3.6 Random Numbers — Monte Carlo Simulations

In many areas of experimental, computational and theoretical physics it is desirable to produce *stochastic* computer models of an experimental setup (including measurement errors) or a physical entity (such as a galaxy or star). Such models are called Monte Carlo simulations because they are non-deterministic, and therefore model real-life processes more closely.

3.6.1 Exercise: Estimation of π

Use Excel to draw a circle of unit radius inside a square with sides measuring two units (both centred on (0,0)). Generate two columns of 1000 random numbers drawn from a uniform distribution spanning the interval [-1,+1]. Consider these two columns to be (x,y) coordinate pairs, and plot these points over the circle and square. The probability that a point lies within the circle is given by:

$$P = \frac{\text{area of circle}}{\text{area of box}} = \frac{\pi r^2}{(2r)^2} = \frac{\pi}{4}$$

Use the Excel COUNTIF() function to estimate π from N pairs of random numbers where N runs from 1 to 1000. $\boxed{\text{Plot your estimate of } \pi \text{ as a function of } N}$.

3.6.2 Exercise: Propagation of Errors

In this exercise we model a simple first-year experiment in which the radius, length and mass of a cylinder are measured in order to determine the density of the material that the cylinder is made from.

1. Generate 150 "synthetic data sets" for length (ℓ), diameter (d) and mass (M) measurement using a gaussian random number generator to add experimental errors with known variances to the "exact" dimensions and mass. Use the following parent distribution parameters to calculate the synthetic data:

$$\begin{aligned} \mu_\ell &= 40 \text{ mm}; \qquad & \sigma_\ell &= 0.05 \text{ mm} \\ \mu_d &= 20 \text{ mm}; \qquad & \sigma_d &= 0.005 \text{ mm} \\ \mu_M &= 90 \text{ g}; \qquad & \sigma_M &= 1 \text{ mg} \end{aligned}$$

2. Consider these 150 data sets to be 15 independent groups of ten measurements. For each group determine the sample mean $\bar{x}_i$, sample standard deviation s_i and the standard deviation of the mean $(s_m)_i$ (the uncertainty in the mean) for each of the three "measured" quantities. $\boxed{\text{Table of } \bar{x}_i,\ s_i \text{ and } (s_m)_i}$

3. Use these 15 sets of mean values to obtain 15 estimates of the required density ρ_i. Find the standard deviation $s(\rho_i)$ of these 15 mean densities (i.e. the standard deviation of the mean), which is the uncertainty in ρ.
$\boxed{\text{Table of 15 values of } \rho_i \text{ and single value of } s(\rho_i)}$

4. Use equation 0-7 to calculate variance s^2 and equation 0-1 to find the uncertainty $(s_m)_i$ of each of the 15 density determinations by propagating the measured variances in the "measurements". Compare these 15 values with the single value obtained in the previous step. $\boxed{\text{Table of } (s_m)_i \text{ compared to } s(\rho_i)}$.

3.6.3 Exercise: Stochastic Physical Model

A certain theoretical model for the distribution of stars in a globular cluster predicts that the stellar density obeys the following probability density functions for the spherical polar coordinates:

$$
\begin{aligned}
P_r(r) &= \begin{cases} \frac{1}{r_0} e^{-r/r_0} & ; r \geq 0 \\ 0 & ; r < 0 \end{cases} \\
P_\phi(\phi) &= \begin{cases} \frac{1}{2\pi} & ; 0 \leq \phi \leq 2\pi \\ 0 & ; \text{otherwise} \end{cases} \\
P_\theta(\theta) &= \begin{cases} \frac{1}{2}\sin(\theta) & ; 0 \leq \theta \leq \pi \\ 0 & ; \text{otherwise} \end{cases}
\end{aligned}
$$

1. $\boxed{\text{Show}}$ that the random variable transformations required are given by:

$$
\begin{aligned}
r &= r_0 \ln\left(\frac{1}{1-u}\right) \\
\theta &= \cos^{-1}(1-2v) \\
\phi &= 2\pi w
\end{aligned}
$$

   where u, v and w are drawn from uniform parent populations spanning [0,1].

2. Generate synthetic (r, θ, ϕ) coordinates for a globular cluster with 1000 stars and a scale-size of $r_0 = 10^5$ km.

3. $\boxed{\text{Plot the projections}}$ of these stars in the (x–y), (y–z) and (x–z) planes. Recall that the transformation from SPC to Cartesian coordinates is given by:

$$
\begin{aligned}
x &= r\cos(\phi)\sin(\theta) \\
y &= r\sin(\phi)\sin(\theta) \\
z &= r\cos(\theta)
\end{aligned}
$$

4. $\boxed{\text{Obtain histograms}}$ of the x, y and z distributions of the stars (plotted on the same set of axes).