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Course Outline

[MM,DD,YY
9/06/22.]

- Multiple integral and their application.
- Differential of Integral
- Line integral
- Analytical functions of complex Variables.
- Second Order differential equation
- Transformation and mapping of Special functions.

Multiple integral and their applications

Double integral.

$\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy$ is a two integral and it is appraise or assess from the inside Outside.

$$\underbrace{\int_{y_1}^{y_2}\underbrace{\left[\int_{x_1}^{x_2} f(x,y)\,dx\right]}_{①} dy}_{②}$$

It can also be in the form $\int_{y_1}^{y_2} dy \int_{x_1}^{x_2} f(x,y)\,dx$

$$\int_{y_1}^{y_2}\left[\int_{x_1}^{x_2} f(x,y)\,dx\right] dy$$

Example 1

$$\int_1^3\int_0^2 (y^3 - xy)\,dy\,dx$$

Solution.

$$\int_1^3 \left[\int_0^2 (y^3 - xy)\,dy\right] dx.$$

$$\int_1^3 \left[\frac{y^4}{4} - \frac{xy^2}{2}\right]_0^2 dx.$$

$$=\int_1^3 \left[\frac{2^4}{4} - \frac{x(2)^2}{2}\right] dx$$

$$=\int_1^3 \left[\cancel{\frac{16}{4}}\ \frac{16}{4} - 2x\right] dx$$

$$=\int_1^3 (4-2x)\,dx$$

$$=\left[4x + \frac{2x^2}{2}\right]_1^3$$

$= [4x - x^2]_1^3$

$= [4(3) - (3)^2] - [4(1) - (1)^2]$

$(12-9) - (4-1)$

$= 3 - (3)$

$= 0$

2. $\int_0^a dx \int_0^{y_1} (x-y)\,dy$ where $y = \sqrt{a^2 - x^2}$

Solution.

$\int_0^a dx \int_0^{y_1} (x-y)\,dy$

$= \int_0^a \left[xy - \frac{y^2}{2}\right]_0^{y_1} dx$

$= \int_0^a \left(xy_1 - \frac{y_1^2}{2}\right) dx$

$\left[\frac{x^2 y_1}{2} - \frac{x y_1^2}{2}\right]_0^a$

$y = \sqrt{a^2 - x^2}$

$\left[\frac{x^2\sqrt{a^2-x^2}}{2} - \frac{x(\sqrt{a^2-x^2})^2}{2}\right]_0^a$

$= \frac{a^2\sqrt{a^2-a^2}}{2} - \frac{a(a^2-a^2)}{2}$

$= \frac{a^2\sqrt{0}}{2} - \frac{a(0)}{2}$

$= 0$

3. $\int_0^{\sqrt{3}+2} \int_0^{\pi/3} (2\cos\theta - 3\sin 3\theta)\,d\theta\,dr$

Solution.

$\int_0^{\sqrt{3}+2} \left[2\sin\theta + \frac{3\cos 3\theta}{3}\right]_0^{\pi/3} dr$

$= \int_0^{\sqrt{3}+2} [2\sin \pi/3 + \cos 3(\pi/3)]\,dr - \int_0^{\sqrt{3}+2} [2\sin 0 + \cos 0]\,dr$

$\cancel{\int_0^{\sqrt{3}+2} (2\sin \pi/3 + \cos \pi)\,dr}$

$= \int_0^{\sqrt{3}+2} [(2\sin \pi/3 + \cos \pi) - (2\sin 0 + \cos 0)]\,dr$

$= \int_0^{\sqrt{3}+2} \left[\left(2\left(\frac{\sqrt{3}}{2}\right) + (-1)\right) - 1\right] dr$

$= \int_0^{\sqrt{3}+2} [$

$= (\sqrt{3} -$

$= 3 -$

$= -1$

N.B Double inte

Find the ar

and $y_2 =$

Point o

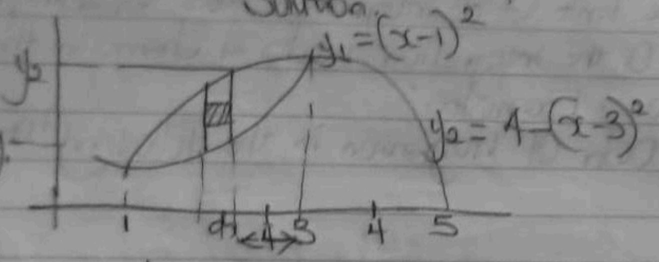
$$= \int_0^{\sqrt{3}+2} \left[\sqrt{3} - 1 - 1 \right] dx$$
$$= \int_0^{\sqrt{3}+2} (\sqrt{3}-2)\,dx$$
$$= \left[(\sqrt{3}-2)x \right]_0^{\sqrt{3}+2}$$
$$= (\sqrt{3}-2)(\sqrt{3}+2)$$
$$= 3 - 4$$
$$= \underline{\underline{-1}}$$

N.B Double integration can be applied in Area

Application of double integral

Find the area of a plane bounded by the circles $y_1 = (x-1)^2$ and $y_2 = 4-(x-3)^2$ at the point of intersection $y_1 = y_2$

Solution



Point of Intersection.

$$y_1 = y_2$$
$$(x-1)^2 = 4-(x-3)^2$$
$$x^2 - 2x + 1 = 4 - (x^2 - 6x + 9)$$
$$x^2 - 2x + 1 = 4 - x^2 + 6x - 9$$
$$2x^2 - 8x + 6 = 0$$
$$2x^2 - 2x - 6x + 6 = 0$$
$$2x(x-1) - 6(x-1) = 0$$
$$(2x-6)(x-1) = 0$$
$$2x - 6 = 0 \text{ or } x - 1 = 0$$
$$x = \frac{6}{2} = 3 \text{ or } x = 1$$

$$\int_{x=1}^{x=3} \int_{y_1}^{y_2} dy\,dx = \int_1^3 \int_{(x-1)^2}^{4-(x-3)^2} dy\,dx$$
$$= \int_1^3 \left[4-(x-3)^2 - (x-1)^2 \right] dx$$
$$= \int_1^3 \left[4-(x^2-6x+9) - (x^2-2x+1) \right] dx$$
$$= \int_1^3 \left[4 - x^2 + 6x - 9 - x^2 + 2x - 1 \right] dx$$
$$\int_1^3 \left[-2x^2 + 8x - 6 \right] dx$$
$$-2 \int_1^3 \left[x^2 - 4x + 3 \right] dx$$

$$-2\left[\frac{x^3}{3}-2x^2+3x\right]_1^3$$
$$= 2^2/_3 \text{ Units}^2$$

Change in the Order Of Integration.

A double integral, the limit Of its integration remain Constant while the Order Of Integration is extraneous provided, the limit of its integration change, appropriately.

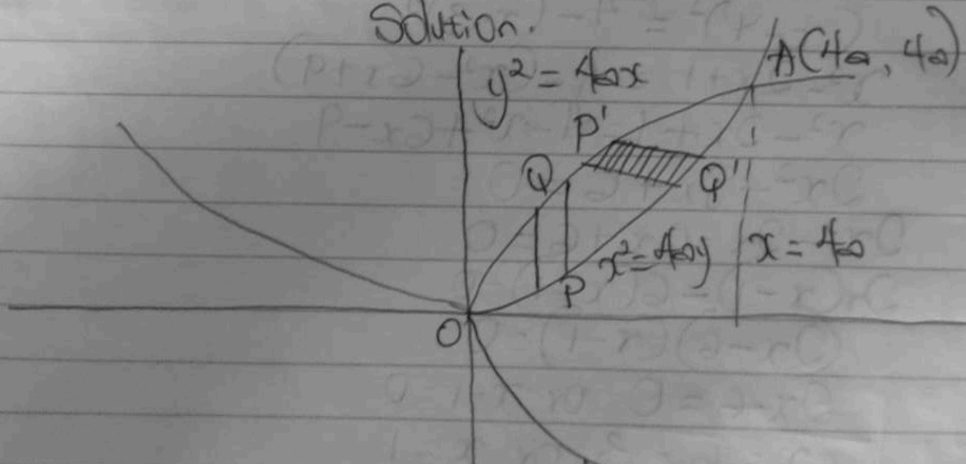
$$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx$$

But if the limit Of its integration are Variables, a change in the Order Of the integration needs a Change in the limit of the integration. Examples:

* Change the Order Of Integration in the following integrals and Evaluate.

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx.$$

Solution.



$y = \frac{x^2}{4a}$ to $y = 2\sqrt{ax}$ and x Varies from $x = 0$ to $x = 4a$

The intersection is first Performed along the Vertical Strips of PQ Which extends along the Parabola.

Hence;

$$y = x^2/4a \Rightarrow x^2 = 4ay$$
$$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax.$$

To change the Order of integration, we divide the region of intersection OPQ/O OPAQO into horizontal strips and we have P'Q' which extend to the vertical PQ. The strips slide from 4a to 0.

i.e $y^2 = 4ax \Rightarrow x = \frac{y^2}{4a}$ to Q' on the parabola

$x^2 = 4ay \Rightarrow x = 2\sqrt{ay}$.

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx = \int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy$$

$$= \int_0^{4a}\left[x\right]_{y^2/4a}^{2\sqrt{ay}} dy = \int_0^{4a}\left(2\sqrt{ay} - \frac{y^2}{4a}\right)dy$$

$$= \left[2\sqrt{a}\cdot\frac{y^{3/2}}{3/2} - \frac{y^3}{12a}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}\cdot(4a)^{3/2}}{3} - \frac{(4a)^3}{12a}$$

$$= \frac{4(8)a^{1/2+3/2}}{3} - \frac{64a^3}{12a}$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

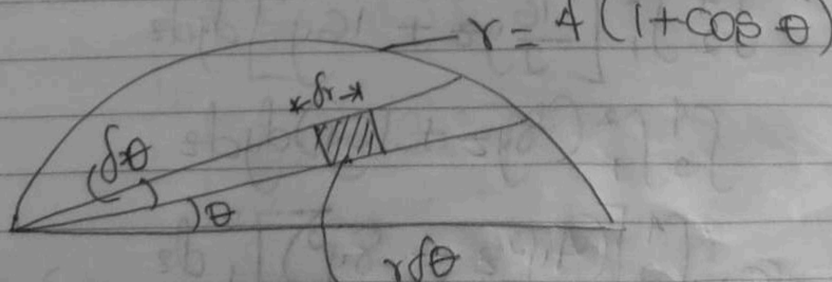
$$= \frac{16a^2}{3}$$

$$= 5\tfrac{1}{3}a^2$$

Polar Coordinates:

Utilizing the double integral, find the area enclosed at $r = 4(1+\cos\theta)$ and the radius at $\theta = 0$ and $\theta = \pi$.

