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1 recursion formular - simple version

Consider generating function for Poisson-Binomial of you case. m Bernoulli trials with success p and n Bernoulli trials with success q.

$$m + n = N \tag{1.1}$$
$$q + p = 1 \tag{1.2}$$
$$G(x) = (q + px)^m \cdot (p + qx)^n \tag{1.3}$$

The derivative of $ln(a(x)$ is given by:

$$[ln(g(x)]' = \frac{g'(x)}{g(x)} = \frac{\left(\sum_{i=0}^{N} a_i x^i\right)^{`}}{\sum_{i=0}^{N} a_i x^i} = \frac{\sum_{i=1}^{N} i \cdot a_i x^{i-1}}{\sum_{i=0}^{N} a_i x^i} \tag{1.4}$$

On the other hand

$$[ln(g(x))]' = (m(q + px) + n(p + qx))^{`} = \frac{mp}{q + px} + \frac{nq}{p + qx} = \frac{xpq(m + n) + mp^2 + nq^2}{x^2 pq + x(p^2 + q^2) + pq} \tag{1.5}$$

Equating both expressions we get

$$\frac{xpqN + mp^2 + nq^2}{x^2 pq + x(p^2 + q^2) + pq} = \frac{\sum_{i=1}^{N} i \cdot a_i x^{i-1}}{\sum_{i=0}^{N} a_i x^i} \tag{1.6}$$
$$(xpqN + mp^2 + nq^2)(\sum_{i=0}^{N} a_i x^i) = (x^2 pq + x(p^2 + q^2) + pq)(\sum_{i=1}^{N} i \cdot a_i x^{i-1}) \tag{1.7}$$

Multiplying and equating terms with same power of x we get:

$$a_i(mp^2 + nq^2) + a_{i-1}pqN = a_{i+1}pq(i + 1) + a_i(p^2 + q^2)i + a_{i-1}pq(i - 1) \tag{1.8}$$
$$a_{i-1}pq(N - i + 1) = a_{i+1}pq(i + 1) + a_i[(p^2 + q^2)i - (mp^2 + nq^2)] \tag{1.9}$$
$$a_{i-1}(N - i + 1) = a_{i+1}(i + 1) + a_i \frac{(p^2 + q^2)i - (mp^2 + nq^2)}{pq} \tag{1.10}$$

Denote the expectation of successes as μ, and note the obvious relationships:

$$mp^2 + nq^2 = mp^2 + (N-m)q^2 = m(p^2 - q^2) + Nq^2 = \tag{1.11}$$
$$m(p-q) + Nq - Nq + Nq^2 = \mu + Nq(q-1) = \mu - Npq \tag{1.12}$$
$$p^2 - q^2 = (p+q)^2 - 2pq = 1 - 2pq \tag{1.13}$$

Replacing the expression above in 1.10, we have:

$$a_{i-1}(N-i+1) = a_{i+1}(i+1) + a_i\frac{(1-2pq)i - (\mu - Npq)}{pq} \tag{1.14}$$
$$a_{i-1}(N-i+1) = a_{i+1}(i+1) + a_i\left(N - 2i - \frac{\mu - i}{pq}\right) \tag{1.15}$$
$$\frac{a_{i-1}}{a_i}(N-i+1) = \frac{a_{i+1}}{a_i}(i+1) + N - 2i - \frac{\mu - i}{pq} \tag{1.16}$$
$$\frac{a_{i-1}}{a_i}(N-i+1) = \frac{a_{i+1}}{a_i}(i+1) + N - i + 1 - i - 1 - \frac{\mu - i}{pq} \tag{1.17}$$
$$\frac{a_{i-1}}{a_i}(N-i+1) = N - i + 1 - (\frac{a_{i+1}}{a_i} - 1)(i+1) - \frac{\mu - i}{pq} \tag{1.18}$$

Rearranging the terms, we arrive to the recursive relationship between probabilistic ratios at i and $i-1$:

$$\frac{N-i+1}{N-i+1+(\frac{a_{i+1}}{a_i} - 1)(i+1) - \frac{\mu - i}{pq}} = \frac{a_i}{a_{i-1}} \tag{1.19}$$

NOTATION:

Denote the distance between μ and i as l. Also denote the probabilistic ratio at i as f_i. Then:

$$\mu - i = l \tag{1.20}$$
$$i = \mu - l \tag{1.21}$$
$$f_{i+1} = f_{l-1} = \frac{a_{i+1}}{a_i} \tag{1.22}$$
$$f_i = f_l = \frac{a_i}{a_{i-1}} \tag{1.23}$$
$$f_{i-1} = f_{l+1} = \frac{a_{i-1}}{a_{i-2}} \tag{1.24}$$
$$\frac{N-i+1}{N-i+1+(f_{l-1}-1)(i+1) - \frac{l}{pq}} = \frac{1}{1 - \frac{1}{N-i+1}\left(\frac{l}{pq} - (f_{l-1}-1)(i+1)\right)} = f_l \tag{1.25}$$
$$\frac{N-i+2}{N-i+2+(f_l-1)i - \frac{l+1}{pq}} = \frac{1}{1 - \frac{1}{N-i+2}\left(\frac{l+1}{pq} - (f_l-1)i\right)} = f_{l+1} \tag{1.26}$$

For some value of m denote corresponding expectation as μ_m. Denote the number of successes equal as i_m, and a corresponding probability ratio at i_m as $f_{m,l}$:

$$\mu_m = mp + (N-m)q \tag{1.27}$$
$$i_m = \mu_m - l \tag{1.28}$$
$$f_{m,l} = \frac{a_{i_m+1}}{a_{i_m}} \tag{1.29}$$

Expressing $i_m = \mu_m - l$, allows to re-write formula 1.26 as follows:

$$\frac{1}{1 - \frac{1}{N-\mu_m+l+2}\left(\frac{l+1}{pq} - (f_{m,l}-1)(\mu_m - l)\right)} = f_{m,l+1} \tag{1.30}$$

Properties of $f_{0,l}$

Consider probability ratio for the case $m = 0$. Since all Bernoulli trials generate successes with probability q, the following relationships hold:

$$\mu_0 = Nq \tag{1.31}$$
$$f_{0,l} = \frac{a_i}{a_{i-1}} = \frac{\binom{N}{i} q^i p^{N-i}}{\binom{N}{i-1} q^{i-1} p^{N-i+1}} = \frac{N-i+1}{i}\frac{q}{p} \tag{1.32}$$

Subtracting 1 from both sides of the equality above gives:

$$f_{0,l} - 1 = \frac{N-i+1}{i}\frac{q}{p} - 1 == \frac{qN - qi + q - pi}{pi} = \frac{\mu_0 - i + q}{pi} = \frac{l+q}{p(\mu_0 - l)} \tag{1.33}$$
$$f_{0,l} - 1 = \frac{l+q}{p(\mu_0 - l)} = \frac{l+q}{pi} \tag{1.34}$$

THEOREM 1

For $l > \sigma$ probability ratio $f_{0,l+1}$ at $l+1$ is higher then probability ratio $f_{m,l}$ for any m.

PROOF:

Suppose that there's a particular value of m such that $f_{m,l+1} > f_{0,l+1}$ for some l and consider $f_{m,l+1}$ and $f_{0,l+1}$ for $m = m$ and $m = 0$:

$$\frac{1}{1 - \frac{1}{N-\mu_m+l+2}\left(\frac{l+1}{pq} - (f_{m,l}-1)(\mu_m - l)\right)} = f_{m,l+1} \tag{1.35}$$
$$\frac{1}{1 - \frac{1}{N-\mu_0+l+2}\left(\frac{l+1}{pq} - (f_{0,l}-1)(\mu_0 - l)\right)} = f_{0,l+1} \tag{1.36}$$

