$\frac{1}{4} - \frac{x^4}{4} - \frac{x^6}{6}$ $\frac{1}{4} - \frac{1}{3}$ $= \frac{3-2}{12} = \frac{1}{12}$

class: Time is a dimension in relativity — Quantum physics
in spring theory, we've 10 dimension

Lecture: GEG 311

Recommended Textbook:
Mathematics of physics and modern engineering – by I.S. Okoni-ter

Mathematics function that depends on several variables

Basic Definitions

Let n be a positive integer and other set of n real numbers $(x_1, x_2, x_3, \dots x_n)$ is called an n dimensional points. If such given points satisfy the rules of some vector algebra, then each point is a vector with n components

Vector Algebra rules, vector space

$x = (x_1, x_2, x_3 \dots x_n)$

$y = (y_1, y_2, y_3 \dots y_n)$

~~A~~ set of n dimensional point is called n dimensional space or n space and is denoted R^n ~~i.e. a set of n~~

When $n = 1$, lines and curves are traced

When $n = 2$, we generate planes & surfaces

$n = 3$, we've volumes and solids (generate shapes of various type)

~~In Spring theory~~

Recall that a metric space is a non-empty set n of point together

Ex 1: Given M is any non-empty set, we can define a distance function such that $d(x, y) = 0$ if $x = y$ otherwise $d(x, y) = 1$

The discrete matrix — Define a matrix such that

M is the set of triples (x_1, x_2, x_3) such that x_1

$x_1^2 + x_2^2 + x_3^2 = 1$ (Everything belongs ~~to~~ a unit square matrix of radius 1)

Euclidean Matrix

Pythagoras theory

~~$z = \sqrt{x^2 + y^2}$~~

~~Complex M~~ Metrix space

Discrete function, Euclidean metric

If P_0 is a point in a set n

$P_0 \in M$

$B_m(P_0, \delta)$ (dimensional sphere m ball = B_m)

center P_0 and radius δ is defined to be the set of all points x in such that the metrics of distance

Mathematically,

$B_m(P_0, \delta) = \{x\} \subseteq M] \cdot d(x, P_0) < \delta$

Epsilon delta

Unless otherwise stated we assume and associate the m ball (B_m) i

$$0 < \sum_{k=1}^{n} (x_k - x_k^0)^2 < \delta^2$$

We can go from $-r$ to r

A delta neighborhood of a point P_0 contained in S_1 is the set of point lying inside the n-dimensional sphere with centre P_0 and radius δ

$P_0 \in S \subset R^n$

~~Pt~~ of ~~x~~ S if every neighborhood of P contain some in S and some do not lie in S

A set S is bounded if it is possible to enclose the whole set within a sufficiently large neighborhood. Two points in a set are connected if there is a path that connects them wholly ~~in a~~ within the set.

Let P

$P(x_1, x_2,$

a function f on R^n

$P = (x_1, x_2, x_3 \cdots x_n) \in R^n$

$f(P) = w \in \Omega' \subseteq \Omega$

f is the image of f of P

domain ~~and~~ and range

Input → Output

Output is always unique (has no multiple value)

Formal Analysis

Metric space

Advance Engineering Maths by Erwin Kreyszig

Mathematics for Engineers & Scientist

Mathematics of Physics and Modern Engineering

Sokolinkoff Redheffer

Second Edition

SSG335 (Laboratory)

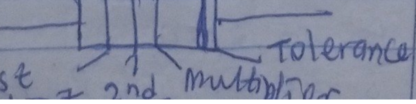
~~[illegible]~~ (Robotics)

~~Resistors~~

Resistors Color Codes

4, 5 and 6

They exist in 4, 5 and 6 band resistor



Resistors
Capacitors
Inductors
Semiconductors
Motors

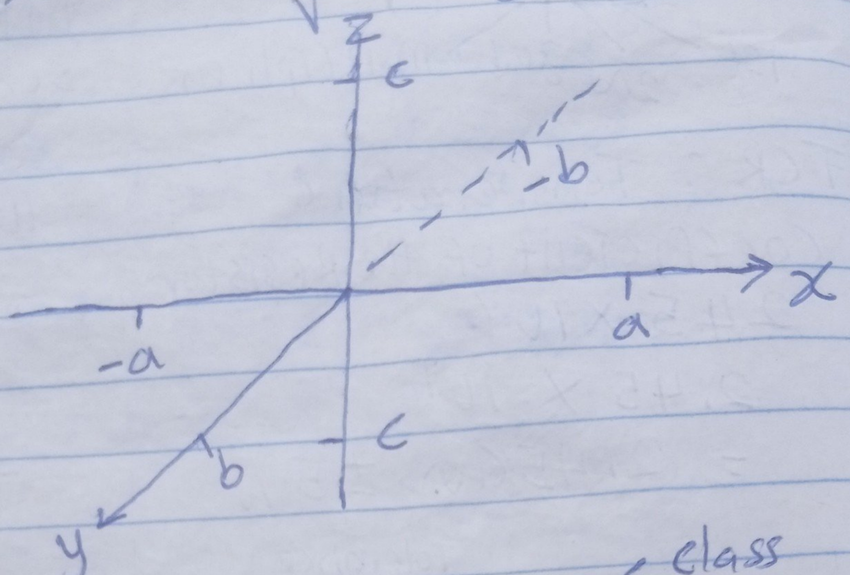
Resistors
Capacitors
Inductors
Transistors
LED
Motors (DC, stepper, servo)
Buzzer
Piezo
Switches
ICs
Transformers
Diodes

Resistors colour codes

	1	2	3	4
B	0	0	0	±1%
B	1	1	1	±2%
R	2	2	2	
O	3	3	3	
Y	4	4	4	±0.5%
G	5	5	5	±0.25%
B	6	6	6	±1.0%
V	7	7	7	±0.05%
G	8	8	8	
W	9	9	9	
G	–	–	–	±5%
S	–	–	–	[illegible]
–	–	–	–	±10%

Black
Brown
Red
Orange
Yellow
Green
Blue
Violet
Grey
White
Gold
Silver

$$x = -a\sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}}$$



Partial Derivatives (class lecture)

Let P be an n-dimensional point and $f(P)$ be a leading function, in domain D, if $f(P)$ is defined at a point P_0 and for all P in some neighborhood of P_0 contained in D, then the P.D of f with respect to any x-scale is conceptually the ordinary derivative w.r.t to that variable, all other independent variable kept constant.

$$P = \{x_1, x_2, \ldots, x_K, \ldots, x_n\}$$

$0 = -2i_2 + 7i_3 - i_4$ $\qquad -5i_2 + 26i_4 - i_3 = 0$

GEG 311

(2) Find the values of n for which $r^n(3\cos^2\theta - 1)$ satisfies

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \text{ is true}$$

Soln

$V(r,\theta) = r^n(3\cos^2\theta - 1)$ satisfies the axisymmetric Laplace equation in spherical coordinates

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \quad \text{—— (1)}$$

Put $V(r,\theta)$ into equ (1)

Let $A(\theta) = 3\cos^2\theta - 1 \Rightarrow V(r,\theta) = r^n A(\theta)$

- radial part

$$\frac{\partial V}{\partial r} = nr^{n-1}A(\theta)$$

$$\Rightarrow r^2\frac{\partial V}{\partial r} = r^2(nr^{n-1}A) = nr^{n+1}A$$

So, $\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = \frac{\partial}{\partial r}(nr^{n+1}A) = n(n+1)r^nA(\theta)$

$A(\theta)$ is independent of r

- Angular part

Compute $A'(\theta)$:

$$A(\theta) = 3\cos^2\theta - 1 \Rightarrow \frac{dA}{d\theta} = 3\cdot 2\cos\theta\cdot(-\sin\theta)$$

$$= -6\sin\theta\cos\theta$$

Then, $\sin\theta\frac{\partial V}{\partial\theta} = \sin\theta\cdot r^nA'(\theta) = r^n\sin\theta(-6\sin\theta\cos\theta)$

$$= -6r^n\sin^2\theta\cos\theta$$

Differentiate w.r.t. θ:

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = -6r^n\frac{d}{d\theta}(\sin^2\theta\cos\theta)$$

$$\frac{d}{d\theta}(\sin^2\theta\cos\theta) = 2\sin\theta\cos\theta\cdot\cos\theta + \sin^2\theta(-\sin\theta)$$

$$= 2\sin\theta\cos^2\theta - \sin^3\theta$$

So, $\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = -6r^n(2\sin\theta\cos^2\theta - \sin^3\theta)$

Divide by $\sin\theta$:

$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)$ ~~$= -6r^n(2\sin\theta\cos^2\theta -$~~

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = -6r^n(2\cos^2\theta - \sin^2\theta)$$

GEG311 Limits and continuity Lecture

EX3 : $f = (3x^2+1)/(1-(y^2+z^2))$ L' hopital's rule

EX4 : Evaluate $\lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x^2 y}{x^4+y^2}\right)$

$L_1 = \lim_{x \to \infty} \left(\frac{2x+4}{x^2+8}\right)$

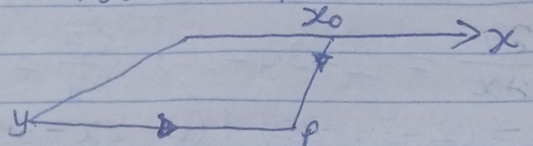
Evaluate $\lim_{\substack{x \to \infty \\ y \to 2}} \left(\frac{xy+4}{x^2+2y^2}\right)$

$= \lim_{x \to \infty} \left(\frac{2+4/x}{x+8/x}\right)$

$$\frac{\frac{y}{x}+\frac{4}{x^2}}{1+\frac{2y^2}{x^2}}$$

Extrema of functions

roots of equations



Homogeneous function

A function $f(x,y)$ is said to be a hom- fn when the power of each term is the same, and it is of order n when the degree of each of its terms in x and y is equal to n

e.g. $f(x,y) = a_0x^n + a_1x^{n-1}y + \dots + a_kx^{n-k}y^k + \dots + a_ny^n$ —(1)

$= x^n\phi\left(\frac{y}{x}\right)$ ϕ = multiply

from (1) $f = x^n\left[a_0 + a_1\left(\frac{y}{x}\right) + \dots a_k\left(\frac{y}{x}\right)^k + \dots a_n\left(\frac{y}{x}\right)^n\right]$

$f = x^2+xy-y^2$ Homogeneous of order two

$= x^2\left(1+\frac{y}{x}-\left(\frac{y}{x}\right)^2\right)$

$$f = \frac{\sqrt{x}-\sqrt{y}}{x^2+y^2} = \frac{\sqrt{x}\left(1-\sqrt{\frac{y}{x}}\right)}{x^2\left(1+\left(\frac{y}{x}\right)^2\right)}$$

$$= \frac{x^{-3/2}\left(1-\sqrt{\frac{y}{x}}\right)}{1+\left(\frac{y}{x}\right)^2}$$

Homogeneous of order $\frac{-3}{2}$

Leonard Euler : $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

theory

Euler's theorem of homogeneous function

Let $f = x^n\phi\left(\frac{y}{x}\right)$ —(2)

$f_x = nx^{n-1}\phi + x^n\left(\frac{-y}{x^2}\right)\phi'$ differentiate wrt x

$\Rightarrow xf_x = nx^n\phi - yx^{n-1}\phi'$ —(3)

$f_y = x^n \cdot \frac{1}{x}\phi'$

$yf_y = x^{n-1}y\phi'$ —(4)

adding (3) and (4)

$xf_x + yf_y = nf$ (Euler's first rule)

Show that $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1)f$

$f_x = nx^{n-1}\phi + x^n\left(\frac{-y}{x^2}\right)\phi = nx^{n-1}\phi - x^{n-2}y\phi$

$f_{xx} = n(n-1)x^{n-2}\phi + nx^{n-1}\phi' + \cancel{nx^{n}}$

$f_y = x^n \cdot \frac{1}{x}\phi' = x^{n-1}\phi'$

$f_{yy} = x^{n-1}\phi''$

$-x^{n-2}y\phi'\left(\frac{y}{x}\right) \qquad x^{n-2}y\phi'\left(\frac{y}{x}\right)$

x

$\cancel{f_{xx}} = f_x = nx^{n-1}\phi\left(\frac{y}{x}\right) + x^n\left(\frac{-y}{x^2}\right)\phi'\left(\frac{y}{x}\right)$

$f_{xx} = n(n-1)x^{n-2}\phi + nx^{n-1}\left(\frac{-y}{x^2}\right)\phi' - (n-2)x^{n-3}y\left(\frac{y}{-x^2}\right)\phi''$

$n(n-1)x^{n-2}\phi\left(\frac{y}{x}\right) + nx^{n-1}\left(\frac{-y}{x^2}\right)\phi' - (n-2)x^{n-3}y\phi'\left(\frac{y}{x}\right)$
$- x^{n-2}y\left(\frac{y}{-x^2}\right)\phi''\left(\frac{y}{x}\right)$

$f_y = x^n \cdot \frac{1}{x}\phi' = x^{n-1}\phi'\left(\frac{y}{x}\right)$

$f_{yy} = x^{n-1} \cdot \frac{1}{x}\phi''$

$x f_{xx} = x\, n(n-1)x^{n-1}\phi + nx^n\left(\frac{-y}{x^2}\right)\phi' + (n-2)x^{n-4}\phi''$

③ $y f_{yy} = x^{n-2}y\phi''$

Adding ③ and ④

Homogeneous function can always be written as $f = x^n\phi\left(\frac{y}{x}\right)$

$f = uv \Rightarrow \frac{df}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Soln

$f_y = x^{n-1}\phi'$, $f_x = nx^{n-1}\phi + x^n\left(\frac{-y}{x^2}\right)\phi'$

$f_{xy} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(x^{n-1}\phi')$

$= x^{n-1}\phi''\left(\frac{-y}{x^2}\right) + \phi'(n-1)x^{n-2}$

$= -yx^{n-3}\phi'' + x^{n-2}(n-1)\phi'$

$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(nx^{n-1}\phi - yx^{n-2}\phi')$

$= nx^{n-1}\left(\frac{-y}{x^2}\right)\phi' + n\phi(n-1)x^{n-2}$
$- \left[y\phi'(n-2)x^{n-3} + yx^{n-2}\phi''\left(\frac{-y}{x^2}\right)\right]$

$= -ynx^{n-3}\phi' - y(n-2)x^{n-3}\phi + n(n-1)\phi x^{n-2} - y^2x^{n-4}\phi''$

differentiate wrt x

$$\frac{\partial}{\partial x}[10] \Rightarrow x f_{xx} + 1.f_x + \frac{\partial}{\partial x}(y f_y) = n f_x$$

$$\Rightarrow x f_{xx} + f_x + y f_{yy} + 0 = n f_x$$

$$x^2 f_{xx} + x f_x + xy f_{xy} = n x f_x$$

$$\frac{\partial}{\partial y}[10] \Rightarrow x f_{xy} + 0 + y f_{yy} + f_y = n f_y$$

$$xy f_{xy} + y^2 f_{yy} + y f_y = n y f_y$$

adding (11) and (12) $\Rightarrow$

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_y = n(x f_x + y f_y) - (x f_y + y f_y)$$

$$= n.nf - nf \quad = \quad n^2 f - nf$$

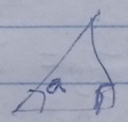
$$\cancel{=} \quad n(n-1)f$$

GEG 311   02/12/25   Lecture

show that using Euler theorem

(1) $f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$

(2) $f(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$



(1) $\cancel{x^2\left[1 \pm \left(\frac{y}{x}\right)^{-10}\right.}$

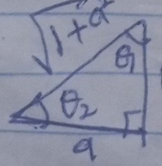
(1) $\cancel{1}$ (1.) $x^{-2}\left[1 + \left(\frac{y}{x}\right)^{-1} + \frac{-\log\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2}\right]$

Homogeneous equation of order -2

(2) ~~f(x,y) = x² tan⁻¹~~

(2) $f(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

$$= x^2\left[\tan^{-1}\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2 \tan^{-1}\left(\frac{y}{x}\right)^{-1}\right]$$



$$\tan\theta_1 = \frac{a}{1}, \quad \tan\theta_2 = \frac{1}{a}$$

$$\theta_1 + \theta_2 = \frac{\pi}{2}$$

$$\tan^{-1} a + \tan^{-1}\frac{1}{a} = \frac{\pi}{2}$$

$$= x^2\left[\tan^{-1}\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{y}{x}\right)\right)\right]$$

