ENGINEERING MATH IV 06/09/2022

Power Series Solutions for differential Equations

① $y = e^{ax}$

$$\frac{dy}{dx} = y' = ae^{ax}$$

$$\frac{d^2y}{dx^2} = y'' = a^2 e^{ax}$$

$$\frac{d^3y}{dx^3} = y''' = a^3 e^{ax}$$

② $y = \sin ax$

$y' = a\cos ax = a\sin(ax + \pi/2)$

from trigonometric identities:

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$y' = a\cos ax = a\sin(ax + \pi/2)$

$\sin ax \cos \frac{\pi}{2} + \cos ax \sin \frac{\pi}{2}$

$= \cos ax$

$y'' = -a^2 \sin ax = -a^2 \sin(ax + 2\pi/2)$

$y''' = -a^3 \cos ax = -a^3 \sin(ax + 3\pi/2)$

$y^n = a^n \sin(ax + \frac{n\pi}{2})$

③ $y = \cos ax$

$y' = -a\sin ax = -a\cos(ax + \pi/2)$

$y'' = -a^2\cos ax = -a^2\cos(ax + 2\pi/2)$

$y''' = a^3 \sin ax = a^3\cos(ax + 3\pi/2)$

$y^n = a^n \cos(ax + n\pi/2)$

④ $y = \ln ax$

$y' =$

$y'' =$

$y''' =$

(5) $y = a^x$

$y' = x a^x$

$y'' = x^2 a^x$

$y''' = x^3 a^x$

(6) $y = \sinh ax$

$y' =$

$y'' =$

$y''' =$

(7) $y = \cosh ax$

$y' =$

$y'' =$

$y''' =$

Leibniz Theorem

$y = uv$

$y' = uv' + vu'$

$y'' = uv'' + v'u' + vu'' + u'v'$

$y'' = uv'' + 2v'u' + vu''$

$y''' = uv''' + u'v'' + 2[v'u'' + v''u'] + vu'''$ $+vu$

$y''' = uv''' + u'v'' + 2v'u'' + 2u''u' + vu''' + v'u''$

$y''' = uv''' + 3uv'' + 3v'u'' + vu'''$

$y^n = (uv)^n$

$= u^n v + n u^{(n-1)} v' + \frac{n(n-1)}{2!} u^{(n-2)} v^2 + \frac{n(n-1)(n-2)}{3!} u^{(n-3)} v^3 +$

Given $y = x^2 e^{3x}$

$v = x^2$

$u = e^{3x}$

$v' = 2x, \quad v^2 = 2, \quad v^3 = 0$

$u' = e^{3x}, \quad u^2 = 9e^{3x}, \quad u^3 = 27e^{3x}$

When $n = 3$ substitute into the eqn

$$y^n = u^n v + n u^{(n-1)} v' + \frac{n(n-1)}{2!} u^{(n-2)} v^2 + \frac{n(n-1)(n-2)}{3!} u^{(n-3)} v^3$$

$$y^3 = (e^{3x})^3 x^2 + 3(e^{3x})^{(3-1)} 2x + \frac{6}{2!}(e^{3x}) 2 + \frac{6}{3!}(e^{3x})^{3-3} + 0$$

$$y^3 = e^{3x})^3 x^2 + (e^{3x})^2 (x + 6(e^{3x})$$

$$y^3 = e^{3x}(e^{3x})^2 x^2 + (e^{3x}) 6x + 6$$

$$= e^{3x}(x e^{3x})^2 + 6(x e^{3x}) + 1$$

$\Rightarrow$ Given $y = x^4 \sin x$

find the 4th derivative and substitute it into the equation

Leibniz - Maclaurin's theorem

Maclaurin's theorem

$$y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + \frac{x^3}{3!}(y''')_0 + \frac{x^4}{4!}(y'''')_0 + \cdots \text{nth}$$

$+ v y'''$

The Steps

1. Differenciate the given equation nth term using the Leibniz theorem
2. Rearrange the result to obtain the recurrent relation at $x = 0$
3. Determine the values of the derivatives at $x = 0$ i.e $(y)_0$, $(y')_0$, $(y'')_0$ ...
4. Substitute in the maclaurin expansion for $y = f(x)$
5. Simplify the result where possible and apply

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$v^3 + \ldots$

boundary Conditions if given.

Determine the power Series Solution of the differential equation $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = 0$ using maclaurin's theorem method given boundaries Condition $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$.

Solution

$y'' + xy' + 2y = 0$

$y^2 + xy' + 2y = 0$

$y^{(n+2)} + xy^{(n+1)} + (n+2)y^n = 0$

$x = 0$

$y^{n+2} + (n+2)y^{(n)} = 0$

$y^{n+2} = -(n+2)y^n = 0$

$n = 0$

$y^2 = (y'')_0 = -2(y)_0$

$n = 1$

$y^3 = (y''')_0 = -3(y')_0$

$n = 2$

$y^4 = (y'''')_0 = -4(y'')_0 = -4(-2(y)_0)$

$= 8(y)_0$

$n = 3$

$y^5 = (y^5)_0 = -5(y''')_0 = -5(-3(y')_0)$

$= 15(y')_0$

$n = 4$

$y^6 = (y^6)_0 = -6(y'''')_0 = -6(-4(y'')_0)$

$= -6(8(y)_0)$

$= -48(y)_0$

$n=5$

$y^7 = (y^7)_0 = -7(y^{vii})_0 = -7(15(y')_0$

$= -7 \times 15(y')_0$

$= -105(y')_0$

$n=6$

$y^8 = (y^8)_0 = -8(y^{''''''})_0 = -7(-48(y)_0)$

$= 336(y)_0$

Now substitute the values into maclaurin's theorem

$$y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(y'')_0 + \frac{x^3}{3!}(y''')_0 + \frac{x^4}{4!}(y'''')_0 + \frac{x^5}{5!}(y''''')_0 + \frac{x^6}{6!}(y'''''')_0$$

