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1 recursion formular - simple version

Consider generating function for Poisson-Binomial of you case. m Bernoulli trials with success p and n Bernoulli trials with success q.

$$m + n = N \tag{1.1}$$
$$q + p = 1 \tag{1.2}$$
$$G(x) = (q + px)^m \cdot (p + qx)^n \tag{1.3}$$

The derivative of $ln(a(x)$ is given by:

$$[ln(g(x)]' = \frac{g'(x)}{g(x)} = \frac{\left(\sum_{i=0}^{N} a_i x^i\right)^{\text{\`{}}}}{\sum_{i=0}^{N} a_i x^i} = \frac{\sum_{i=1}^{N} i \cdot a_i x^{i-1}}{\sum_{i=0}^{N} a_i x^i} \tag{1.4}$$

On the other hand

$$[ln(g(x))]' = (m(q + px) + n(p + qx))^{\text{\`{}}} = \frac{mp}{q + px} + \frac{nq}{p + qx} = \frac{xpq(m + n) + mp^2 + nq^2}{x^2pq + x(p^2 + q^2) + pq} \tag{1.5}$$

Equating both expressions we get

$$\frac{xpqN + mp^2 + nq^2}{x^2pq + x(p^2 + q^2) + pq} = \frac{\sum_{i=1}^{N} i \cdot a_i x^{i-1}}{\sum_{i=0}^{N} a_i x^i} \tag{1.6}$$
$$(xpqN + mp^2 + nq^2)(\sum_{i=0}^{N} a_i x^i) = (x^2pq + x(p^2 + q^2) + pq)(\sum_{i=1}^{N} i \cdot a_i x^{i-1}) \tag{1.7}$$

Multiplying and equating terms with same power of x we get:

$$a_i(mp^2 + nq^2) + a_{i-1}pqN = a_{i+1}pq(i + 1) + a_i(p^2 + q^2)i + a_{i-1}pq(i - 1) \tag{1.8}$$
$$a_i(mp^2 + nq^2) + a_{i-1}pqN = a_{i+1}pqi + a_i(p^2 + q^2)i + a_{i-1}pqi + (a_{i+1} - a_{i-1}) \tag{1.9}$$

Ignore the difference of $a_{i+1} - a_{i-1}$, and denote the expectation of successes as μ. Then the expression simplifies to:

$$\frac{N-i}{\frac{a_{i+1}}{a_i}i - (\frac{\mu-Npq}{pq} - \frac{p^2+q^2}{pq}i)} = \frac{a_i}{a_{i-1}} \tag{1.10}$$

$$\frac{N-i}{N-i+(\frac{a_{i+1}}{a_i}-1)i - \frac{\mu-i}{pq}} = \frac{a_i}{a_{i-1}} \tag{1.11}$$

$$\tag{1.12}$$

Denote the distance between i and μ as l. Then:

$$\mu - i = l \tag{1.13}$$

$$i = \mu - l \tag{1.14}$$

$$\frac{a_{i+1}}{a_i} = f_l \tag{1.15}$$

$$\frac{a_i}{a_{i-1}} = f_{l+1} \tag{1.16}$$

$$\frac{N-\mu+l}{N-\mu+l+(f_l-1)(\mu-l)-\frac{l}{pq}} = f_{l+1} \tag{1.17}$$

$$f_{l+1} = \frac{1}{1-\frac{1}{N-\mu+l}(\frac{l}{pq}-(f_l-1)(\mu-l))} \tag{1.18}$$

Note that when $l = 0$ and $f_0 = 1$, $f_1 = 1$ for all μ. When $l = 1$, f_2 is given by

$$f_2 = \frac{1}{1-\frac{1}{N-\mu+1}\frac{1}{pq}} \tag{1.19}$$

Clearly, f_2 is largest for the smallest μ which is reached when $m = 0$ and the smallest when $m = N$. Denote μ_0 and μ_x as expectations at $m = 0$ and $m = x$ respectively. Obviously $\mu_0 < \mu_x$. Suppose that for some $l+1$, the corresponding ratio f_l of distribution with μ_x becomes larger then that of distribution with μ_0.

$$f^x_{l+1} > f^0_{l+1} \tag{1.20}$$

$$\tag{1.21}$$

Using the recursion formula, we can express f^x_{l+1} and f^0_{l+1} as:

$$f^x_{l+1} = \frac{1}{1-\frac{1}{N-\mu_x+l}(\frac{l}{pq}-(f^x_l-1)(\mu_x-l))} \tag{1.22}$$

$$f^0_{l+1} = \frac{1}{1-\frac{1}{N-\mu_0+l}(\frac{l}{pq}-(f^0_l-1)(\mu_0-l))} \tag{1.23}$$