Homogeneous equation of degree 2

(3) $f(x,y) = \frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}$

for an exact diff: $\frac{\partial z}{\partial x} = 2xy$ Integrating both sides

$$\int \frac{\partial z}{\partial x}\, dx = \int 2xy\, dx$$

[C(y) means its a function of y it depends only]

$$z = x^2 y + C(y)$$

However, $\frac{\partial z}{\partial y} = x^2$

$$\Rightarrow \frac{\partial}{\partial y}\left[x^2 y + C(y)\right] = x^2$$

$$x^2 + \frac{dC}{dy} = x^2$$

$$\frac{dC}{dy} = 0 \Rightarrow C(y) = k$$

$$z = x^2 y + k$$

(3) $dz = (2x\sin y + xy^2e^x + e^x y^2)\,dx + (2yxe^x + x^2\cos y + 1)\,dy$

Soln:

$A(x,y) = 2x\sin y + xy^2e^x + e^x y^2$

$B(x,y) = 2yxe^x + x^2\cos y + 1$

$$\frac{\partial A}{\partial y} = 2x\cos y + 2xye^x + 2ye^x$$

$$\frac{\partial B}{\partial x} = 2y(xe^x + e^x.1) + 2x\cos y$$
$$= 2xye^x + 2ye^x + 2x\cos y$$

obtain z

$$z = \int \frac{\partial z}{\partial x}\, dx = \int (2x\sin y + xy^2e^x + e^x y^2)\, dx$$

$$= 2\sin y \int x\,dx + y^2\int xe^x\,dx + y^2\int e^x\,dx$$

$$z = x^2\sin y + y^2\left[xe^x - \int 1.e^x\,dx\right] + e^x y^2 + C(y)$$

$$= x^2\sin y + xy^2e^x - y^2e^x + y^2e^x + C(y)$$

$$z = x^2\sin y + xe^x y^2 + C(y)$$

but $\frac{\partial z}{\partial y} = x^2\cos y + 2yxe^x + \frac{dC}{dy} = 2xye^x + x^2\cos y + 1$

$$\Rightarrow \frac{dC}{dy} = 1 \qquad C = \int dy = y + k$$

$$z = x^2\sin y + xe^x y^2 + y + k$$

$$\Rightarrow y\frac{\partial z}{\partial u} = e^u \sin v\left(e^u \cos v \frac{\partial z}{\partial x} + e^u \sin v \frac{\partial z}{\partial y}\right)$$
$$e^{2u}\sin v\cos v\frac{\partial z}{\partial x} + e^{2u}\sin^2 v\frac{\partial z}{\partial y}$$
$$x\frac{\partial z}{\partial v} = e^u\cos v\left(-e^u\sin v\frac{\partial z}{\partial x} + e^u\cos v\frac{\partial z}{\partial y}\right)$$
$$-e^{2u}\sin v\cos v\frac{\partial z}{\partial x} + e^{2u}\cos^2 v\frac{\partial z}{\partial y}$$
adding $\Rightarrow y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\sin^2 v Z_y + e^{2u}\cos^2 v Z_y$

Exercise: Given $z = z(x,y)$

$x = e^u\cos v$, $y = e^u \sin v$

Show that: $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

Ex2: Given that $z(x,y) = 0$ where

$z = x^3 - xy + x^2y^2 - 1$ find $\frac{dy}{dx}$

$z = 0 \Rightarrow dz = 0$

$$\Rightarrow dz = \frac{\partial z}{\partial x}\partial x + \frac{\partial z}{\partial y}\partial y = 0$$

or $-Z_y dy = Z_x dx$

$$\Rightarrow \frac{dy}{dx} = \frac{-Z_x}{Z_y} \quad \text{provided } Z_y \neq 0$$

implicit differentiation

$Z_x = 3x^2 - y + 2xy^2$

$Z_y = -x + 2yx^2$

$$\Rightarrow \frac{dy}{dx} = \frac{-(3x^2 - y + 2xy^2)}{x(1-2yx)}$$

[Exist everywhere except at $x = 0, \frac{1}{2y}$]

Exist everyw

Jacobians and Transformation

Consider the 3D case $n=3$. If you're given that x

$x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ — (1)

this means a point in u,v,w 3D space is mapped into some other point in the x,y,z space/axis; if this transformation is unique (one-one mapping) then the above can be solved for inverses.

$u = u(x,y,z)$, $v = v(x,y,z)$, $w = w(x,y,z)$ — (2)

From (1)

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw \\ dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw \\ dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw \end{aligned}\right\} \text{eqn } (3)$$

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \begin{bmatrix} du \\ dv \\ dw \end{bmatrix}$$

~~Frame of reference~~

~~The~~ Frame of reference

* Thin wire

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \begin{bmatrix} du \\ dv \\ dw \end{bmatrix} \quad \text{——— (4)}$$

$$\Rightarrow J = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} \neq 0 \quad \text{——— (5)}$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)}$$

If it is true that the determinant is not zero, equ 2 exist (inverse transfer matrix) Furthermore, the jacobian of x, y, z relative to u, v, w times the jacobian of u, v, w relative to x, y, z must be one

~~If u, v, w are functions of r, s, t~~

$$\cancel{\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{\partial(u,v,w)}{\partial(x,}}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1 \quad \text{——— (6)}$$

If u, v, w are fns of r, s, t

$$\frac{\partial(x,y,z)}{\partial(r,s,t)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(r,s,t)} \quad \text{——— (7)}$$

Example 2: For what values of a will the transform below have a unique inverse?

$x = 2u - v + wa$, $y = au + av + w$, $z = u + v - w$

Soln: when $\frac{\partial(x,y,z)}{\partial(u,v,w)} \neq 0$

$$\begin{vmatrix} 2 & -1 & a \\ a & a & 1 \\ 1 & 1 & -1 \end{vmatrix} \neq 0 \quad \Rightarrow -3a - 3 \neq 0$$

$$a \neq -1$$

$$2\begin{vmatrix} a & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} a & 1 \\ 1 & -1 \end{vmatrix} + a\begin{vmatrix} a & a \\ 1 & 1 \end{vmatrix}$$

$$2(-a-1) + 1(-a-1) + a(a-a)$$

$$-2a - 2 - a - 1 + 0 = -3a - 3$$

Transforming Polar to cartesian coordinates

~~Show that the Laplacian~~

$x = r\cos\theta,\ y = r\sin\theta,$

$r \geq 0,\ 0 < \theta < 2\pi$

Show that the Laplacian

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ becomes

$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

Soln: $\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$

$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}$

$x = r\cos\theta$

$1 = \frac{\partial}{\partial x}(r\cos\theta)$

$= -r\frac{\partial \theta}{\partial x}\sin\theta + \cos\theta\frac{\partial r}{\partial x}$

~~$y = r\cos\theta$~~ —(1)

$y = r\sin\theta$

differentiate w.r.t x —(2)

$0 = r\frac{\partial \theta}{\partial x}c + s\frac{\partial r}{\partial x}$

$1 = -rs\theta_x + cr_x$

$1 = -rs\theta_x + \frac{c(-r\theta_x c)}{s}$

$s = -rs^2\theta_x - c^2 r\theta_x$

$= -r\theta_x(s^2 + c^2)$

$\Rightarrow \theta_x = \frac{-s}{r}$

$-rc\cdot\left(\frac{-s}{r}\right) = sr_x$

$\frac{cs}{s} = r_x = c$

$s = \sin\theta,\ c = \cos\theta$

~~$c^2 + s^2 = 1 = \left(\frac{x}{y}\right)^2$~~ Equation of circle

$c^2 + s^2 = 1 = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2$

$\Rightarrow r^2 = x^2 + y^2$

$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$

$\Rightarrow$ Inverse transform:

$r = (x^2 + y^2)^{1/2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$2r\frac{\partial r}{\partial x} = 2x$

$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = c$

$\frac{\partial r}{\partial y} = s$

$\frac{-y}{x^2} = \frac{\partial \theta}{\partial x}\left(1 + \frac{y^2}{x^2}\right)$