$\Rightarrow$ $r = 4(1+\cos\theta)$ at $\theta = 0$ and $\theta = \pi$.



to

$\theta = \pi \quad r = r_1$

$$A = \sum_{\theta=0}\sum_{r=0} r\,\delta\theta\,\delta\theta$$

$$= \int_0^{\pi}\int_0^{r_1} r\,dr\,d\theta$$

But $r_1 = f(\theta) = 4(1+\cos\theta)$

$= \int_0^\pi \left[\frac{r^2}{2}\right]_0^{r_1} d\theta$

$= \int_0^\pi 8(1+\cos\theta)^2\, d\theta$

$= \int_0^\pi 8(1+2\cos\theta+\cos^2\theta)\, d\theta$

$= 8\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^\pi$

$8[\pi + \pi/2] - (0)$

$= 8\pi + 4\pi = 12\pi$

Triple Integral:

$$I = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dx\,dy\,dz$$

$$\boxed{\int_a^b \boxed{\int_c^d \boxed{\int_e^f f(x,y,z)\,dx}_{1}\,dy}_{2}\,dz}_{3}$$

Evaluate

$$I = \int_2^4 \int_1^2 \int_0^4 xy(z+2)\,dx\,dy\,dz$$

Solution

$\int_2^4 \int_1^2 \int_0^4 (xyz + 2xy)\,dx\,dy\,dz$

$\int_2^4 \int_1^2 \left[\frac{x^2yz}{2} + x^2y\right]_0^4 dy\,dz$

$\int_2^4 \int_1^2 \left[\frac{16}{2}yz + 16y\right] dy\,dz$

$\int_2^4 \int_1^2 (8yz + 16y)\,dy\,dz$

$\int_2^4 \left[(4y^2z + 8y^2)\right]_1^2 dz$

$\int_2^4 \left[(4(2)^2z + 8(2)^2) - (4(1)^2z + 8(1)^2)\right] dz$

$\int_2^4 (16z + 32 - (4z + 8))\,dz$

$$= \int_2^4 (16z + 32 - 4z - 8)\,dz$$

$$\int_2^4 (12z + 24)\,dz$$

$$\left[6z^2 + 24z\right]_2^4$$

$$6(4)^2 + 24(4) - (6(2)^2 + 24(2))$$

$$= 6(16) + 96 - (24 + 48)$$

$$= 96 + 96 - 72$$

$$= 96 + 24$$

$$= \underline{\underline{120}}$$

Application Of Triple integral:

Unlike the double integral, it is applied to volume

* A Solid is enclosed by the plane $z=0$, $x=0$, $x=3$, $y=1$, $y=3$ and the Surface $z = x^2 + xy$, Calculate the Volume.

$$\int_0^3 dx \int_1^3 dy \int_0^{x^2+xy} dz$$

Solution.

$$= \int_0^3 dx \int_1^3 dy\,(x^2 + xy)$$

$$= \int_0^3 dx \left[x^2y + \frac{xy^2}{2}\right]_1^3$$

$$= \int_0^3 \left[\left(x^2(3) + \frac{x(3)^2}{2}\right) - \left(x^2(1) + x(1)^2/2\right)\right] dx$$

$$= \int_0^3 \left(3x^2 + \frac{9}{2}x - \left(x^2 + \frac{x}{2}\right)\right) dx$$

$$= \int_0^3 \left(3x^2 + \frac{9}{2}x - x^2 - \frac{x}{2}\right) dx$$

$$= \int_0^3 \left(2x^2 + \frac{8x}{2}\right) dx$$

$$= \int_0^3 (2x^2 + 4x)\,dx$$

$$= \left[\frac{2}{3}x^3 + 2x^2\right]_0^3$$

$$= \frac{2(3)^3}{3} + 2(3)^2$$

$$= 2(3)^2 + 2(9)$$

$$= 18 + 18 = \underline{\underline{36}} \text{ Units}^3$$

Polar Cordinate:

$$\int_0^{\pi}\int_0^{\pi/2}\int_0^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{\pi}\int_0^{\pi/2}\left[\frac{r^3}{3}\right]_0^{r} \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{\pi}\int_0^{\pi/2} \frac{r^3}{3} \sin\theta \, d\theta \, d\phi$$

$$= \int_0^{\pi}\left[-\frac{r^3}{3}\cos\theta\right]_0^{\pi/2} d\phi$$

$$\int_0^{\pi}\left[-\frac{r^3}{3}\cos\pi/2 - \left(-\frac{r^3}{3}\cos 0\right)\right] d\phi$$

$$\int_0^{\pi}\left(-\frac{r^3}{3}(0) + \frac{r^3}{3}\right) d\phi$$

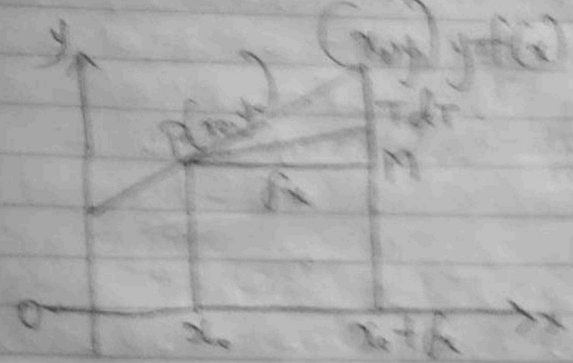
$$= \int_0^{\pi} \frac{r^3}{3} \, d\phi$$

$$= \left[\frac{r^3}{3}\phi\right]_0^{\pi}$$

$$= \frac{r^3 \pi}{3} = \frac{\pi r^3}{3}$$

Differentials.

This is normally used to denote small increments in the values of variables.
This technique makes the writing of the calculus integral very short.



Example

* $y = x^5$
  $dy = 5x^4 dx$.

* $y = \sin 3x$
  $dy = 3\cos 3x\,dx$

* $y = e^{4x}$
  $dy = 4e^{4x} dx$.

* $y = \cosh 2x$
  $dy = 2\sinh 2x\,dx$.

* $y = e^{2x} \sin 4x$
  U V → Product rule

$\frac{dy}{dx} = V\frac{du}{dx} + U\frac{dv}{dx}$.

$U = e^{2x}$

$\frac{du}{dx} = 2e^{2x}$

$V = \sin 4x$

$\frac{dv}{dx} = 4\cos 4x$

$\frac{dy}{dx} = \sin 4x(2e^{2x}) + e^{2x}(4\cos 4x)$

$$\frac{dy}{dx} = e^{2x}(2\sin 4x + 4\cos 4x)$$

$$dy = 2e^{2x}(\sin 4x + 2\cos 4x)\,dx.$$

* $y = \frac{\cos 2t}{t^2} = \frac{u}{v}$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ → Quotient rule.

$u = \cos 2t$

$\frac{du}{dt} = -2\sin 2t$

$v = t^2$

$\frac{dv}{dt} = 2t$

$$\frac{dy}{dt} = \frac{t^2(-2\sin 2t) - \cos 2t(2t)}{(t^2)^2}$$

$$d\left(\frac{u}{v}\right) = \frac{t[(-2t\sin 2t) - 2\cos 2t]}{t^4}$$

$$dy = \frac{-2(t\sin 2t + \cos 2t)\,dt}{t^3}.$$

$$dy = \frac{-2}{t^3}(t\sin 2t + \cos 2t)\,dt.$$

Integration Of two Or more Variables.

1 $z = x^3 \sin 2y$

2 $z = (2x-1)e^{3y}$

3 $z = x^3 y^2 w$

$= f(x,y)$

$= \frac{\partial z}{\partial x}\partial x + \frac{\partial z}{\partial y}\partial y$

$$\partial z = \frac{\partial z}{\partial x}\partial x + \frac{\partial z}{\partial y}\partial y$$

Similarly,

$z = f(x,y,w)$

$$\partial z = \frac{\partial z}{\partial x}\partial x + \frac{\partial z}{\partial y}\partial y + \frac{\partial z}{\partial w}\partial w$$

1. $\partial z = 3x^2 \sin 2y \, \partial x + 2x^3 \cos 2y \, \partial y$
$= x^2(3 \sin 2y \, \partial x + 2x \cos 2y \, \partial y)$.

Integration Of Exact differentials.

$\partial z = P\partial x + Q\partial y$

where $P = \frac{\partial z}{\partial x}$ $Q = \frac{\partial z}{\partial y}$

$z = \int P \partial x$ and $z = \int Q \partial y$

$\partial z = (6x^2 + 8xy^3)\partial x + (12x^2y^2 + 12y^3)\partial y$

$P = \frac{\partial z}{\partial x} = 6x^2 + 8xy^3$.

* $z = \int (6x^2 + 8xy^3)\partial x = \frac{6x^3}{3} + \frac{8x^2y^3}{2} = 2x^3 + 4x^2y^3$

$y = \frac{\partial z}{\partial y} = 12x^2y^2 + 12y^3$

$z = \int (12x^2y^2 + 12y^3)\partial y$

$= \frac{12x^2y^3}{3} + \frac{12y^4}{4}$

* $= 4x^2y^3 + 3y^4$ {H. C. F}

$z = 2x^3 + 4x^2y^3 + 3y^4$.

$\partial z = (\sinh y + y \sinh x)\partial x + (x\cosh y + \cosh x)\partial y$.

$P = \frac{\partial z}{\partial x}$ * $z = \int (\sinh y + y \sinh x)\partial x$.

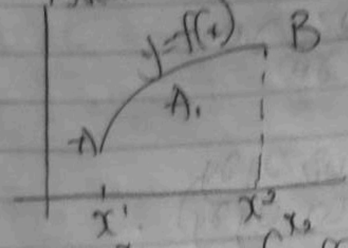
$= x\sinh y + y\cosh x$

$Q = \frac{\partial z}{\partial y}$; $z = \int (x\cosh y + \cosh x)\partial y$

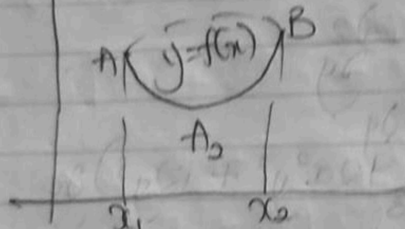
* $= x\sinh y + \sinh xy \cdot x\cosh x$

$z = x\sinh y + y\cosh x$.

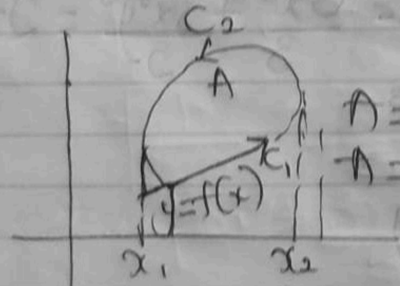
Area enclosed by a Circle.



$$A_1 = \int_{x_1}^{x_2} y\,dx = \int_{x_1}^{x_2} f(x)\,dx$$



$$A_2 = \int_{x_1}^{x_2} f(x)\,dx.$$



$$A = A_1 - A_2.$$

$$A = \int_{x_1}^{x_2} f(x)\,dx - \int_{x_1}^{x_2} f(x)\,dx.$$

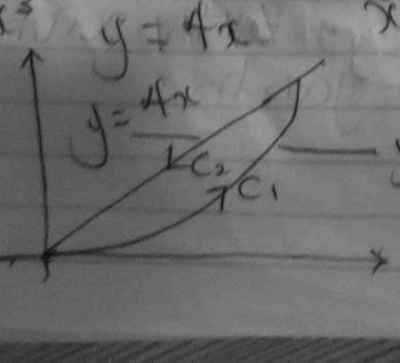
$$A = -\oint y\,dx.$$

$$= -\left[\int_{x_1}^{x_2} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx\right]$$

Determine the area enclosed by the graph $y = x^3$ and $y = 4x$ for $x \geq 0$

Solution.

