Assignment-3 (PCS)

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Problem 1 Answer the following

- Let $Y(t) = X(t-d)$, where d is a constant delay and $X(t)$ is WSS. Find $R_{YX}(\tau)$, $S_{YX}(f)$, $R_Y(\tau)$ and $S_Y(f)$.
- Let $X(t) = A\cos(2\pi t)$, where A is a random variable. Find the expressions for mean and autocorrelation of $X(t)$.
- Let $Y(t)$ consist of a desired signal $X(t)$ plus noise $N(t)$: $Y(t) = X(t) + N(t)$. Find the cross-correlation between $Y(t)$ and $X(t)$ assuming that $X(t)$ and $N(t)$ are independent random processes.
- Sinusoid with Random Phase: Let $X(t) = A\cos(2\pi f t + \theta)$, where θ is uniformly distributed in the interval $[0, 2\pi]$. Find $S_X(f)$.
- Let the process X_n be a sequence of uncorrelated random variables with zero mean and variance σ^2. Find $S_X(f)$.

Ans-1} (a) $y(t) = x(t-d)$

$x(t) \rightarrow$ WSS

$\therefore$ clearly $y(t)$ is also WSS

$$R_y(\tau) = \overline{y(t)\,y(t+\tau)}$$
$$= \overline{x(t-d)\,x(t+\tau-d)}$$

Let $t-d \rightarrow t_0$

so it is just shift of origin and since $x(t)$ is WSS

Here shift of origin doesn't matter

$$\therefore R_y(\tau) = \overline{x(t_0)\,x(t_0+\tau)} = R_x(\tau)$$

Now,

$$R_{yx}(\tau) = \overline{y(t)\,x(t+\tau)} = \overline{x(t-d)\,x(t+\tau)}$$

similarly just like as above

we have

$$R_{yx}(\tau) = \overline{x(t-d)\,x(t-d+\tau+d)} = R_x(\tau+d)$$

Now as we know that

$$R_x(\tau) \xrightarrow{F.T} S_x(f)$$

Thus →
$$R_y(\tau) \xrightarrow{F.T} S_y(f)$$
$$R_{yx}(\tau) \xrightarrow{F.T} S_{yx}(f)$$

and using $R_y(\tau)$ & $R_{yx}(\tau)$ we have;

and using $R_Y(\tau)$ & $R_{YX}(\tau)$ we have;

$$S_Y(f) = S_X(f)$$

and $S_{YX}(f) = FT\{R_X(\tau+d)\} = S_X(f)\, e^{j2\pi f d}$

(b) $x(t) = A\cos(2\pi f t)$

↳ Mean at some t = $\overline{x(t)} = \overline{A\cos 2\pi f t} = \bar{A}\cos(2\pi f t)$

where $\bar{A} = \int_{-\infty}^{\infty} a\, p_A(a)\, da$

$$R_X(t_1,t_2) = \overline{A^2\cos(2\pi f t_1)\cos(2\pi f t_2)} = \left(\overline{A^2}\right) \times \frac{1}{2} \times \left[\cos\left(2\pi f(t_1+t_2)\right) + \cos\left(2\pi f(t_1-t_2)\right)\right]$$

where $\overline{A^2} = \int_{-\infty}^{\infty} a^2\, p_A(a)\, da$

(c) $Y(t) = x(t) + N(t)$

↳ where $x(t)$ & $N(t)$ are independent Random process

$$R_{YX}(t_1,t_2) = \overline{Y(t_1)\,x(t_2)} = \underbrace{\overline{x(t_1)\,x(t_2)}}_{R_X(t_1,t_2)} + \overline{x(t_2)\,N(t_1)}$$

Now, since $x(t)$ and $N(t)$ are independent

we have $p(x_2, N_1) = p_{X_2}\, p_{N_1}$

using this we get

$$R_{YX}(t_1,t_2) = R_X(t_1,t_2) + \left[\overline{N(t_1)} \times \overline{X(t_2)}\right]$$

(d) $x(t) = A\cos(2\pi f t + \theta)$ $\quad \theta \rightarrow$ U.D. in $[0, 2\pi]$

$$R_X(t_1,t_2) = \int_0^{2\pi} A^2\cos(2\pi f t_1 + \theta)\cos(2\pi f t_2 + \theta) \times \frac{1}{2\pi}\, d\theta$$

$$= \frac{A^2}{4\pi}\int_0^{2\pi} \left\{\cos\left(2\pi f(t_2+t_1) + 2\theta\right) + \cos\left(2\pi f(t_2-t_1)\right)\right\} d\theta$$

