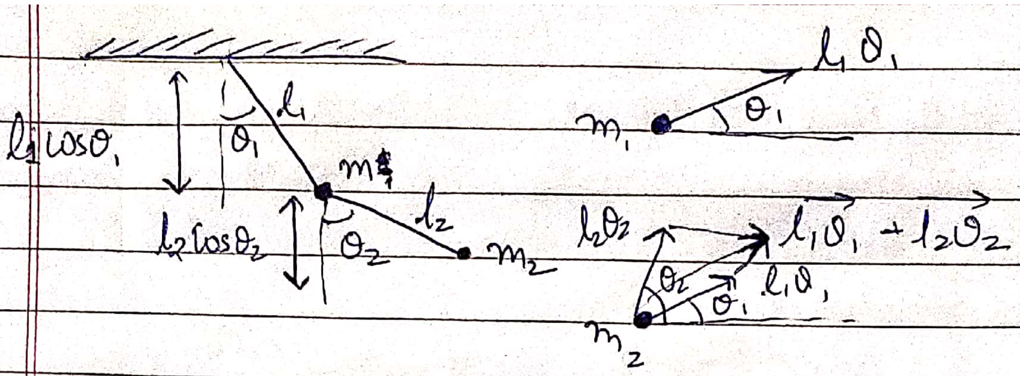
Q3



Let the double Pendulum system consist of a simple pendulum with length of rod l_1 and mass of bob m_1, to which is connected simple pendulum with length l_2 and mass m_2. The angles are as show $\in [-\pi, \pi]$. We observe that each physical state of the pendulum can be represented by a unique value of (θ_1, θ_2)

a) Hence, the generalized coordinates for the system is $\{\theta_1, \theta_2\}$

Let k_1, U_1 be the kinetic and Potential Energy of the bob of mass m_1 and k_2, U_2 for the bob with mass m_2. $\therefore$ $k_1 = \frac{1}{2} m_1 l_1^2 \dot\theta_1^2$, $U_1 = m_1 g l_1 (1-\cos\theta_1)$

$$k_2 = \frac{1}{2} m_2 [l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1-\theta_2)]$$

$$U_2 = m_2 g [l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)]$$

Let $k = k_1 + k_2$

$U = U_1 + U_2$

$$k = \frac{(m_1+m_2)}{2} l_1^2\dot\theta_1^2 + \frac{m_2}{2} l_2^2 \dot\theta_2^2 + m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1-\theta_2)$$

$$U = (m_1+m_2) g l_1 (1-\cos\theta_1) + m_2 g l_2 (1-\cos\theta_2)$$

The Lagrangian $L = k - U$

b) Hence, $L = \frac{(m_1+m_2)}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2-\theta_1)$

$- (m_1+m_2) g l_1 (1-\cos\theta_1) - m_2 g l_2 (1-\cos\theta_2)$

For θ_1,

$\frac{\partial L}{\partial \theta_1} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) \Rightarrow m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2-\theta_1) - (m_1+m_2) g l_1 \sin\theta_1 =$
$\frac{d}{dt}\left((m_1+m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_2-\theta_1)\right)$

$\Rightarrow m_2 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - (m_1+m_2) g \sin\theta_1 = (m_1+m_2) l_1 \ddot{\theta}_1 +$
$m_2 \ddot{\theta}_2 \cos(\theta_2-\theta_1) l_2 -$
$m_2 l_2 \dot{\theta}_2 \sin(\theta_2-\theta_1)(\dot{\theta}_2-\dot{\theta}_1)$

$\Rightarrow (m_1+m_2)[l_1\ddot{\theta}_1 + g\sin\theta_1] + m_2 l_2 \ddot{\theta}_2 \cos(\theta_2-\theta_1) = m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2-\theta_1)$

For θ_2

$\frac{\partial L}{\partial \theta_2} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) \Rightarrow -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2-\theta_1) - m_2 g l_2 \sin\theta_2 =$
$\frac{d}{dt}\left(m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_2-\theta_1)\right)$

$\Rightarrow -l_1 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2-\theta_1) - g\sin\theta_2 = l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_2-\theta_1)$
$- l_1 \dot{\theta}_1 \sin(\theta_2-\theta_1)(\dot{\theta}_2-\dot{\theta}_1)$

$\Rightarrow l_2 \ddot{\theta}_2 + g\sin\theta_2 + l_1 \ddot{\theta}_1 \cos(\theta_2-\theta_1) + l_1 \dot{\theta}_1^2 \sin(\theta_2-\theta_1) = 0$

c) Hence, the Euler-Lagrange equations are

i) $(m_1+m_2)[l_1\ddot{\theta}_1 + g\sin\theta_1] + m_2 l_2 \ddot{\theta}_2 \cos(\theta_2-\theta_1) = m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2-\theta_1)$

ii) $l_2 \ddot{\theta}_2 + g\sin\theta_2 + l_1 \ddot{\theta}_1 \cos(\theta_2-\theta_1) + l_1 \dot{\theta}_1^2 \sin(\theta_2-\theta_1) = 0$