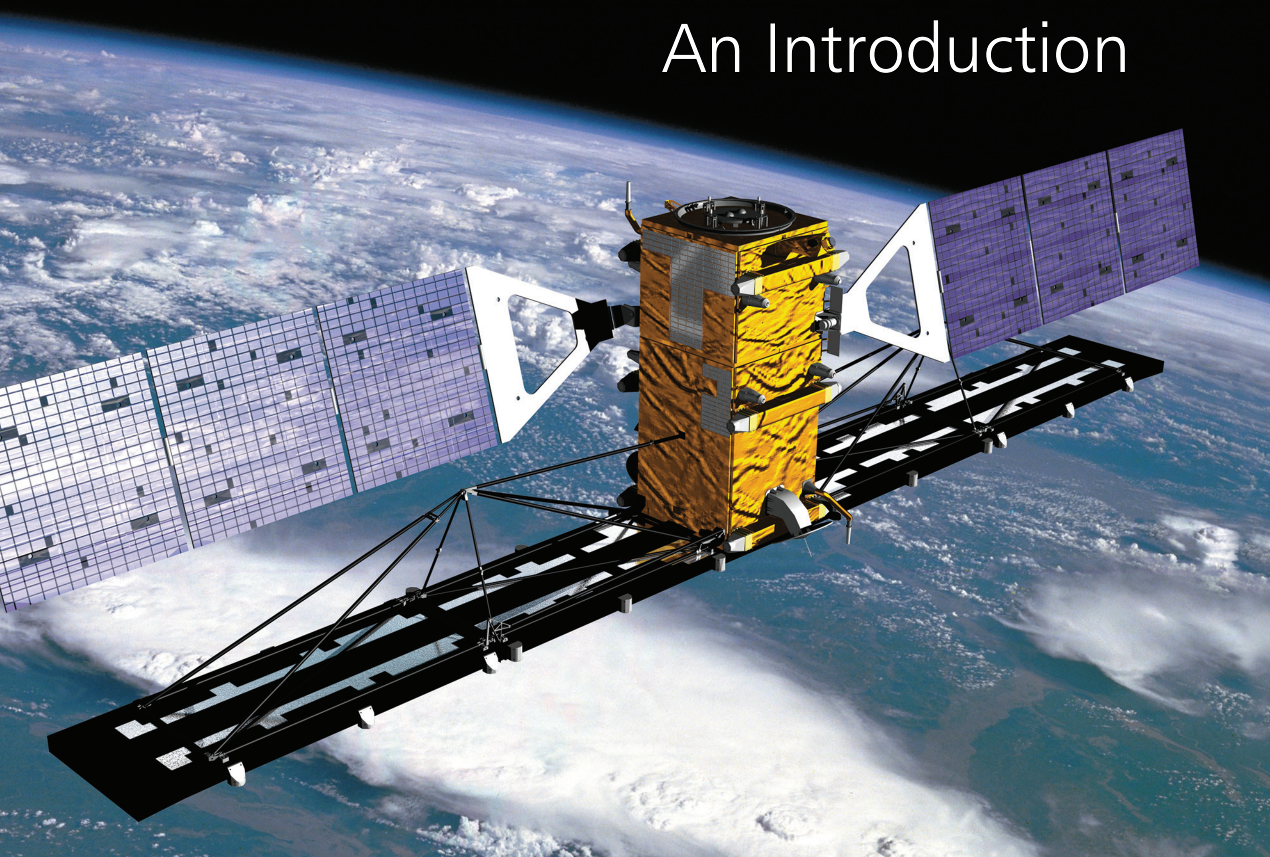
ANTON H.J. DE RUITER | CHRISTOPHER J. DAMAREN | JAMES R. FORBES

# Spacecraft Dynamics and Control

## An Introduction



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# SPACECRAFT DYNAMICS AND CONTROL

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## AN INTRODUCTION

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To Janice, Thomas, Benjamin, Therese and Marie
*A.dR*

To Yvonne, Gwen, and Georgia
*C.J.D*

For Allison
*J.R.F*

# Contents

# Preface

This book presents a fundamental introduction to spacecraft orbital and attitude dynamics as well as its control. There are several excellent books related to these subjects. It is not our intention to compete with these well-established texts. However, many of them assume relatively significant backgrounds on behalf of the reader, which can make them difficult to follow for the beginner. It is our hope that this book will fill that void, and that by studying this book, more advanced texts on the subject will become more accessible to the reader. This book is suitable for first courses in spacecraft dynamics and control at the upper undergraduate level or at the beginning graduate level. The book is naturally split between orbital mechanics, and spacecraft attitude dynamics and control. It could therefore be used for two one semester courses, one on each subject. It could also be used for self-study.