$\frac{-y}{x^2} = \theta_x \cdot \frac{r^2}{x^2}$

$\theta_x = \frac{-y}{r}\cdot\frac{1}{r} = \frac{-s}{r}$

$\frac{1}{x}\cdot 1$

—(A)

$\frac{\partial^2}{\partial x^2} = \left(c\frac{\partial}{\partial r} - \frac{s}{r}\frac{\partial}{\partial \theta}\right)\left(c\frac{\partial}{\partial r} - \frac{s}{r}\frac{\partial}{\partial \theta}\right)$

$\frac{\partial^2}{\partial y^2} = \left(s\frac{\partial}{\partial r} + \frac{c}{r}\frac{\partial}{\partial \theta}\right)\left(s\frac{\partial}{\partial r} + \frac{c}{r}\frac{\partial}{\partial \theta}\right)$ —(B)

$c\frac{\partial}{\partial r}\left(c\frac{\partial}{\partial r}\right) - c\frac{\partial}{\partial r}\left(\frac{s}{r}\frac{\partial}{\partial \theta}\right)$

$-\frac{s}{r}\frac{\partial}{\partial \theta}\left(c\frac{\partial}{\partial r}\right) + \frac{s}{r}\frac{\partial}{\partial \theta}\left(\frac{s}{r}\frac{\partial}{\partial \theta}\right)$

$c^2\frac{\partial^2}{\partial r^2} - cs\left(\frac{1}{r}\frac{\partial^2}{\partial r\partial \theta} - \frac{1}{r^2}\frac{\partial}{\partial \theta}\right)$

$-\frac{s}{r}\left(-s\frac{\partial}{\partial r} + c\frac{\partial^2}{\partial \theta\partial r}\right)$

$+\frac{s}{r^2}\left(c\frac{\partial}{\partial \theta} + s\frac{\partial^2}{\partial \theta^2}\right)$

$\Rightarrow \frac{\partial^2}{\partial x^2} = c^2\frac{\partial^2}{\partial r^2} - \frac{2cs}{r}\frac{\partial^2}{\partial r\partial \theta} + \frac{2cs}{r^2}\frac{\partial}{\partial \theta} + \frac{s^2}{r}\frac{\partial}{\partial r} + \frac{s^2}{r^2}\frac{\partial^2}{\partial \theta^2}$

Exerci
spheric
transfor

$\frac{\partial^2}{\partial y^2} = s^2$

$-\frac{2cs}{r^2}$

addin

$\nabla^2 =$

Ass
Exerc
spheric
transfo
$x =$
$y =$
$z =$
where
$0 \leq$
(a) sho
(b) Fin

in sp

Show
and

Exam 1998

Exercise: Find the Laplacian in spherical to cartesian transformation is

$$\frac{\partial^2}{\partial y^2} = s^2\frac{\partial^2}{\partial r^2} + \frac{2cs}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{2cs}{r^2}\frac{\partial}{\partial\theta} + \frac{c^2}{r}\frac{\partial}{\partial r} + \frac{c^2}{r^2}\frac{\partial^2}{\partial\theta^2}$$

adding

$$\nabla^2 = (c^2+s^2)\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]$$

Ass'

Exercise: Find the Laplacian spherical to cartesian transformation if

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

where $r > 0$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$

(a) show that $J = r^2\sin\theta$

(b) Find $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in spherical coordinate

Show the Laplacian (b) and the Jacobian (a)

1/1 → Battery

indicators

th pressure

319

reads

Nov, 25 2025

→ Ellipsoid

$= 1$

$\frac{z^2}{c^2}$

$\frac{z^2}{c^2}$

$\to x$

class lecture)

sional

ling function,

defined at

in some

ined in D,

respect to

ally the

t to that

dent

$, x_n\}$

$$F_{x_f} = \frac{\partial F}{\partial x_k} = \lim_{\Delta x_k \to 0} \left[\cancel{F(x_1, x_2, \ldots} \right.$$

$$= \lim_{\Delta x_k \to 0} \left[\frac{f(x_1, x_2, \ldots \Delta x_k, \ldots x_n) - f(P)}{\Delta x_k} \right]$$

* Higher order P.D with mixed or unmixed derivative.

~~Ex1~~: $F(x,y,z) = x + x^2 \sin y + e^{-z}$ and $P_0 = (1, \pi/2, 0)$ first principle method

Evaluate the F.F at P_0:

F_x, F_y, F_{xy}, F_{xx}, F_{xyz}, F_{zzz}

soln:

(a) $F_x = 1 + 2x \sin y$

At P_0: $F_x = 1 + 2(1)\sin\left(\frac{\pi}{2}\right)$

$= 3$ [radians, Natural log]

(b) $F_y = x^2 \cos y$

at P_0: $F_y = 0$

(c) $F_{xy} = \frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(1 + 2x \sin y)$

$= 2x \cos y$

At P_0: $F_{xy} = 0$

(d) $F_{xx} = \frac{\partial}{\partial x} F_x = \frac{\partial}{\partial x}(1 + 2x \sin y)$

$= 2 \sin y$

at P_0: $F_{xx} = 2$

(e) $F_{xyz} = \frac{\partial F_{xy}}{\partial z}$

$\frac{\partial}{\partial z}(2x \cos y) = 0$

(f) $F_{zzz} = \frac{\partial^2 F_z}{\partial z^2}$

$F_z = -e^{-z}$

$\frac{\partial^2}{\partial z^2}(-e^{-z}) = \frac{\partial}{\partial z}(e^{-z})$

$= -e^{-z}$

At P_0: $F_{zzz} = -1$

Ex2: Prove that $u = F(x + at) + g(x - at)$ satisfies the ~~$\frac{\partial^2 u}{\partial t^2}$~~

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$ [This is a wave equation]

soln: True as long as functions F and g are +ve or -ve respectively or vice versa

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[F(x + at) + g(x - at)\right]$$

$$= aF'() - ag'()$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left[aF'() - ag'()\right]$$

$$= a^2F''() + a^2g''()$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[F(x + at) + g(x - at)\right]$$

$$= F'() + g'()$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[F'() + g'()\right] = F''() + g''()$$

$$\Rightarrow a^2 \frac{\partial^2 u}{\partial x^2} = a^2F''() + a^2g''()$$

$$= \frac{\partial^2 u}{\partial t^2} \quad \text{[Proven]}$$

Exercise:

(1) If $u = \log(x^2 + y^2 + z^2 - 3xyz)$

Find $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u$

Ans: $\frac{-9}{(x + y + z)^2}$

(2) Find the values of n for which $r^n(3\cos^2\theta - 1)$ satisfies

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0$$

is true

$$\overline{x^{1/2}\left(1+\left(\frac{y}{x}\right)^{1/2}\right)}$$

Homogeneous equation of degree $\frac{1}{6}$

Total Differentials Lecture

Recall, if $y=f(x)$, $\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$ and $dy = y'dx$

Let $f = f(x_1, x_2, \ldots x_n)$ then the ~~df~~ df is the sum of the partially induced changes in the x_i

$$df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots \frac{\partial f}{\partial x_n}dx_n \quad \text{——— (1)}$$

From eqn (1) we deduce the notion of :

(i) Integration of a total differential

(ii) Composite differentiation

(iii) Jacobian of transformation

Given; ~~$dz = A(x,y)dx$~~ + Two variables Idea can be extended to multiple variables

$$dz = A(x,y)dx + B(x,y)dy \quad \text{——— (2)}$$

we could find $z = z(x,y)$ if the ~~right~~ RHS is ~~the~~ an exact differential.

