MPI* Info

Logique 1

TD13

PICQUET Augustine

1 Prouver les séquents suivants en logique intuitionniste :

(a) $B \vdash A \to (B \vee C)$

(b) $A \to (B \to C) \vdash (A \wedge B) \to C$

(c) $(A \wedge B) \to C \vdash A \to (B \to C)$

(d) $(A \vee B) \wedge (A \vee C) \vdash A \vee (B \wedge C)$

(e) $\vdash (((A \wedge B \to C) \wedge B) \to (A \to C)$

2 Prouver le séquent suivant en logique classique : $A \to \neg B \vdash \neg A \vee \neg B$

3 Parmi les séquents suivants, trois sont dérivables en logique intuitionniste et un l'est uniquement en logique classique. Identifier lequel en exhibant des preuves adaptées.

(a) $\neg(A \vee B) \vdash \neg A \wedge \neg B$

(b) $\neg(A \wedge B) \vdash \neg A \vee \neg B$

(c) $\neg A \wedge \neg B \vdash \neg(A \vee B)$

(c) $\neg A \vee \neg B \vdash \neg(A \wedge B)$

4 Les règles suivantes permettent d'introduire des connecteurs dans la partie gauche d'un séquent. Montrer que toutes ces règles sont dérivables.

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \wedge_g$$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C} \vee_g$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A, A \to B \vdash C} \to_g$$

$$\frac{}{\Gamma, A \neg A \vdash \bot} \neg_g$$

TD Logique (1)

1a)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,r}{B, A \vdash B}\,\text{hom}}{B, A \vdash B \vee C}\,I_{\vee 1}}{B \vdash A \to B \vee C}\,I_{\to}$$

1b)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\;}\,r}{\Gamma \vdash A \to (B \to C)}\,\text{Hom} \quad \dfrac{\dfrac{}{\Gamma \vdash A \wedge B}\,\text{Hom+r}}{\Gamma \vdash A}\,e_{\wedge}}{A \to B \to C, (A \wedge B) \vdash B \to C}\,MP \quad \dfrac{\dfrac{\dfrac{}{A \wedge B \vdash A \wedge B}\,r}{A \wedge B, A \to B \to C \vdash A \wedge B}\,\text{Hom}}{(A \wedge B), A \to B \to C \vdash B}\,E e_{\wedge}}{\dfrac{A \to (B \to C), (A \wedge B) \vdash C}{A \to (B \to C) \vdash (A \wedge B) \to C}\,I_{\to}}\,MP$$

$\Gamma = A \to B \to C, (A \wedge B)$.

1c)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A \wedge B) \to C \vdash (A \wedge B) \to C}\,r}{(A \wedge B) \to C, A, B \vdash (A \wedge B) \to C}\,\text{Hom} \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,r}{\Gamma \vdash A}\,\text{Hom} \quad \dfrac{\dfrac{}{B \vdash B}\,r}{\Gamma \vdash B}\,\text{Hom}}{\Gamma \vdash A \wedge B}\,i_{\wedge}}{(A \wedge B) \to C, A, B \vdash C}\,MP}{\dfrac{(A \wedge B) \to C, A \vdash B \to C}{(A \wedge B) \to C \vdash A \to (B \to C)}\,i_{\to}}\,i_{\to}$$

$\Gamma = (A \wedge B) \to C, A, B$.

1d)

$\Gamma = (A \vee B) \wedge (A \vee C)$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (A \vee B) \wedge (A \vee C)}\,r}{\Gamma \vdash A \vee B}\,e_{\wedge} \quad \dfrac{\dfrac{}{\Gamma, A \vdash A}\,r}{\Gamma, A \vdash A \vee (B \wedge C)}\,i_{\vee} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, B \vdash (A \vee C) \wedge (A \vee B)}\,r}{\Gamma, B \vdash A \vee C}\,e_{\wedge} \quad \dfrac{\dfrac{}{\Gamma, B, A \vdash A}\,r}{\Gamma, B, A \vdash A \vee (B \wedge C)} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, B, C \vdash B}\,r \quad \dfrac{}{\Gamma, B, C \vdash C}}{\Gamma, B, C \vdash B \wedge C}}{\Gamma, B, C \vdash A \vee (B \wedge C)}}{\Gamma, B \vdash A \vee (B \wedge C)}}{(A \vee B) \wedge (A \vee C) \vdash A \vee (B \wedge C)}$$

