1 The triangulated prism

The basic building block that we used in our 4–manifold reconstruction is the triangulated n–prism with identical walls, $n \leq 4$. By n–prism, we mean the n–disc seen as $\Delta^{n-1} \times [-1, 1]$. The prism has canonical "top" and "bottom" faces which are $\Delta^{n-1} \times \{1\}$ and $\Delta^{n-1} \times \{-1\}$ respectively as well as n identical "walls" which are $F^i(\Delta^{n-1}) \times [-1, 1]$ for each $i = 0, \ldots, n-1$. By identical walls, we mean walls with identical triangulations. If a pair of Δ^{n-1} are glued over a face, then we may take these simplices to be the bottom faces of a pair of prisms. We can glue these prisms together over their corresponding walls. This effectively provides the two Δ^{n-1} with an "n–dimensional thickening." If we can do this to any pair of simplices, then we can do this to an entire triangulation.

Seeing that $F^i(\Delta^{n-1})$ is an $(n-2)$–simplex, it is reasonable to build our prisms iteratively. The intuition is that we build an $(n-1)$–sphere whose walls are $(n-2)$–prisms, themselves with identical walls. Then, we cone the $(n-1)$–sphere to make a triangulated n–prism.

The 1–prism is just an edge. This is the base case, so no coning is needed.

To make the 2–prism, we take two copies of the 1–simplex: $e\times\{1\}$ and $e\times\{-1\}$. The two face maps of e give us the two vertices of e. Then $F^0(e) = v_1$ and $F^1(e) = v_0$, so we now have the 1–sphere triangulated by the top face $e \times \{1\}$, bottom face $e \times \{-1\}$, and walls $v_1 \times I$ and $v_0 \times I$. Coning gives us a triangulation of D^2 whose walls are single edges and whose top and bottom faces are 1–simplices.

To make the 3–prism, we take two copies of the 2–simplex: $t\times\{1\}$ and $t\times\{-1\}$. The three face maps of t give us the three edges of t. Then $F^0(t) = [v_1, v_2]$, $F^1(t) = [v_0, v_2]$ and $F^2(t) = [v_1, v_2]$, so we now have the 2–sphere triangulated by the top face $t \times \{1\}$, bottom face $t \times \{-1\}$, and walls $[v_1, v_2] \times I$, $[v_0, v_2] \times I$ and $[v_1, v_2] \times I$. The walls are 2–prisms with the previous triangulation. The walls glue to the top and bottom triangles in the obvious way, and are glued to each other over their identical 1–dimensional walls. Coning gives us a triangulation of D^3 whose walls are the 2–prism found above and whose top and bottom faces are 2–simplices.

The 4–prism construction is nearly identical to the 3–prism construction, but it's not a bad idea to write it down. To make the 4–prism, we take two copies of the 3–simplex T: $T \times \{1\}$ and $T \times \{-1\}$. The four face maps of T give us the four triangles of T. Then $F^0(T) = [v_1, v_2, v_3]$, $F^1(T) = [v_0, v_2, v_3]$, $F^2(T) = [v_0, v_1, v_3]$ and $F^3(T) = [v_0, v_1, v_2]$. We now have a triangulated 3–sphere with top face $T\times\{1\}$, bottom face $T \times \{-1\}$, and walls $[v_1, v_2, v_3] \times I$, $[v_0, v_2, v_3] \times I$, $[v_0, v_1, v_3] \times I$ and $[v_0, v_1, v_2] \times I$. The walls are 3–prisms with the previous triangulation. The walls glue to the top and bottom faces in the obvious way, and are glued to each other over their identical 2–dimensional walls. This triangulation is seen to be S^3 because it is constructed by gluing together three basic pieces. The three pieces are the top face (a 3–ball), which is glued to the union of the walls (an $S^2 \times I$), which is glued to the bottom face (a 3–ball). Coning gives us a triangulation of D^4 whose walls are each the 3–prism found above and whose top and bottom faces are 3–simplices.

