Complexity and Big-O Notation

Contents

Introduction

An important question is: How efficient is an algorithm or piece of code? Efficiency covers lots of resources, including:

- CPU (time) usage
- memory usage
- disk usage
- network usage

All are important but we will mostly talk about CPU time in 367. Other classes will discuss other resources (e.g., disk usage may be an important topic in a database class).

Be careful to differentiate between:

1. Performance: how much time/memory/disk/... is actually used when a program is run. This depends on the machine, compiler, etc. as well as the code.
2. Complexity: how do the resource requirements of a program or algorithm scale, i.e., what happens as the size of the problem being solved gets larger.

Complexity affects performance but not the other way around.

The time required by a method is proportional to the number of "basic operations" that it performs. Here are some examples of basic operations:

- one arithmetic operation (e.g., +, *).
- one assignment
- one test (e.g., x == 0)
- one read
- one write (of a primitive type)

Some methods perform the same number of operations every time they are called. For example, the *size* method of the *List* class always performs just one operation: `return numItems`; the number of operations is independent of the size of the list. We say that methods like this (that always perform a fixed number of basic operations) require **constant time**.

Other methods may perform different numbers of operations, depending on the value of a parameter or a field. For example, for the array implementation of the *List* class, the *remove* method has to move over all of the items that were to the right of the item that was removed (to fill in the gap). The number of moves depends both on the position of the removed item and the number of items in the list. We call the important factors (the parameters and/or fields whose values affect the number of operations performed) the **problem size** or the **input size**.

When we consider the complexity of a method, we don't really care about the **exact** number of operations that are performed; instead, we care about how the number of operations relates to the problem size. If the problem size doubles, does the number of operations stay the same? double? increase in some other way? For constant-time methods like the *size* method, doubling the problem size does not affect the number of operations (which stays the same).

Furthermore, we are usually interested in the **worst case**: what is the **most** operations that might be performed for a given problem size (other cases -- best case and average case -- are discussed below). For example, as discussed above, the *remove* method has to move all of the items that come after the removed item one place to the left in the array. In the worst case, **all** of the items in the array must be moved. Therefore, in the worst case, the time for *remove* is proportional to the number of items in the list, and we say that the worst-case time for *remove* is **linear** in the number of items in the list. For a linear-time method, if the problem size doubles, the number of operations also doubles.

TEST YOURSELF #1

Assume that lists are implemented using an array. For each of the following *List* methods, say whether (in the worst case) the number of operations is independent of the size of the list (is a constant-time method), or is proportional to the size of the list (is a linear-time method):

- the constructor
- *add* (to the end of the list)
- *add* (at a given position in the list)
- *isEmpty*
- *contains*
- *get*

solution

Constant and linear times are not the only possibilities. For example, consider method *createList*:

```
List createList( int N ) {
  List L = new List();
  for (int k=1; k<=N; k++) L.add(0, new Integer(k));
  return L;
}
```

Note that, for a given N, the for-loop above is equivalent to:

```
L.add(0, new Integer(1) );
L.add(0, new Integer(2) );
L.add(0, new Integer(3) );
    ...
L.add(0, new Integer(N) );
```

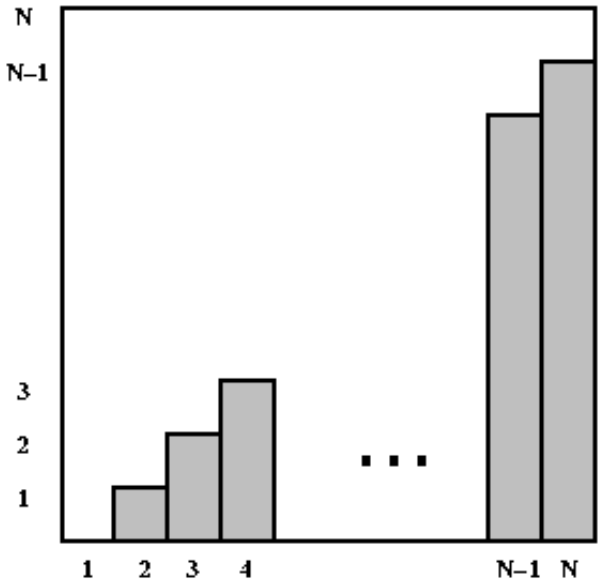
If we assume that the initial array is large enough to hold N items, then the number of operations for each call to *add* is proportional to the number of items in the list when *add* is called (because it has to move every item already in the array one place to the right to make room for the new item at position 0). For the N calls shown above, the list lengths are: 0, 1, 2, ..., N-1. So what is the total time for all N calls? It is proportional to 0 + 1 + 2 + ... + N-1.

Recall that we don't care about the exact time, just how the time depends on the problem size. For method *createList*, the "problem size" is the value of N (because the number of operations will be different for different values of N). It is clear that the time for the N calls (and therefore the time for method *createList*) is **not** independent of N (so *createList* is not a constant-time method). Is it proportional to N (linear in N)? That would mean that doubling N would double the number of operations performed by *createList*. Here's a table showing the value of 0+1+2+...+(N-1) for some different values of N:

N	0+1+2+...+(N-1)

4	6
8	28
16	120

Clearly, the value of the sum does more than double when the value of N doubles, so *createList* is not linear in N. In the following graph, the bars represent the lengths of the list (0, 1, 2, ..., N-1) for each of the N calls.