The primary objective of this book is to educate, and the structure of the book reflects this. This book could also be used by the professional looking to refresh some of the fundamentals. We have made this book as self-contained as possible. In each chapter we develop a subject at a fundamental level (perhaps drawing on results from previous chapters). As a result, the reader should not only understand what the key mathematical results are, but also how they were obtained and what their limitations are (if applicable). At the end of each chapter we provide a few recommended references should the reader have interest in exploring the subject further.

The assumed reader background is minimal. Junior undergraduate level mathematics and mechanics taught in standard engineering programs should be sufficient. While a background in classical control theory would help, it is not necessary to have it in order to be able to follow the treatment of spacecraft attitude control. The presentation in this book on spacecraft attitude control is completely self-contained, and it could in fact be used as a substitute for a complete first (undergraduate level) course in classical control. The reader without a control background will learn classical control theory motivated by a real system to be controlled, namely, a spacecraft (as opposed to some abstract transfer functions). The reader with a prior control background may gain new appreciation of the theory by seeing it presented in the context of an application.

In Chapters 1 and 2, we present the vector kinematics and rigid body dynamics required to be able to describe spacecraft motion. Chapters 3 to 10 contain the orbital mechanics component of this book. Topics include the two-body problem, preliminary orbit determination, orbital and interplanetary maneuvers, orbital perturbations, low-thrust trajectory design, spacecraft formation flying, and the restricted three-body problem. Chapter 11 presents a high level overview of both passive and active means of spacecraft attitude stabilization, and provides an introduction to control systems. Chapters 12 to 16 present aspects of spacecraft attitude dynamics (disturbance torques and a solution for torque-free motion), and more detailed treatments of passive means of spacecraft attitude stabilization. Chapters 17 to 23 present active means of spacecraft attitude control using classical control techniques. Chapters 24 and 25 present introductions to some more advanced topics, namely nonlinear spacecraft attitude control and spacecraft attitude determination. These chapters also provide a brief introduction to nonlinear control theory and state estimation. Chapter 26 presents an overview of practical issues that must be dealt with in designing a spacecraft attitude control system, namely different spacecraft attitude sensor and actuator types, digital control implementation issues and effects of unmodeled dynamics on spacecraft attitude control systems. Finally, Appendices A and B contain some background reference material.

After finishing this book, the reader should have a strong understanding of the fundamentals of spacecraft orbital and attitude dynamics and control, and should be aware of important practical issues that need to be accounted for in spacecraft attitude control design. The reader will be well-prepared for further study in the subject.

The first author would like to express his deep gratitude to the Department of Mechanical and Aerospace Engineering at Carleton University in Ottawa, Canada, for the opportunity to develop and teach courses in orbital mechanics and spacecraft dynamics and control. The notes developed for these courses were the starting point for much of this book.

The reader will notice that this book contains no exercises. This was a decision made in order to keep the page count down. However, the reader will find a full set of exercises accompanying the book, as well as other supplementary material on the book's companion website: *http://arrow.utias.utoronto.ca/damaren/book/*.

Anton H.J. de Ruiter
Christopher J. Damaren
James R. Forbes

# 1

# Kinematics

Spacecraft are free bodies, possessing both translational and rotational motion. The translational component is the subject of *orbital dynamics*, the rotational component is the subject of *attitude dynamics*. It will be seen that the two classes of motion are essentially uncoupled, and can be treated separately.

To be able to study the motion of a spacecraft mathematically, we need a framework for describing it. For this purpose, we need to have a solid understanding of vectors and reference frames, and the associated calculus.

## 1.1 Physical Vectors

A *physical vector* is a three-dimensional quantity that possesses a *magnitude* and a *direction*. A physical vector will be denoted as $\vec{\mathbf{r}}$, for example. It can be represented graphically by an arrow. Vector addition is defined head-to-tail as shown in Figure 1.1. Multiplication of a vector $\vec{\mathbf{r}}$ by a scalar $a$ scales the magnitude by $|a|$. If $a$ is positive, the direction is unchanged, and if $a$ is negative, the direction is reversed. It is also useful to define a zero-vector denoted by $\vec{\mathbf{0}}$, which has magnitude 0, but no specified direction.