Remark 6.1.1. Before moving further with construction, we should revisit the idea of an "n–dimensional thickening." The 3–thickening of the Möbius band under the definition above gives the solid Klein bottle and, in general, the thickening as described produces the trivial interval bundle over the base triangulation. Our goal in later sections is to produce an interval bundle over a possibly nonorientable 3–handlebody H whose total space is orientable. For a smooth bundle, this would amount to identifying all 1–handles whose attachments cause H to be nonorientable, and defining the fiber bundle so that a transition function on each such 1–handle is the reflection diffeomorphism in the fiber direction. Our prisms are symmetric through switching of "top" and "bottom," so such a strategy can be employed here as well.

The above argument will be dealt with in more detail in the following sections.

6.2 Build a handlebody

This algorithm takes as input a shadow $S = (P, \mathtt{gl})$ as produced in Chapter 5. In particular, all boundary edges of P are coloured false, and gl is defined only the regions of P whose closures do not touch the boundary of P. This algorithm gives as output a 4–dimensional orientable manifold which is a disc bundle over a regular neighbourhood of $\mathrm{Sing}(P) \setminus \partial P$. All that remains at that point is to extend the disc bundle over the removed regions.

The singular set $\mathrm{Sing}(P)$ is a graph with 4–valent interior vertices and 3–valent boundary graph. Let $\mathrm{Sing}'(P)$ denote $\mathrm{Sing}(P)$ minus the boundary edges of $\mathrm{Sing}(P)$ Take note, though, that a boundary vertex of P is 4–valent in $\mathrm{Sing}(P)$. We take N to be a regular neighbourhood of $\mathrm{Sing}(P)$. This is P minus a shrunken copy of each region of P and can be decomposed into nice blocks with gluing instructions. The blocks associated to the neighbourhoods of interior vertices and edges are the standard vertex and edge blocks from Figure {FIGURE: Let's get a new figure in here}. The blocks associated to the neighbourhoods of boundary vertices and edges are the standard edge block and I^2 respectively. Each of these blocks is contractible, so a disc bundle over these blocks is also contractible. The prisms of the previous section allow us to take interval bundles quite easily, so we form our disc bundle by first taking an interval bundle over N then taking another interval bundle over that space. The interval bundle over N must be taken as $N \times I$ so that we may keep track of the attaching spheres for our 2–handles later. The space $N \times I$ may be nonorientable, but we may take an interval bundle over $N \times I$ in such a way that the total space is orientable, as discussed in the previous section.

We take the interval bundle over a vertex block to be a single tetrahedron. Let v

be a vertex of N and Δ_v the tetrahedron associated to the interval bundle over the vertex block of v. We call Δ_v a *vertex bundle block*. Each interior vertex is incident to four edges and six regions, so if Δ_v comes from an interior vertex then to each edge of Δ_v we associate a region of P and each face of Δ_v we associate an edge of P in a way that makes combinatorial sense. Each boundary vertex is incident to four edges and three regions, but three of those edges are coloured false and all three of the regions are boundary regions. By the discussion at the beginning of the chapter, false edges along with the vertices and regions touching them contribute contractible 4–balls to the disc bundle over P. So if v is boundary, we do not construct a vertex bundle block for it.

{FIGURE: v AS IT SITS IN P AND THE RESULTING Δ_v WITH EDGE AND FACE COLOURING}

We take the interval bundle over an edge block to be a 3–prism with identical walls. Let e be an edge of N and Π_e the prism associated to the interval bundle over the edge block of e. We call Π_e an *edge bundle block*. Each interior edge is incident to two vertices and three regions, so if Π_e comes from an interior edge then to each of the three walls of Π_e we associate a region of P and to the top and bottom faces of Π_e we associate a vertex of P. Again, we can discard any edge bundle blocks that would come from false edges.