$4\pi \quad 0 \quad t$

$$= \frac{A^2}{4\pi}\left[0 + \int_0^{2\pi} \cos(2\pi f(t_2 - t_1))\, d\theta\right]$$

$$= \frac{A^2}{2}\cos 2\pi f(t_2 - t_1) = R_x(t_2 - t_1) = R_x(\tau)$$

$\therefore R_x(\tau) = \frac{A^2}{2}\cos(2\pi f \tau)$ where $\tau = t_2 - t_1$

and

$$S_x(f) = F.T\{R_x(\tau)\}$$

Note → Let the frequency of $R_x(\tau)$ be 'f_1' instead of 'f'

(Just a Notation change :))

$$S_x(f) = \frac{A^2}{2}\int_{-\infty}^{\infty} \cos 2\pi f_1 \tau \, e^{-j2\pi f \tau}\, d\tau$$

$$= \frac{A^2}{4} F.T\{e^{j2\pi f_1 t} + e^{-j2\pi f_1 t}\}$$

$$\therefore S_x(f) = \frac{A^2}{4}\left(\delta(2\pi(f - f_1)) + \delta(2\pi(f + f_1))\right)$$

(e) X_n → Sequence of Uncorrelated Random Variables of zero Mean and Variance → σ^2

so we can say that say for some time t_1 and t_2 we record 2 random variable outputs of X_n say x_1 and x_2 where $\overline{x_1} = \overline{x_2} = 0$ with x_1 uncorrelated of x_2

with $var(x_1) = var(x_2) = \sigma^2$

or $\overline{x_1^2} - 0 = \sigma^2$

↳ $\overline{x_1^2} = \overline{x_2^2} = \sigma^2$

$$\therefore R_{x_n}(t_1, t_2) = \overline{x_1 x_2} = \overline{x_1}\,\overline{x_2} = 0 \quad \{\text{where } t_1 \neq t_2\}$$

$\tau \neq 0$

when $t_1 = t_2$

$$R_{x_n}(t_1, t_1) = \overline{x_1^2} = \sigma^2 \; ; \text{ when } \tau = 0$$

$$\therefore R_{x_n}(\tau) = \begin{cases} \sigma^2 & ; \tau = 0 \\ 0 & ; \text{else} \end{cases}$$

$$S_{x_n}(f) = F.T\{R_{x_n}(\tau)\}$$

$$= F.T\{\text{box fn (with } A=\sigma^2 \text{ and } T\to 0)\}$$

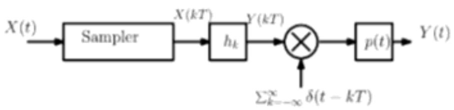
$$= \frac{AT}{\pi f T}\sin(\pi f T)\Big|_{T\to 0,\, A=\sigma^2}$$

$$\therefore S_{x_n}(f) = \frac{\sigma^2}{\pi f}\lim_{T\to 0}\frac{T\sin(\pi f T)}{T} = \frac{\sigma^2}{\pi f}\times 0 = 0$$

Problem 2 Sampling Random Processes: We have already covered the sampling theorem for deterministic signals. Now we state the sampling theorem for random processes. Let $X(t)$ be a WSS process with autocorrelation function $R_X(\tau)$ and power spectral density $S_X(f)$. Suppose that $S_X(f)$ is bandlimited, that is, $S_X(f) = 0$ for $|f| > W$. Sampling theorem can be extended to $X(t)$ as follows. If

$$\hat{X}(t) = \sum_{-\infty}^{\infty} X(nT)p(t-nT), \quad p(t) = \frac{sin(\pi t/T)}{\pi t/T},$$

the $\hat{X}(t) = X(t)$ in the mean square sense. The proof of above is beyond the scope of this course. Consider



the block diagram given below. Considering $h_k(f) = \sum_{n=0}^{p}\beta_n e^{-j2\pi fn}$, the output of the digital filter is given by

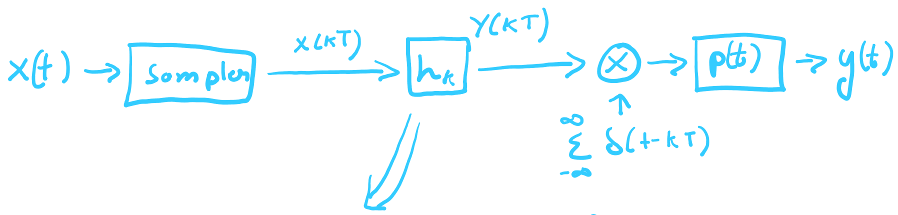
$$Y(kT) = \sum_{n=0}^{p}\beta_n X((k-n)T)$$

Derive the autocorrelation function $R_Y(kT)$. Find the autocorrelation of $Y(t)$, $R_Y(\tau)$ using the interpolation formula. Using $R_Y(\tau)$, compute the output power spectral density $S_Y(f)$, in terms of $h_k(f)$.