Under these definitions, physical vectors satisfy the following rules for addition:

$$
\begin{aligned}
(\vec{\mathbf{a}} + \vec{\mathbf{b}}) + \vec{\mathbf{c}} &= \vec{\mathbf{a}} + (\vec{\mathbf{b}} + \vec{\mathbf{c}}),\\
\vec{\mathbf{a}} + \vec{\mathbf{b}} &= \vec{\mathbf{b}} + \vec{\mathbf{a}},\\
\vec{\mathbf{a}} + \vec{\mathbf{0}} &= \vec{\mathbf{a}},\\
\vec{\mathbf{a}} + (-\vec{\mathbf{a}}) &= \vec{\mathbf{0}},
\end{aligned}
$$

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Anton H.J. de Ruiter, Christopher J. Damaren and James R. Forbes.

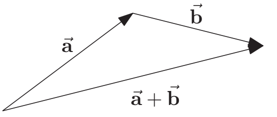


**Figure 1.1** Physical vector addition

and the following rules for scalar multiplication:

$$a(b\vec{\mathbf{c}}) = (ab)\vec{\mathbf{c}},$$
$$(a+b)\vec{\mathbf{c}} = a\vec{\mathbf{c}} + b\vec{\mathbf{c}},$$
$$a(\vec{\mathbf{b}}+\vec{\mathbf{c}}) = a\vec{\mathbf{b}} + a\vec{\mathbf{c}},$$
$$1\vec{\mathbf{a}} = \vec{\mathbf{a}},$$
$$0\vec{\mathbf{a}} = \vec{\mathbf{0}}.$$

**It is very important to note that the concept of a physical vector is** *independent* **of a coordinate system.**

### 1.1.1 *Scalar Product*

Given vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, the scalar (or dot) product between the two vectors is defined as

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} \stackrel{\Delta}{=} |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\cos\theta,$$

where $0 \le \theta \le 180^\circ$ is the small angle between the two vectors, as shown in Figure 1.2. By this definition, the scalar product is commutative, that is

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \vec{\mathbf{b}} \cdot \vec{\mathbf{a}}.$$

As demonstrated in Figure 1.2, the scalar product $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$ is just the projection of $\vec{\mathbf{a}}$ onto $\vec{\mathbf{b}}$ multiplied by $|\vec{\mathbf{b}}|$. Projections are additive, as shown in Figure 1.3, therefore, the scalar product is also distributive, that is

$$(\vec{\mathbf{a}} + \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}. \tag{1.1}$$

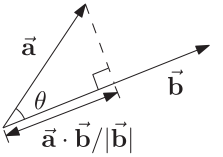


**Figure 1.2** Scalar product geometry

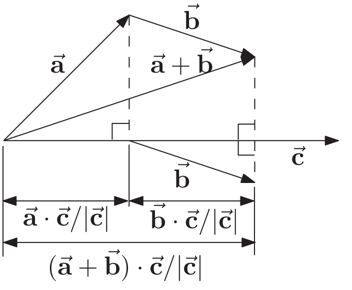


**Figure 1.3** Distributivity of scalar product

The following properties are also readily verified from the definition

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{a}} = |\vec{\mathbf{a}}|^2 \geq 0, \tag{1.2}$$

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{a}} = 0 \Leftrightarrow \vec{\mathbf{a}} = \vec{\mathbf{0}}, \tag{1.3}$$

$$\vec{\mathbf{a}}\cdot(c\vec{\mathbf{b}}) = c\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}, \tag{1.4}$$

$$\vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = 0 \Leftrightarrow \vec{\mathbf{a}} \perp \vec{\mathbf{b}} \text{ or } \vec{\mathbf{a}} = \vec{\mathbf{0}} \text{ or } \vec{\mathbf{b}} = \vec{\mathbf{0}}. \tag{1.5}$$

### *1.1.2 Vector Cross Product*

Given vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, the cross-product is defined as a vector $\vec{\mathbf{c}}$, denoted by $\vec{\mathbf{c}} = \vec{\mathbf{a}}\times\vec{\mathbf{b}}$ with magnitude

$$|\vec{\mathbf{c}}| = |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}|\sin\theta,$$

with a direction perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, chosen according to the right-hand rule, as shown in Figure 1.4. Note that $0 \leq \theta \leq 180^\circ$ is again the small angle between the two vectors.