It is an Exact differential :

If and Only if $A(x,y) = \frac{\partial z}{\partial x}$ and $B(x,y) = \frac{\partial z}{\partial y}$

This requires that :

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \Rightarrow \frac{\partial B}{\partial x} = \frac{\partial A}{\partial y}$$

$$z = z(x,y)$$

Ex Find z, given

(1) $dz = (x^2+y)dx + e^x dy$ (2) $dz = 2xy\,dx + x^2 dy$

Soln :

$$A(x,y) = x^2+1$$
$$B(x,y) = e^x$$

$$\frac{\partial A}{\partial y} = 1, \quad \frac{\partial B}{\partial x} = e^x$$

$$\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$$

~~so~~ since $\frac{\partial A}{\partial y} \neq \frac{\partial B}{\partial x}$) we can't determine z with the info given

(2) $dz = 2xy\,dx + x^2dy$

$$A(x,y) = 2xy$$
$$B(x,y) = x^2$$

$$\frac{\partial A}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$$

$$\frac{\partial B}{\partial x} = \frac{\partial}{\partial x}(x^2) = 2x$$

Biased region

$\exp\left(\frac{V_2 - V_1}{nV_T}\right)$

ense temperature:

uple, Thermopile

on current, I_s

al characteristics:

own Region

breakdown Voltage

1

QUIZ 2

5V

8nA

g The Diode

d characteristics

ponential Model

ether to see where

tersect

taneous approach

line

$= 5V$ (at all time)

Resistance model

s the minimum

e needed for the

to start Conducting

Model

ant-Voltage Drop ~~Model~~

ery Model

l diode Model (Assume $V_D = 0$)

$\frac{5-0}{1} = 5mA$

Small-Signal Model

nction under reverse

condition

GEG311 Lecture

Given $z = z(x,y)$

we deduce that

$$dz = \frac{\partial z}{\partial x}\partial x + \frac{\partial z}{\partial y}\partial y \quad \text{—— (1)}$$

furthermore if $\cancel{x = x(u,v)},$

$\cancel{y = y(u,v)}$

$\cancel{x = x(u,v),\ y =}$

$$x = x(u,v);\ y = y(u,v) \quad \text{—— (2)}$$

the chain rule gives

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad \text{—— (3a)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \quad \text{—— (3b)}$$

Example: Given $z = z(x,y)$;

$x = e^u \cos v$, $y = e^u \sin v$

Show that $y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}$

Soln

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot x_u + \frac{\partial z}{\partial y}\cdot y_u$$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(e^u \cos v) = e^u \cos v$$

$$\frac{\partial y}{\partial u} = e^u \sin v$$

$$\cancel{\Rightarrow y\frac{\partial z}{\partial u} = e^u \sin v}$$

$$\cancel{y\frac{\partial z}{\partial u} = e^u \sin v\left(e^u \cos v \cdot \frac{\partial z}{\partial x}\right.} +$$

Class

* Separation - eg
*

① If $dw = A(x,y,z)dx + B(x,y,z)dy$ (1) $+ C(x,y,z)dz$;
is it integratable?

* Given $dw = (2xy - 2z)dx + (x^2+2y)dy + (e^z - 2x)dz$

Soln

* For a fxn to be integratable;
$A(x,y)dx + B(x,y)dy$
$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}$$
in this case;
$$\frac{\partial^2 A}{\partial y \partial z} = \frac{\partial^2 B}{\partial x \partial z} = \frac{\partial^2 C}{\partial x \partial y}$$
$A = 2xy - 2z$, $B = x^2 + 2y$
$C = e^z - 2x$
$$\frac{\partial A}{\partial z} = -2 \; ; \; \frac{\partial^2 A}{\partial y \partial z} = 0$$
$$\frac{\partial B}{\partial z} = 0 \; ; \; \frac{\partial^2 B}{\partial x \partial z} = 0$$
$$\frac{\partial C}{\partial y} = 0 \; ; \; \frac{\partial^2 C}{\partial x \partial y} = 0$$
$$\frac{\partial^2 A}{\partial y \partial z} = \frac{\partial^2 B}{\partial x \partial z} = \frac{\partial^2 C}{\partial x \partial y} = 0 \checkmark$$
$\therefore$ The function is integratable

finding w;

$$\int dw = (2xy - 2z)dx$$
$$w = x^2y - 2xz + c(y) + k(z) + c_1$$
but $\frac{\partial w}{\partial y} = x^2 + c'(y)$

$\frac{\partial w}{\partial y} = x^2 + 2y$; $c'(y) = 2y$;

$\int c'(y) = \int 2y$; $c(y) = y^2$

$$\frac{\partial w}{\partial z} = -2x + k'(z);$$

but $\frac{\partial w}{\partial y} = e^z - 2x$; $\therefore k'(z) = e^z$

$\int k'(z) = \int e^z$; $k(z) = e^z$

$$\therefore w = x^2y - 2xz + y^2 + e^z + c_1$$

Correction
$$w = \int A(x,y,z)dx + D(y,z)$$

② If $w = w(x,y)$ and $x = u\cosh v$
$y = u\sinh v$

Show that $\left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial w}{\partial v}\right)^2$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial x}{\partial u} = \cosh v$$

$$\frac{\partial x}{\partial v} = u\sinh v; \quad \frac{\partial y}{\partial u} = \sinh v$$

$$\frac{\partial y}{\partial v} = u\cosh v$$

$$\frac{\partial w}{\partial u} = \cosh v\frac{\partial w}{\partial x} + \sinh v\frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial v} = u\sinh v\frac{\partial w}{\partial x} + u\cosh v\frac{\partial w}{\partial y}$$

$$\left(\frac{\partial w}{\partial u}\right)^2 = \left(\cosh v\frac{\partial w}{\partial x} + \sinh v\frac{\partial w}{\partial y}\right)^2$$

$$= \cosh^2 v\left(\frac{\partial w}{\partial x}\right)^2 + \cosh v\frac{\partial w}{\partial x}\left(\sinh v\frac{\partial w}{\partial y}\right)$$

$$+ \sinh v\frac{\partial w}{\partial y}\left(\cosh v\frac{\partial w}{\partial x}\right) + \sinh^2 v\left(\frac{\partial w}{\partial y}\right)^2$$

$$\left(\frac{\partial w}{\partial u}\right)^2 = \cosh^2 v\left(\frac{\partial w}{\partial x}\right)^2 + \cosh v\sinh v\frac{\partial^2 w}{\partial x\partial y}$$

$$+ \sinh v\cosh v\frac{\partial^2 w}{\partial y\partial x} + \sinh^2 v\left(\frac{\partial w}{\partial y}\right)^2$$

$$\left(\frac{\partial w}{\partial v}\right)^2 = \left(u\sinh v\frac{\partial w}{\partial x} + u\cosh v\frac{\partial w}{\partial y}\right)^2$$

$$= u^2\sinh^2 v\left(\frac{\partial w}{\partial x}\right)^2 + u^2\sinh v\cosh v\frac{\partial^2 w}{\partial x\partial y}$$

$$+ u^2\sinh v\cosh v\frac{\partial^2 w}{\partial x\partial y} + u^2\cosh^2 v\frac{\partial^2 w}{\partial y^2}$$

$$\left(\frac{\partial w}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial w}{\partial v}\right)^2 = \cosh^2 v\left(\frac{\partial w}{\partial x}\right)^2 +$$

$$2\cosh v\sinh v\frac{\partial^2 w}{\partial x\partial y} + \sinh^2 v\left(\frac{\partial w}{\partial y}\right)^2 - \Big[$$

$$\Big[\sinh^2 v\left(\frac{\partial w}{\partial x}\right)^2 - 2\sinh v\cosh v\frac{\partial^2 w}{\partial x\partial y} -$$

$$\cosh^2 v\left(\frac{\partial^2 w}{\partial y^2}\right)^2$$

$$= \left(\cosh^2 v - \sinh^2 v\right)\left(\frac{\partial w}{\partial x}\right)^2 - \left(\cosh^2 v - \sinh^2 v\right)$$

but $\cosh^2 v - \sinh^2 v = 1$;

$$\therefore \left(\frac{\partial w}{\partial x}\right)^2 - \left(\frac{\partial^2 w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial w}{\partial v}\right)^2$$

GEG 311

Vector Field Theorem

(1) $\phi = \phi(x, y, z)$ and

$\bar{V} = V_1\bar{i} + V_2\bar{j} + V_3 k$

where $V_s = V_s(x, y, z)$

Given a scalar function ϕ and a vector function V in a 3 D

we say ϕ and V define a scalar and a vector field respectively since each is a function of point in space.

$\phi(x, y, z)$ as the temp. of a body is density of potential and $\bar{V}$ as the velocity of a body is momentum etc.