$y = x^3 \quad y = 4x \quad x \geq 0$



NB $C = \text{Curve}\{C_1, C_2\}$

$C_1 \Rightarrow y = x^3 \Rightarrow x = 0,\ x = 2$

$C_2 \Rightarrow y = 4x \Rightarrow x = 2,\ x = 0$

$A = -\oint y\,dx = -\left[\int_0^2 x^3\,dx + \int_2^0 4x\,dx\right]$

$= -\left[\left[\frac{x^4}{4}\right]_0^2 + \left[\frac{4x^2}{2}\right]_2^0\right]$

$= -\left[\frac{2^4}{4} + \left[2(0)^2 - 2(2)^2\right]\right]$

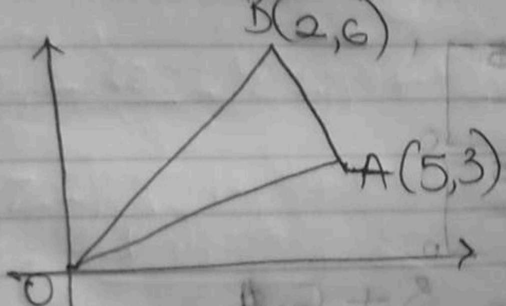
$= -[4-8]$

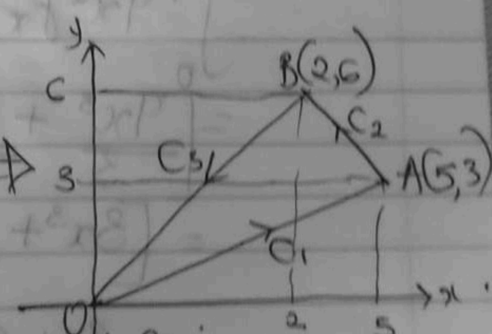
$= -4+8$

$= \underline{4}$

Find the area Of a Triangle With Vertices $(0,0)$, $(5,3)$ and $(2,6)$.

Solution.





$y = mx + c$

$C_1 = OA: y = \frac{3}{5}x$.

$C_2 = BA: y = 8 - x$

$C_3 = OB: y = 3x$

$A = -\oint y\,dx$.

$= -\left[\int_0^5 \frac{3}{5}x\,dx + \int_5^2 (8-x)\,dx + \int_2^0 3x\,dx\right]$

$= \underline{12}$

Line Integral

In line integral, we express the line in:

$$I = \int_{AB} f_t \, ds = \int_c f_t \, ds.$$

$$= \int f_t \, ds = \int_c (P\,dx + Q\,dy)$$

Evaluate

$$I = \int_c [(x^2 + 2y)\,dx + xy\,dy]$$

$O(0,0)$ + $B(1,4)$ along the curve $y = 4x^2$.

Solution.

$$I = \int_c [(x^2 + 2y)\,dx + xy\,dy]$$

$y = 4x^2 \quad dy = 8x\,dx.$

$$= \int_c [x^2 + 2(4x^2)]\,dx + x(4x^2)(8x)\,dx]$$

$$\int_0^1 9x^2\,dx + 32x^4\,dx.$$

$$= \left[\frac{9x^3}{3} + \frac{32x^5}{5}\right]_0^1$$

$$= \left[3x^3 + \frac{32x^5}{5}\right]_0^1$$

$$= 3 + \frac{32}{5} = 3 + 6.4$$

$$= 9.4$$

Solve.

$I = \int_c (x^2 + 2y)\,dy + xy\,dy$ from $O(0,0)$ to $A(1,0)$ along line $y = 0$ and then from $A(1,0)$ to $B(1,4)$ along line $x = 1$.

Solution.

$$I_{OA} = \int_c (x^2 + 2y)\,dx + xy\,dy \quad \text{from } O(0,0) \text{ to } A(1,0)$$

line $y = 0$, $dy = 0$

$$\int_c [x^2 + 2(0)]\,dx + x(0)(0)]$$

$$= \int_c x^2 \delta x = \int_0^1 x^2 \delta x = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$I_{AB} = \int_c (x^2 + 2y)\delta x + xy\delta y$ from $A(1,0)$ to $OB(1,4)$

$x = 1$, $\delta x = 0$

$$\int_c \left((1)^2 + 2y\right)0 + 1(y)\delta y$$

$$\int_0^4 y\delta y = \left[\frac{y^2}{2}\right]_0^4 = \frac{4^2}{2}$$

$$= 8$$

$$I = I_{OA} + I_{AB}$$

$$= \frac{1}{3} + 8$$

$$= 8\tfrac{1}{3}$$

$= 8/3$

$\Rightarrow$

Properties of line integral

1. $\int_C f\,ds = \int_C (P\,dx + Q\,dy)$

2. $\int_{AB} F\,ds = -\int_{BA} f\,ds$ and $\int_{AB} (P\,dx + Q\,dy)$
$= -\int_{BA} (P\,dx + Q\,dy)$

That is the sign of a line integral is reverse when the direction of the integration along the line is reverse.

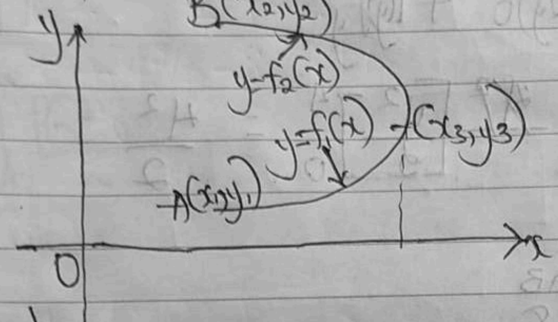
3(a) For a path of integration parallel to the y-axis i.e $x = k$
$dx = 0 \quad \therefore \int_C P\,dx = 0 \quad \therefore I_c = \int_C Q\,dy$

(b) For a path of integration parallel to the x-axis $y = k$;
$dy = 0 \quad \therefore \int_C Q\,dy = 0 \quad \therefore I_c = \int P\,dx$.

4 If the path of integration, C, joining A to B is divided into two parts AK and KB, then

$I_c = I_{AK} + I_{KB}$

5 If the function y is equal to x; $y=f(x)$ that describes the path of integration C is not single value for part of its extent the path is divided into two sections



Example

Solve $I = \int_c (x+y)\,dx$ from $A(0,1)$ to $B(0,-1)$ along the semi-circle $x^2+y^2=1$ for $x \geq 0$

Solution

$y^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2}$

$x^2 = 1 - y^2$

$x = \pm\sqrt{1-y^2}$

$y = \sqrt{1-x^2}$ from A to K

$y = -\sqrt{1-x^2}$ from K to B

$\int_c P\,dx = \int_0^1 (x+\sqrt{1-x^2})\,dx + \int_1^0 (x-\sqrt{1-x^2})\,dx$

$= \int_0^1 \left[(x+\sqrt{1-x^2}) - x + \sqrt{1-x^2}\right] dx$

$= 2\int_0^1 \sqrt{1-x^2}\,dx$

$x = \sin\theta$, $dx = \cos\theta\, d\theta$

$\Rightarrow \sqrt{1-x^2} = \cos\theta$

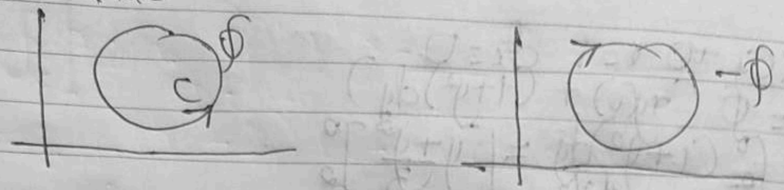
limit $x=0$; $\theta=0$; $x=1$; $\theta=\pi/2$

$I = 2\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$

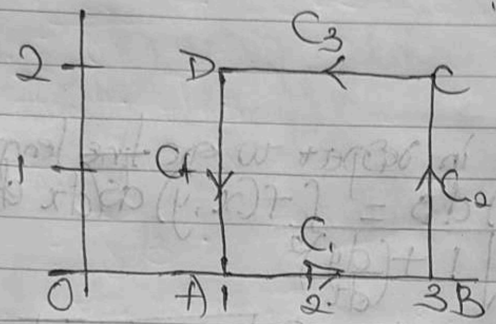
$= \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

$= \frac{\pi}{2} + 0 = \frac{\pi}{2}$

Line integral has a closed curve, the first curve moves in the direction of anti-clockwise with a positive value where the second curve moves in the direction of clockwise with a negative value



Solve $I = \oint (xy\,dx + (1+y^2)\,dy)$ where C is the boundary of the rectangle joining $A(1,0)$, $B(3,0)$, $C(3,2)$ and $D(1,2)$



$I_1 = AB$

$I_2 = BC$

$I_3 = CD$

$I_4 = DA$

(a) ABC, to $y=0$

$\therefore dy = 0$

$I_1 = \oint [x(0)\,dx + (1+(0)^2)]\,dy$

$I_1 = 0$

(b) $BC: C_2$ to $x = 3$

$dx = 0$

$I_2 = \oint (xy(0) + (1+y^2))\,dy$

$= \oint (1+y^2)\,dy$

$I_2 = \int_0^2 (1+y^2)\,dy$

$= \left[y + \frac{y^3}{3}\right]_0^2 = 4^0/3$

(c) $CD: C_3$ to $y = 2$

$dy = 0$

$I_3 = \oint (xy\,dx + (1+y^2)0)$

$= \oint (2x\,dx)$

$= \int_3^1 2x\,dx = [x^2]_3^1$

$= -8$

(d) DA: C_4 to $x=1$ $dx=0$

$$I_4 = \oint (x,y(0) + (1+y^2)dy)$$

$$= \int_2^0 (1+y^2)dy = \left[y + \frac{y^3}{3}\right]_2^0$$

$$= -4^2/3$$

$$I = I_1 + I_2 + I_3 + I_4$$

$$= 0 + 4^2/3 - 8 - 4^2/3$$

$$= -8$$

The line integral in respect to arc the length

$$I = \int_C f(x,y)\,ds = \int_C f(x,y)\,ds/dx\,dx.$$

where $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\int_C f(x,y)\,ds = \int_{x_1}^{x_2} f(x,y)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

The path length of a parabola by $y = x^2$ between values $x=0$ and $x=2$ is given by the integral:

$$I = \int_C ds$$

$$= \int_0^2 \sqrt{1+(dy/dx)}\,dx$$

$y = x^2$

Where $\frac{dy}{dx} = 2x$

$$= \int_0^2 \sqrt{1+2x}\,dx$$

let $u = 1+2x$.

$$\frac{du}{dx} = 2$$

$du = 2dx \Rightarrow \frac{1}{2}du = dx$

$$\int_0^2 U^{1/2}\left(\frac{1}{2}du\right)$$

$$\frac{1}{2}\int_0^2 U^{1/2}\,du$$

$$\frac{1}{2}\left[\frac{2U^{3/2}}{3}\right]_0^2$$

$$\frac{1}{2}\left[\frac{2(1+2x)^{3/2}}{3}\right]_0^2$$

Parametric equations.