From the definition of the cross-product, it is clear that changing the order simply reverses the direction of the cross-product, that is

$$\vec{\mathbf{a}}\times\vec{\mathbf{b}} = -\vec{\mathbf{b}}\times\vec{\mathbf{a}}.$$

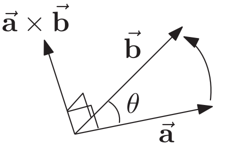


**Figure 1.4** Vector cross product

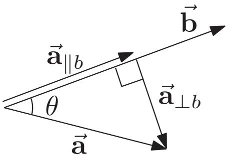


**Figure 1.5** Parallel and perpendicular vector components

Now, as shown in Figure 1.5, the vector $\vec{\mathbf{a}}$ can be decomposed into two mutually perpendicular vectors $\vec{\mathbf{a}} = \vec{\mathbf{a}}_{\perp b} + \vec{\mathbf{a}}_{\parallel b}$, where $\vec{\mathbf{a}}_{\perp b}$ is perpendicular to $\vec{\mathbf{b}}$, and $\vec{\mathbf{a}}_{\parallel b}$ is parallel to $\vec{\mathbf{b}}$. These components are given by

$$\vec{\mathbf{a}}_{\parallel b} = \frac{(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})}{|\vec{\mathbf{b}}|^2}\vec{\mathbf{b}},$$

which is the projection of $\vec{\mathbf{a}}$ onto the direction of $\vec{\mathbf{b}}$, and

$$\vec{\mathbf{a}}_{\perp b} = \vec{\mathbf{a}} - \vec{\mathbf{a}}_{\parallel b} = \vec{\mathbf{a}} - \frac{(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})}{|\vec{\mathbf{b}}|^2}\vec{\mathbf{b}}.$$

Since $|\vec{\mathbf{a}}_{\perp b}| = |\vec{\mathbf{a}}| \sin\theta$ (see Figure 1.5), and $\vec{\mathbf{a}}_{\perp b}$ is perpendicular to $\vec{\mathbf{b}}$, $|\vec{\mathbf{a}}_{\perp b} \times \vec{\mathbf{b}}| = |\vec{\mathbf{a}}|\,|\vec{\mathbf{b}}| \sin\theta$. Since $\vec{\mathbf{a}}_{\perp b}$ lies in the plane defined by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, and points to the same side of $\vec{\mathbf{b}}$ as $\vec{\mathbf{a}}$, $\vec{\mathbf{a}}_{\perp b} \times \vec{\mathbf{b}}$ has the same direction as $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$. Therefore,

$$\vec{\mathbf{a}}_{\perp b} \times \vec{\mathbf{b}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}. \tag{1.6}$$

Now, we are in a position to show a distributive property of the cross-product. Consider three vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$. First of all, note that

$$\begin{aligned}
(\vec{\mathbf{a}} + \vec{\mathbf{b}})_{\perp c} &= (\vec{\mathbf{a}} + \vec{\mathbf{b}}) - \frac{\left((\vec{\mathbf{a}} + \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}}\right)}{|\vec{\mathbf{c}}|^2}\vec{\mathbf{c}} \\
&= (\vec{\mathbf{a}} + \vec{\mathbf{b}}) - \frac{\left(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}} + \vec{\mathbf{b}} \cdot \vec{\mathbf{c}}\right)}{|\vec{\mathbf{c}}|^2}\vec{\mathbf{c}} \\
&= \left(\vec{\mathbf{a}} - \frac{(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})}{|\vec{\mathbf{c}}|^2}\vec{\mathbf{c}}\right) + \left(\vec{\mathbf{b}} - \frac{(\vec{\mathbf{b}} \cdot \vec{\mathbf{c}})}{|\vec{\mathbf{c}}|^2}\vec{\mathbf{c}}\right) \\
&= \vec{\mathbf{a}}_{\perp c} + \vec{\mathbf{b}}_{\perp c}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\left(\vec{\mathbf{a}} + \vec{\mathbf{b}}\right) \times \vec{\mathbf{c}} &= \left(\vec{\mathbf{a}} + \vec{\mathbf{b}}\right)_{\perp c} \times \vec{\mathbf{c}} \\
&= \left(\vec{\mathbf{a}}_{\perp c} + \vec{\mathbf{b}}_{\perp c}\right) \times \vec{\mathbf{c}}.
\end{aligned}$$

