2. Perceptron & Adaline

COMP3314
Machine Learning

Outline

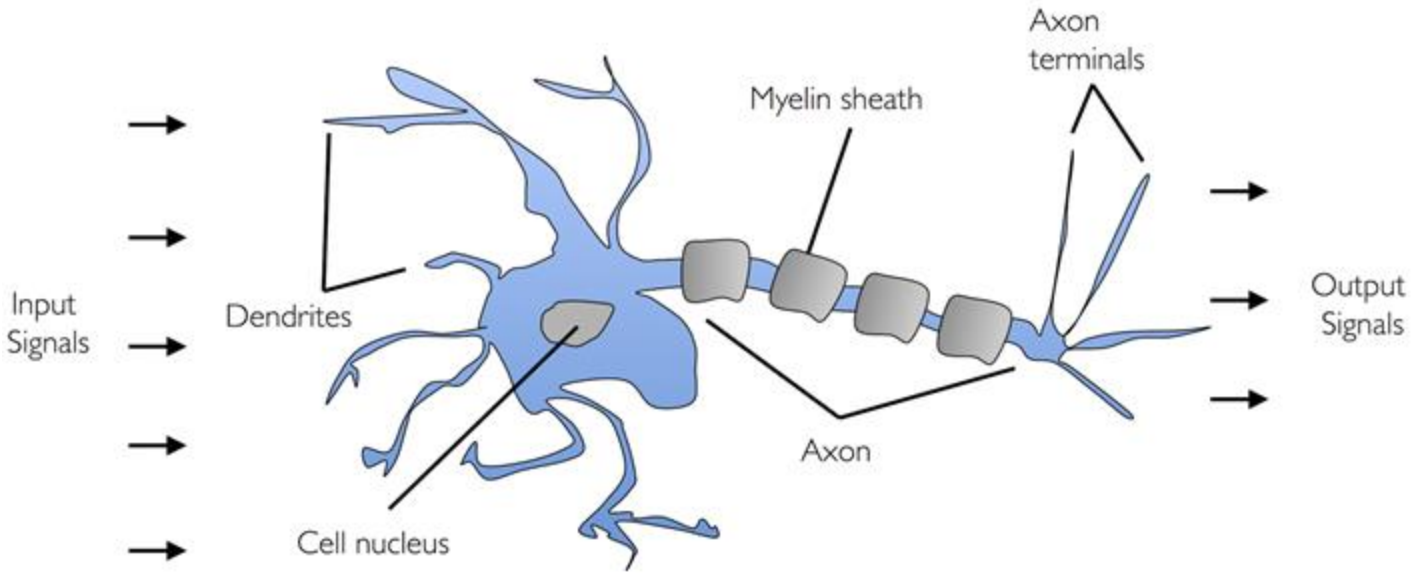
- In this chapter we will implement two of the first published machine learning algorithms for classification
  - Perceptron
  - Adaptive Linear Neurons (Adaline)
- This will lay the groundwork for using more powerful classifiers with the scikit-learn library
- We will
  - Build an intuition for machine learning algorithms
  - Use pandas, NumPy, and Matplotlib to read in, process, and visualize data
  - Implement linear classification algorithms in Python

Outline

- Neuron
- Artificial Neuron
  - History
  - Definition
- Perceptron
  - Perceptron Learning Rule
  - Linearly Separable
  - Implementation
- Adaptive Linear Neuron (Adaline)
  - Implementation
  - Feature Scaling
  - Stochastic Gradient Descent / Mini-Batch Learning

Neuron

- Neurons are interconnected nerve cells in the brain that are involved in the processing and transmitting of chemical and electrical signals



Artificial Neuron - History

- McCulloch and Pitts described the first artificial neuron in 1943 (aka MCP neuron) as a simple logic gate with binary outputs
  - Signals arrive at the dendrites, are integrated into the cell body, and, if the accumulated signal exceeds a certain threshold, an output signal is generated at the axon
- A few years later, in 1958, Rosenblatt published the first concept of the perceptron learning rule based on the MCP neuron
  - Automatically learn the optimal weight coefficients that are then multiplied with the input features

Artificial Neuron - Definition

- Consider a binary classification task where we refer to our two classes as 1 (positive class) and -1 (negative class)
- We can define a decision function

$$\phi(z) = \begin{cases} 1 \ \ if \ z \geq \theta \\ -1 \ \ otherwise \end{cases}$$

where θ is a threshold and z is the net input

$$z = w_1 x_1 + \ldots + w_m x_m$$

of the input values **x** and the corresponding weight vector **w**

$$\boldsymbol{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}, \ \boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

Artificial Neuron - Definition

- For simplicity, we can bring the threshold θ to the left side of the equation and define

$$z = w_0 x_0 + w_1 x_1 + \ldots + w_m x_m = \boldsymbol{w}^T \boldsymbol{x}$$

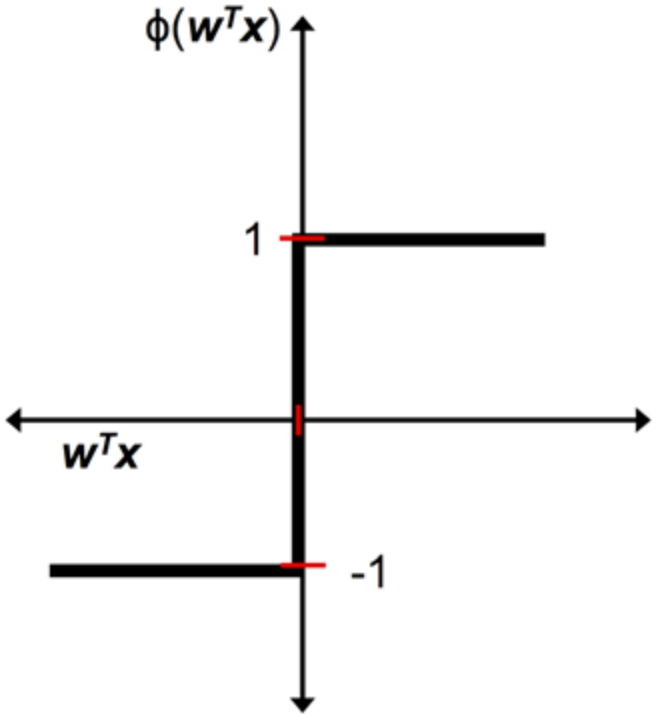
and

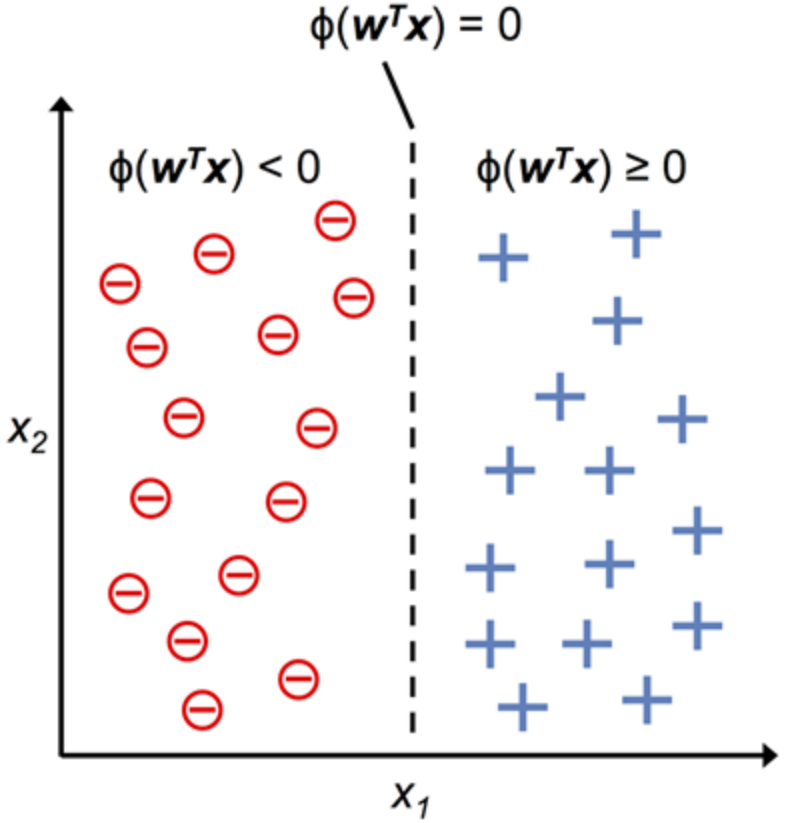
$$\phi(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

where $w_0 = -\theta$ and $x_0 = 1$

- w_0 is called the bias unit

Artificial Neuron - Definition





Perceptron Learning Rule

- Rosenblatt's perceptron rule can be summarized by the following steps
  a. Initialize the weights
  b. For each training sample $x^{(i)}$
     i. Compute the output value $\hat{y}$,
     i.e. the class label predicted by unit step function
     i. Update the weights

$$w_j := w_j + \Delta w_j$$

The value of Δw_j is calculated as follows

$$\Delta w_j = \eta \left(y^{(i)} - \hat{y}^{(i)} \right) x_j^{(i)}$$

Where η is the learning rate, $y^{(i)}$ is the true class label of the ith training sample, and $\hat{y}^{(i)}$ is the predicted class label

