

K-Randomization

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1 Differential Privacy

The typical setting for differential privacy is the following. We consider a **database** as a collection of records. The records are elements of some space D , and a database \mathbf{x} is a vector of n records: $\mathbf{x} \in D^n$.

We wish to release information based on the database by applying a **query** to it. This is a function A mapping the database into another space: $A : D^n \rightarrow \mathbf{S}$. If the function A is random, i.e., $A(\mathbf{x}) = A(\mathbf{x}, X)$ for a random element X , then the output $A(\mathbf{x})$ is a random element of \mathbf{S} .

In considering the differential privacy of A , we compare the result of applying A to two very similar databases $\mathbf{x}, \mathbf{x}' \in D^n$. We say the databases **differ in one row** if $\sum_{i=1}^n \mathbf{1}_{\{x_i \neq x'_i\}} = 1$. The random query A is said to be ϵ -**differentially private** if, for any two databases $\mathbf{x}, \mathbf{x}' \in D^n$ differing in one row,

$$P[A(\mathbf{x}) \in S] \leq \epsilon \cdot P[A(\mathbf{x}') \in S]$$

for all $S \subset \mathbf{S}$ (measurable). An alternative notion of differing in one row that is sometimes used is that $\mathbf{x} \in D^n$, $\mathbf{x}' \in D^{n+1}$, and $x_i = x'_i$ for $i = 1, \dots, n$. In other words, \mathbf{x}' includes an additional record that is not in \mathbf{x} .

If \mathbf{S} is countable, then we can write

$$P[A(\mathbf{x}) \in S] = \sum_{s \in S} P[A(\mathbf{x}) = s].$$

Hence,

$$\frac{P[A(\mathbf{x}) \in S]}{P[A(\mathbf{x}') \in S]} = \frac{\sum_{s \in S} P[A(\mathbf{x}) = s]}{\sum_{s \in S} P[A(\mathbf{x}') = s]} \leq \max_{s \in S} \frac{P[A(\mathbf{x}) = s]}{P[A(\mathbf{x}') = s]}$$

by the Lemma (need reference).

Furthermore, if A randomizes each record in the database independently, i.e., $A(\mathbf{x}) = A(\mathbf{x}, \mathbf{X}) := (A_0(x_1, X_1), \dots, A_0(x_n, X_n))$ where X_i are independent, then $\mathbf{S} = \mathbf{S}_0^n$ and $s = (s_1, \dots, s_n)$ with $s_i \in \mathbf{S}_0$. In this case $P[A(\mathbf{x}) = s] = P[A_0(x_1) = s_1, \dots, A_0(x_n) = s_n] = \prod P[A_0(x_i) = s_i]$. If \mathbf{x} and

\mathbf{x}' differ in one row (wlog $x_1 \neq x'_1$ and $x_i = x'_i$ for $i = 2, \dots, n$), then

$$\frac{P[A(\mathbf{x}) = s]}{P[A(\mathbf{x}') = s]} = \frac{P[A_0(x_1) = s_1]}{P[A_0(x'_1) = s_1]}.$$

Therefore, in this case, the query A will satisfy differential privacy if

$$P[A_0(x) = s] \leq \epsilon \cdot P[A_0(x') = s]$$

for all $x, x' \in D$ and $s \in \mathcal{S}_0$. This is the formulation used in the RAPPOR paper that applies to differences between individual records rather than collections differing on a single element.

Consider a collection T of bit vectors, and write $T_v = T \setminus \{v\}$. The randomization procedure R is ϵ -differentially private if

$$\log \left(\frac{P[R(T) \in S]}{P[R(T_v) \in S]} \right) \leq \epsilon$$

for any set of bit vectors S .

Anonymity:

$$A_p = \min_{v \in T, s \in \{0,1\}^L} \frac{P[s \in R(T_v)]}{P[s = R(v)]}$$

2 Theoretical setup

In the following we work with data in the form of bit vectors. A **bit vector** is a vector $v \in \{0, 1\}^L$.

First we define the randomization procedure we will be applying.

Definition. The randomization procedure R with **lie probability** $0 < q < 1/2$ flips a bit with probability q , and leaves it as-is with probability $1 - q$. In other words, for a bit $b \in \{0, 1\}$,

$$R(b) = R(b; X) = (1 - b) \cdot X + b \cdot (1 - X) \quad \text{where } X \sim \text{Ber}(q).$$

When applied to a vector, each bit is randomized independently:

$$R(v) = R(v; (X_1, \dots, X_L)) = (R(v_1; X_1), \dots, R(v_L; X_L)) \quad \text{where } X \stackrel{\text{iid}}{\sim} \text{Ber}(q).$$

Remark. The randomization R reports the original bit value with probability $1 - q > q$, and lies with probability q . This is equivalent to the randomized response procedure where the value is reported as-is with probability $1 - f$, and with probability f the reported value is the outcome of the toss of a fair coin. In this case, $q = f/2$.

Remark. If $q = 1/2$, then $R(0) \stackrel{d}{=} R(1)$, and the reported value is “completely” randomly generated, i.e., independently of the original value.

Distribution of $R(v)$.

For a bit b , the randomization lies iff $R(b) \neq b$:

$$P[R(b) = s] = q^{\mathbf{1}_{\{b \neq s\}}} (1 - q)^{\mathbf{1}_{\{b = s\}}}$$

Hence, for a bit vector v ,

$$P[R(v) = s] = q^{\sum \mathbf{1}_{\{b_i \neq s_i\}}} (1 - q)^{\sum \mathbf{1}_{\{b_i = s_i\}}} = q^{L - m(v, s)} (1 - q)^{m(v, s)},$$

where $m(v, s) = |\{i : v_i = s_i\}|$. Note that this probability is maximized when $m(v, s) = L$ (the reported vector s is identical to the original vector v), and minimized when $m(v, s) = 0$. In other words, the most likely outcome of randomizing a bit vector is obtaining an identical vector.

For a collection T ,

$$P[s \in R(T)] = 1 - P[s \notin R(T)] = 1 - \prod_{v \in T} P[R(v) \neq s] = 1 - \prod_{v \in T} [1 - q^{L - m(v, s)} (1 - q)^{m(v, s)}].$$