For α to have a unique inverse $a \neq -1$

Transforming polar to Cartesian coordinates

Example 2.5 $x = r\cos\theta$, $y = r\sin\theta$, $r \geq 0$, $0 \leq \theta < 2\pi$

Show that the laplacian $\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}}$ becomes $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

called the laplacian.

Soln

(B) To find $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

$x=r\cos\theta,\ y=r\sin\theta$

$r\geq 0,\ 0\leq\theta\leq 2\pi$

$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

$J=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix}=\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$

$|J|=\cos\theta(r\cos\theta)-\sin\theta(-r\sin\theta)$
$=r$

$J^{-1}=\frac{1}{r}\begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}=\frac{\partial(r,\theta)}{\partial(x,y)}$

$r_x=\cos\theta,\ r_y=\sin\theta$

$\theta_x=-\frac{1}{r}\sin\theta,\ \theta_y=\frac{1}{r}\cos\theta$

$\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}=\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}$

$\frac{\partial}{\partial y}=\frac{\partial r}{\partial y}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}=\sin\theta\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}$

$$\frac{\partial^2}{\partial x^2} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial}{\partial r} - \frac{1}{r}\sin\theta \frac{\partial}{\partial \theta}\right)$$

$$\cancel{= \cos\theta\left[\cos\theta \frac{\partial^2}{\partial r^2}\right] \cos\theta}$$

$$= \cos^2\theta \frac{\partial^2}{\partial r^2} - \cos\theta\sin\theta\left(\frac{1}{r}\frac{\partial^2}{\partial\theta\partial r} + \frac{-1}{r^2}\frac{\partial}{\partial\theta}\right)$$

$$- \frac{1}{r}\sin\theta\left(\cos\theta \frac{\partial^2}{\partial r\partial\theta} + -\sin\theta\frac{\partial}{\partial r}\right) + \frac{1}{r}\sin\theta\cdot\frac{1}{r}\left(\sin\theta\frac{\partial^2}{\partial\theta^2} + \cos\theta\frac{\partial}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} - \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta} \cancel{+ \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}} - \frac{1}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta}$$

$$+ \frac{1}{r}\sin^2\theta\frac{\partial}{\partial r} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2}{\partial\theta^2} \cancel{+ \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}}$$

$$= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2}{\partial\theta^2} - \frac{2}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta} + \frac{1}{r}\sin^2\theta\frac{\partial}{\partial r} + \frac{2}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}$$

$$\frac{\partial^2}{\partial y^2} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right)\left(\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}\right)$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2} + \sin\theta\cos\theta\left(-\frac{1}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}\right) + \frac{1}{r}\cos\theta\left(\sin\theta\frac{\partial^2}{\partial r\partial\theta}\right.$$

$$\left. + \cos\theta\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\cos\theta\left(\cos\theta\frac{\partial^2}{\partial\theta^2} - \sin\theta\frac{\partial}{\partial\theta}\right)$$

$$= \sin^2\theta\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos^2\theta\frac{\partial^2}{\partial\theta^2} + \frac{2}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta} + \frac{1}{r}\cos^2\theta\frac{\partial}{\partial r}$$

$$- \frac{2}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r}$$