1e)

$$
\dfrac{
  \dfrac{
    \dfrac{\dfrac{}{\Gamma \vdash \Gamma}\,\text{ax}}{\Gamma \vdash ((A\wedge B)\to C)\wedge B}\,\text{mon}
  }{\Gamma \vdash (A\wedge B)\to C}
  \qquad
  \dfrac{
    \dfrac{\dfrac{}{A \vdash A}\,\text{r}}{\Gamma \vdash A}\,\text{mon}
    \quad
    \dfrac{\dfrac{\dfrac{}{(A\wedge B)\to C\wedge B \vdash (A\wedge B)\to C\wedge B}\,\text{r}}{\Gamma \vdash ((A\wedge B)\to C)\wedge B}\,\text{Hom}}{\Gamma \vdash B}
  }{\Gamma \vdash A\wedge B}
}{
  \dfrac{((A\wedge B)\to C)\wedge B,\ A \vdash C}{
    \dfrac{(((A\wedge B)\to C)\wedge B) \vdash A\to C}{\vdash (((A\wedge B)\to C)\wedge B)\to(A\to C)}\,\text{I}\to
  }\,\text{I}\to
}\,\text{mp}
$$

2)

$$\Gamma = A\to\neg B,\ A.$$

$$
\dfrac{
  \dfrac{}{A\to\neg B \vdash A\vee\neg A}\,\text{te}
  \qquad
  \dfrac{
    \dfrac{\dfrac{}{A\vdash A}\,\text{ax}}{\Gamma\vdash A}\,\text{mon}
    \quad
    \dfrac{}{\Gamma\vdash A\to\neg B}\,\text{ax}
  }{A\to\neg B,\ A\vdash \neg A\vee\neg B}\,\text{mp}
  \qquad
  \dfrac{\dfrac{\dfrac{}{\neg A\vdash\neg A}\,\text{r}}{A\to\neg B,\ \neg A\vdash\neg A}\,\text{mon}}{A\to\neg B,\ \neg A\vdash\neg A\vee\neg B}\,\text{iv}
}{A\to\neg B\vdash\neg A\vee\neg B}\,\text{iv}
$$

3)
(a)

__Note__ $\Gamma_1 = \neg(A\vee B), A \qquad \Gamma_2 = \neg(A\vee B), B$.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\Gamma_1\vdash\neg(A\vee B)}\,\text{r}
      \quad
      \dfrac{\dfrac{}{\Gamma\vdash A}\,\text{r}}{\Gamma_1\vdash A\vee B}\,\cancel{\text{Hom}}\ \text{iv}
    }{\neg(A\vee B),\ A\vdash\bot}\,\text{e}\neg
  }{\neg(A\vee B)\vdash\neg A}\,\text{i}\neg
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{\Gamma_2\vdash\neg(A\vee B)}\,\text{r}
      \quad
      \dfrac{\dfrac{}{\Gamma_2\vdash B}\,\text{r}}{\Gamma_2\vdash A\vee B}\,\text{Hom}
    }{\neg(A\vee B),\ B\vdash\bot}\,\text{e}\neg
  }{\neg(A\vee B)\vdash\neg B}\,\text{i}\neg
}{\neg(A\vee B)\vdash\neg A\wedge\neg B}\,\text{i}\wedge
$$

TD Logique 3.

3b) C'est ce séquent qui ne peut être démontré en logique naturelle, en effet, on a besoin du tiers-exclu (te)

$$\dfrac{\dfrac{}{\neg(A\wedge B)\vdash A\vee\neg A}\,te \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg(A\wedge B),A,B\vdash\neg(A\wedge B)}\,r \quad \dfrac{\dfrac{}{\neg(A\wedge B),A,B\vdash A}\,r\quad\dfrac{}{\neg(A\wedge B),A,B\vdash B}\,r}{\neg(A\wedge B),A,B\vdash A\wedge B}\,i_\wedge}{\neg(A\wedge B),A,B\vdash\bot}\,e_\neg}{\neg(A\wedge B),A\vdash\neg B}\,i_\neg}{\neg(A\wedge B),A\vdash\neg A\vee\neg B}\,i_\vee \qquad \dfrac{\dfrac{}{\neg(A\wedge B),\neg A\vdash\neg A}\,ax}{\neg(A\wedge B),\neg A\vdash\neg A\vee\neg B}\,i_{\vee d}}{\neg(A\wedge B)\vdash\neg A\vee\neg B}\,e_\vee$$