The value of the sum (0+1+2+...+(N-1)) is the sum of the areas of the individual bars. You can see that the bars fill about half of the square. The whole square is an N-by-N square, so its area is N^2; therefore, the sum of the areas of the bars is about $N^2/2$. In other words, the time for method *createList* is proportional to the **square** of the problem size; if the problem size doubles, the number of operations will quadruple. We say that the worst-case time for *createList* is **quadratic** in the problem size.

TEST YOURSELF #2

Consider the following three algorithms for determining whether anyone in the room has the same birthday as you.

Algorithm 1: You say your birthday, and ask whether anyone in the room has the same birthday. If anyone does have the same birthday, they answer yes.

Algorithm 2: You tell the first person your birthday, and ask if they have the same birthday; if they say no, you tell the second person your birthday and ask whether they have the same birthday; etc, for each person in the room.

Algorithm 3: You only ask questions of person 1, who only asks questions of person 2, who only asks questions of person 3, etc. You tell person 1 your birthday, and ask if they have the same birthday; if they say no, you ask them to find out about person 2. Person 1 asks person 2 and tells you the answer. If it is no, you ask person 1 to find out about person 3. Person 1 asks person 2 to find out about person 3, etc.

Question 1: For each algorithm, what is the factor that can affect the number of questions asked (the "problem size")?

Question 2: In the worst case, how many questions will be asked for each of the three algorithms?

Question 3: For each algorithm, say whether it is constant, linear, or quadratic in the problem size in the worst case.

solution

Big-O Notation

We express complexity using **big-O notation**. For a problem of size N:

- a constant-time method is "order 1": O(1)
- a linear-time method is "order N": O(N)
- a quadratic-time method is "order N squared": $O(N^2)$

Note that the big-O expressions do not have constants or low-order terms. This is because, when N gets large enough, constants and low-order terms don't matter (a constant-time method will be faster than a linear-time method, which will be faster than a quadratic-time method). See below for an example.

Formal definition:

> A function T(N) is O(F(N)) if for some constant c and for all values of N greater than some value n_0:
>
> $$T(N) <= c * F(N)$$

The idea is that T(N) is the **exact** complexity of a method or algorithm as a function of the problem size N, and that F(N) is an upper-bound on that complexity (i.e., the actual time/space or whatever for a problem of size N will be no worse than F(N)). In practice, we want the smallest F(N) -- the **least** upper bound on the actual complexity.

For example, consider T(N) = 3 * N^2 + 5. We can show that T(N) is O(N^2) by choosing c = 4 and n_0 = 2. This is because for all values of N greater than 2:

$$3 * N^2 + 5 <= 4 * N^2$$

T(N) is **not** O(N), because whatever constant c and value n_0 you choose, I can always find a value of N greater than n_0 so that 3 * N^2 + 5 is greater than c * N.

How to Determine Complexities

In general, how can you determine the running time of a piece of code? The answer is that it depends on what kinds of statements are used.

1. Sequence of statements

```
statement 1;
statement 2;
  ...
statement k;
```

(Note: this is code that really is exactly k statements; this is **not** an unrolled loop like the N calls to *add* shown above.) The total time is found by adding the times for all statements:

total time = time(statement 1) + time(statement 2) + ... + time(statement k)

If each statement is "simple" (only involves basic operations) then the time for each statement is constant and the total time is also constant: O(1). In the following examples, assume the statements are simple unless noted otherwise.

2. if-then-else statements

```
if (condition) {
    sequence of statements 1
}
else {
    sequence of statements 2
}
```

Here, either sequence 1 will execute, or sequence 2 will execute. Therefore, the worst-case time is the slowest of the two possibilities: max(time(sequence 1), time(sequence 2)). For example, if sequence 1 is O(N) and sequence 2 is O(1) the worst-case time for the whole if-

then-else statement would be O(N).

3. for loops

```
for (i = 0; i < N; i++) {
    sequence of statements
}
```

The loop executes N times, so the sequence of statements also executes N times. Since we assume the statements are O(1), the total time for the for loop is N * O(1), which is O(N) overall.

4. Nested loops

First we'll consider loops where the number of iterations of the inner loop is independent of the value of the outer loop's index. For example:

```
for (i = 0; i < N; i++) {
    for (j = 0; j < M; j++) {
        sequence of statements
    }
}
```

The outer loop executes N times. Every time the outer loop executes, the inner loop executes M times. As a result, the statements in the inner loop execute a total of N * M times. Thus, the complexity is O(N * M). In a common special case where the stopping condition of the inner loop is `j < N` instead of `j < M` (i.e., the inner loop also executes N times), the total complexity for the two loops is $O(N^2)$.