For $f_{m,l+1} > f_{0,l+1}$ to hold the difference of denominators in the expressions for $f_{0,l+1}$ and $f_{0,m+1}$ must be positive, hence:

$$\frac{1}{N-\mu_m+l+2}\left(\frac{l+1}{pq}-(f_{m,l}-1)(\mu_m-l)\right)-\frac{1}{N-\mu_0+l+2}\left(\frac{l+1}{pq}-(f_{0,l}-1)(\mu_0-l)\right)>0 \tag{1.37}$$

$$\frac{l+1}{pq}(\frac{1}{N-\mu_m+l+2}-\frac{1}{N-\mu_0+l+2})+\frac{(f_{0,l}-1)(\mu_0-l)}{N-\mu_0+l+2}-\frac{(f_{m,l}-1)(\mu_m-l)}{N-\mu_m+l+2}>0 \tag{1.38}$$

$$\frac{l+1}{pq}\frac{\mu_m-\mu_0}{(N-\mu_m+l+2)(N-\mu_0+l+2)}+\frac{(f_{0,l}-1)(\mu_0-l)}{N-\mu_0+l+2}-\frac{(f_{m,l}-1)(\mu_m-l)}{N-\mu_m+l+2}>0 \tag{1.39}$$

Suppose $f_{m,l} \geq f_{0,l}$, then we can replace $f_{m,l}-1$ with $f_{0,l}-1$, and the inequality should still hold since we subtract a lesser value.

$$\frac{l+1}{pq}\frac{\mu_m-\mu_0}{(N-\mu_m+l)(N-\mu_0+l)}+\frac{(f_{0,l}-1)(\mu_m-l)}{N-\mu_m+l}-\frac{(f_{0,l}-1)(\mu_0-l)}{N-\mu_0+l}>0 \tag{1.40}$$

$$\frac{l+1}{pq}\frac{\mu_m-\mu_0}{(N-\mu_m+l+2)(N-\mu_0+l+2)}+(f_{0,l}-1)\left[\frac{\mu_0-l}{N-\mu_0+l+2}-\frac{\mu_m-l}{N-\mu_m+l+2}\right]>0 \tag{1.41}$$

The expression inside the brackets simplifies to:

$$\frac{\mu_0-l}{N-\mu_0+l+2}-\frac{\mu_m-l}{N-\mu_m+l+2}=-\frac{(N+2)(\mu_m-\mu_0)}{(N-\mu_m+l)(N-\mu_0+l)} \tag{1.42}$$

From here:

$$\frac{l+1}{pq}-(f_{0,l}-1)(N+2)>0 \tag{1.43}$$

$$\frac{l+1}{pq}>(f_{0,l}-1)(N+2) \tag{1.44}$$

From properties of $f_{0,l}$ we have:

$$\frac{l+1}{pq}>(f_{0,l}-1)(N+2) \tag{1.45}$$

$$\frac{l+1}{pq}>\frac{l+q}{p(\mu_0-l)}(N+2)>\frac{l+q}{p(\mu_0-l)}N \tag{1.46}$$

$$(l+1)(\mu_0-l)>Nq(l+q) \tag{1.47}$$

$$(l+1)(\mu_0-l)>\mu_0(l+q) \tag{1.48}$$

After simplification we have:

$$-l^2 - l + Npq > 0 \tag{1.49}$$

Since the standard deviation is $\sigma = \sqrt{Npq}$, this condition clearly does not hold for $l > \sigma$. Which proves an important point. When l exceeds σ, $f_{m,l+1}$ could only become greater than $f_{0,l+1}$ if at the previous iteration $f_{m,l}$ is less than $f_{0,l}$. And conversely, if $f_{m,l} > f_{0,l}$ at l, the next iteration $f_{m,l+1} < f_{0,l+1}$. We now can prove the main statement of the theorem.

Choose a particular value for l such that $l > \sigma$. For those values of m such that $f_{0,l} > f_{m,l}$ the theorem holds already, since $f_{0,l+1} > f_{0,l}$. Now consider those m for which $f_{0,l} < f_{m,l}$ at l. In the next iteration the probability ratio $f_{m,l+1}$ for such m has to be become less than $f_{0,l+1}$. Hence:

$$f_{0,l+1} > f_{m+1,l} > f_{m,l}$$

Which proves the theorem.

2 Derivative magic

Taking deirvate by μ in formula 1.30, we have

$$F'(\mu) = -\frac{(f_{m,l} - 1)(N + 2)pq - (l + 1)}{[pq(N + 2 + (f_{m,l} - 2)(\mu_m - l)) - (l + 1)]^2}$$

The denominator is always positive, hence the derivate is always negative when:

$$(f - 1)(N + 2)pq > (l + 1)$$

Which the exact condition for oscillating ratios (see formula 1.45). Why?