Weight Update

- All weights in the weight vector are updated simultaneously, which means that we don't recompute the $\hat{y}^{(i)}$ before all of the weights Δw_j are updated
  - I.e., for a two-dimensional dataset, we would write the update as

$$\Delta w_0 = \eta \left(y^{(i)} - output^{(i)} \right)$$

$$\Delta w_1 = \eta \left(y^{(i)} - output^{(i)} \right) x_1^{(i)}$$

$$\Delta w_2 = \eta \left(y^{(i)} - output^{(i)} \right) x_2^{(i)}$$

Weight Update

- In the two scenarios where the perceptron predicts the class label correctly, the weights remain unchanged

$$\Delta w_j = \eta\left(-1-\left(-1\right)\right)x_j^{(i)} = 0$$

$$\Delta w_j = \eta\left(1-1\right)x_j^{(i)} = 0$$

- However, in the case of a wrong prediction, the weights are being pushed towards the direction of the positive or negative target class

$$\Delta w_j = \eta\left(1--1\right)x_j^{(i)} = \eta\left(2\right)x_j^{(i)}$$

$$\Delta w_j = \eta\left(-1-1\right)x_j^{(i)} = \eta\left(-2\right)x_j^{(i)}$$

Weight Update

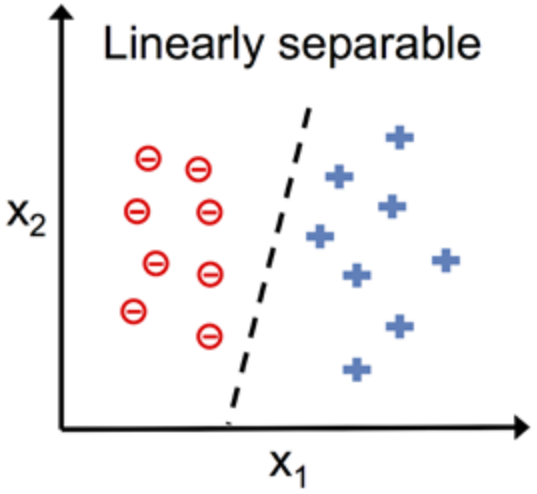
- Note that the weight update is proportional the value of $x_j^{(i)}$

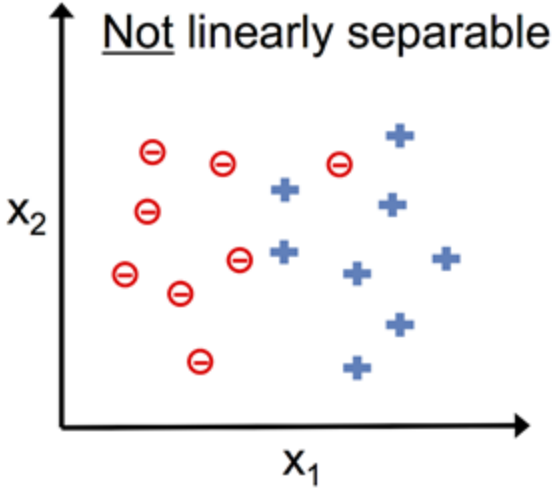
$$\Delta w_j = \eta \left(y^{(i)} - \hat{y}^{(i)} \right) x_j^{(i)}$$

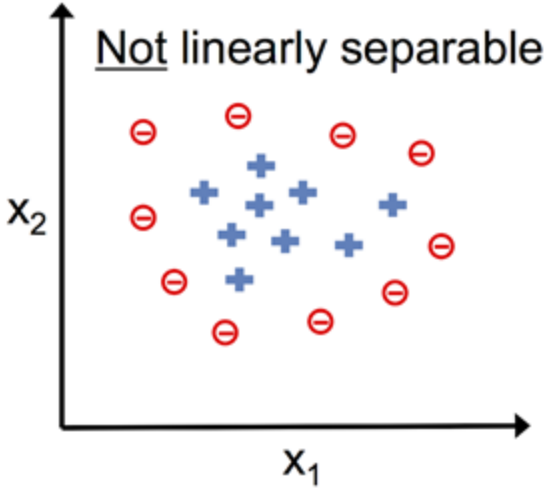
Linearly Separable

- It was shown that the convergence of the perceptron is only guaranteed if the two classes are linearly separable and the learning rate is sufficiently small
- If the two classes can't be separated by a linear decision boundary, we can set a maximum number of passes over the training dataset (epochs) and/or a threshold for the number of tolerated misclassifications
  - The perceptron would never stop updating the weights otherwise

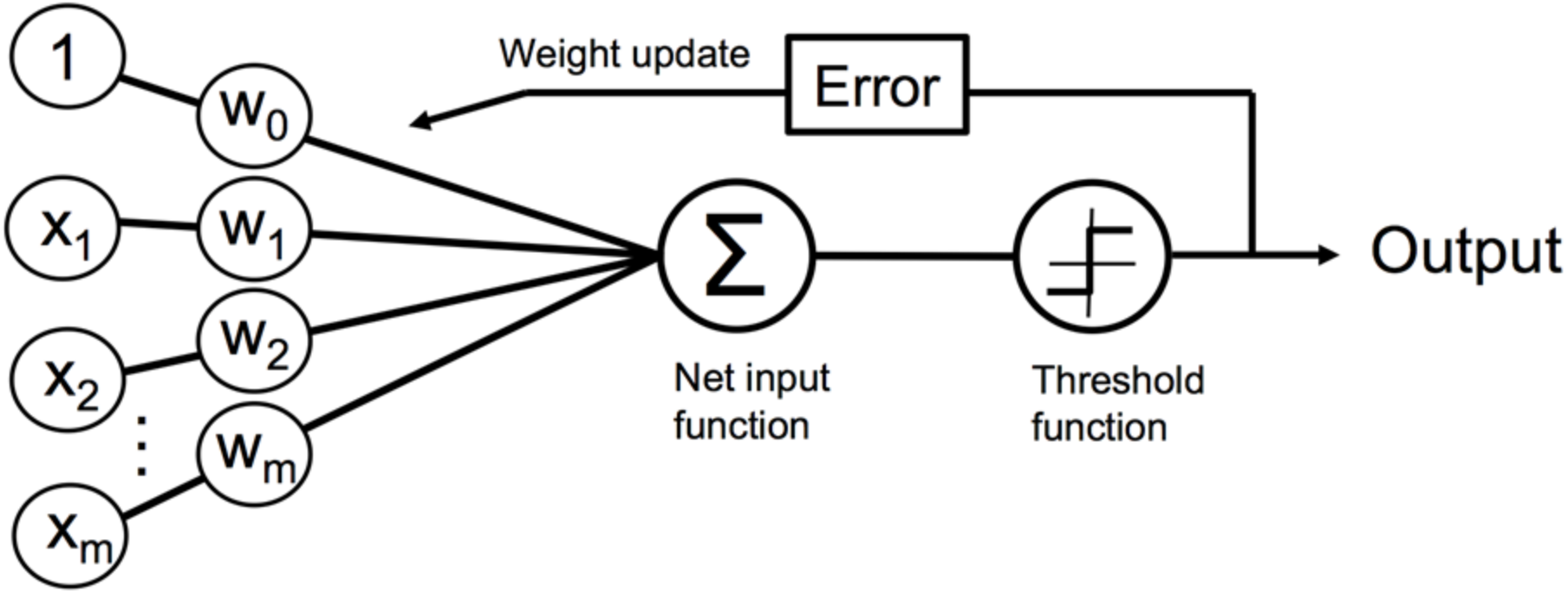
Linearly Separable vs. Not Linearly Separable







Perceptron


1
w_0
x_1
w_1
w_2
x_2
w_m
x_m
Weight update
Error
Σ
Net input function
Threshold function
Output

Iris Dataset - Loading

```
import pandas as pd
df = pd.read_csv('https://archive.ics.uci.edu/ml/machine-learning-databases/iris/iris.data', header=None)
df.head()
```