Ans-2} $x(t) \Rightarrow$ WSS with $S_x(f) = 0$ for $|f| > W$ | $p(t) = \dfrac{\sin(\pi t/T)}{\pi t/T}$

$\to$ Sampling theorem on $x(t) \Rightarrow \hat{x}(t) = \sum_{-\infty}^{\infty} x(nT)p(t-nT)$

then $\hat{x}(t) = x(t)$ in mean square sense



$$h_k(f) = \sum_{n=0}^{p}\beta_n e^{-j2\pi fn}$$

Using this we can see $\to$ $Y(kT) = \sum_{n=0}^{p}\beta_n x((k-n)T)$

Now, $Y(t) = \left(\sum_{k=-\infty}^{\infty} Y(kT)\delta(t-kT)\right) * p(t)$

using interpolation formula we have $\Rightarrow$

using interpolation formula we have $\Rightarrow$

$$y(t) = \sum_{k=-\infty}^{\infty} y(kT)\, p(t-kT)$$

$$= \sum_{k=-\infty}^{\infty} \left[\sum_{n=0}^{p} \beta_n\, x((k-n)T) \right] p(t-kT)$$

$$= \sum_{n=0}^{p} \beta_n \sum_{k=-\infty}^{\infty} x((k-n)T)\, p(t-kT)$$

$$= \sum_{n=0}^{p} \beta_n \sum_{k=-\infty}^{\infty} x((k-n)T)\, p(t-nT-(k-n)T)$$

$$= \sum_{n=0}^{p} \beta_n\, x(t-nT)$$

$$R_y(kT) = \overline{\sum_{n_1=0}^{p} \sum_{n_2=0}^{p} \beta_{n_1} \beta_{n_2}\, x(t-n_1T)\, x(t+(k-n_2)T)}$$

$$= \sum_{n_1=0}^{p} \sum_{n_2=0}^{p} \beta_{n_1} \beta_{n_2}\, \overline{x(t-n_1T)\, x(t-n_1T+(k+n_1-n_2)T)}$$

$$= \sum_{n_1=0}^{p} \sum_{n_2=0}^{p} \beta_{n_1} \beta_{n_2}\, R_x((k+n_1-n_2)T)$$

Now, $R_y(\tau) = \overline{\sum_{n=0}^{p} \beta_n\, x(t-nT) \sum_{n=0}^{p} \beta_n\, x(t+\tau-nT)}$

$$= \sum_{n_1=0}^{p} \sum_{n_2=0}^{p} \beta_{n_1} \beta_{n_2}\, \overline{x(t-n_1t)\, x(t+\tau-n_2T)}$$

$$= \sum_{n_1=0}^{p} \sum_{n_2=0}^{p} \beta_{n_1} \beta_{n_2}\, \overline{x(t-n_1T)\, x(t-n_1T+\tau-n_2T+n_1T)}$$

$$= \sum_{n_1=0}^{p} \sum_{n_2=0}^{p} \beta_{n_1} \beta_{n_2}\, R_x(\tau+(n_1-n_2)T)$$

Now, to calculate $S_y(f)$ in terms of $h_k(f)$

we know that

$$S_y(f) = |p(f)|^2\, S_M(f)$$

where $M(t) = Y(kT) \sum_{k=-\infty}^{\infty} \delta(t-kT)$

and $Y(kT) = h_k(t) * X(kT)$

$k=-\infty$

and $\quad y(kT) = h_k(t) * x(kT)$

and $\quad x(kT) = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$

Now when

$$A(t) = B(t)\, C(t)$$

$R_A(t) = R_B(t)\, C(t)$ if $\quad C \rightarrow$ Not a Random process

and by taking fourier transform on both the sides, we get,

$$S_A(f) = S_B(f) * C(f)$$

with $\quad C(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT) \longrightarrow C(f) = \sum_{k=-\infty}^{\infty} e^{j2\pi f kT}$