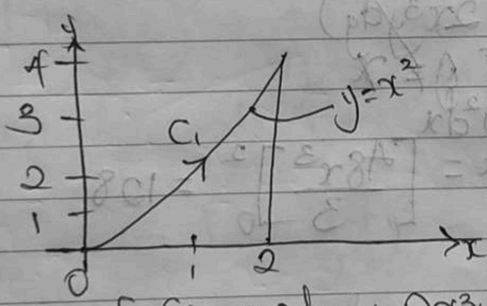
$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$I = \int_c f(x,y)\, ds = \int_{t_1}^{t_2} f(x,y) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example

$I = \oint_c [3x^2y^2dx + 2x^3y\,dy]$ between $O(0,0)$ and $A(2,4)$



$$I = \int_c (3x^2y^2dx + 2x^3y\,dy)$$

The path C_1 to $y = x^2$

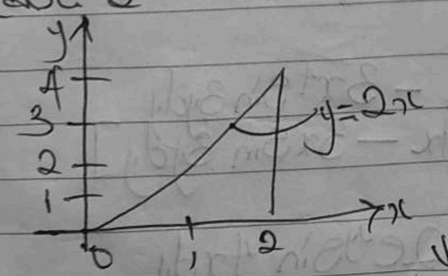
$dy = 2x\,dx$.

$$I_1 = \int_0^2 [3x^2(x^2)^2dx + 2x^3x^2\,\cancel{dy}^{\,2x\,dx}]$$

$$= \int_0^2 7x^6 dx = [x^7]_0^2$$

$$= 128$$

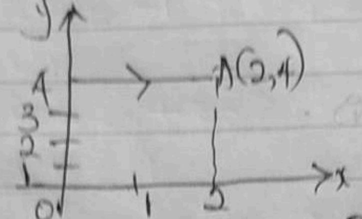
Case 2



$y = 2x \quad dy = 2$

$$I_2 = \int_0^2 (3x^2 4x^2 dx + 2x^3 2x\, 2dx)$$
$$= \int_0^2 20x^4 dx$$
$$= \left[\frac{20x^5}{5}\right]_0^2$$

Case 3



from $(0,0)$ to $(0,4)$
$x=0$
$dx=0$
$I_3 = 0$
from $(0,4)$ to $(2,4)$
$y=4$
$dy=0$
$$I = \oint_c (3x^2y^2 dx + 2x^3 y\, dy)$$
The path c_1 for $y \neq x$
$$= \oint_c 3x^2(4)^2 dx$$
$$\oint_c 48x^2 dx = \left[\frac{48x^3}{3}\right]_0^2 = 128$$

Exact differential

If $(Pdx + Qdy)$ is an exact differential
$$\oint (Pdx + Qdy) = 0$$

Example
(a) $z = x^4 \cos 3y$
(b) $z = e^{2y} \sin 4x$
(c) $z = x^2 y w^3$
(a) $dz = 4x^3 \cos 3y\, dx - 3x^4 \sin 3y\, dy$
$= x^3[4\cos 3y\, dx - 3x \sin 3y\, dy]$
(b) $dz = 4e^{2y} \cos 4x\, dx + 2e^{2y} \sin 4x\, dy$

$= 2e^{2y}[2\cos 4x\,dx + \sin 4x\,dy]$

(c) $dz = 2xyw^3dx + x^2w^3dy + 3x^2yw^2dw$

⇒ Integration Of Exact differentials.

Surface Integral.

$~~S = \int\int d\phi = \int\int~~$

$I = \int_S \phi(x,y,z)\,ds$

$I = \int_R\int \phi(x,y,z)\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}\,dxdy$

$z = F(x,y)$

$S = \int_S ds = \int_R\int\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}\,dxdy$

$z = \sqrt{x^2+y^2}$ Over the region bounded by $x^2+y^2=4$

$S = \int_R\int\sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2}\,dxdy$

$z = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$

$\frac{dz}{dx} = \frac{1}{2}(x^2+y^2)^{-1/2}\times 2x = \frac{x}{\sqrt{x^2+y^2}}$

$1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2 = 1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{x^2+y^2}\right)^2$

Where $\frac{dz}{dy} = \frac{1}{2}(x^2+y^2)^{-1/2}\times 2y = \frac{y}{\sqrt{x^2+y^2}}$

$= 1 + \frac{x^2+y^2}{x^2+y^2} = 1+1 = 2$

$\Rightarrow \sqrt{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2} = \sqrt{2}$

$S = \sqrt{2}\int_R\int dxdy$

But from Coordinate Geometry (Ellipse)

$x^2+y^2=4$

$\frac{x^2}{4}+\frac{y^2}{4}=1$

$a^2=4$ and $b^2=4$

$a=2$; $b=2$

$\Rightarrow A = \pi ab = \pi\times 2\times 2 = 4\pi$

$I = \sqrt{2}\pi$

$= 4\sqrt{2}\pi$

Green Theorem

$$\iint_R \left(-\frac{dP}{dy} - \frac{dQ}{dx}\right) dx dy = -\oint_C (P dx + Q dy)$$

$$I = \oint_C (2x-y) dx + (2y+x) dx$$

$$x^2 + 9y^2 = 16$$

$$I = \int_C (P dx + Q dy)$$

$$P = 2x - y$$

$$\frac{dP}{dy} = -1$$

$$Q = 2y + x$$

$$\frac{dQ}{dx} = 1$$

$$I = -\int_R \int (-1-1) dx dy$$

$$= 2 \iint_R dx dy$$

$\int_R \int dx dy$ = area of the region

$$I = 2A$$

$$x^2 + 9y^2 = 16$$

$$\frac{x^2}{16} + \frac{9y^2}{16} = 1$$

By comparison

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} =$$

$$a^2 = 16 \;;\; a = \sqrt{16} = 4$$

$$b^2 = \frac{16}{9} \;;\; b = \frac{4}{3} = 1\tfrac{1}{3}$$

$$A = \pi ab = \pi\left(4 \times \frac{4}{3}\right)$$

$$A = \frac{16}{3}\pi$$

$$I = 2A = 2A = 2 \times \frac{16}{3}\pi = \frac{32}{3}\pi$$

Double Integral

$x = f(u,v)$; $y = g(u,v)$

$$dA = \left|\frac{d(x,y)}{d(u,v)}\right| du\,dv \;;\; j(u,v) = \frac{d(x,y)}{d(u,v)}$$

$$= \begin{vmatrix} \frac{dx}{du} & \frac{dy}{du} \\ \frac{dx}{dv} & \frac{dy}{dv} \end{vmatrix}$$

$$= \int_R\int f(x,y)\,dx\,dy$$

$$= \int_R\int f(u,v), g(u,v) \left|\frac{d(x,y)}{d(u,v)}\right| du\,dv$$

For Triple integral

$x = f(u,v,w)$

$y = g(u,v,w)$

$z = h(u,v,w)$

$$j(u,v,w) = \frac{d(x,y,z)}{d(u,v,w)} = \begin{vmatrix} \frac{dx}{du} & \frac{dy}{du} & \frac{dz}{du} \\ \frac{dx}{dv} & \frac{dy}{dv} & \frac{dz}{dv} \\ \frac{dx}{dw} & \frac{dy}{dw} & \frac{dz}{dw} \end{vmatrix}$$

$$I = \int_R\int f(x,y,z)\,dx\,dy\,dz$$

$$= \int_R\int G(u,v,w) \left|\frac{d(x,y,z)}{d(u,v,w)}\right| du\,dv\,dw.$$

Express the Integral $I = \int_R\int (x-y)\,dx\,dy$ in the form uv where $x = u(1+v)$ and $y = u-v$

$x = u + uv$

$\frac{dx}{du} = 1+v$ $\frac{dx}{dv} = u$

$y = u - v$

$\frac{dy}{du} = 1$ $\frac{dy}{dv} = -1$

$$\therefore j(u,v) = \frac{d(x,y)}{d(u,v)} = \begin{vmatrix} dx/du & dy/du \\ dx/dv & dy/dv \end{vmatrix}$$

$$= \begin{vmatrix} 1+v & 1 \\ u & -1 \end{vmatrix}$$

$$= 1+v-(1-u)$$

$$= 1+u+v$$

But $x-y = u+uv-u+v$

$$= v(1+u)$$

$$I = \int_R \int v(1+u)(1+u+v)\,dv\,du$$

Assignment

1 $I = \iiint \left(\frac{x+z}{y}\right) dx\,dy\,dz$ in terms of u, v, w using the equations

$$x = u+v+w$$

$$y = v^2 w$$

$$z = u-w$$

Relationship between Cartesian and Cylindrical coordinates

$x = r\cos\theta \Rightarrow r = \sqrt{x^2+y^2}$

$y = r\sin\theta \Rightarrow \theta = \arctan(y/x)$

$z = z \Rightarrow z = z$

Assignment

2 Transform $(4, 2, 3)$ to Cylindrical coordinate

Relationship between Cartesian and Spherical coordinates

$x = r\sin\theta\cos\phi \Rightarrow r = \sqrt{x^2+y^2+z^2}$

$y = r\sin\theta\sin\phi \Rightarrow \theta = \operatorname{Arccos}(x/r)$

$z = r\cos\theta \Rightarrow \phi = \arctan(y/x)$

Assignment

3. Transform $(3, 1, 5)$ to Spherical coordinates (r, θ, ϕ)

Second Order differential Equations 18/10/22.

$$\frac{ad^2y}{dx^2} + \frac{bdy}{dx} + cy = f(x).$$ Homogeneous eqn $f(x) = 0$.

* Case 1

$f(x) = 0$

$$\frac{ad^2y}{dx^2} + \frac{bdy}{dx} + cy = 0$$

⇒ Real and Different roots to the auxiliary eqn.

$y = Ae^{m_1 x} + Be^{m_2 x}$

⇒ Real and the Same roots to the auxiliary eqn.

$y = (A + Bx)e^{m_1 x}$.

$m = m_1 = m_2$.

Ex 1:> $$\frac{d^2y}{dx^2} + \frac{5dy}{dx} + 6y = 0$$

$m^2 + 5m + 6 = 0$

$= (m+2)(m+3) = 0$

$m = -2$ or $m = -3$ (different roots)

$y = Ae^{m_1 x} + Be^{m_2 x}$

$y = Ae^{-2x} + Be^{-3x}$

Where e = exponential.

Ex 2:> $$\frac{d^2y}{dx^2} + \frac{3dy}{dx} - 10y = 0$$

$m^2 + 3m - 10 = 0$

$(m-2)(m+5) = 0$

$m = 2$ or -5 (different roots)

$y = Ae^{2x} + Be^{-5x}$.