Let ϕ and V be defined & differentiable over a certain region of space

~~Let ∂ or Laplace~~:

Let Del or nabla or Hamiltonian operator be define as ∇

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

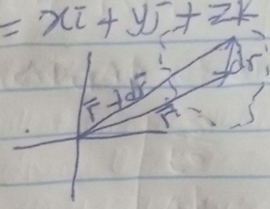
(2) $\text{grad}\,\phi = \nabla\phi$

$\text{grad}\,\phi$ = gradient of ϕ and it represent the normal to a surface represent by ϕ constant.

If ϕ is a constant and any point on ϕ can be represented as

$$\phi = c$$

$$F = xi + yj + zk$$



$$F = xi + yj + zk$$

$$\nabla\phi . d\bar{r} =$$

$$\left(i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right).(idx + jdy + kdz)$$

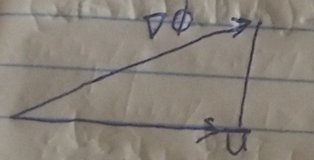
$$= \phi_x dx + \phi_y dy + \phi_z dz$$

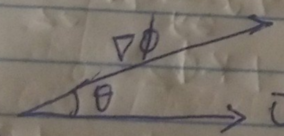
If the product of any two function is zero it means they are orthogonal.

$$\nabla_u \phi . \hat{u}$$

In general, this is a maximum at:

$$[\nabla_u \phi]_{max} = |\nabla\phi|$$





Eg

$$\cos\theta = \frac{\nabla_u \phi . \hat{u}}{|\nabla_u \phi| . |\bar{u}|}$$

3a The divergence of $\bar{V}$ is del. V

$$\text{Div } \bar{V} = \nabla . \bar{V}$$

3b. $\nabla^2\phi = \text{div}(\text{grad}\,\phi)$

$\nabla^2 \rightarrow$ Laplacian

$\nabla^4 \rightarrow$ Bi-harmonic

(4) The curl or rotation of V is del. V

$\nabla \times \bar{V}$ or write it as $\nabla \wedge \bar{V}$

Note: (1) When a vector field satisfy $dv = 0$ that vector field is called sonianoidal

(2) If curl V equals zero, then it's called irrational or conservative it means there exist a scalar potential such that

$$\bar{V} = \text{grad}\,\phi$$

(3) If curl $\bar{V} = 0 \Rightarrow \bar{V} = \text{grad}\,\phi$

When a function satisfies $\nabla^2\phi = 0$, ϕ is said to be harmonic

(4) In general, curl of grad ϕ equals zero and div of curl A equals 1

You can only take the cross product of two vectors. curl

$$\nabla \wedge (\nabla\phi) = 0$$

and $\nabla . (\nabla \wedge \bar{A}) = 0$

(5) $\nabla \wedge (\nabla \wedge \bar{A}) = \nabla(\nabla . \bar{A} - \nabla^2 \bar{A})$

(6) $\text{grad}(\phi_1 \pm \phi_2)$

(7) Applications abound in engineering modelling

compatibility Equation of elastic

(a) $$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{2\partial^2 \varepsilon_{xy}}{\partial x \partial y}$$

(b) Deflection of ~~...~~ in strength of material

$$\nabla^4 w = \frac{-P}{EI}$$

Ex: Given $\phi = x^2 + 9y^2 - 11$ at $P_0 = (1,2)$ along the vector $\bar{u} = 8i + 6j$ find the directional derivative.

Soln: $\nabla_u \phi = ?$

$\nabla\phi \cdot \hat{u}$

$$\text{grad}\,\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$= 2x\,i + 18y\,j$

at P_0, $\text{grad}\,\phi = 2i + 36j$

$$\hat{u} = \frac{8i + 6j}{\sqrt{8^2 + 6^2}}$$

$\hat{u} = 0.8i + 0.6j$

$\Rightarrow \nabla_u \phi = (2i + 36j)\cdot(0.8i + 0.6j)$

$= 23.2$

Find the divergence of $\bar{V}$ at $P_0 = (6,1)$ given that $\bar{V} = x^2 y\, i + j\cos 2x$

Soln: $\text{div}\,\bar{V} = \nabla\cdot\bar{V}$

$$\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right)\cdot(x^2 y\, i + j\cos x)$$

$= 2xy + 0$

at P_0, $\text{div}\,\bar{V} = 2.6.1 = 12$

Exercise: Show that if $\phi = \phi(x,y,z)$ and $\bar{A} = A(x,y,z) = A_1 i + A_2 j + A_3 k$ then $\text{curl}(\phi\bar{A}) = \phi\,\text{curl}\,\bar{A} + (\nabla\phi)\wedge\bar{A}$

Soln: ~~Curl($\phi\bar{A}$~~

$\text{curl}(\phi\bar{A}) = \nabla\wedge\phi\bar{A}$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$i\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi A_2 & \phi A_3 \end{vmatrix} - j\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_3 \end{vmatrix}$$

Note: (Use product rule for $\phi A_1, \phi A_2, \phi A_3$)

$$+ k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \phi A_1 & \phi A_2 \end{vmatrix}$$

$$i\frac{\partial}{\partial y}\phi A_3 - i\frac{\partial}{\partial z}\phi A_2 - j\frac{\partial}{\partial x}\phi A_3 + k\frac{\partial}{\partial z}\phi A_1 + k\frac{\partial}{\partial x}\phi A_2 - k\frac{\partial}{\partial y}\phi A_1$$

$i\cdot i = 1$

$i\cdot j = 0$

$i(A_3\phi_y A_{3y}) -$

$i(\cancel{A_3}\phi_y A_{3y}$

$i(A_3\phi_y + \phi A_{3y}) - i(A_2\phi_z + \phi A_{2z}) - j(A_3\phi_x + \phi A_{3x}) + k(A_1\phi_z + \phi A_{1x}) + k(A_2\phi_x + \phi A_{2x}) - k(A_1\phi_x + \phi A_{1x})$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$i\phi_x + j\phi_y + k\phi_z$

$(\nabla\phi)\wedge\bar{A} = (i\phi_x + j\phi_y + k\phi_z) \times (A_1 i + A_2 j + A_3 k)$

$\cancel{\phi A_1} =$

$A_1\phi + A_2\phi + A_3\phi$

Find the a...
sphere x^2+...
the parabolo...
at the point...
Soln...
The angle bt...
B equal to t...
The $\angle$ btw...
$\phi_1 = x^2+$...
$\phi_2 = x^2+$...
~~Grad~~
Normal = ...
$\bar{N}_1 = \nabla$...
$\bar{N}_2 =$ ...
Derivative $\bar{N}_1 = 2$...
~~N_2~~ at ...
~~$\bar{N}_2 = 2x$~~
$\bar{N}_2 = 2$...
at P_0 $\bar{N}_2$...
$\cos\theta =$ ...
$|\bar{N}_1| =$ ...
$|\bar{N}_2| = \sqrt{}$...
~~$\cos\theta =$~~
Note: ...
eqn of a n...
equa...
Given...
$\nabla V =$ ...
Show...
non-...
~~that~~
the...