For $f^x_{l+1} > f^0_{l+1}$ to hold, the following must hold:

$$\frac{1}{N-\mu_x+l}(\frac{l}{pq}-(f_l^x-1)(\mu_x-l)) > \frac{1}{N-\mu_0+l}(\frac{l}{pq}-(f_l^0-1)(\mu_0-l)) \tag{1.24}$$

$$\frac{(f_l^0-1)(\mu_0-l)}{N-\mu_0+l} - \frac{(f_l^x-1)(\mu_x-l)}{N-\mu_x+l} > \frac{1}{N-\mu_0+l}\frac{l}{pq} - \frac{1}{N-\mu_x+l}\frac{l}{pq} \tag{1.25}$$

$$\frac{(f_l^0-1)(\mu_0-l)}{N-\mu_0+l} - \frac{(f_l^x-1)(\mu_x-l)}{N-\mu_x+l} > -\frac{l}{pq}\frac{\mu_x-\mu_0}{((N-\mu_x+l)(N-\mu_0+l)} \tag{1.26}$$

$$\frac{(f_l^x-1)(\mu_x-l)}{N-\mu_x+l} - \frac{(f_l^0-1)(\mu_0-l)}{N-\mu_0+l} < \frac{l}{pq}\frac{\mu_x-\mu_0}{((N-\mu_x+l)(N-\mu_0+l)} \tag{1.27}$$

Suppose $f_l^x \geq f_l^0$, then we can replace $f_l^x - 1$ with $f_l^0 - 1$ and the inequality should still hold since we reduce the left part.

$$\frac{(f_0^x-1)(\mu_x-l)}{N-\mu_x+l} - \frac{(f_l^0-1)(\mu_0-l)}{N-\mu_0+l} < \frac{l}{pq}\frac{\mu_x-\mu_0}{(N-\mu_x+l)(N-\mu_0+l)} \tag{1.28}$$

$$(f_l^0-1)\left[\frac{\mu_x-l}{N-\mu_x+l} - \frac{\mu_0-l}{N-\mu_0+l}\right] < \frac{l}{pq}\frac{\mu_x-\mu_0}{(N-\mu_x+l)(N-\mu_0+l)} \tag{1.29}$$

The expression inside the brackets simplifies to:

$$\frac{\mu_x-l}{N-\mu_x+l} - \frac{\mu_0-l}{N-\mu_0+l} = \frac{N(\mu_x-\mu_0)}{(N-\mu_x+l)(N-\mu_0+l)} \tag{1.30}$$

From here:

$$(f_l^0-1)\frac{N(\mu_x-\mu_0)}{(N-\mu_x+l)(N-\mu_0+l)} < \frac{l}{pq}\frac{\mu_x-\mu_0}{(N-\mu_x+l)(N-\mu_0+l)} \tag{1.31}$$

$$(f_l^0-1)Npq < l \tag{1.32}$$

$$(f_l^0-1) < \frac{l}{Npq} \tag{1.33}$$

Now consider the values of probabilistic ratio for μ_0. Since $m = 0$, all Bernoulli trials generate successes with probability q, hence the ratio at any given number of successes i is:

$$\frac{a_i}{a_{i-1}} = \frac{\binom{N}{i} q^i p^{N-i}}{\binom{N}{i-1} q^{i-1} p^{N-i+1}} = \frac{N-i+1}{i}\frac{q}{p} \tag{1.34}$$

$$f_l^0 = \frac{N-\mu_0+l+1}{\mu_0-l}\frac{q}{p} = \frac{N-qN+l+1}{qN-l}\frac{q}{p} \tag{1.35}$$

$$f_l^0 - 1 = \frac{N-qN+l+1}{qN-l}\frac{q}{p} - 1 = \frac{(N-qN+l+1)q-(qN-l)p}{(qN-l)p} = \tag{1.36}$$

$$\frac{Nq-q^2N+lq+q-pqN+pl}{(qN-l)p} = \frac{Nq(1-p)-q^2N+l(q+p)+q}{(qN-l)p} = \tag{1.37}$$

$$\frac{Nq^2-q^2N+l+q}{(qN-l)p} = \frac{l+q}{(qN-l)p} \tag{1.38}$$

$$f_l^0 - 1 = \frac{l+q}{(qN-l)p} \tag{1.39}$$

Now, consider the inequality:

$$f_l^0 - 1 = \frac{l+q}{(qN-l)p} > \frac{l}{Npq} \tag{1.40}$$

$$\frac{l+q}{qN-l} > \frac{l}{qN} \tag{1.41}$$

Since this inequality holds for all l, we arrived at contradiction. Then f_x must be strictly less than f_0. That's interesting, for it means that if $f_l^x > f_l^0$ and iteration l, then at iteration $l+1$, $f_l^0 > f_l^x$. Basically, f_l^0 may only loose it maximum status for a single iteration. This could be sufficient for our purpose, but i am still looking for a better prove.