this Rule quite holds

-> By putting every thing together we get

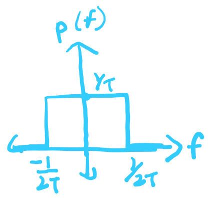
$$\therefore \quad S_y(f) = |p(f)|^2 \left\{ \left[|h_k(f)|^2 \left(S_x(f) * \sum_{k=-\infty}^{\infty} e^{j2\pi f kT} \right) \right] * \sum_{k=-\infty}^{\infty} e^{j2\pi f kT} \right\}$$

where $\quad p(t) = \frac{A}{\pi t} \sin(\pi t T) \Big|_{\substack{A \to 1/T \\ T \to 1/T}}$

by using duality

$$p(f) = \frac{1}{T}\left[u\left(f + \frac{1}{2T}\right) - u\left(f - \frac{1}{2T}\right) \right]$$

or visually



[Note-> we can now compute $S_y(f)$ from here in terms of $h_k(f)$ clearly but since $S_x(f)$ is unknown I am leaving expression here only]

Problem 3 Amplitude Modulation by Random Signals: Let $A(t)$ be a WSS random process that represents an information signal. In general $A(t)$ will be lowpass, that is, its power spectral density will be concentrated at low frequencies. An amplitude modulation (AM) system generates a transmit signal $X(t) = A(t)cos(2\pi f_c t + \theta)$. We assume that θ is a random variable that is uniformly distributed in the interval $[0, 2\pi]$, and θ and $A(t)$ are independent. Find the expression for the autocorrealtion function of $X(t)$. Is $X(t)$ WSS? What is the power spectral density $S(f)$?

Ans-3} $X(t) = A(t)\cos(2\pi f_c t + \theta)$

$$R_x(t_1, t_2) = \overline{A(t_1)A(t_2)\cos(2\pi f_c t_1 + \theta)\cos(2\pi f_c t_2 + \theta)}$$

and since $A(t)$ & θ are independent $p(A,\theta) = P_A P_\theta$

and

$$R_x(t_1, t_2) = (\overline{A(t_1)A(t_2)}) \times \overline{\cos(2\pi f_c t_1 + \theta)\cos(2\pi f_c t_2 + \theta)}$$

$$= R_A(t_1, t_2)\int_0^{2\pi} \cos 2\pi f_c t_1 + \theta \cos 2\pi f_c t_2 + \theta \times \frac{1}{2\pi} d\theta$$

$$= \frac{1}{4\pi} R_A(t_1, t_2)\int_0^{2\pi} \cos(2\pi f_c(t_2 + t_1) + 2\theta) + \cos 2\pi f_c (t_2 - t_1)\, d\theta$$

$$= \frac{1}{4\pi} R_A(t_1, t_2)\left[0 + \cos 2\pi f_c(t_2 - t_1) \times \int_0^{2\pi} d\theta\right]$$

$$= \frac{R_A(t_1, t_2)}{2}\cos 2\pi f_c(t_2 - t_1)$$

and since $A(t) \rightarrow$ WSS

$$R_x(t_1, t_2) = \frac{R_A(t_2 - t_1)}{2}\cos 2\pi f_c(t_2 - t_1)$$

and Now since $R_x(t_1, t_2)$ depends only on $t_2 - t_1 = \tau$

$x(t) \rightarrow$ WSS

with $R_x(\tau) = \frac{R_A(\tau)}{2}\cos(2\pi f_c \tau)$

Using $R_x(\tau) \xrightarrow{F.T} S_x(f)$, $R_A(\tau) \rightarrow S_A(f)$

we have

$$S_x(f) = \frac{1}{2} F.T\left\{R_A(\tau)\left(\frac{e^{j2\pi f_c \tau} + e^{-j2\pi f_c \tau}}{2}\right)\right\}$$

also by using freq^n shift property we get:-

$$S_x(f) = \frac{1}{4}\left[S_A(f - f_c) + S_A(f + f_c)\right]$$

Problem 4 Bandpass random signals arise in communication systems when wide-sense stationary white noise is filtered by bandpass filters. Let $N(t)$ be such a process with power spectral density $S_N(f)$. By the virtue of being bandpass, $N(t)$ can be represented as

$$N(t) = N_c(t)cos(2\pi f_c t) - N_s(t)sin(2\pi f_c t)$$

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$$N(t) = N_c(t)cos(2\pi f_c t) - N_s(t)sin(2\pi f_c t)$$

