3.8 Fitting Harmonic Functions — The FFT

For this section of the practical you will find the Signals and Systems textbook and handout *Sampled Signals, the DTFT and the DFT* useful. In an experimental situation we often measure a time-series which we want to Fourier transform to obtain a spectrum, or a frequency-dependent quantity which we wish to inverse Fourier transform in order to retrieve a time-dependent quantity. This requires us to perform a Discrete Fourier Transform (DFT), which is usually calculated using the Fast Fourier Transform (FFT) algorithm (Brigham). The DFT (and therefore the FFT) work on a finite number N data samples, and return the same number of transform samples. The FFT algorithm places restrictions on the possible values of N, and the FFT algorithm used by the Excel FFT tool restricts us to $N = 2^n$, i.e. N must be an integer power of 2 (e.g. 64, 512, 2048).

3.8.1 Exercise: The Spectrum of a Truncated and Sampled Signal

1. $\boxed{\text{What is}}$ the Fourier transform of the signal:

$$x_1(t) = \cos(\omega_1 t) + \sin(\omega_2 t)$$

   where $\omega_1 = 2\pi \times 10^3$ rad/s and $\omega_2 = 4\pi \times 10^3$ rad/s?

2. $\boxed{\text{What is}}$ the Fourier transform of the signal if its is forced to be causal and is truncated after 16 ms, i.e. the signal:

$$x_2(t) = x_1(t) \times \left[u(t) - u(t - 16 \times 10^{-3})\right] = x_1(t) \times \sqcap\left(\frac{t - 8 \times 10^{-3}}{16 \times 10^{-3}}\right)$$

3. Use an Excel spreadsheet to generate 512 samples of the signal $x_2(t)$ using a sampling interval of $T = 3.125 \times 10^{-5}$ s, i.e. generate:

$$x_2[n] = x_2(nT) \quad n = 0 \cdots 511$$

   $\boxed{\text{Plot this signal on suitable axes.}}$

4. Use the Excel FFT tool to obtain the DFT $X[k]$ of the sampled signal. Notice that the returned array is in cartesian complex form, with the index k running from 0 through to $N - 1$. Convert the returned array to polar form and $\boxed{\text{plot the magnitude}}$ on appropriate axes. The equivalent frequency spacing of the returned DFT samples is $\Delta\omega = 2\pi/(NT)$. Use this relationship to scale the frequency axis appropriately. Recall that the DFT is a finite set of samples of the DTFT, which in turn is related to the aliased CTFT spectrum, i.e.:

$$X[k] \approx \frac{1}{NT} X(\omega)\big|_{\omega = 2\pi k/(NT)}$$

   You must now reorganize the returned FFT samples $X[k]$ to give a more physically realistic appearance to the spectrum (Recall that the DFT is periodic with a period N, i.e. $X[k] = X[k + N]$). Thus, the 512 samples are $X[0] \cdots X[511]$, of which the first 256 are $X[0] \cdots X[255]$. The remaining 256 samples $X[256] \cdots X[511]$ must now be copied to positions $X[-256] \cdots X[-1]$. Note that this places the "DC" component corresponding to $k = 0$ at position 257. The point $X[256]$ (now at position 1) is called the "folding point" $\boxed{\text{Plot } X[-256] \cdots X[255] \text{ vs } \omega.}$

5. Repeat the above exercise using a sampling period $T = 3.0 \times 10^{-5}$. $\boxed{\text{Plot } X[-256] \cdots X[255] \text{ vs } \omega.}$ Can you $\boxed{\text{explain}}$ the difference in the resulting DFT spectrum? Which signal is more likely to be obtained in a real experiment?

3.8.2 Exercise: The Aliased Spectrum of a Signal

1. $\boxed{\text{What is}}$ the Fourier transform of the signal:

$$x_3(t) = \sqcap\left(\frac{t - 25 \times 10^{-3}}{50 \times 10^{-3}}\right)$$

2. $\boxed{\text{What is}}$ the Fourier transform of this signal after impulse train sampling with a sampling period of $T = 10^{-3}$ s, i.e.:

$$x_4(t) = x_3(t) \times \sum_{\ell=-\infty}^{+\infty} \delta(t - \ell \times 10^{-3})$$

3. Use an Excel spreadsheet to generate and $\boxed{\text{plot 512 samples}}$ of the signal $x_3(t)$ using a sampling period of $T = 10^{-3}$ s, i.e.:

$$x_4[n] = x_3(nT) \quad n = 0 \cdots 511$$

4. Use Excel to find the DFT spectrum $X(\omega)$, $k = 0 \cdots 511$, re-order to give a 512 point spectrum centred near zero, plotted against ω, $k = -256, \cdots 0, \cdots 255$, and compare this DFT spectrum with the analytic spectrum of the signal $x_3(t)$. $\boxed{\text{Why}}$ are there differences?

3.8.3 Exercise: The Inverse DFT

1. What is the inverse Fourier Transform of the signal:

$$X_5(\omega) = \sqcap\left(\frac{\omega}{4\pi \times 10^3}\right)$$

2. Use an Excel spreadsheet to generate 512 samples of the function $X_5(\omega)$, which might be considered to be a measured frequency response of an electronic filter, i.e.:

$$X[k] = X_5(k\Delta\omega) \qquad k = -\frac{N}{2} \cdots \frac{N}{2} - 1$$

Use a frequency sampling interval of $\Delta\omega = 40\pi$ rad/s. Notice that this choice of N and $\Delta\omega$ implies a total frequency coverage of:

$$-\left(\frac{N}{2}\right) \times \Delta\omega \leq \omega \leq +\left(\frac{N}{2} - 1\right) \times \Delta\omega$$

Note $k = 0$ will be the 257th sample. Re-order the samples prior to inverse DFT processing so that the index k runs from 0 to $N-1$. Ensure that the samples are Hermitian to ensure that the transformed function is purely real.

3. Use the Excel inverse DFT tool to determine and $\boxed{\text{plot } x_5(t),}$ which is the impulse response for the hypothetical filter, usually denoted $h(t)$.
   $\boxed{\text{Compare}}$ the DFT result with the expected analytic result (Be careful to use the correct frequency width to generate the analytic result).

4. As an optional exercise, include a linear phase factor in the hypothetical discrete frequency response and see how it affects the computed impulse response.

3.9 References

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