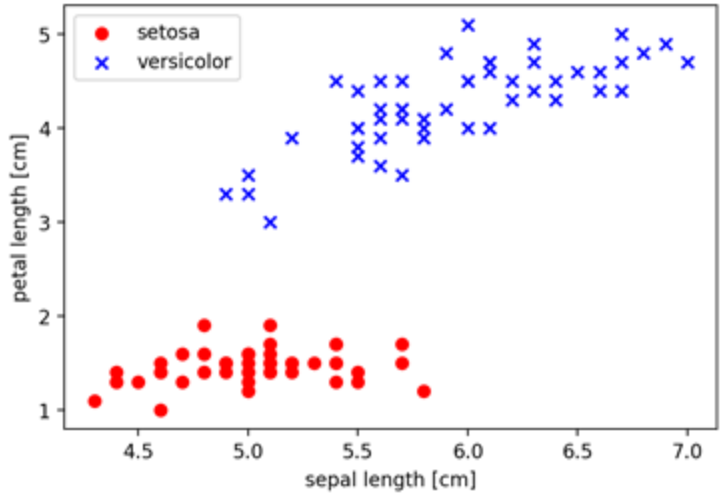
	0	1	2	3	4
0	5.1	3.5	1.4	0.2	Iris-setosa
1	4.9	3.0	1.4	0.2	Iris-setosa
2	4.7	3.2	1.3	0.2	Iris-setosa
3	4.6	3.1	1.5	0.2	Iris-setosa
4	5.0	3.6	1.4	0.2	Iris-setosa

Code is here

Iris Dataset - Preprocessing and Plotting

```
import numpy as np
y = df.iloc[0:100, 4]
y = np.where(y == 'Iris-setosa', -1, 1)
X = df.iloc[0:100, [0, 2]].values
```

```
import matplotlib.pyplot as plt
plt.scatter(X[:50, 0], X[:50, 1],
            color='red', marker='o', label='setosa')
plt.scatter(X[50:100, 0], X[50:100, 1],
            color='blue', marker='x', label='versicolor')
plt.xlabel('sepal length [cm]')
plt.ylabel('petal length [cm]')
plt.legend(loc='upper left')
plt.show()
```



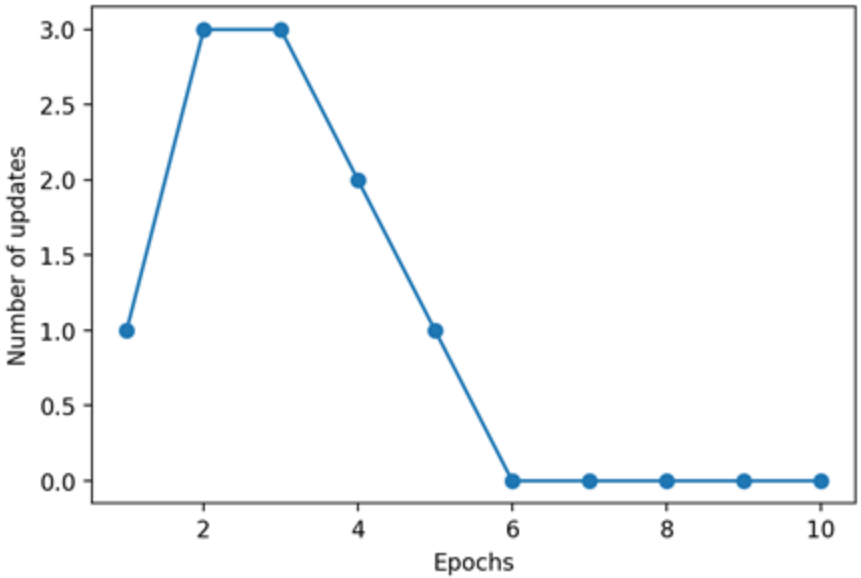
Perceptron - Implementation

$$\Delta w_j = \eta\left(y^{(i)} - \hat{y}^{(i)}\right)x_j^{(i)}$$

```python
import numpy as np
class Perceptron(object):
    def __init__(self, eta=0.01, n_iter=50, random_state=1):
        self.eta = eta
        self.n_iter = n_iter
        self.random_state = random_state
    def fit(self, X, y):
        rgen = np.random.RandomState(self.random_state)
        self.w_ = rgen.normal(loc=0.0, scale=0.01, size=1 + X.shape[1])
        self.errors_ = []
        for _ in range(self.n_iter):
            errors = 0
            for xi, target in zip(X, y):
                update = self.eta * (target - self.predict(xi))
                self.w_[1:] += update * xi
                self.w_[0] += update
                errors += int(update != 0.0)
            self.errors_.append(errors)
        return self
    def net_input(self, X):
        return np.dot(X, self.w_[1:]) + self.w_[0]
    def predict(self, X):
        return np.where(self.net_input(X) >= 0.0, 1, -1)
ppn = Perceptron(eta=0.1, n_iter=10)
ppn.fit(X, y)
```

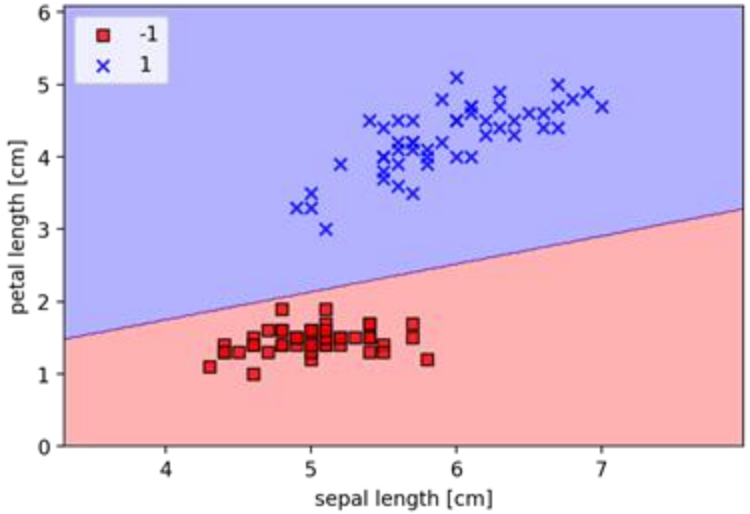
Perceptron - Training

```
plt.plot(range(1, len(ppn.errors_) + 1), ppn.errors_, marker='o')
plt.xlabel('Epochs')
plt.ylabel('Number of updates')
plt.show()
```



Perceptron - Plotting Decision Region

```
from matplotlib.colors import ListedColormap
def plot_decision_regions(X, y, classifier, resolution=0.02):
    markers = ('s', 'x', 'o', '^', 'v')
    colors = ('red', 'blue', 'lightgreen', 'gray', 'cyan')
    cmap = ListedColormap(colors[:len(np.unique(y))])
    x1_min, x1_max = X[:, 0].min() - 1, X[:, 0].max() + 1
    x2_min, x2_max = X[:, 1].min() - 1, X[:, 1].max() + 1
    xx1, xx2 = np.meshgrid(np.arange(x1_min, x1_max, resolution),
                           np.arange(x2_min, x2_max, resolution))
    Z = classifier.predict(np.array([xx1.ravel(), xx2.ravel()]).T)
    Z = Z.reshape(xx1.shape)
    plt.contourf(xx1, xx2, Z, alpha=0.3, cmap=cmap)
    plt.xlim(xx1.min(), xx1.max())
    plt.ylim(xx2.min(), xx2.max())
    for idx, cl in enumerate(np.unique(y)):
        plt.scatter(x=X[y == cl, 0], y=X[y == cl, 1], alpha=0.8,
                    c=colors[idx],  marker=markers[idx],
                    label=cl, edgecolor='black')
plot_decision_regions(X, y, classifier=ppn)
plt.xlabel('sepal length [cm]')
plt.ylabel('petal length [cm]')
plt.legend(loc='upper left')
plt.show()
```



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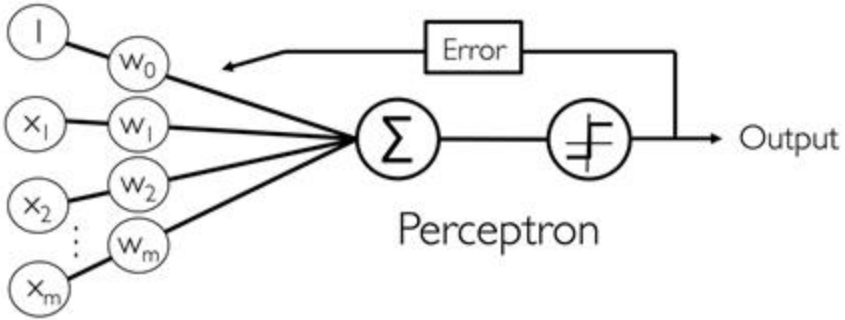
Adaptive Linear Neuron