$\frac{\partial}{\partial z}$ $\Big| -j \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \phi A_1 & \phi A_3 \end{vmatrix}$

ϕA_3

...duct rule for $\phi A_1, \phi A_2, \phi A_3$)

$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \phi A_1 & \phi A_2 \end{vmatrix}$

$\ldots_3 - i\frac{\partial}{\partial z}\phi A_2 - j\frac{\partial}{\partial x}\phi A_3$

$\ldots \phi A_1 + k\frac{\partial}{\partial x}\phi A_2$

$k\frac{\partial}{\partial y}\phi A_1$ $\qquad i.i = 1$

$i.j = 0$

$\ldots A_{3y}) -$

$\ldots_{3y}$

$\ldots + \phi A_{3y}) - i(A_2\phi_z + \phi A_{2z})$

$\ldots (A_3\phi_x + \phi A_{3x}) +$

$\ldots \phi A_{1x}) + k(A_2\phi_x$

$\ldots x) - k(A_1\phi_x + \phi A_{1x})$

$i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\ldots j\phi_y + k\phi_z$

$= (i\phi_x + j\phi_y + k\phi_z)$

$\ldots A_1 i + A_2 j + A_3 k)$

$\ldots \phi + A_3\phi$

Find the angle btw the sphere $x^2+y^2+z^2=9$ and the paraboloid $z=x^2+9y^2-11$ at the points $P_0 = (2,-1,2)$

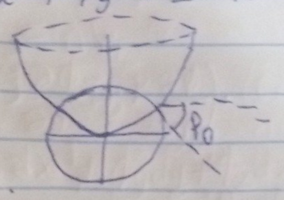
Soln

The angle btw any two surfaces is equal to the normal

The $\angle$ btw the

$\phi_1 = x^2+y^2+z^2-9=0$

$\phi_2 = x^2+9y^2-z-11=0$



Normal = N

$\bar{N}_1 = \nabla\phi_1$

$\bar{N}_2 = \nabla\phi_2$

Derivative $\bar{N}_1 = 2xi+2yj+2zk$

$\cancel{\bar{N}_1}$ at $P_0 = 4i-2j+4k$

$\cancel{\bar{N}_2 = 2xi+18y}$

$\bar{N}_2 = 2xi+18yj-k$

at P_0 $\bar{N}_2 = 4i-18y-k$

$$\cos\theta = \frac{\bar{N}_1 \cdot \bar{N}_2}{|\bar{N}_1| \cdot |\bar{N}_2|}$$

$|\bar{N}_1| = \sqrt{4^2+(-2)^2+4^2}$

$= 6$

$|\bar{N}_2| = \sqrt{4^2+(-18)^2+1^2}$

$= \sqrt{341}$

$$\cos\theta = \frac{48}{6\sqrt{341}} \Rightarrow \theta = \cos^{-1}\left(\frac{8}{\sqrt{341}}\right)$$

Make findings on:

Note: For a surface $\phi = 0$

eqn of a normal line

equation of a tangent

Given

$V = 2xyzi + x^2zj + x^2yk$

Show that curl V is the non-vector, hence ~~show that~~ find the scalar of the potential of V

~~Limits~~ Multivariable calculus limits

GEG311

Limits and continuity

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

Path 1 $\to x = y$

$$\to \lim_{(x,x)\to(0,0)} \frac{xx}{x^2+x^2}$$

$$\lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2}$$

$$\lim_{(x,x)\to(0,0)} \frac{1}{2} = \frac{1}{2}$$

Path 2: $x \to 0 \to \lim_{(0,y)} \frac{0y}{0^2+y^2}$

$$= 0$$

Path 1 $\neq$ Path 2 $\to$ Limit does not exist

Find $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$

$\Rightarrow \frac{0^2}{0+0} = \frac{0}{0}$ undefined

Soln

use Polar coordinate

$x = r\cos\theta,\ y = r\sin\theta$

$$\lim_{r\to 0} \frac{(r\cos\theta)^2 r\sin\theta}{(r\cos\theta)^2 + (r\sin\theta)^2}$$

$$\lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r^2(\cos^2\theta+\sin^2\theta)}$$

$$\lim_{r\to 0} r\cos^2\theta\sin\theta = 0$$

$0 \leq |r\cos^2\theta\sin\theta|$ $\quad |r| \to 0$

$|\cos\theta| \leq 1,\ |\sin\theta| \leq 1$

$\boxed{\text{Squeez theorem}}$ → Learn

Ways to implement multiple variable limit

(1) Just plug in

$$\lim_{(x,y)\to(1,\pi)} \frac{x\cos y}{x^2+2}$$

$$= \frac{1\cdot\cos(\pi)}{1^2+2} = \frac{-1}{3}$$

(2) Do Algebra

$$\lim_{(x,y)\to(4,0)} \frac{\sqrt{x}-\sqrt{y+4}}{x-y-4}$$

$$\frac{\sqrt{4}-\sqrt{0+4}}{4-0-4} = \frac{0}{0}$$

$\frac{0}{0}$ undefined

$$\lim_{(x,y)\to(4,0)} \frac{\sqrt{x}-\sqrt{y+4}}{(x-y-4)} \cdot \frac{(\sqrt{x}+\sqrt{y+4})}{\sqrt{x}+\sqrt{y+4}}$$

$$\lim_{(x,y)\to(4,0)} \frac{(\sqrt{x})^2-(\sqrt{y+4})^2}{(x-y-4)(\sqrt{x}+\sqrt{y+4})}$$

$$\lim_{(x,y)\to(4,0)} \frac{x-y-4}{(x-y-4)(\sqrt{x}+\sqrt{y+4})}$$

$$= \frac{1}{\sqrt{4}+\sqrt{0+4}} = \frac{1}{4}$$

(3) Substitution

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2y)}{x^2y}$$

Let $t = x^2y$

Since x and y is 0 t is also 0

$$\lim_{t\to 0} \frac{\sin t}{t} \quad [\text{L'Hopital's rule}]$$

$$= 1$$

(4) Separable

$$\lim_{(x,y)\to(0,0)} \frac{e^{2x}\ln(2y+1) - \ln(2y+1)}{x\ln(3y+1)}$$

$$\lim_{(x,y)\to(0,0)} \frac{(e^{2x}-1)\ln(2y+1)}{x\ln(3y+1)}$$

$$\lim_{x\to 0} \frac{e^{2x}-1}{x} \cdot \lim_{y\to 0} \frac{\ln(2y+1)}{\ln(3y+1)}$$

For $\lim_{x\to 0} \frac{e^{2x}-1}{x}$

represent $f'(0)$

where $f(x) = e^{2x}$

$f'(x) = 2e^{2x}$

$f'(0) = 2e^{2(0)} = 2$

L'Hopital's rule

GEG311 Lecture

TAYLOR AND MACLAURIN SERIES

Introduction

Let $x = f(x)$ be a defined and continuous function over a domain d with f having continuous nth order derivatives at a point a in d. then f can be expanded in a power series of $x - a$ and degree n when $a = 0$ it is known as a Maclaurin series and for a not zero it is Taylor series.

Prove: Since every polynomial is continuous we can approximate

Let $f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \dots + A_n(x-a)^n \quad -(1)$

the last term which is approximately chosen to acct. for neglected terms

If (1) is true

$f'(x) = A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + \dots$

$f''(x) = 2A_2 + 6A_3(x-a)$

at $x = a$,

$f(a) = A_0$

$f'(a) = A_1$

$f''(a) = 2A_2$

In general,

$$A_r = \frac{f^r(a)}{r!}$$

where r is the rth derivative of $f(a)$

$$\Rightarrow \sum_{r=0}^{n-1} \frac{f^r(a)}{r!}(x-a)^r + R_n$$

where $R_n = \dfrac{f^n(\xi)(x-a)^n}{n!}$

$a < \xi < x$

Example: Expand $\log(1+x)$ about $x = 0$ $\quad -1 < x < 1$

$f = \log(1+x)$

$f' = \frac{1}{1+x} = (1+x)^{-1}$

$f'' = -(1+x)^{-2}$

$f''' = -(-2)(1+x)^{-3}$

$f^r = (-1)^{r-1}(r-1)!(1+x)^{-r}$

at $x = 0$

$f(0) = \log 1 = 0$

$f'(0) = 1$

$f''(0) = -1$

$f'''(0) = 2$

$$\log(1+x) = 0 + \frac{(-1)^0 1!x}{1!} + \frac{(-1)^1 1!x^2}{2!} + \frac{(-1)^2 2!x^3}{3!} = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

Expand $\cos x$ about $x = \frac{\pi}{2}$ and obtain

(a) $\cos x = -(x - \frac{\pi}{2}) + \frac{1}{3!}(x - \frac{\pi}{2})^3 - \frac{1}{5}(x - \frac{\pi}{2})^5 + \dots$

(b) Substitute $t = \frac{\pi}{2} - x$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots \cancel{\frac{(-1)^m t^{2m+1}}{(2m+1)!}} + \frac{(-1)^m t^{2m+1}}{(2m+1)!}$$

(c) $\cos t = ?$

verify the result by expanding $\cos t$ about 0

TAYLOR SERIES FOR A FN OF M VARIABLES

If $f = f(x_1, x_2, \dots, x_m)$ have continuous derivatives

$P_0 = (a_1, a_2, \dots, a_m)$

in an open region containing P_0 the capital f can be expanded in a mix power series