Substituting we have

$$y = (y)_0 + x(y')_0 + \frac{x^2}{2!}(-2(y)_0) + \frac{x^3}{3!}(-3(y')_0) + \frac{x^4}{4!}(8(y)_0) + \frac{x^5}{5!}(15(y')_0) + \frac{x^6}{6!}(-48(y)_0)$$

$$y = (y)_0 + x(y')_0 - x^2(y)_0 - \frac{x^3}{2}(y')_0 + \frac{x^4}{3}(y)_0 + \frac{x^5}{8}(y')_0 - \frac{x^6}{15}(y)_0$$

Collecting like terms.

$$y = (y)_0\{1 - x^2(y)_0 + \frac{x^4}{3}(y)_0 - \frac{x^6}{15}(y)_0 + x(y')_0 - \frac{x^3}{2}(y')_0 + \frac{x^5}{8}(y')_0]$$

Inputting the boundary conditions.

at $x = 0$

$(y)_0 = 1$

$(y')_0 = 2$

$$y = \left[1 - x^2 + \frac{x^4}{3} - \frac{x^6}{15} + 2x - \frac{2x^3}{2} + \frac{2x^5}{8}\right]$$

$\Rightarrow \quad y'' + y' + xy$

Power Series Solution by Frobenius 07/09/2022

$y'' + Py' + Qy = 0$

where P and Q are both functions of x.

Step I

Assume a trial solution of the form $y = x^c[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_r x^r + \ldots]$

Step II

Differentiate the trial solution.

Step III

Substitute the results in the given differential equation.

Step IV

Equate coefficients of corresponding powers of the variable on each side of the equation. Obtain the equation which enables us to form $y = x^c[\ldots\ldots$

Example.

Determine the ~~equation which enables us form~~ general power series solution using the differential equation.

$$3x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$$

$$3xy'' + y' - y = 0$$

i. $y = x^c[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_r x^r$

$= a_0 x^c + a_1 x^{c+1} + a_2 x^{c+2} + a_3 x^{c+3} + \ldots + a_r x^{c+r} + \ldots]$

(ii) $y' = a_0 c x^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + a_3(c+3)x^{c+2} + \ldots + a_r(c+r)x^{c+r-1} + \ldots$

(iii) $y'' = a_0 c(c-1)x^{c-2} + a_1 c(c+1)x^{c-1} + a_2(c+2)x^c + a_3(c+2)(c+3)x^{c+1} + \ldots + a_r(c+r)(c+r-1)x^{c+r-2}\ldots$

$3xy''$	$3a_0c(c-1)x^{c-1} + 3a_1c(c+1)x^c + 3a\,c(c+1)(c+2)$ $x^{c+1} + 3a_3(c+2)(c+3)x^{c+2} + \dots + 3a_r(c+r)$ $(c+r-1)x^{c+r-1}$
y'	$a_0cx^{c-1} + a_1(c+1)x^c + a_2(c+2)x^{c+1} + a_3(c+3)$ $x^{c+2} + \dots + a_r(c+r)x^{c+r-1}$
$-y$	$-a_0x^c - a_1x^{c+1} - a_2x^{c+2} - a_3x^{c+3} + \dots$ $a_rx^{c+r} + \dots$

$3a_0c(c-1) + a_0c = 0 \rightarrow ① \rightarrow$ Indicial equation

$3a_0c^2 - 3a_0c + a_0c = 0$

$3a_0c^2 - 2a_0c = 0$

$3a_0c(c - 2/3) = 0$

$3a_0c = 0$ or $c - 2/3 = 0$

$c = 0$ or $c = 2/3$

Equating Coefficients.

$3a_1c(c+1 + a_1(c+1) - a_0 = 0$

$3a_1(3c^2 + 3c + c + 1) - a_0 = 0$

$3a_1(3c^2 + 4c + 1) - a_0 = 0$

$a_1(3c+1)(c+1) - a_0 = 0$ --- ②

Equate Coefficients of x^{c+r}

at $r \rightarrow r+1$

$3a_{r+1}(c+r+1)(c+r) + a_{r+1}(c+r+1) - a_r$

$a_{r+1}[(c+r+1)(3c+3r+1)] - a_r = 0$ --- ③

When $c = 0$

from eqn ②

$a_1 - a_0 = 0$

$a_1 = a_0$

from eqn ③

$a_{r+1}[(r+1)(3r+1)] - a_r = 0$

$a_{r+1}(r+1)(3r+1) = a_r$

$a_{r+1} = \dfrac{a_r}{(r+1)(3r+1)}$

wher $r = 1$

$$a_2 = \frac{a_1}{2\times 4} = \frac{a_0}{2\times 4} = \frac{a_0}{8}$$

when $r = 2$

$$a_3 = \frac{a_2}{3\times 7} = \frac{a_0}{2\times 3\times 4\times 7} = \frac{a_0}{168}$$

when $r = 3$

$$a_4 = \frac{a_3}{4\times 10} =$$