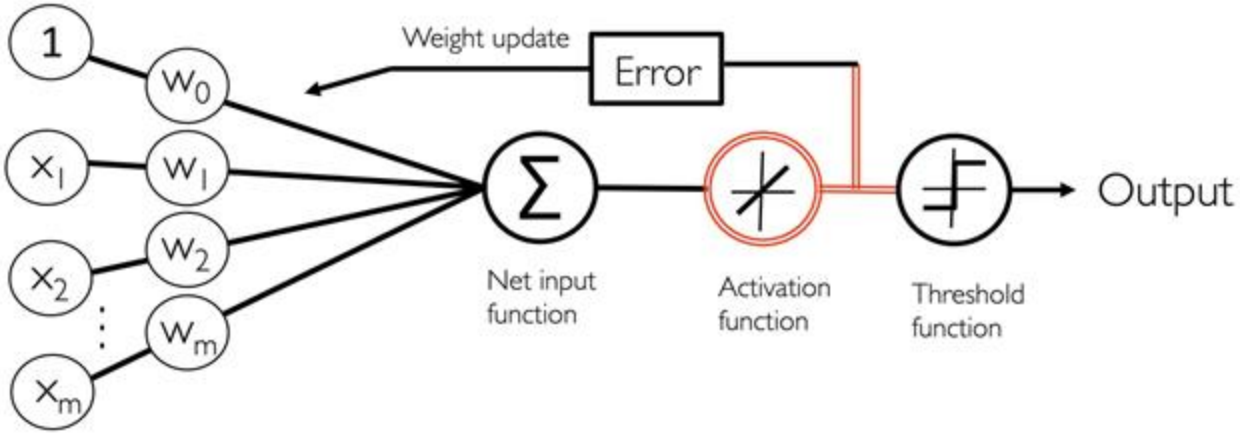
- ADAptive LInear NEuron: Adaline
- Improvement on Perceptron algorithm
- Published in 1960 by Bernard Widrow and Ted Hoff
- In Adaline the weights are updated based on a linear activation function

$$\phi\left(\boldsymbol{w}^T \boldsymbol{x}\right) = \boldsymbol{w}^T \boldsymbol{x}$$

- While the linear activation function is used for learning the weights, we still use a threshold function to make the final prediction

Adaptive Linear Neuron





Adaptive Linear Neuron (Adaline)

Objective Function

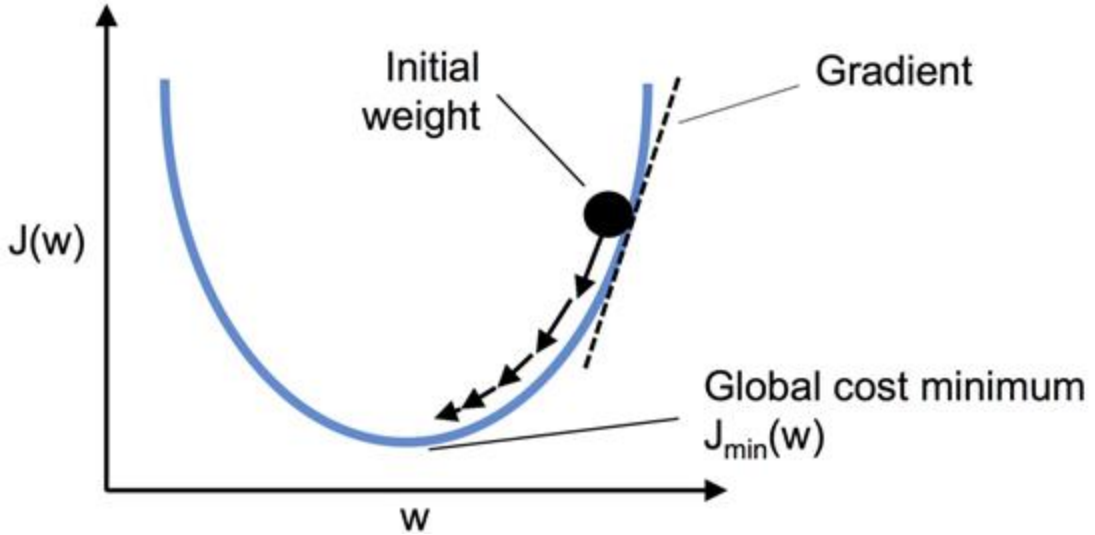
- One of the key ingredients of supervised ML algorithms is the objective function that is to be optimized
  - E.g., cost function that we want to minimize
- In Adaline the cost function *J* is defined as the sum of squared errors (SSE) between the calculated and true class label

$$J(\boldsymbol{w}) = \frac{1}{2}\sum_i \left(y^{(i)} - \phi\left(z^{(i)}\right)\right)^2$$

- It can be shown that *J* is differentiable and convex
  - It will be easy to minimize (using e.g. gradient descent)

Gradient Descent

- A generic optimization algorithm
  - Capable of finding optimal solutions to a wide range of problems
- Main idea
  - Tweak parameters iteratively in order to minimize a cost function
  - Measures the local gradient of the error function with regard to the parameter vector and goes in the direction of descending gradient
    - Once the gradient is zero, it has reached a minimum



Weight Update

- Update the weights by taking a step in the opposite direction of the gradient $\nabla J(\boldsymbol{w})$ of our cost function $J(\boldsymbol{w})$

$$\boldsymbol{w} := \boldsymbol{w} + \Delta \boldsymbol{w}$$

- The weight change $\mathbf{\Delta w}$ is defined as the negative gradient multiplied by the learning rate η

$$\Delta \boldsymbol{w} = -\eta \nabla \mathrm{J}\left(\boldsymbol{w}\right)$$

$$\frac{\partial J}{\partial w_j} = \frac{\partial}{\partial w_j} \frac{1}{2} \sum_i \left(y^{(i)} - \phi\left(z^{(i)} \right) \right)^2$$

$$= \frac{1}{2} \frac{\partial}{\partial w_j} \sum_i \left(y^{(i)} - \phi\left(z^{(i)} \right) \right)^2$$

$$= \frac{1}{2} \sum_i 2 \left(y^{(i)} - \phi\left(z^{(i)} \right) \right) \frac{\partial}{\partial w_j} \left(y^{(i)} - \phi\left(z^{(i)} \right) \right)$$

$$= \sum_i \left(y^{(i)} - \phi\left(z^{(i)} \right) \right) \frac{\partial}{\partial w_j} \left(y^{(i)} - \sum_k \left(w_k x_k^{(i)} \right) \right)$$

$$= \sum_i \left(y^{(i)} - \phi\left(z^{(i)} \right) \right) \left(-x_j^{(i)} \right)$$

$$= -\sum_i \left(y^{(i)} - \phi\left(z^{(i)} \right) \right) x_j^{(i)}$$

Weight Update

- To compute the gradient of the cost function, we compute the partial derivative of the cost function with respect to each weight w_j

$$\frac{\partial J}{\partial w_j} = -\sum_i \left(y^{(i)} - \phi\left(z^{(i)} \right) \right) x_j^{(i)}$$