Ex 3 $$\frac{d^2y}{dx^2} - \frac{12dy}{dx} + 36y = 0$$

$m^2 - 12m + 36 = 0$

$(m-6)(m-6) = 0$

$m = 6$ (twice) → Same roots.

$y = (A + Bx)e^{6x}$.

Ex 4:> $$\frac{d^2y}{dx^2} + \frac{10dy}{dx} + 25y = 0$$

$m^2 + 10m + 25 = 0$

$(m+5)(m+5) = 0$

$m = -5$ (twice) → Same root.

$y=(A+Bx)e^{-5x}$.

Complex root to the Auxiliary equation.

The complex root:

$y=e^{\alpha x}[A\cos\beta x+B\sin\beta x]$

If $m=-2\pm j3$

$m=\alpha\pm j\beta$

$y=e^{-2x}[A\cos 3x+B\sin 3x]$

If $m=5\pm j2$

$y=e^{5x}[A\cos 2x+B\sin 2x]$

Ex 1 $\frac{d^2y}{dx^2}+\frac{4dy}{dx}+9y=0$

$m^2+4m+9=0$

Use; $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\frac{-4\pm\sqrt{4^2-4(1)(9)}}{2(1)}=\frac{-4\pm\sqrt{16-36}}{2}$

$=-2\pm\frac{\sqrt{-20}}{2}=-2\pm\frac{2\sqrt{5}}{2}$

$=-2\pm\sqrt{-5}$

$=-2\pm j\sqrt{5}$

$y=e^{-2x}[A\cos\sqrt{5}x+B\sin\sqrt{5}x]$

Ex 2 ⇒ $\frac{d^2y}{dx^2}+\frac{2dy}{dx}+10y=0$

$m^2+2m+10=0$

Using; $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$=\frac{-2\pm\sqrt{2^2-4(1)(10)}}{2}$

$=\frac{-2\pm\sqrt{4-40}}{2}=\frac{-2\pm\sqrt{-36}}{2}$

$=\frac{-2\pm 6j}{2}=-1\pm j3$.

$\therefore y=e^{-x}[A\cos 3x+B\sin 3x]$

Consider;

$\frac{d^2y}{dx^2}+n^2y=0$

But we know that

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

When $b = 0$

$$a\frac{d^2y}{dx^2} + cy = 0$$

$$\frac{d^2y}{dx^2} + \frac{c}{a}y = 0$$

$\Rightarrow$ Case 1

$$\frac{d^2y}{dx^2} + n^2y = 0$$

$m^2 + n^2 = 0$

$m^2 = -n^2$

$m = \pm jn$

$m = \alpha \pm j\beta$

$\therefore y = A\cos nx + B\sin nx$.

$\Rightarrow$ Case 2.

$$\frac{d^2y}{dx^2} - n^2y = 0$$

$m^2 - n^2 = 0$

$m^2 = n^2$

$m = \pm n$

*$\Rightarrow$ $\cosh nx = \dfrac{e^{nx} + e^{-nx}}{2}$ $\rightarrow$ Hyperbolic function.

$e^{nx} + e^{-nx} = 2\cosh nx$.

*$\Rightarrow$ $\sinh nx = \dfrac{e^{nx} - e^{-nx}}{2}$

$e^{nx} - e^{-nx} = 2\sinh nx$.

$e^{nx} = \cosh nx + \sinh nx$.

$\Rightarrow y = Ce^{nx}$.

$y = C(\cosh nx + \sinh nx) + D(\cosh nx - \sinh nx)$

$\Rightarrow (C+D)\cosh nx + (C-D)\sinh nx$.

$\therefore y = A\cosh nx + B\sinh nx$.

Ex 1 $\dfrac{d^2y}{dx^2} + 16y = 0$

$m^2 + 16 = 0 \Rightarrow m^2 = -16 \Rightarrow m \pm \sqrt{-16}$

$\therefore m = \pm j4$

$y = A\cos 4x + B\sin 4x \rightarrow$ In accordance with Case 1

Ex 2 $\frac{d^2y}{dx^2} - 3y = 0$

$m^2 - 3 = 0 \Rightarrow m^2 = 3$

$m = \pm\sqrt{3}$

$\therefore y = A\cosh\sqrt{3}\,x + B\sinh\sqrt{3}x \rightarrow$ In accordance with Case 2

Ex 3:- $\underbrace{\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y}_{L.H.S} = \underbrace{24}_{R.H.S}$

* Complementary Function (CF)

L.H.S = 0

$\Rightarrow \frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = 0$

$m^2 - 5m + 6 = 0$

$(m-2)(m-3)$

$m = 2$ or 3

$y = Ae^{2x} + Be^{3x}$.

* Particular Integral (PI)

Inhomogeneous eqn $f(x) \neq 0$

$f(x) = 24 \quad y = C$

$\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = 24$

$\frac{d^2y}{dx^2} = 0 \quad \frac{dy}{dx} = 0$

$0 - 0 + 6(C) = 24$

$6C = 24$

$C = 4$

Now taking; CF + PI

$\Rightarrow y = Ae^{2x} + Be^{3x} + 4$

Ex 4 $\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = 2\sin 4x$.

Complimentary function (CF)

L.H.S = 0

$\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = 0$

$m^2 - 5m + 6 = 0$

$(m-2)(m-3)=0$

$m = 2$ or 3

$y = Ae^{2x} + Be^{3x}$.

* Particular Integral (PI)

Here, $y = C\cos 4x + D\sin 4x$

$\frac{dy}{dx} = -4C\sin 4x + 4D\cos 4x$.

$\frac{d^2y}{dx^2} = -16C\cos 4x - 16D\sin 4x$

$\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = 2\sin 4x$.

$-16C\cos 4x - 16D\sin 4x + 20C\sin 4x - 20D\cos 4x$
$+ 6C\cos 4x + 6D\sin 4x = 2\sin 4x$.

Bring like terms together:

$\Rightarrow 20C\sin 4x - 16D\sin 4x + 6D\sin 4x - 16C\cos 4x - 20D$
$\cos 4x + 6C\cos 4x = 2\sin 4x$.

$\Rightarrow 20C\sin 4x - 10D\sin 4x - 10C\cos 4x - 20D\cos 4x = 2\sin 4x$

$\Rightarrow [20C - 10D]\sin 4x - [10C + 20D]\cos 4x = 2\sin 4x$

$20C - 10D = 2$ ——— (i)

$10C + 20D = 0$ ——— (ii)

Multiplying 2 in eqn(i), ~~by~~ then add ~~subtract~~ (i) and ~~from~~ (ii)

$40C - 20D = 4$

$+\ 10C + 20D = 0$

$50C = 4$

$C = \frac{4}{50} = \frac{2}{25}$

Subst. C into eqn(i)

$20\left(\frac{2}{25}\right) - 10D = 2$

$-10D = 2 - \frac{40}{25}$

$-10D = \frac{10}{25}$

$D = -\frac{1}{25}$

$\Rightarrow y = \frac{2}{25}\cos 4x - \frac{1}{25}\sin 4x$

$C_F + P_I$

$Ae^{2x} + Be^{3x} + 1/25[2\cos 4x + \sin 4x]$

Assignment.

1. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} - 8y = 3e^{-2x}$

2. $\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 5y = 13e^{3x}$

3. $\frac{d^2y}{dx^2} - \frac{3dy}{dx} + 2y = x^2$

4. $\frac{d^2y}{dx^2} + \frac{6dy}{dx} + 10y = 2\sin 2x$.

* Assume,

If $f(x) = k$	$y = C$
$f(x) = kx$	$y = Cx + D$
$f(x) = kx^2$	$y = Cx^2 + Dx + E$
$f(x) = k\sin x$ Or $k\cos x$	$y = C\cos x + D\sin x$.
$f(x) = k\sinh x$ or $k\cosh x$	$y = C\cosh x + D\sinh x$.
$f(x) = e^{kx}$	$y = Ce^{kx}$

$y = CF + PI$

Solution.

1. $\frac{d^2y}{dx^2} - \frac{2dy}{dx} - 8y = 3e^{-2x}$

Considering L.H.S

$m^2 - 2m - 8 = 0$

$(m+2)(m-4) = 0$

$m = -2$ or 4

C.F

$y = Ae^{-2x} + Be^{4x}$

P/I

$\cancel{y = Ce^{kx} = Ce^{-2x}}$

$\cancel{\frac{dy}{dx} = -2Ce^{-2x}}$

$\cancel{\frac{d^2y}{dx^2} = 4Ce^{-2x}}$

$\cancel{4Ce^{-2x} - 2(-2Ce^{-2x}) - 8(Ce^{-2x}) = 3e^{-2x}}$

$\cancel{4Ce^{-2x} + 4Ce^{-2x} - 8Ce^{-2x} = 3e^{-2x}}$

P.I

$$y = Cxe^{-2x}$$

$$\frac{dy}{dx} = -2Cxe^{-2x} + Ce^{-2x}.$$

$$\frac{d^2y}{dx^2} = 4Cxe^{-2x} - 2Ce^{-2x} - 2Ce^{-2x}$$

$$= 4Cxe^{-2x} - 4Ce^{-2x}.$$

$$(4Cxe^{-2x} - 4Ce^{-2x}) - 2(-2Cxe^{-2x} + Ce^{-2x}) - 8(Cxe^{-2x}) = 3e^{-2x}$$

$$\cancel{4C}xe^{-2x} - 4Ce^{-2x} + \cancel{4C}xe^{-2x} - 2Ce^{-2x} - \cancel{8C}xe^{-2x} = 3e^{-2x}$$

$$-6Ce^{-2x} = 3e^{-2x}.$$

$$-6C = 3$$

$$C = -\frac{1}{2}$$

$$y = CF + PI$$

$$y = Ae^{-2x} + Be^{4x} + -\tfrac{1}{2}(xe^{-2x})$$

$$y = Ae^{-2x} + Be^{4x} - \frac{1}{2}xe^{-2x}.$$

2 $$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 5y = 13e^{3x}.$$

considering L.H.S

$$m^2 + 4m + 5 = 0$$

CF

$\cancel{(m+5)(m+}$ $$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2} = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$m = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm j2}{2} = -2 \pm j$$

$$y = e^{-2x}[A\cos x + B\sin x]$$

P.I

$$y = Ce^{3x}$$

$$\frac{dy}{dx} = 3Ce^{3x}$$

$$\frac{d^2y}{dx^2} = 9Ce^{3x}.$$

$$9Ce^{3x} + 4(3Ce^{3x}) + 5Ce^{3x} = 13e^{3x}$$

$$9Ce^{3x} + 12Ce^{3x} + 5Ce^{3x} = 13e^{3x}$$

$$26Ce^{3x} = 13e^{3x}$$

$$C = \frac{13}{26} \Rightarrow C = \frac{1}{2}$$

$$y = \frac{1}{2}e^{3x}$$

$$y = C.F + P.I$$

$$= e^{-2x}[A\cos x + B\sin x] + \frac{1}{2}e^{3x}.$$

3. $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = x^2$$

Considering L.H.S

$$m^2 - 3m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 1 \text{ or } 2.$$

$$y = Ae^{x} + Be^{2x}$$

P.I

$$y = Cx^2 + Dx + E$$

$$\frac{dy}{dx} = 2Cx + D$$

$$\frac{d^2y}{dx^2} = 2C$$

$$2C - 3(2Cx + D) + 2(Cx^2 + Dx + E) = x^2.$$

$$2C - 6Cx - 3D + 2Cx^2 + 2Dx + 2E = x^2.$$

$$2Cx^2 - 6Cx + 2Dx + 2C - 3D + 2E = x^2.$$

By comparison

$$2C = 1$$

$$C = \frac{1}{2}$$

$$-6C + 2D = 0$$

$$2D = 6C$$

$$D = 3C = 3\left(\frac{1}{2}\right) = \frac{3}{2}$$

$$2C - 3D + 2E = 0$$

$$2E = 3D - 2C$$

$$2E = 3\left(\frac{3}{2}\right) - 2\left(\frac{1}{2}\right)$$

$$2E = \frac{9}{2} - 1$$

$$2E = \frac{9-2}{2}$$

$$E = \frac{7}{4}$$

$$y = \frac{1}{2}x^2 + \frac{3}{2}x + \frac{7}{4}$$

$$y = C.F + P.I$$

$$y = Ae^{x} + Be^{2x} + \frac{x^2}{2} + \frac{3}{2}x + \frac{7}{4}$$

4 $\frac{d^2y}{dx^2} + \frac{6dy}{dx} + 10y = 2\sin 2x$.