{FIGURE: e AS IT SITS IN P AND THE RESULTING Π_e WITH EDGE/FACE COLOURING}

There are clear gluing maps between the Δ_v and Π_e which form a possibly nonorientable 3–handlebody which can be seen as the trivial interval bundle over N. The top triangular face of a given Π_e is associated to the vertex v of $\mathrm{Sing}(P)$. We call that face t. If v is a boundary vertex then we colour t "false." Otherwise, the walls of Π_e are associated to regions r_1, r_2, r_3, and we call those triangulated walls by w_1, w_2 and w_3. A face of t intersects one w_i nontrivially. We call that face of t by t_i. There is a Δ_v with face $F^e(\Delta_v)$ associated with e. The three edges of $F^e(\Delta_v)$ are associated with the regions r_1, r_2, r_3, so we identify t to $F^e(\Delta_v)$ by their edge.

At this point, we have not glued any faces together. If we were to glue the faces based on their identifications, though, we would have the 3–handlebody described above. Instead, let M be a maximal spanning tree of the graph $\mathrm{Sing}'(P)$. The subcomplex of N associated with M is M_c. We freely glue together the Δ_v and Π_e associated to $\mathrm{Sing}'(P)$. The result is the 3–ball $H = M_c \times I$, which is orientable, and a collection of edge bundle blocks Π_e, one for each edge e not in M. Iterating through the remaining edge bundle blocks, if the attachment of a given P_e to H would cause the resulting 3–handlebody to be orientable, we attach both faces of P_e to H. Otherwise, we attach exactly one of the faces of P_e to H, and remember the attaching map between the other two faces. The orientation preservation condition can be easily checked by comparing the orientation of H with the attaching faces of P_e. The edge bundle block P_e is attached to H over the oriented triangular top and bottom faces of Π_e, and a pair of oriented triangular faces of H. Recall that if both attaching maps are orientation preserving or orientation reversing, then the resulting body is nonorientable and if one map is orientation preserving and the other is orientation reversing, then the resulting body is orientable. Here, an attaching map between oriented triangles is completely described by an element of Σ_3, the symmetric group on three points. An even element of Σ_3 corresponds to an orientation preserving attaching map and an odd element corresponds to an orientation reversing attaching map. The result is an orientable simplicial manifold H along with some additional gluing information. Furthermore, every region of P is associated to either an annulus or strip on the boundary of H built from the walls of the 3–prisms. For each region R, call its annulus or strip by R_f, where f stands for "framing." These objects are significant because they allow us to define 0–framings for our 2–handle attachments.

To perform 4–thickening, we take two copies of H: $H \times \{1\}$ and $H \times \{-1\}$. For every tetrahedron Δ^3 of H, take the tetrahedra $\Delta^3 \times \{1\}$ and $\Delta^3 \times \{-1\}$ to be the "top" and "bottom" tetrahedra of a 4–prism with identical walls. We glue the walls of every prism together in the way we designed the walls to do. A pair of unglued edge bundle block faces are triangles $t_{1,2}$ along with a gluing map $g : t_1 \to t_2$. The 4–thickening of these triangles is a pair of triangulated 3–balls $b_{1,2}$. Each of $b_{1,2}$ has the triangulation of a 3–prism with identical walls, complete with the triangles $t_{1,2} \times \{\pm 1\}$.

{FIGURE: DRAW THESE SPECIFIC 3–PRISMS}

The reason we did not glue $t_{1,2}$ in the first place was because g corresponded to an odd permutation between the vertices of the triangles according to their orientations as induced by the orientation of H. We may now glue $b_{1,2}$ by first gluing $t_1 \times \{1\}$ to $t_2 \times \{-1\}$ and $t_1 \times \{-1\}$ to $t_2 \times \{1\}$ using the map g, then by gluing over the tetrahedra of $b_{1,2}$ as forced by this identification. One may view this map as multiplication by -1 in the $[-1, 1]$ factor of the object $\Delta^2 \times [-1, 1]$. The result is a compact orientable triangulated 4–handlebody which has N as its shadow.