Now let's consider nested loops where the number of iterations of the inner loop depends on the value of the outer loop's index. For example:

```
for (i = 0; i < N; i++) {
    for (j = i+1; j < N; j++) {
        sequence of statements
    }
}
```

Now we can't just multiply the number of iterations of the outer loop times the number of iterations of the inner loop, because the inner loop has a different number of iterations each time. So let's think about how many iterations that inner loop has. That information is given in the following table:

Value of i	Number of iterations of inner loop
0	N

1	N-1
2	N-2
...	...
N-2	2
N-1	1

So we can see that the total number of times the sequence of statements executes is: N + N-1 + N-2 + ... + 3 + 2 + 1. We've seen that formula before: the total is $O(N^2)$.

TEST YOURSELF #3

What is the worst-case complexity of the each of the following code fragments?

1. Two loops in a row:

```
for (i = 0; i < N; i++) {
    sequence of statements
}
for (j = 0; j < M; j++) {
    sequence of statements
}
```

   How would the complexity change if the second loop went to N instead of M?

2. A nested loop followed by a non-nested loop:

```
for (i = 0; i < N; i++) {
    for (j = 0; j < N; j++) {
        sequence of statements
    }
}
for (k = 0; k < N; k++) {
    sequence of statements
}
```

3. A nested loop in which the number of times the inner loop executes depends on the value of the outer loop index:

```
for (i = 0; i < N; i++) {
    for (j = N; j > i; j--) {
```

```
            sequence of statements
        }
    }
```

solution

---

5. Statements with method calls:

   When a statement involves a method call, the complexity of the statement includes the complexity of the method call. Assume that you know that method *f* takes constant time, and that method *g* takes time proportional to (linear in) the value of its parameter *k*. Then the statements below have the time complexities indicated.

```
f(k);  // O(1)
g(k);  // O(k)
```

   When a loop is involved, the same rule applies. For example:

```
for (j = 0; j < N; j++) g(N);
```

   has complexity (N^2). The loop executes N times and each method call `g(N)` is complexity O(N).

TEST YOURSELF #4

For each of the following loops with a method call, determine the overall complexity. As above, assume that method *f* takes constant time, and that method *g* takes time linear in the value of its parameter.

```
1. for (j = 0; j < N; j++) f(j);

2. for (j = 0; j < N; j++) g(j);

3. for (j = 0; j < N; j++) g(k);
```

solution

Best-case and Average-case Complexity

Some methods may require different amounts of time on different calls, even when the problem size is the same for both calls. For example, consider the *add* method that adds an item to the end of the list. In the worst case (the array is full), that method requires time proportional to the number of items in the list (because it has to copy all of them into the new, larger array). However, when the array is not full, *add* will only have to copy one value into the array, so in that case its time is independent of the length of the list; i.e., constant time.

In general, we may want to consider the **best** and **average** time requirements of a method as well as its worst-case time requirements. Which is considered the most important will depend on several factors. For example, if a method is part of a time-critical system like one that controls an airplane, the worst-case times are probably the most important (if the plane is flying towards a mountain and the controlling program can't make the next course correction until it has performed a computation, then the best-case and average-case times for that computation are not relevant -- the computation needs to be guaranteed to be fast enough to finish before the plane hits the mountain).

On the other hand, if occasionally waiting a long time for an answer is merely inconvenient (as opposed to life-threatening), it may be better to use an algorithm with a slow worst-case time and a fast average-case time, rather than one with so-so times in both the average and worst cases.

Note that calculating the average-case time for a method can be tricky. You need to consider all possible values for the important factors, and whether they will be distributed evenly.

When do Constants Matter?

Recall that when we use big-O notation, we drop constants and low-order terms. This is because when the problem size gets sufficiently large, those terms don't matter. However, this means that two algorithms can have the **same** big-O time complexity, even though one is always faster than the other. For example, suppose algorithm 1 requires N^2 time, and algorithm 2 requires $10 * N^2 + N$ time. For both algorithms, the time is $O(N^2)$, but algorithm 1 will always be faster than algorithm 2. In this case, the constants and low-order terms **do** matter in terms of which algorithm is actually faster.

However, it is important to note that constants do *not* matter in terms of the question of how an algorithm "scales" (i.e., how does the algorithm's time change when the problem size doubles). Although an algorithm that requires N^2 time will always be faster than an algorithm that requires $10*N^2$ time, for **both** algorithms, if the problem size doubles, the actual time will quadruple.

When two algorithms have **different** big-O time complexity, the constants and low-order terms only matter when the problem size is small. For example, even if there are large constants involved, a linear-time algorithm will always eventually be faster than a quadratic-time algorithm. This is illustrated in the following table, which shows the value of 100*N (a time that is linear in N) and the value of $N^2/100$ (a time that is quadratic in N) for some values of N. For values of N less than 10^4, the quadratic time is smaller than the linear time. However, for all values of N greater than 10^4, the linear time is smaller.

N	100*N	$N^2/100$
10^2	10^4	10^2
10^3	10^5	10^4
10^4	10^6	10^6
10^5	10^7	10^8
10^6	10^8	10^{10}
10^7	10^9	10^{12}

Answers to Self-Study Questions