Considering L.H.S

$$m^2 + 6m + 10 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{6^2 - 4(1)(10)}}{2(1)}$$

$$= \frac{-6 \pm \sqrt{36-40}}{2} = \frac{-6 \pm \sqrt{-4}}{2} = \frac{-6 \pm j2}{2}$$

$$= -3 \pm j$$

$$y = e^{-3x}[A\cos x + B\sin x]$$

P.I

$$y = C\cos 2x + D\sin 2x.$$

$$\frac{dy}{dx} = -2C\sin 2x + 2D\cos 2x.$$

$$\frac{d^2y}{dx^2} = -4C\cos 2x + -4D\sin 2x$$

$$= -4C\cos 2x - 4D\sin 2x.$$

$$-4C\cos 2x - 4D\sin 2x + 6(-2C\sin 2x + 2D\cos 2x) + 10(C\cos 2x + D\sin 2x) = 2\sin 2x.$$

$$-4C\cos 2x - 4D\sin 2x - 12C\sin 2x + 12D\cos 2x + 10C\cos 2x + 10D\sin 2x = 2\sin 2x.$$

$$(-4C + 12D)\cos 2x + 10C\cos 2x + (-4D - 12C + 10D)\sin 2x = 2\sin 2x.$$

$$(6C + 12D)\cos 2x + (6D - 12C)\sin 2x = 2\sin 2x.$$

$6c + 12D = 0$ —— (1) $\times 2$

$-12c + 6D = 2$ —— (2)

$$
\begin{aligned}
+\ 12c + 24D &= 0 \\
-12c + 6D &= 2 \\
30D &= 2 \\
D = \frac{2}{30} &= \frac{1}{15}
\end{aligned}
$$

$6c + 12\left(\frac{1}{15}\right) = 0$

$\cancel{6}c = -\frac{\cancel{12}^{2}}{15}$

$c = \frac{-2}{15}$

$y = \frac{-2\cos 2x}{15} + \frac{1}{15}\sin 2x$

$y = C.F + P.I$

$y = e^{-3x}[A\cos x + B\sin x] \frac{-2}{15}\cos 2x + \frac{1}{15}\sin 2x$

Scheme of Work. 25/10/22

1. Integral function.
   * (A) gamma function and (B) Beta function
2. Mapping
3. Analytical function
4. Complex Variables

Integral function.

Gamma function $\Gamma(x)$

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ —— (1)

$\Gamma(x+1) = x \int_0^\infty t^{x-1} e^{-t} dt$ —— (2)

$\Gamma(x+1) = x\Gamma(x)$ —— (3)

$\Gamma(x) = \frac{\Gamma(x+1)}{x}$ —— (4)

$\Gamma(x-1) = \frac{\Gamma(x)}{x-1}$ —— (5)

$\Gamma(x+1) = x\Gamma(x)$
$\Gamma(x+1) = x(x-1)\Gamma(x-1)$
$\Gamma(x+1) = x(x-1)(x-2)\Gamma(x-3)$
$\Gamma(x+1) = x(x-1)(x-2)(x-3)\Gamma \ldots \Gamma(1)$
$\Gamma(1) = 1$
$\Gamma(x+1) = x!$ ——— (6)

Example 1
Evaluate integral $\int_0^\infty x^7 e^{-x} dx$

Solution.

$\int_0^\infty x^7 e^{-x} dx = \int_0^\infty x^{8-1} e^{-x} dx$
$\Gamma(8) = \int_0^\infty x^{8-1} e^{-x} dx$
$\Gamma(7+1) = 7!$
$= 5,040$

DIY

1. $\int_0^\infty y^6 e^{-y} dy$
2. $\int t^5 e^{-t} dt$.

Gamma functions of Fractions. (Positive Fraction).

$\Gamma(1/2) = \sqrt{\pi}$
$\Gamma(x+1) = x\Gamma(x)$
$\Gamma(3/2) = \Gamma(1/2+1) = 1/2\,\Gamma(1/2)$
$= \frac{1\sqrt{\pi}}{2}$.

$\Gamma(5/2) = \Gamma(3/2+1) = 3/2\,\Gamma(3/2)$
$= \frac{3}{2}\left(\frac{\sqrt{\pi}}{2}\right)$
$= \frac{3\sqrt{\pi}}{4}$

DIY

$\Gamma(7/2)$, $\Gamma(9/2)$, $\Gamma(11/2)$

$\Gamma(7/2) = \Gamma(5/2+1) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$

$= \frac{5}{2}\left(\frac{3\sqrt{\pi}}{4}\right) = \frac{15}{8}\sqrt{\pi}$

$$\Gamma(9/2) = \Gamma(7/2+1) = 7/2\,\Gamma(7/2)$$
$$= \frac{7}{2}\left(\frac{15\sqrt{\pi}}{8}\right)$$
$$= \frac{105\sqrt{\pi}}{16}$$

$$\Gamma(11/2) = \Gamma(9/2+1) = 9/2\,\Gamma(9/2)$$
$$= \frac{9}{2}\left(\frac{105\sqrt{\pi}}{16}\right)$$
$$= \frac{945\sqrt{\pi}}{32}.$$

Gamma functions of Negative fractions.

$$\Gamma(-1/2) = \Gamma(1/2 - 1)$$
$$\Gamma(x-1) = \frac{\Gamma(x)}{x-1}$$
$$\Gamma(-1/2) = \frac{\Gamma(1/2)}{(1/2-1)} = \frac{\sqrt{\pi}}{-1/2} = -2\sqrt{\pi}.$$
$$\Gamma(-3/2) = \Gamma(-1/2-1) = \frac{\Gamma(-1/2)}{(-1/2-1)}$$
$$= \frac{-2\sqrt{\pi}}{-3/2} = \frac{4}{3}\sqrt{\pi}.$$
$$\Gamma(-5/2) = \Gamma(-3/2-1) = \frac{\Gamma(-3/2)}{-3/2-1}$$
$$= \frac{4/3\sqrt{\pi}}{-5/2} = -\frac{8}{15}\sqrt{\pi}.$$
$$\Gamma(-7/2) = \Gamma(-5/2-1) = \frac{\Gamma(-5/2)}{-5/2-1}$$
$$= \frac{-8/15\sqrt{\pi}}{-7/2} = \frac{16\sqrt{\pi}}{105}$$
$$\Gamma(-9/2) = \Gamma(-7/2-1) = \frac{\Gamma(-7/2)}{-7/2-1} = \frac{16/105\sqrt{\pi}}{-9/2}$$
$$= \frac{-32\sqrt{\pi}}{945}.$$

Beta Function $B(m,n)$

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \rightarrow$ Algebraic

$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \rightarrow$ Trigonometry.

but

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

Example I

Evaluate:

$$\int_0^1 x^5(1-x)^4 dx$$

Solution

$$\int_0^1 x^5(1-x)^4dx = \int_0^1 x^{6-1}(1-x)^{5-1}dx$$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$\therefore \quad m = 6 \quad , n = 5$

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{(6-1)!(5-1)!}{(6+5-1)!}$$

$$= \frac{5!\,4!}{10!} = \frac{120 \times 24}{3628800}$$

$$= \frac{1}{1260}$$

Example II

Evaluate:

$$\int_0^{\pi/2} \sin^5\theta \cos^4\theta \, d\theta.$$

Solution

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{B(m,n)}{2}$$

$2m-1 = 5$

$m = 3$

$2n-1 = 4$

$2n = 5$

$n = 2.5$

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{(3-1)!(2.5-1)!}{(3+2.5-1)!}$$

$$= \frac{2!\,1.5!}{4.5!}$$

$$\int_0^{\pi/2} \sin^5\theta \cos^4\theta \, d\theta = \frac{1}{2}\left(\frac{2!\,1.5!}{4.5!}\right) = \frac{1}{2}B(3, 5/2)$$

Relationship between Gamma function and Beta function

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\int_0^{\pi/2} \sin^5\theta \cos^4\theta \, d\theta = \frac{1}{2} B(3, 5/2)$$

$$\frac{1}{2}B(3, 5/2) = \frac{\Gamma(3)\,\Gamma(5/2)}{\Gamma(3+5/2)}$$

$$= \frac{2!\,(3/4\sqrt{\pi})}{\Gamma(11/2)} = \frac{\frac{6}{4}\sqrt{\pi}}{\frac{945}{32}\sqrt{\pi}}$$

$$= \frac{6\cancel{\sqrt{\pi}}}{\cancel{4}} \times \frac{\cancel{32}\,8}{945\cancel{\sqrt{\pi}}}$$

$$= \frac{16}{315}$$

Assignment

* Evaluate $B(1/2, 1/2)$

315

Assignment

* Evaluate $B(1/2, 1/2)$.

Solution.

DIY

① $\int_0^\infty y^6 e^{-y} dy$

$\int_0^\infty t^{x-1} e^{-t} dt$

$= \int_0^\infty y^{7-1} e^{-y} dy = \Gamma(7)$

$\Gamma(7) = \Gamma(6+1) = 6!$

$= \underline{\underline{720}}$

② $\int_0^\infty t^5 e^{-t} dt$.

$\int_0^\infty t^{6-1} e^{-t} dt = \Gamma(6)$

$\Gamma(6) = \Gamma(5+1) = 5!$

$= \underline{\underline{120}}$

Assignment.

$B(1/2, 1/2)$

$B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$\Gamma(1/2) = \sqrt{\pi}$

$m = n = \sqrt{\pi}$.