Let P be a nearby point

in R such that both point can be connected in a str line for some constant b_1, b_2 up to b_A

$$x_j = a_j + b_j t, \quad 0 \le t \le 1 \quad \text{---(1)}$$

$$F = f(a_1 + b_1 t, a_2 + b_2 t, \ldots a_m + b_m t) \equiv F(t) \quad \text{---(2)}$$

Thus, f is a composite fxn of that can be expanded

$$F(t) = F(0) + \frac{F'(0)t}{1!} + \frac{F''(0)t^2}{2!} + \ldots + \frac{F^n(\theta)t^n}{n!} \quad \text{---(3)}$$

$$0 < \theta < 1$$

Lagrange form of the remainder $\frac{F^n(\theta)t^n}{n!}$

from (2) $\rightarrow$ At $t=0$,

$$F(0) = f(P_0) = f(a_1, a_2, \ldots a_m)$$

from (1) $\frac{dx_j}{dt} = b_j$ ---(4)

Now

$$\frac{dF}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \ldots \frac{\partial f}{\partial x_m}\frac{dx_m}{dt}$$

$$\Rightarrow \frac{d}{dt} = \sum_{j=1}^{m} b_j \frac{\partial}{\partial x_j} \quad \text{---(4b)}$$

$$\frac{d^r F}{dt^r} = \left[\sum_{j=1}^{m} b_j \frac{\partial}{\partial x_j}\right]^r f \rightarrow (4c)$$

Given th

Be cause the b_j are constant operating it ~~on~~ r th times

4c can be expanded by binomial theorem, equation 3 becomes:

$$\cancel{F'(t) = \sum_{r=0}^{n-1} \frac{d^r f}{dt^r}\Big|_{t=0} + R_n} \quad \text{---(5)}$$

$$F(t) = \sum_{r=0}^{n-1} \frac{d^r f}{dt^r}\Big|_{t=0} \cdot \frac{t^r}{r!} + R_n \quad \text{---(5)}$$

$$= \sum_{r=0}^{n-1}\left(\sum_{j=1}^{m}\left(b_j\frac{\partial}{\partial x_j}\right)^r f\Big|_{t=0}\frac{t^r}{r!}\right) + R_n$$

$$= \sum_{r=0}^{n-1} \frac{1}{r!}\left[\sum_{j=1}^{m}(x_j - a_j)\frac{\partial}{\partial x_j}\right]^r f\Big|_{P=P_0} + R_n$$

where $R_n = \frac{1}{n!}\sum_{j=1}^{m}\left(x_j - a_j\frac{\partial}{\partial x_j}\right)^n f\Big|_{P=\xi}$

we know that the expansion of RHS of are obtained by binomial equation. If the limit of R_n tends to infinity the eqn 7 gives an infinite convergence

Example: Given $0 \le x, y < 2\pi$ and that power greater than zero degree are negligible show that the Taylor expansion of $f(x,y) = \cos x \sin y$ in the nbhd of the neighbourh of $(0, \frac{\pi}{2})$ is

$$f = 1 - \frac{x^2}{2} - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2$$

Soln: $m = 2$, $x_1 = x$, $x_2 = y$

$P_0 = (a_1, a_2) = (0, \frac{\pi}{2}) \in D\,[\mathbb{R}^2]$

since $n \ge 3$ are negligible

$$\cancel{f = \sum_{r=0}^{2}}$$

$$\cancel{f = \frac{1}{}}$$

$$f = \sum_{r=0}^{2}\frac{1}{r!}\left[x\frac{\partial}{\partial x} + \left(y - \frac{\pi}{2}\right)\frac{\partial}{\partial y}\right]^r f\Big|_{P=P_0}$$

$$f = \frac{1}{0!}f_{P_0} + \frac{1}{1!}\left[x\frac{\partial}{\partial x} + \left(y - \frac{\pi}{2}\right)\frac{\partial}{\partial y}\right]f + \frac{1}{2!}\left[x\frac{\partial}{\partial x} + \left(y - \frac{\pi}{2}\right)\frac{\partial}{\partial y}\right]^2 f\Big|_{P_0} + R_3$$

$$f_{P_0} + x f_x\big|_{P_0} + \left(y - \frac{\pi}{2}\right)f_y\big|_{P_0}$$

$$+ \frac{1}{2}\left[x^2 f_{xx} + 2x\left(y - \frac{\pi}{2}\right)f_{xy} + \left(y - \frac{\pi}{2}\right)^2 f_{yy}\right]_{P_0}$$

$$f_{P_0} = \cos 0 \cdot \sin\frac{\pi}{2} = 1$$

$$f_x = -\sin x \sin y\big|_{P_0} = 0$$

$f_y = \cos x \cos y|_{P_0} = 0$

$f_{xx} = -\cos x \sin y|_{P_0} = -1$

$f_{xy} = -\sin x \cos y|_{P_0} = 0$

$f_{yy} = -\cos x \sin y|_{P_0} = -1$

$f = 1 + 0 + 0 + \frac{1}{2}\left[x^2(-1) + 2x(y-\frac{\pi}{2})0 + (y-\frac{\pi}{2})^2(-1)\right]$

$= 1 - \frac{1}{2}x^2 - \frac{1}{2}(y-\frac{\pi}{2})^2$

Ex: Given $-1 < xy \leq 1$ expand $z = \log(1+xy)$ about $P_0 = (-\frac{1}{2}, \frac{1}{2})$ up to 2nd degree of terms, also expand about $(0,1)$

$\frac{\text{Soln}}{(0,1)}$

(5)

$n = 2,\ x_1 = x,\ x_2 = y$

$P_0 = (a_1, a_2) = (0,1)$

since $n \geq 2$

$f = \sum_{r=0}^{1} \frac{1}{r!}\left[x\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right]^r f_{P_0}$

$= \frac{1}{0!}f_{P_0} + \frac{1}{1!}\left[x\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right]^1 f$

$\cancel{+ \frac{1}{2!}\left[x\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right]^2 f_{P_0}} + R$

$f_{P_0} + xf_x + (y-1)f_y +$

$\cancel{\frac{1}{2}\left[x^2 f_{xx} + 2x(y-1)f_{xy} + (y-1)^2 f_{yy}\right]_{P_0}}$

$z = f(x,y) = \log(1+xy)$

$z_{P_0} = \log 1 = 0$

$z_x^{\circ} = f_x^{\circ} = \frac{y}{1+xy} = 1$

$z_y^{\circ} = \frac{x}{1+xy} = 0$

$\cancel{z_{xx}^{\circ}}$

$f_{P_0} + xf_x + (y-1)f_y + R$

$f = 0 + x1 + (y-1)0$

$f = x$

MEE 319

Formulas for Efficiency (η) of screw thread

using effort:

$\eta = \frac{\text{Ideal effort}}{\text{Actual effort}} = \frac{\cancel{\text{effort}}\ \text{effort without friction}}{\text{effort with friction}}$

$= \frac{W\tan\alpha}{W\tan(\alpha+\phi)} = \frac{\tan\alpha}{\tan(\alpha+\phi)}$

using torque

$\eta = \frac{\text{Torque without friction}}{\text{Torque with friction}}$

$= \frac{P.\frac{d}{2}}{P.\frac{d}{2} + \mu WR}$

Efficiency using Mechanical Advantage (MA) and Velocity Ratio (VR)

$MA = \frac{\text{Load}}{\text{Effort}} = \frac{W}{P_L}$ (P_L is from $T_L = P_L \times L$)

$\frac{W \times L}{\frac{d}{2}.W\tan(\alpha+\phi)} = \frac{2L}{d\tan(\alpha+\phi)}$

$VR = \frac{\text{Distance moved by effort}}{\text{Distance moved by Load}}$

$= \frac{2\pi L}{P_i}$

$\tan\alpha = \frac{nP_i}{\pi d} \Rightarrow P_i = \frac{\pi d\tan\alpha}{n}$

$VR = \frac{2\pi Ln}{\pi d\tan\alpha} = \frac{2Ln}{d\tan\alpha}$

~~Efficiency =~~ $\cancel{\frac{MA}{}}$

Assume $n = 1$

$VR = \frac{2L}{d\tan\alpha}$