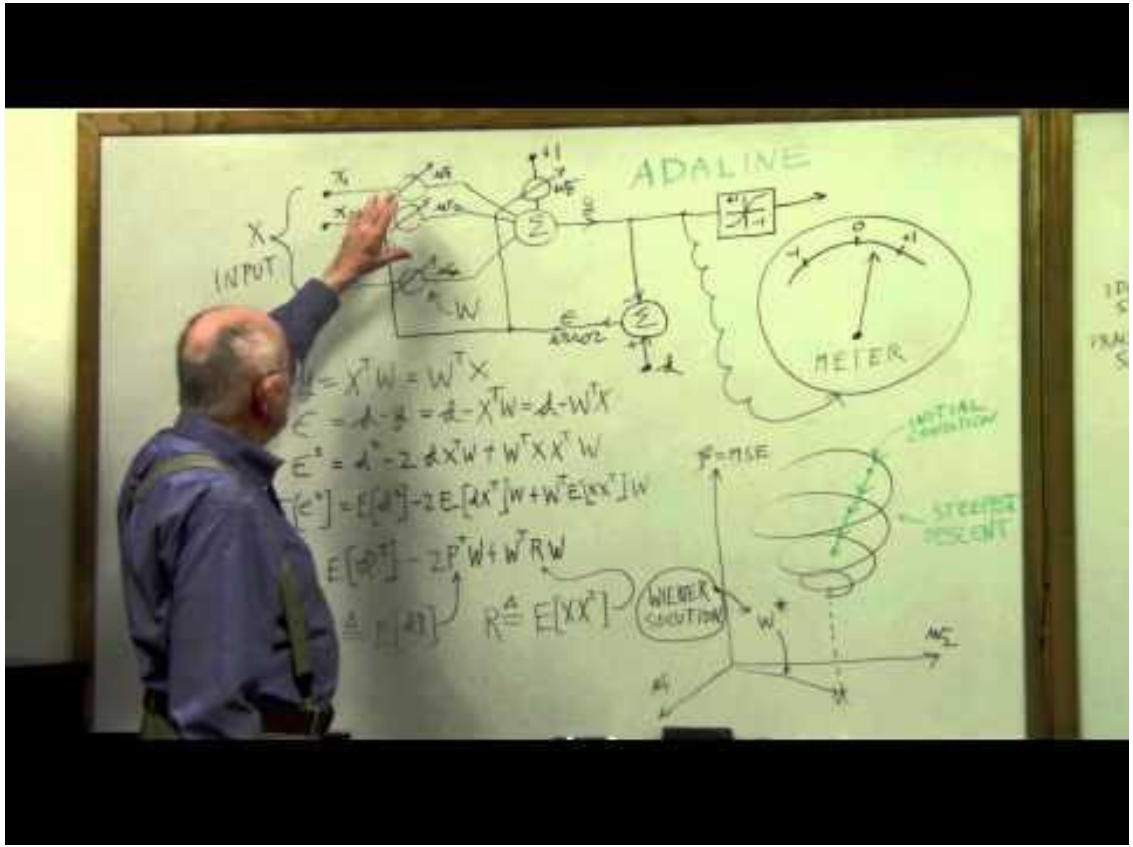
- The update of weight w_j can then be written as

$$\Delta w_j = -\eta \frac{\partial J}{\partial w_j} = \eta \sum_i \left(y^{(i)} - \phi\left(z^{(i)} \right) \right) x_j^{(i)}$$

- Batch gradient descent
  - Note that the weight update is calculated based on all samples in the training set (instead of updating the weights incrementally after each sample)

Learn from the Author

- Bernard Widrow and Ted Hoff published a paper introducing Adaline in 1959
- Bernard Widrow has a youtube channel
  - He has a video talking about Adaline :-)



Adaline - Implementation

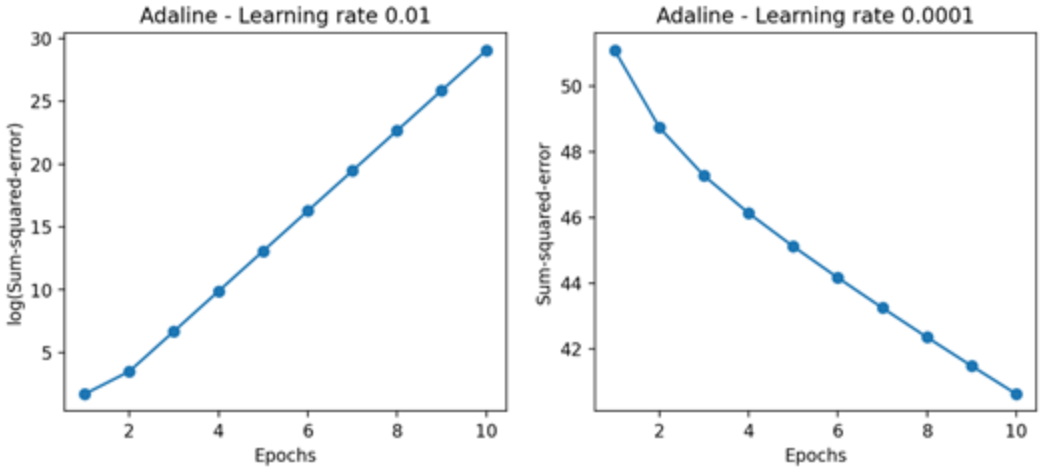
```
class AdalineGD(object):
    def __init__(self, eta=0.01, n_iter=50, random_state=1):
        self.eta = eta
        self.n_iter = n_iter
        self.random_state = random_state
    def fit(self, X, y):
        rgen = np.random.RandomState(self.random_state)
        self.w_ = rgen.normal(loc=0.0, scale=0.01, size=1 + X.shape[1])
        self.cost_ = []
        for i in range(self.n_iter):
            net_input = self.net_input(X)
            output = self.activation(net_input)
            errors = (y - output)
            self.w_[1:] += self.eta * X.T.dot(errors)
            self.w_[0] += self.eta * errors.sum()
            cost = (errors**2).sum() / 2.0
            self.cost_.append(cost)
        return self
    def net_input(self, X):
        return np.dot(X, self.w_[1:]) + self.w_[0]
    def activation(self, X):
        return X
    def predict(self, X):
        return np.where(self.activation(self.net_input(X)) >= 0.0, 1, -1)
ada = AdalineGD(n_iter=30, eta=0.0001).fit(X, y)
```

$$\Delta w_j = -\eta \frac{\partial J}{\partial w_j} = \eta \sum_i \left(y^{(i)} - \phi\left(z^{(i)}\right) \right) x_j^{(i)}$$

$$J(\boldsymbol{w}) = \frac{1}{2} \sum_i \left(y^{(i)} - \phi\left(z^{(i)}\right) \right)^2$$

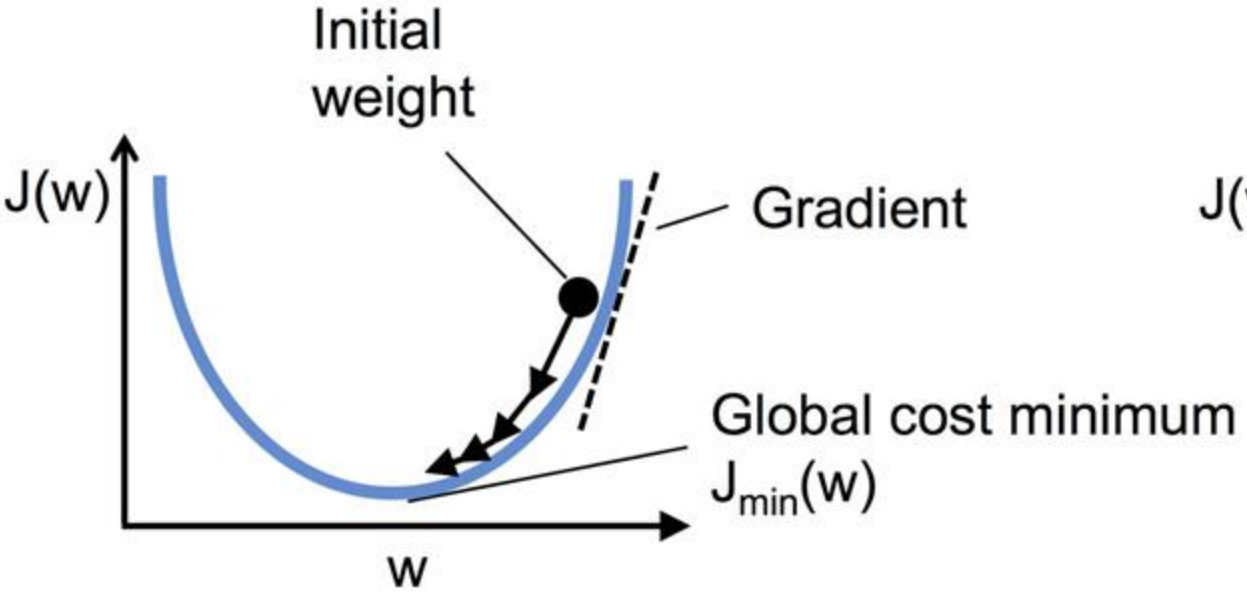
→ Code is here ←

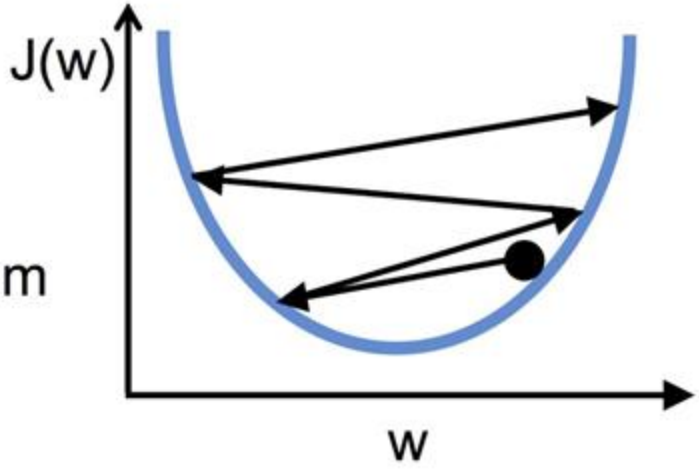
```
import matplotlib.pyplot as plt
fig, ax = plt.subplots(nrows=1, ncols=2, figsize=(10, 4))
ada1 = AdalineGD(n_iter=10, eta=0.01).fit(X, y)
ax[0].plot(range(1, len(ada1.cost_) + 1), np.log10(ada1.cost_), marker='o')
ax[0].set_xlabel('Epochs')
ax[0].set_ylabel('log(Sum-squared-error)')
ax[0].set_title('Adaline - Learning rate 0.01')
ada2 = AdalineGD(n_iter=10, eta=0.0001).fit(X, y)
ax[1].plot(range(1, len(ada2.cost_) + 1), ada2.cost_, marker='o')
ax[1].set_xlabel('Epochs')
ax[1].set_ylabel('Sum-squared-error')
ax[1].set_title('Adaline - Learning rate 0.0001')
plt.show()
```