$\Gamma(m+n) = ?$

$m+n = \frac{1}{2} + \frac{1}{2} = 1$

$\Gamma(\frac{1}{2}+\frac{1}{2}) = \Gamma(1) = 1$

$B(m,n) = B(\frac{1}{2},\frac{1}{2}) = \frac{(\sqrt{\pi})(\sqrt{\pi})}{1}$

$= \sqrt{\pi^2} = \pi$

Complex Variable

$A = a + ib$

Example I

If $A = 3 + i4$

$B = 9 + i12$

Find:

(a) $A+B$

(b) $2A-B$

(c) AB

(d) A/B

(e) A^2

(f) $\sqrt[3]{B}$

Solution

(a) $A+B = (3+i4)+(9+i12)$

$= (3+9)+(i4+i12)$

$= 12+i16$

(b) $2A-B = 2(3+i4)-(9+i12)$

$= (6+i8)-9-i12$

$= 6-9+(i8-i12)$

$= -3-i4$

(c) AB

$A = 3+i4 = 5\angle 53.1$

$B = 9+i12 = 15\angle 53.1$

$AB = (5\angle 53.1)(15\angle 53.1)$

$= 75\angle 106.2^\circ$

(d) $\frac{A}{B} = \frac{5\angle 53.1}{15\angle 53.1} = 0.33\angle 0^\circ$

(e) $A^2 = (5\angle 53.1)^2$
$= 5^2\angle(53.1\times 2)$
$= 25\angle 106.2$

(f) $\sqrt[3]{B}$ $\sqrt[3]{9+12}$
$= \sqrt[3]{15\angle 53.1^\circ}$
$= \sqrt[3]{15}\angle \frac{53.1^\circ}{3}$

$= 2.47\angle 17.7^\circ$ → 1st answer

$\frac{360}{3} = 120^\circ$

$2.47\angle 17.7 + 120^\circ$
$2.47\angle 137.7^\circ$ → 2nd answer

$2.47\angle 137.7^\circ + 120^\circ$
$2.47\angle 257.7^\circ$ → 3rd answer

N.B When taking the roots (square root, cube root, ...) of complex numbers, the number of answers correspond to the root.

Evaluate the following

(1) $\sqrt{1-i}$ (2) $\sqrt[3]{3+4i}$ (3) $\sqrt[3]{343}$
(4) $\sqrt{-4}$ (5) $\sqrt[4]{i}$ (6) $\sqrt[3]{1}$ (7) $\sqrt{-1}$
(8) $z^4 + 324 = 0$ (9) $z^2 - z + 1 + i = 0$ (10) $z^4 - 6iz^2 + 16 = 0$ (11) $z^2 - (6-2i)z + 17 - 6i = 0$

Solution

(1) $\sqrt{1-i} = \sqrt{1.41\angle -45^\circ}$
$= \sqrt{1.41}\angle \frac{-45^\circ}{2}$

$= 1.19\angle -22.5^\circ$
$= 1.19\angle(-22.5^\circ + 180^\circ)$
$= 1.19\angle 157.5^\circ$ $\frac{360^\circ}{2} = 180^\circ$

8 $z^4 + 324 = 0$

$z^4 = -324$

let $z^2 = P$

$(z^2)^2 = -324$

$P^2 = -324$

$P = \sqrt{-324}$

$P = \sqrt{-1} \times \sqrt{324}$

$P = i18$

$P = 18\angle 90^\circ$

$P = 18\angle 270^\circ$

but $z^2 = P$

$z = \sqrt{P}$

$z = \sqrt{18\angle 90^\circ}$

$= 4.24\angle 45^\circ$

$= 4.24\angle 225^\circ$

$z = \sqrt{18\angle 270^\circ}$

$= 4.24\angle 135^\circ$

$= 4.24\angle 315^\circ$

Or

$z = \sqrt[4]{-324}$

$= \sqrt[4]{324\angle 180^\circ}$

$= 4.24\angle 45^\circ$

$= 4.24\angle 135^\circ$

$= 4.24\angle 225^\circ$

$= 4.24\angle 315^\circ$.

9 $z^2 - z + 1 + i = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1, b = -1, c = 1 + i$

$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1+i)}}{2(1)}$

$z = \frac{1 \pm \sqrt{1 - 4(1+i)}}{2} = \frac{1 \pm \sqrt{1 - 4 - 4i}}{2}$

$z = \frac{1 \pm \sqrt{-3 - 4i}}{2}$

$\sqrt{5\angle -126.87} = \sqrt{5} \angle 53.13/2$ -> Work on this. 2.

2.25

$\dfrac{1 \pm \sqrt{5\angle -106.9}}{2}$ 1 *

$= \dfrac{1 \pm 2.04\angle -63.45}{2}$

$= \dfrac{1 \pm (1 - i2)}{2}$

$z = \dfrac{1 + (1 - i2)}{2} = \dfrac{1 + 1 - i2}{2} = \dfrac{2 - i2}{2}$ 2

$\underline{z = 1 - i = 1\angle -45^\circ}$

$z = \dfrac{1 - (1 - i2)}{2} = \dfrac{1 - 1 + i2}{2} = \dfrac{i2}{2}$

$= i = \underline{1\angle 90^\circ}$

De Moivre's Theorem.

De moivres theorem state that:

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

N.B $\dfrac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$

$z = \cos n\theta + i\sin n\theta$

$\dfrac{1}{z} = \cos n\theta - i\sin n\theta$

$z + \dfrac{1}{z} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$z + \dfrac{1}{z} = 2\cos n\theta$

$z - \dfrac{1}{z} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$

$z - \dfrac{1}{z} = \cancel{\cos} n\theta + i\sin n\theta - \cancel{\cos} n\theta + i\sin n\theta$

$z - \dfrac{1}{z} = i2\sin n\theta$

Exercise:

1. $\dfrac{(\cos 3\theta + i\sin 3\theta)^4 (\cos 4\theta - i\sin 4\theta)^5}{(\cos 4\theta + i\sin 4\theta)^3 (\cos 5\theta + i\sin 5\theta)^{-4}}$

2. $\dfrac{(\cos 5\theta - i\sin 5\theta)^2(\cos 7\theta + i\sin 7\theta)^{-3}}{(\cos 4\theta - i\sin 4\theta)^9(\cos\theta + i\sin\theta)^5}$

Solution

1 * $\dfrac{(\cos\theta + i\sin\theta)^{3\times4}(\cos\theta + i\sin\theta)^{-1(4\times5)}}{(\cos\theta + i\sin\theta)^{3\times4}(\cos\theta + i\sin\theta)^{-4\times5}}$

$= \dfrac{(\cos\theta + \cancel{i\sin\theta})^{12}(\cos\theta + \cancel{i\sin\theta})^{-20}}{(\cos\theta + \cancel{i\sin\theta})^{12}(\cos\theta + \cancel{i\sin\theta})^{-20}}$

$= \dfrac{1}{1} = \underline{1}$

2 $\dfrac{(\cos\theta + i\sin\theta)^{-1(5\times2)}(\cos\theta + i\sin\theta)^{-3\times7}}{(\cos\theta + i\sin\theta)^{-1(9\times4)}(\cos\theta + i\sin\theta)^5}$

$= \dfrac{(\cos\theta + i\sin\theta)^{-10}(\cos\theta + i\sin\theta)^{-21}}{(\cos\theta + i\sin\theta)^{-36}(\cos\theta + i\sin\theta)^5}$

$= \dfrac{(\cos\theta + i\sin\theta)^{-31}}{(\cos\theta + i\sin\theta)^{-31}} = \dfrac{1}{1} = \underline{1}$

Expand $\cos^8\theta$ in a Series of cosines of multiple of θ.

Solution.

$\cos^8\theta = (\cos\theta)^8$

Applying $z + \dfrac{1}{z} = 2\cos n\theta$

$2\cos n\theta = z + \dfrac{1}{z}$

$\{2\cos\theta\}^8 = 2^8\cos 8\theta = \left(z + \dfrac{1}{z}\right)^8$

$256\cos 8\theta = \left(z + \dfrac{1}{z}\right)^8$

Applying binomial expansion.

$(a+b)^8 = a^8 + {}^8C_1 a^7 b + {}^8C_2 a^6 b^2 + {}^8C_3 a^5 b^3 + \dots$

$\left(z + \dfrac{1}{z}\right)^8 = z^8 + {}^8C_1 z^7\left(\dfrac{1}{z}\right) + {}^8C_2 z^6\left(\dfrac{1}{z}\right)^2 + {}^8C_3 z^5\left(\dfrac{1}{z}\right)^3$

$+ {}^8C_4(z)^4\left(\dfrac{1}{z}\right)^4 + {}^8C_5(z)^3\left(\dfrac{1}{z}\right)^5 + {}^8C_6(z)^2\left(\dfrac{1}{z}\right)^6$

$+ {}^8C_7(z)\left(\dfrac{1}{z}\right)^7 + {}^8C_8(z)^0\left(\dfrac{1}{z}\right)^8$

$= z^8 + 8z^6 + 28z^4 + 56z^2 + 70 + 56\dfrac{1}{z^2} + 28\dfrac{1}{z^4} + 8\left(\dfrac{1}{z^6}\right)$

$+\frac{1}{z^8}$

$= z^8 + 8z^6 + 28z^4 + 56z^2 + 70 + 56z^{-2} + 28z^{-4} + 8z^{-6} + z^{-8}.$

DIY

2) $\sqrt[3]{3+4i}$

$= \sqrt[3]{5\angle 53.13^\circ}$

$= \sqrt[3]{5} \angle \frac{53.13^\circ}{3}$

$= 1.71 \angle 17.71^\circ$

$\frac{360}{3} = 120^\circ$

$= 1.71 \angle 137.71$

$1.71 \angle 257.71^\circ$

3) $\sqrt[3]{343}$

$\sqrt[3]{343\angle \cancel{180} 0^\circ}$

$\sqrt[3]{343} \angle \frac{0^\circ}{3}$

$= 7\angle 0^\circ$

$\frac{360}{3} = 120^\circ$

$= 7\angle 120^\circ$

$= 7\angle 240^\circ$

4) $\sqrt[4]{-4}$

$\sqrt[4]{4\angle 180^\circ}$

$\sqrt[4]{4} \angle \frac{180^\circ}{4}$

$\frac{360^\circ}{4} = 90^\circ$

$= 1.41\angle 45^\circ$

$= 1.41\angle 135^\circ$

$= 1.41\angle 225^\circ$

$= 1.41\angle 315^\circ$

5) $\sqrt[4]{1}$ ...

6) $\sqrt[8]{1}$ ...

7) $\sqrt{-}$ ...

