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1 MSTACK QUESTION

Bounding ratio of probabilities of Poisson-Binomial Distribution.

There are N Bernoulli trials, where m trails have probability of success p and $N - m$ trails have probability of success $q = 1 - p$. Assume $p > q$. The number of successes is a random variable S expressed as a sum of two binomial random variables:

$$S = Bin(p, m) + Bin(q, N - m) \tag{1.1}$$

The distribution of S is known to be a Poisson-Binomial Distribution. I am studying the behavior of the ratio between P(S=k) and P(S=k-1) with respect to m and k. For a given m and number of successes k, denote such ratio as $R(k, m)$:

$$R(k, m) = \frac{P(S = k \mid m)}{P(S = k - 1 \mid m)} \tag{1.2}$$

I am particularly interested in the behavior of this ratio for small quantiles, and ran numerical simulation for $R(k, m)$ when $k << mean$. It appears that for values of k that are equally distant from the mean, the following holds:

Let $\mu_m = m \cdot (p - q) + N \cdot q$ be the mean of the corresponding distribution and α be the distance away from the mean. Then for $k = \mu_m - \alpha$:

$$R(\mu_0 - \alpha, 0) > R(\mu_1 - \alpha, 1) > \cdots > R(\mu_{m-1} - \alpha, m - 1) > R(\mu_m - \alpha, m) > \cdots > R(\mu_N - \alpha, N) \tag{1.3}$$

The ratio seems to bounded by two binomial distribution for $m = 0$ and $m = N$ respectively.

It is easy to see why $R(\mu_0 - \alpha, 0) > R(\mu_N - \alpha, N)$.

$$R(k,0) = \frac{\binom{N}{k} q^k p^{N-k}}{\binom{N}{k-1} q^{k-1} p^{N-k+1}} = \frac{N-k+1}{k} \cdot \frac{q}{p} \tag{1.4}$$

$$R(k,N) = \frac{\binom{N}{k} q^k p^{N-k}}{\binom{N}{k-1} q^{k-1} p^{N-k+1}} = \frac{N-k+1}{k} \cdot \frac{p}{q} \tag{1.5}$$

Setting $k = \mu - \alpha$ for each case, we have

$$R(k,0) = \frac{N - qN + \alpha + 1}{qN - \alpha} \cdot \frac{q}{p} \approx \frac{qp + \frac{\alpha q}{N}}{qp - \frac{\alpha p}{N}} \tag{1.6}$$

$$R(k,N) = \frac{N - pN + \alpha + 1}{pN - \alpha} \cdot \frac{p}{q} \approx \frac{qp + \frac{\alpha p}{N}}{qp - \frac{\alpha q}{N}} \tag{1.7}$$

$$\frac{qp + \frac{\alpha q}{N}}{qp - \frac{\alpha p}{N}} > \frac{qp + \frac{\alpha p}{N}}{qp - \frac{\alpha q}{N}} \tag{1.8}$$

$$(qp + \frac{\alpha q}{N})(qp - \frac{\alpha q}{N}) > (qp + \frac{\alpha p}{N})(qp - \frac{\alpha p}{N}) \tag{1.9}$$

$$(qp)^2 - \left(\frac{\alpha q}{N}\right)^2 > (qp)^2 - \left(\frac{\alpha p}{N}\right)^2 \tag{1.10}$$

Since $p > q$, the above inequality holds.

But I am unable to express $R(k,m)$ analytically, no prove that it is bounded between ratios corresponding to the limiting binomial distributions (although I see it in simulations). Any help with proving this and/or pointers to related papers will be much appreciated.