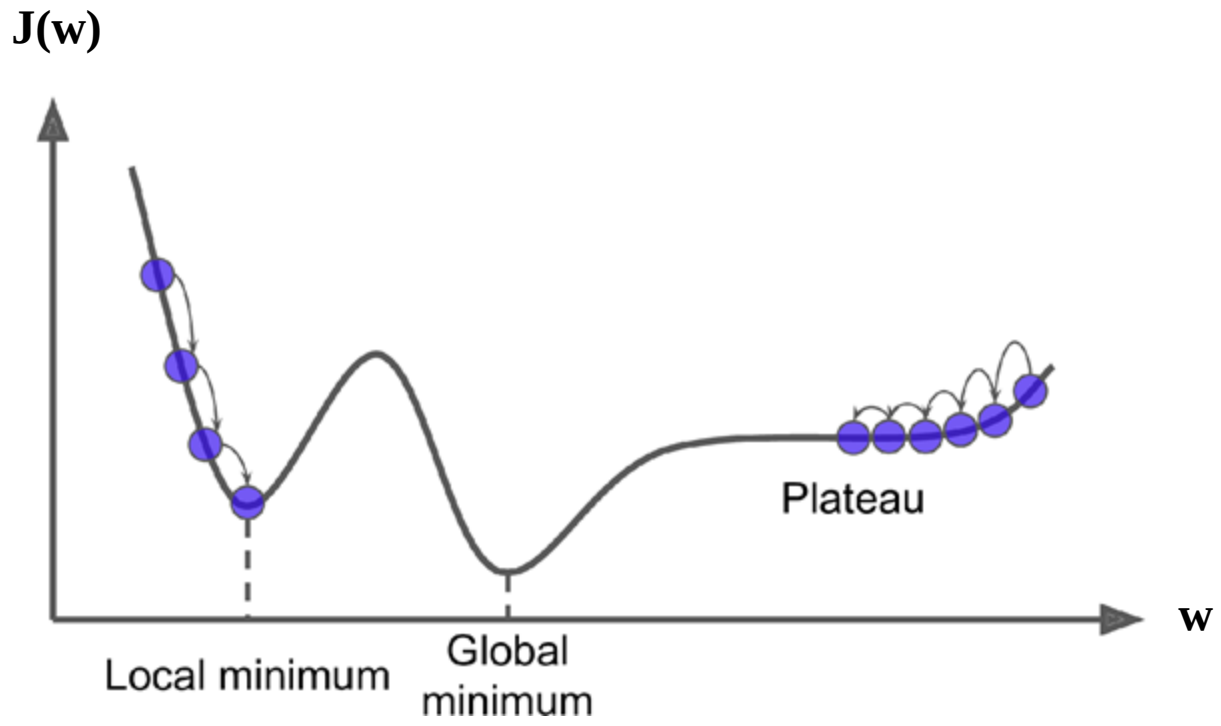
Learning Rate

- Important hyperparameter of Gradient Descent
- If too small
  - Requires excessive many iterations to converge
    - Takes a long time
- If too large
  - Might jump across the valley and end up on the other side
    - Possibly even higher up than before
    - Algorithm may diverge, with larger and larger values


Initial
weight
J(w)
Gradient
Global cost minimum
$J_{min}(w)$
w

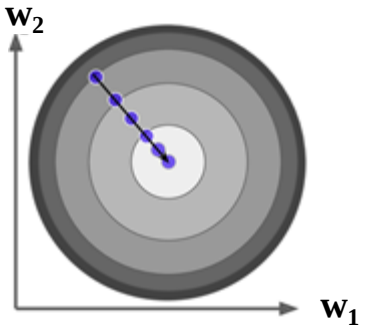

J(w)
w

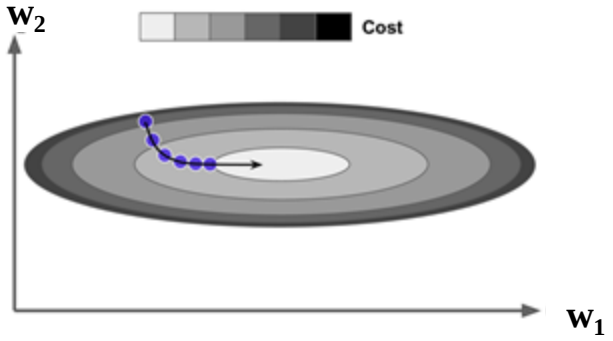
Gradient Descent Pitfalls



Feature Scaling

- Many ML algorithms require feature scaling for optimal performance
  - E.g., gradient descent converges more quickly if our data follows a standard distribution
- Standardization
  - A feature scaling method
  - After standardization, feature have
    - a mean value of 0
    - a standard deviation of 1





Standardization

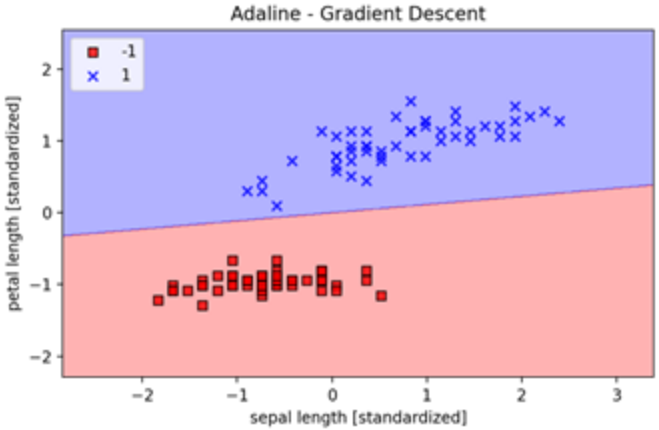
- For instance, to standardize the *j*th feature, we can simply subtract the sample mean μ_j from every training sample and divide it by its standard deviation σ_j

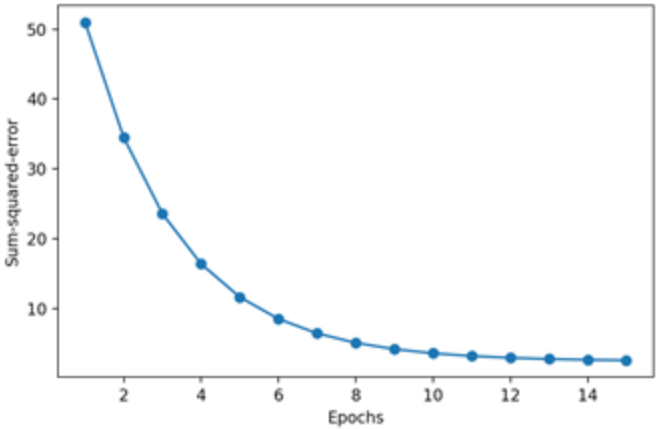
$$x'_j = \frac{x_j - \mu_j}{\sigma_j}$$

- Here, x_j is a vector consisting of the *j*th feature values of all training samples n
- This standardization technique is applied to each feature *j* in our dataset

```
X_std = np.copy(X)
X_std[:, 0] = (X[:, 0] - X[:, 0].mean()) / X[:, 0].std()
X_std[:, 1] = (X[:, 1] - X[:, 1].mean()) / X[:, 1].std()
```

```
ada = AdalineGD(n_iter=15, eta=0.01)
ada.fit(X_std, y)
plot_decision_regions(X_std, y, classifier=ada)
plt.title('Adaline - Gradient Descent')
plt.xlabel('sepal length [standardized]')
plt.ylabel('petal length [standardized]')
plt.legend(loc='upper left')
plt.tight_layout()
plt.show()
plt.plot(range(1, len(ada.cost_) + 1), ada.cost_, marker='o')
plt.xlabel('Epochs')
plt.ylabel('Sum-squared-error')
plt.tight_layout()
plt.show()
```