where $N_c(t)$ and $N_s(t)$ are jointly wide-sense stationary processes with

$$S_{N_c}(f) = S_{N_s}(f) = S_N(f - f_c) + S_N(f + f_c).$$

The received signal in an amplitude modulated system is $Y(t) = A(t)cos(2\pi f_c t + \theta) + N(t)$, where $N(t)$ is a bandlimited white noise process with spectral density $S_N(f) = \frac{N_0}{2}$ for $|f \pm f_c| \leq W$, and zero elsewhere. Find the signal-to-noise ratio of the recovered signal.
Hint: Follow these steps: First represent the received signal in terms of the bandpass representation of noise. Next step, use the AM demodulation procedure, i.e., multiply the received signal by $2cos(2\pi f_c t + \theta)$, and pass the signal through a low pass filter. Next, compute the power in the resulting signal and noise terms at the output, and then compute the ratio of signal and noise power.

Ans-4 }

$N(t) = N_c(t)\cos(2\pi f_c t) - N_s \sin(2\pi f_c t) \rightarrow S_N(f)$

↳ $N_c(t)$ & $N_s(t)$ ⇒ Jointly are WSS

$S_{N_c}(f) = S_{N_s}(f) = S_N(f-f_c) + S_N(f+f_c)$

Recived signal → $Y(t) = A(t)\cos(2\pi f_c t + \theta) + N(t)$

↳ in an AM system

$N(t)$ → Bandlimited with White Noise with $S_N(f) = \begin{cases} \frac{N_0}{2} & ; |f \pm f_c| \leq W \\ 0 & ; \text{else} \end{cases}$

→ step-1
{ Recived signal in banpass representation of $N(t)$ }

$Y(t) = A(t)\cos(2\pi f_c t + \theta) + N_c(t)\cos(2\pi f_c t) - N_s(t)\sin(2\pi f_c t)$

→ step-2
{ AM Demodulation → $\times 2\cos(2\pi f_c t + \theta)$ }

$Y_{DM}(t) = A(t)2\cos^2(2\pi f_c t + \theta) + N_c(t)\,2\cos(2\pi f_c t + \theta)\cos(2\pi f_c t)$
$- N_s(t)\,2\cos(2\pi f_c t + \theta)\sin 2\pi f_c t$

using trigrometric Identities this could be reduced to

$Y_{DM}(t) = A(t) + A(t)\cos(4\pi f_c t + 2\theta) - N_s(t)(\sin(4\pi f_c t + \theta) + \sin\theta)$
$+ N_c(t)(\cos 4\pi f_c t + \cos\theta)$

→ step-3
{Passing $Y_{DM}(t)$ through Low Pass filter and geting $Y_{LP}(t)$ }

$Y_{LP}(t) = A(t) + N_c(t)\cos\theta - N_s(t)\sin\theta$

(TRICK IS TO Remove higher freq^n terms)

→ Step-4
{To Find SNR using $Y_{LP}(t)$ }

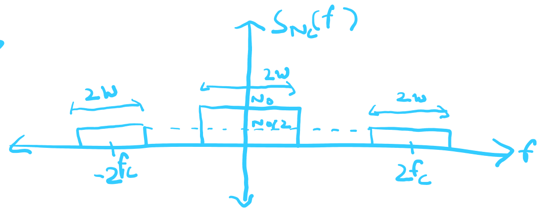
using sum of Random Process we have

$$S_{Y_{LP}}(f) = S_A(f) + S_{N_c}(f)\cos\theta - S_{N_s}(f)\sin\theta$$

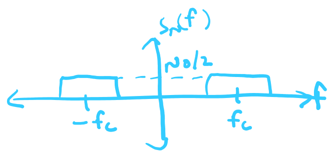
$$= S_A(f) + (\cos\theta - \sin\theta)\underbrace{\left[S_N(f-f_c) + S_N(f+f_c)\right]}_{\hookrightarrow S_{N_c}(f)}$$

$$SNR = \frac{S_{Y_{LP}}(f)}{S_N(f)} = \frac{S_A(f)}{S_N(f)} + (\cos\theta - \sin\theta)\frac{S_{N_c}(f)}{S_N(f)}$$

where $S_{N_c}(f) \Rightarrow$



and $S_N(f) \Rightarrow$



and $\theta \rightarrow \theta_{recived}$ (a constant)

So since we don't know $S_A(f)$ we can't calculate SNR

further though here we know $\theta \to$ constn, $S_N(f)$ and $S_{N_c}(f)$

— x — x —

END of Assignment

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— x —

Thank You

— x —