3c) $\boxed{\Gamma_1 = \neg A\wedge\neg B, A\vee B}$

$$\dfrac{\dfrac{\dfrac{}{\Gamma\vdash A\vee B}\,r \quad \dfrac{\dfrac{\dfrac{}{\Gamma,A\vdash\neg A\wedge\neg B}\,r}{\Gamma,A\vdash\neg A}\,e_\wedge \quad \dfrac{}{\Gamma,A\vdash A}\,r}{\Gamma,A\vdash\bot}\,e_\neg \quad \dfrac{\dfrac{}{\Gamma,B\vdash B}\quad \dfrac{\dfrac{}{\Gamma,B\vdash\neg A\wedge\neg B}\,r}{\Gamma,B\vdash\neg B}\,e_\wedge}{\Gamma,B\vdash\bot}\,e_\neg}{\neg A\wedge\neg B, A\vee B\vdash\bot}\,e_\vee}{\neg A\wedge\neg B\vdash\neg(A\vee B)}\,i_\neg$$

3d) $\boxed{\Gamma_1 = \neg A\vee\neg B, A\wedge B}$

$$\dfrac{\dfrac{\Gamma\vdash\neg A\vee\neg B \quad \dfrac{\dfrac{\dfrac{}{\Gamma,\neg A\vdash A\wedge B}\,r}{\Gamma,\neg A\vdash A}\,e_\wedge \quad \dfrac{}{\Gamma,\neg A\vdash\neg A}\,r}{\Gamma,\neg A\vdash\bot}\,e_\neg \quad \dfrac{\dfrac{\dfrac{}{\Gamma,\neg B\vdash A\wedge B}\,r}{\Gamma,\neg B\vdash B}\,e_\wedge \quad \dfrac{}{\Gamma,\neg B\vdash\neg B}\,r}{\Gamma,\neg B\vdash\bot}\,e_\neg}{\neg A\vee\neg B, A\wedge B\vdash\bot}\,e_\vee}{\neg A\vee\neg B\vdash\neg(A\wedge B)}\,i_\neg$$

4a)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma,A,B\vdash C}{\Gamma,A,B,A\wedge B\vdash C}\,Hon}{\Gamma,A,A\wedge B\vdash B\to C}\,i_\to \quad \dfrac{\dfrac{}{\Gamma,A,A\wedge B\vdash A\wedge B}\,r}{\Gamma,A,A\wedge B\vdash B}\,e_\wedge}{\Gamma,A\wedge B,A\vdash C}\,MP}{\Gamma,A\wedge B\vdash A\to C}\,i_\to \quad \dfrac{\dfrac{}{\Gamma,A\wedge B\vdash A\wedge B}\,r}{\Gamma,A\wedge B\vdash A}\,e_\wedge}{\Gamma,A\wedge B\vdash C}\,MP$$

TD logique 4

4b)

$$\dfrac{}{\Gamma, A\vee B \vdash A\vee B}\,r \qquad \dfrac{\dfrac{}{\Gamma, B \vdash C}\,ax}{\Gamma, A\vee B, B \vdash C}\,Hom$$

$$\dfrac{\quad}{\Gamma, A\vee B \vdash C}\,e_\vee \qquad \dfrac{\dfrac{}{\Gamma, A \vdash C}\,ax}{\Gamma, A\vee B, A \vdash C}$$

4c)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A, B \vdash C}\,ax}{\Gamma, A, B, A\to B \vdash C}\,Hom}{\Gamma, A, A\to B \vdash B\to C}\,i_\to \qquad \dfrac{\dfrac{}{\Gamma, A, A\to B \vdash A\to B}\,r \quad \dfrac{}{\Gamma, A, A\to B \vdash A}\,r}{\Gamma, A, A\to B \vdash B}\,MP}{\Gamma, A, A\to B \vdash C}\,MP$$

4d)

$$\dfrac{\dfrac{}{\Gamma, A, \neg A \vdash A}\,ax \qquad \dfrac{}{\Gamma, A, \neg A \vdash \neg A}\,ax}{\Gamma, A, \neg A \vdash \bot}\,\neg e$$