Stochastic Gradient Descent

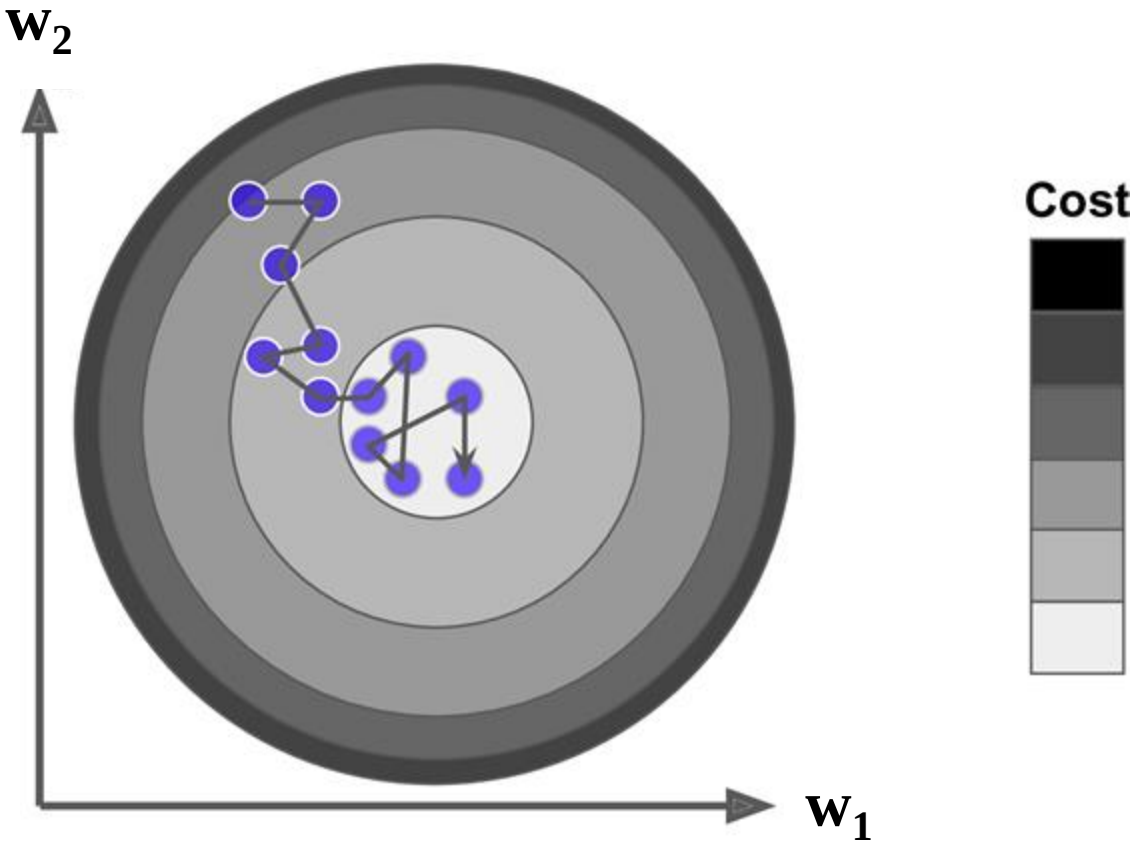
- Imagine we have a very large dataset with millions of data points
  - Not uncommon in ML applications
  - Running batch gradient descent can be computationally costly in such scenarios since we need to reevaluate the whole training dataset each time we take one step
- Stochastic Gradient Descent is a popular alternative
  - Instead of updating the weights based on the sum of the accumulated errors over all samples $\boldsymbol{x}^{(i)}$

$$\Delta \boldsymbol{w} = \eta \sum_i \left(y^{(i)} - \phi\left(z^{(i)} \right) \right) \boldsymbol{x}^{(i)}$$

  we update all the weights incremental for each training sample

Stochastic Gradient Descent

- Typically reaches convergences faster because of the more frequent weight updates
- Can escape shallow local minima more readily if we are working with nonlinear cost functions
- It is important to present it training data in a random order
- In addition shuffle the training set for every epoch to prevent cycles
- Another advantage of stochastic gradient descent is that we can use it for online learning
    - In online learning, our model is trained on the fly as new training data arrives


w_2
w_1
Cost

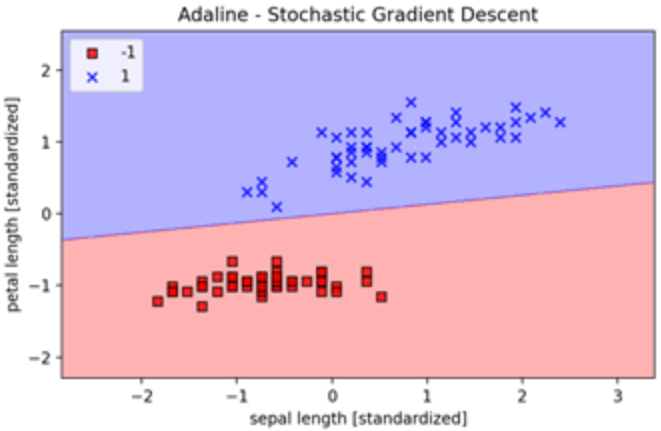
Mini-Batch Learning

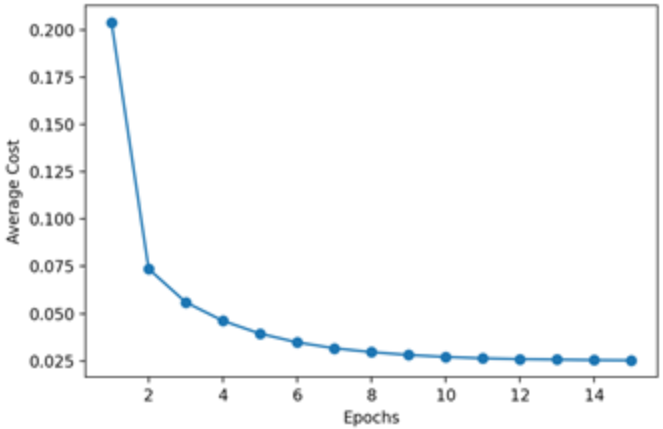
- A compromise between batch gradient descent and stochastic gradient descent
- Apply batch gradient descent to smaller subsets of the training data
  - E.g., 32 samples at a time
- The advantage over batch gradient descent is that convergence is reached faster via mini-batches because of the more frequent weight updates
- Furthermore, mini-batch learning allows us to replace the for loop over the training samples in stochastic gradient descent with vectorized operations