Lets examine what has happened to the objects R_f. First, the case where R_f is an annulus. Choose a triangulated boundary circle C of R_f. In the 4–thickening of H, R_f is thickened to an open solid torus V, and C is a longitude of this open solid torus. We take C to be the 0–framing of the core of V.

Next, the case where R_f is a strip in H. The gluing map g associated with R_f would make R_f into a Möbius strip, so the boundary of R_f can be easily decomposed into four pieces: two pieces C, C' which form the boundary circle of the Möbius strip and a pair of edges which are glued together by g. We took our 4–thickening to be orientable by treating the strips R_f differently, so the 4–thickening of R_f is an open solid torus V. Then C sits on the boundary of V as a curve which runs once around the longitudinal direction of V, but has endpoints which are diametrically opposed in a meridinal disc. Choose C^+ to be the 0–framing of the core of V, where C^+ is formed from C by connecting the endpoints of C by a positive one half twist around the boundary of V, where positive is defined by the orientation of V. The gleam of R is a half–integer in this case, so the alteration to the 0–framing we've performed here is corrected by decreasing the gleam of R by $1/2$.

{REF: COSTANTINO, INTRO TO SHADOWS?}

The triangulation of V is built from a number of 3–dimensional prisms with identical walls equal to the number of edges m in the boundary of R, where R is the region of P to which V corresponds. Such an open solid torus has easily defined curves which intersect a given 0–framing only at vertices in exactly n points, where $n \leq k$. Thus, an n–framing of such a torus with respect to our given 0–framing is realizable.

6.3 Attach 2-handles

This algorithm takes as input:

1. a triangulated open solid torus V which is a subtriangulation of a triangulated 3–manifold M, itself the boundary of a triangulated 4–manifold W,
2. a triangulated longitude of V represented by a curve z in the boundary of V which is defined as the 0–framing of the core of V in M,
3. a triangulated longitude of V represented by a curve N in the boundary of V which intersects z only at vertices in exactly n points.

This algorithm gives as output the manifold $W \cup_\varphi h$, where h is a 2–handle which is attached to W over the attaching region V and whose attaching sphere has framing datum associated with the element n of $\pi_1(O(2))$ with respect to the 0–framing of z.

First, we detail the completion of an open solid torus into a 4–ball. Let V be a triangulated open solid torus, and λ a triangulated longitude of V. Then $\partial V|\lambda$ is the boundary torus of V cut along the curve λ. This object is an annulus A whose boundary circle have the same triangulation C. The cone $C(A)$ on A is not a manifold, but the only place at which $C(A)$ is degenerate is the coning point. We fix this by letting B be the 3–ball triangulated by a number of tetrahedron equal to the number of edges is C, each of which share a common edge e. Then $\texttt{lk}(()e)$ in B is C, and the boundary of B is made of two triangulated discs which are equal to the cone on C. We attach B to $C(A)$ in the obvious way. The result is the open solid torus U which has boundary whose triangulation is equal to the triangulation of the boundary of V. The curve λ of U is a meridian of U, so it bounds a disc inside of U. Gluing U to V along their identical boundaries produces a 3–sphere $sph(V, \lambda) = U \cup V$. Then the 0–framing of the core of V in $sph(V, \lambda)$ is given by λ, as V is unknotted in $sph(V, \lambda)$, and λ bounds a Seifert surface which is a disc in $sph(V, \lambda)$. The cone on $sph(V, \lambda)$, $C(sph(V, \lambda))$, is a 4–ball whose prescribed 2–handle structure is defined by a core, which is the Seifert surface disc that λ bounds, and attaching region V. Then we denote $C(sph(V, \lambda))$ by $cmpl(V, \lambda)$ and call this object the *4–dimensional 2–handle of the pair* (V, λ).

With the input data, we attach the 4–dimensional 2–handle $cmpl(V, N)$ to M via the attaching map that identifies the copy of V in $cmpl(V, N)$ with the copy V in M.

Chapter 7

Conclusion

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