5) $\sqrt[4]{i}$

$\sqrt[4]{1\angle 90^\circ}$

$\sqrt[4]{1}\ \angle\frac{90^\circ}{4}$

$= 1\angle 22.5^\circ$

$\frac{360}{4} = 90^\circ$

$= 1\angle 112.5^\circ$

$= 1\angle 202.5^\circ$

$= 1\angle 292.5^\circ$

6) $\sqrt[8]{1}$

$\sqrt[8]{1\angle 0}$

$\sqrt[8]{1}\angle\frac{0}{8}$

$\frac{360^\circ}{8} = 45^\circ$

$= 1\angle 0$

$= 1\angle 45^\circ$

$= 1\angle 90^\circ$

$= 1\angle 135^\circ$

$= 1\angle 180^\circ$

$= 1\angle 225^\circ$

$= 1\angle 270^\circ$

$= 1\angle 315^\circ$

7) $\sqrt{-1}$

$\sqrt{1\angle 180^\circ}$

$\sqrt{1}\angle\frac{180^\circ}{2}$

$\sqrt{1}\angle 90^\circ$

$= 1\angle 90^\circ$

$= 1\angle 270^\circ$

10 $z^4 - 6iz^2 + 16 = 0$

$z^2 = p$

$p^2 - 6ip + 16 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$b = -6i$

$a = 1$

$c = 16$

$p = \frac{-(-6i) \pm \sqrt{(-6i)^2 - 4(1 \times 16)}}{2(1)}$

$= \frac{6i \pm \sqrt{36i^2 - \cancel{x}64}}{2}$

$p = \frac{6i \pm \sqrt{-36-64}}{2}$

$p = \frac{6i \pm \sqrt{-100}}{2} = \frac{6i \pm \sqrt{100\angle 180^\circ}}{2}$

$p = \frac{6i \pm (\sqrt{100\angle 90^\circ})}{2} = \frac{6i \pm (10\angle 90^\circ)}{2}$

$p = \frac{6i \pm 10i}{2}$

$p = \frac{6i + 10i}{2}$ or $\frac{6i - 10i}{2}$

$p = \frac{16i}{2}$ or $\frac{-4i}{2}$

$p = 8i$ or $-2i$

$p = z^2$

$z^2 = 8i$ or $z^2 = -2i$

$z = \sqrt{8i}$ $z = \sqrt{-2i}$

$z = \sqrt{8\angle 90^\circ}$ $z = \sqrt{2\angle -90^\circ}$

$z = \sqrt{8}\angle 45^\circ$ $z = 1.41\angle -45^\circ$

$= 2.83\angle 45^\circ$

$= 2.83\angle 225^\circ$ $z = 1.41\angle 135^\circ$

11.

11. $z^2 - (6-2i)z + 17 - 6i = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b = -(6-2i)$

$c = 17 - 6i$

$a = 1$

$$\cancel{z} = \frac{-[-(6-2i)] \pm \sqrt{[\cancel{-(}6-2i)]^2 - 4(1)(17-6i)}}{2(1)}$$

$$= \frac{6-2i \pm \sqrt{(6-2i)^2 - 4(17-6i)}}{2}$$

$$= \frac{6-2i \pm \sqrt{36 - \cancel{24i} - 4 - 68 + \cancel{24i}}}{2}$$

$$= \frac{6-2i \pm \sqrt{-36}}{2}$$

$$= \frac{6-2i \pm i6}{2}$$

$$= \frac{6-i2+i6}{2} \text{ or } \frac{6-i2-i6}{2}$$

$$z = \frac{6+i4}{2} \text{ or } \frac{6-i8}{2}$$

$z = 3 + i2$ or $3 - i4$

$z = 3.61\angle 33.69°$ or $5\angle -53.13$

16/11/20.

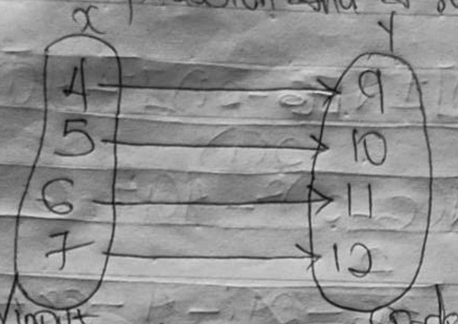
Mapping, Transformation and Special functions.

Mapping :- it is used to describe the relationship between two sets. E.g

$x = \{4, 5, 6, 7\}$, $Y = \{9, 10, 11, 12\}$

N.B the relationship between the two sets is describe using mathematical expression and a rule must be assign to all.

1.



9 is an image of 4.

Domain/input co-domain/output/range

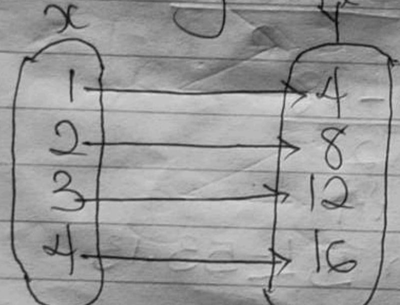
$Y = x + 5$ → The relationship.

Paired number

(4,9) (5,10) (6,11) (7,12) → Cartesian form.

2. $x = \{1, 2, 3, 4\}$ $Y = \{4, 8, 12, 16\}$

$y = 4x$



Cartesian form: (1,4) (2,8) (3,12) (4,16).

3. $x = \{2, 3, 4, 5\}$ and $Y = \{7, 10, 13, 16\}$

$y = 3x + 1$

We can only look at the relationship when certain conditions are satisfy,

1. Two sets must be involved.
2. There must be a clear rule describing the relationship.
3. There must be a directional property.

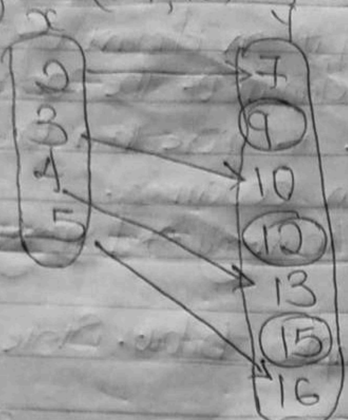
Codomain and Range.

$x = \{2, 3, 4, 5\}$

$Y = \{7, 9, 10, 12, 13, 15, 16\}$

* Firstly define your relationship.

4. Codomain is $Y = \{7, 9, 10, 12, 13, 15, 16\}$
Range $= \{7, 10, 13, 16\}$



$y = 3x + 1$

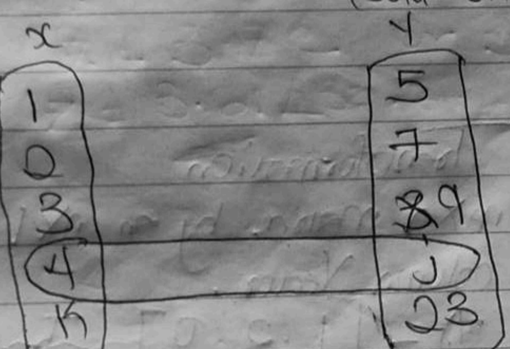
Range is members of codomain that are image of the domain.

Eg 5 :- A relation is represented by the Ordered pair Shown below

$(1,5)\ (2,7)\ (3,9)\ (4,j)\ (k,23)$

(a) State the rule of the relation
(b) What is the Image of 4
(c) What is the input for an Output 23.

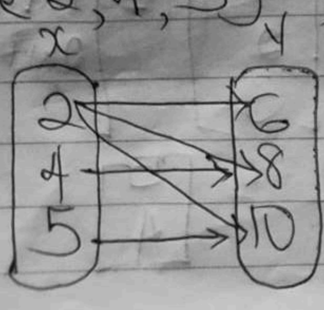
Solution



(a) $y = 2x + 3$
(b) $j = (2 \times 4) + 3$
$= 11$
(c) $23 = 2k + 3$
$23 - 3 = 2k$
$20 = 2k$
$k = \frac{20}{2}$
$k = 10$

Types Of relation.

$x = \{2, 4, 5\}$, $Y = \{6, 8, 10\}$



more than One element of x {2, and 5} is associated with One element of Y {10} More than One element of x {2, and 4} is associated with One element of Y {8}

An element of $x\{2\}$ is connected to more than One element of $Y\{6, 8, 10\}$

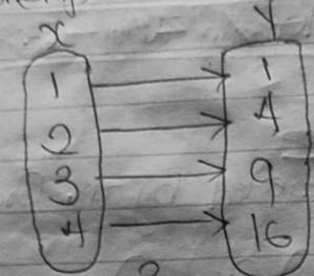
1) A relation is One-to many if an element of the domain is mapped unto more than One element of the codomain.

2) A relation is Many-to One. If more than One element of domain has the same image in codomain.

3) If the relation has both One-to many and many to One, the relation is many to many.

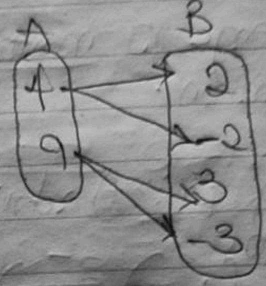
E.g 6 => In each of the examples below. State the type of relationship.

(1)



One-to One

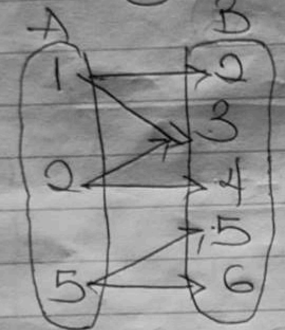
(2)



(4,2)
(4,-2)
(9,3)
(9,-3)

One ~~...~~ many to many

(3)



many to ~~...~~ many

Linear Transformation

If you multiply an $m \times n$ matrix by an $n \times 1$ vector, the result is an $m \times 1$ column vector.

E.g Using a Matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}_{2\times3}$ Transform Vector $R^3 \to R^2$

$$R^3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad 3\times1$$

$$R^2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\times1+2\times2+0\times3 \\ 2\times1+1\times2+0\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+0 \\ 2+2+0 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Def

let R^n and R^m be a Vector Spaces and $T: R^n \to R^m$ be function, then T is a linear transformation if

(a) $T(x_1 + x_2) = T(x_1) + T(x_2)$ for all $x_1, x_2 \in R$

(b) $T(Kx) = KT(x)$ for all $x \in R$, K is Scalar.

$V_1 = R_3(x)$ $\quad T: V_1 \to V_2$

$V_2 = R_3(x)$ Verify if T is a linear Transformation.

$R_3 \to$ Polynomial with degree 3.

$$T(P(x)) = P'(x)$$

$$T(P(x) + q(x)) = (P(x) + q(x))'$$

$$= P'(x) + q'(x)$$

$$= T(P(x)) + T(q(x))$$

$$T(KP(x)) = (KP(x))'$$

$$= KP'(x)$$

$$T(KP(x)) = KT(P(x))$$

$$T: R^2 \to R^2$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Verify that T is a linear transformation.

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} z \\ w \end{pmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\left[\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} z \\ w \end{pmatrix}\right]$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x+z \\ y+w \end{bmatrix}$$

$$= \begin{bmatrix} x+z+2y+2w \\ 0-y-w \end{bmatrix} = \begin{bmatrix} x+z+2y+2w \\ -(y+w) \end{bmatrix}$$

$$T\begin{pmatrix} x \\ y \end{pmatrix} + T\begin{pmatrix} z \\ w \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix}$$

$$= \begin{pmatrix} x+2y \\ 0-y \end{pmatrix} + \begin{pmatrix} z+2w \\ 0-w \end{pmatrix} = \begin{bmatrix} x+z+2(y+w) \\ -(y+w) \end{bmatrix}$$