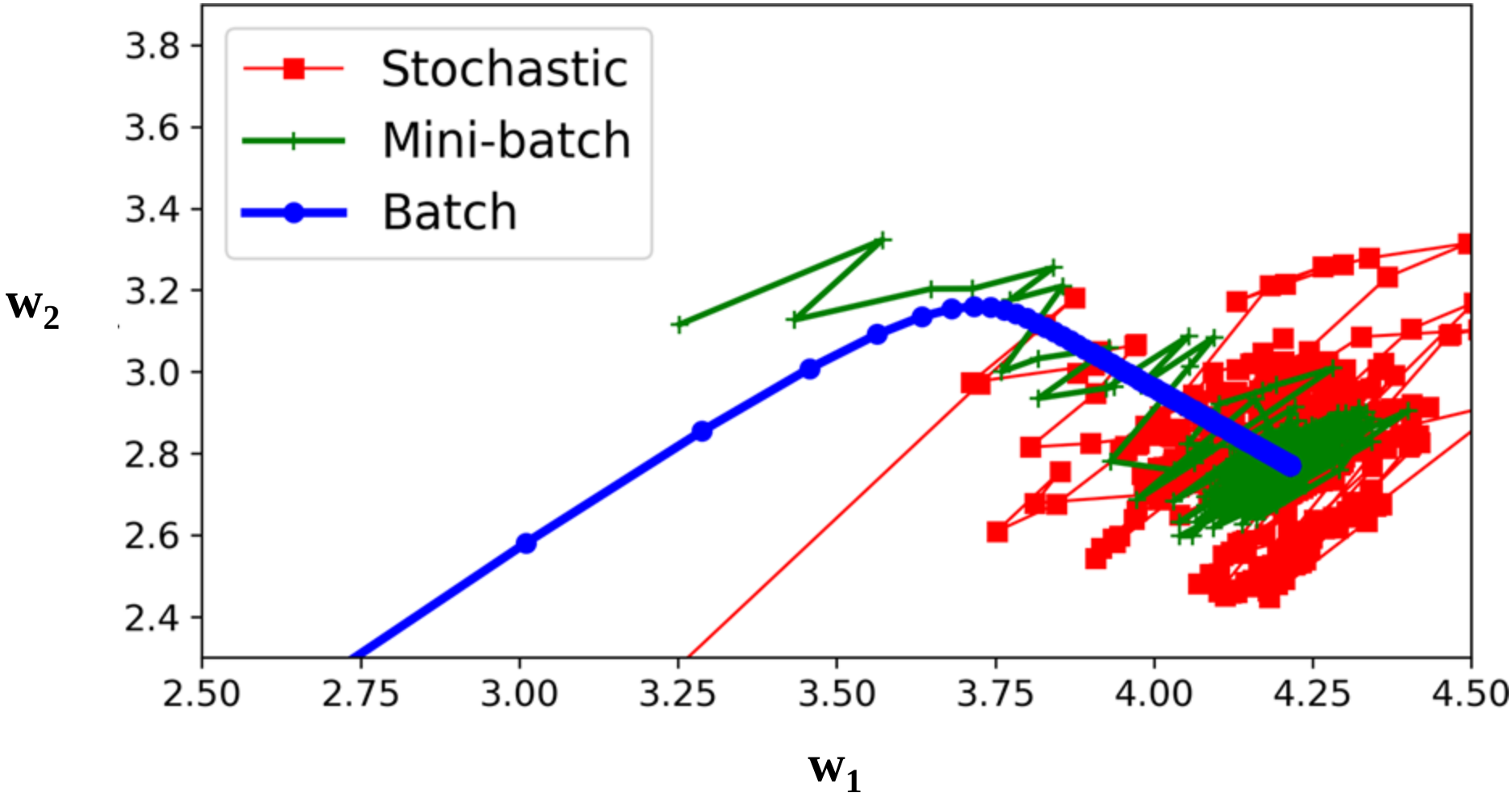
```
class AdalineSGD(object):
    def __init__(self, eta=0.01, n_iter=10, shuffle=True, random_state=None):
        self.eta = eta
        self.n_iter = n_iter
        self.w_initialized = False
        self.shuffle = shuffle
        self.random_state = random_state
    def fit(self, X, y):
        self._initialize_weights(X.shape[1])
        self.cost_ = []
        for i in range(self.n_iter):
            if self.shuffle:
                X, y = self._shuffle(X, y)
            cost = []
            for xi, target in zip(X, y):
                cost.append(self._update_weights(xi, target))
            avg_cost = sum(cost) / len(y)
            self.cost_.append(avg_cost)
        return self
    def partial_fit(self, X, y):
        if not self.w_initialized:
            self._initialize_weights(X.shape[1])
        if y.ravel().shape[0] > 1:
            for xi, target in zip(X, y):
                self._update_weights(xi, target)
        else:
            self._update_weights(X, y)
        return self
```

```
    def _shuffle(self, X, y):
        r = self.rgen.permutation(len(y))
        return X[r], y[r]
    def _initialize_weights(self, m):
        self.rgen = np.random.RandomState(self.random_state)
        self.w_ = self.rgen.normal(loc=0.0, scale=0.01, size=1 + m)
        self.w_initialized = True
    def _update_weights(self, xi, target):
        output = self.activation(self.net_input(xi))
        error = (target - output)
        self.w_[1:] += self.eta * xi.dot(error)
        self.w_[0] += self.eta * error
        cost = 0.5 * error**2
        return cost
    def net_input(self, X):
        return np.dot(X, self.w_[1:]) + self.w_[0]
    def activation(self, X):
        return X
    def predict(self, X):
        return np.where(self.activation(self.net_input(X)) >= 0.0, 1, -1)
```

```
ada = AdalineSGD(n_iter=15, eta=0.01, random_state=1)
ada.fit(X_std, y)
plot_decision_regions(X_std, y, classifier=ada)
plt.title('Adaline - Stochastic Gradient Descent')
plt.xlabel('sepal length [standardized]')
plt.ylabel('petal length [standardized]')
plt.legend(loc='upper left')
plt.tight_layout()
plt.show()
plt.plot(range(1, len(ada.cost_) + 1), ada.cost_, marker='o')
plt.xlabel('Epochs')
plt.ylabel('Average Cost')
plt.tight_layout()
plt.show()
```





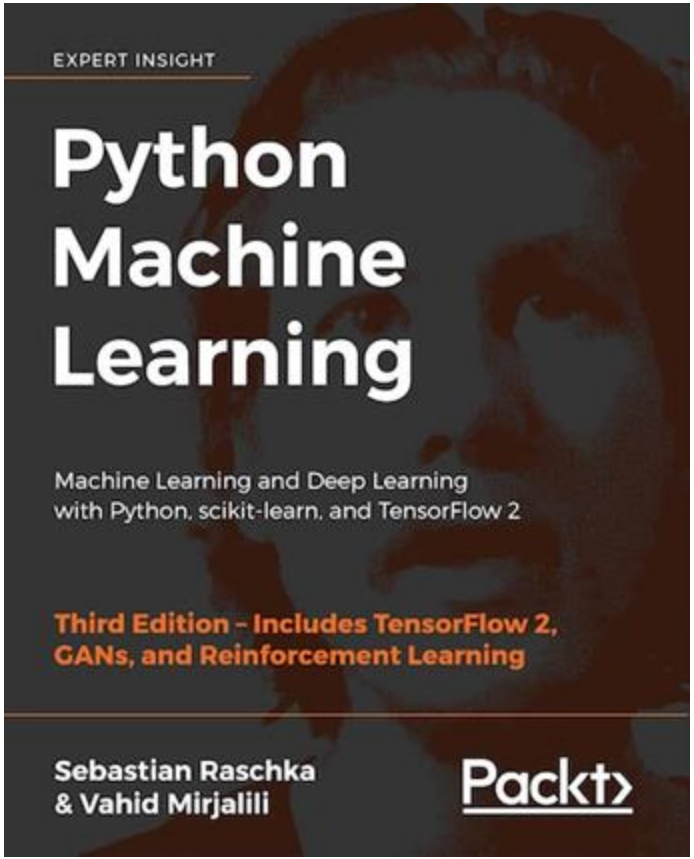

Stochastic
Mini-batch
Batch
3.8
3.6
3.4
3.2
3.0
2.8
2.6
2.4
2.50
2.75
3.00
3.25
3.50
3.75
4.00
4.25
4.50
w_2
w_1

Summary

- Gained a good understanding of the basic concepts of linear classifiers for supervised learning
- Implemented
  - Perceptron
  - Adaline
- Efficient training via a vectorized implementation of gradient descent and online learning via stochastic gradient descent

References

- Most materials in this chapter are based on
  - Book
  - Code



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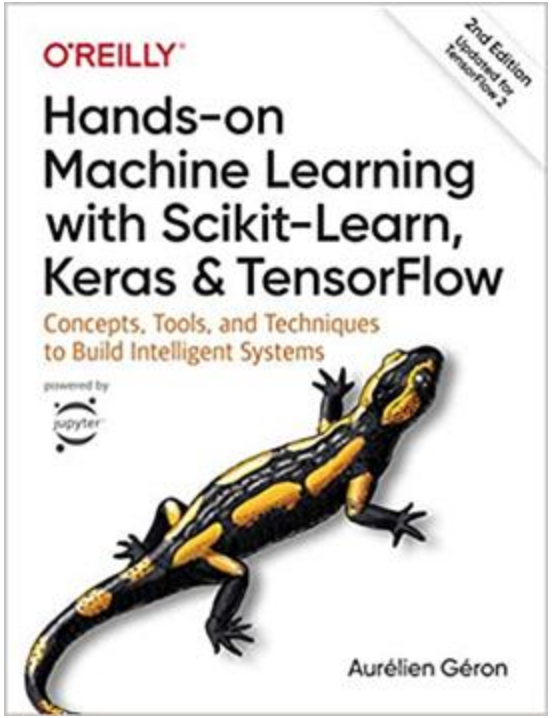

Exercise 1

- Suppose the features in your training set have very different scales
  - What can you do about it?
- Can Gradient Descent get stuck in a local minimum when training a Logistic Regression model?
- Do all Gradient Descent algorithms lead to the same model, provided you let them run long enough?
- Suppose you use Batch Gradient Descent and you plot the validation error at every epoch
  - If you notice that the validation error consistently goes up, what is likely going on? How can you fix this?
- Is it a good idea to stop Mini-batch Gradient Descent immediately when the validation error goes up?