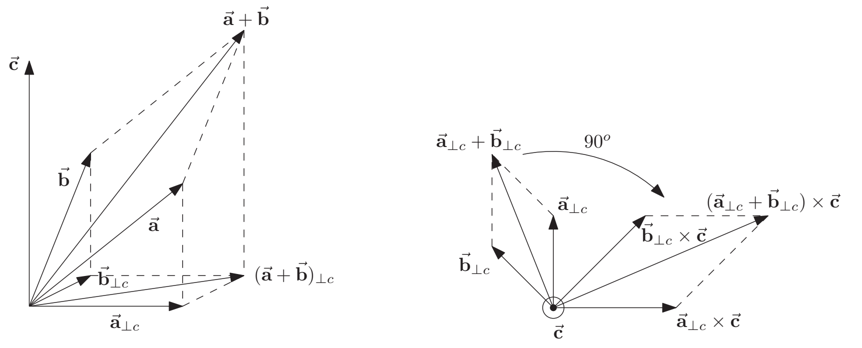


**Figure 1.6** Distributivity of vector cross product

Now, the vectors $\vec{\mathbf{a}}_{\perp c}$, $\vec{\mathbf{b}}_{\perp c}$ and $\vec{\mathbf{a}}_{\perp c} + \vec{\mathbf{b}}_{\perp c}$ all are perpendicular to $\vec{\mathbf{c}}$. Therefore,

$$|\vec{\mathbf{a}}_{\perp c} \times \vec{\mathbf{c}}| = |\vec{\mathbf{a}}_{\perp c}|\, |\vec{\mathbf{c}}|,$$

$$|\vec{\mathbf{b}}_{\perp c} \times \vec{\mathbf{c}}| = |\vec{\mathbf{b}}_{\perp c}|\, |\vec{\mathbf{c}}|,$$

$$\left|\left(\vec{\mathbf{a}}_{\perp c} + \vec{\mathbf{b}}_{\perp c}\right) \times \vec{\mathbf{c}}\right| = \left|\left(\vec{\mathbf{a}}_{\perp c} + \vec{\mathbf{b}}_{\perp c}\right)\right| \left|\vec{\mathbf{c}}\right|.$$

Since the vectors $\vec{\mathbf{a}}_{\perp c}$, $\vec{\mathbf{b}}_{\perp c}$ and $\vec{\mathbf{a}}_{\perp c} + \vec{\mathbf{b}}_{\perp c}$ are all perpendicular to $\vec{\mathbf{c}}$, the cross-products $\vec{\mathbf{a}}_{\perp c} \times \vec{\mathbf{c}}$, $\vec{\mathbf{b}}_{\perp c} \times \vec{\mathbf{c}}$ and $\left(\vec{\mathbf{a}}_{\perp c} + \vec{\mathbf{b}}_{\perp c}\right) \times \vec{\mathbf{c}}$ are all simply the vectors $\vec{\mathbf{a}}_{\perp c}$, $\vec{\mathbf{b}}_{\perp c}$ and $\vec{\mathbf{a}}_{\perp c} + \vec{\mathbf{b}}_{\perp c}$ rotated by $90^\circ$ about the vector $\vec{\mathbf{c}}$, and then scaled by the factor $|\vec{\mathbf{c}}|$, as shown in Figure 1.6. What this shows is that

$$\left(\vec{\mathbf{a}}_{\perp c} + \vec{\mathbf{b}}_{\perp c}\right) \times \vec{\mathbf{c}} = \vec{\mathbf{a}}_{\perp c} \times \vec{\mathbf{c}} + \vec{\mathbf{b}}_{\perp c} \times \vec{\mathbf{c}},$$

and therefore by (1.6),

$$\left(\vec{\mathbf{a}} + \vec{\mathbf{b}}\right) \times \vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{c}} + \vec{\mathbf{b}} \times \vec{\mathbf{c}}, \tag{1.7}$$

which is the distributive property we wanted to show. Finally, the following results are also readily derived from the definition:

$$\vec{\mathbf{a}} \times \vec{\mathbf{a}} = \vec{\mathbf{0}}, \tag{1.8}$$

$$(a\vec{\mathbf{b}}) \times \vec{\mathbf{c}} = a(\vec{\mathbf{b}} \times \vec{\mathbf{c}}). \tag{1.9}$$

### 1.1.3 Other Useful Vector Identities

Some other useful vector identities are:

$$
\begin{aligned}
&\vec{\mathbf{a}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}, \\
&\vec{\mathbf{a}} \cdot \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) = \vec{\mathbf{b}} \cdot (\vec{\mathbf{c}} \times \vec{\mathbf{a}}) = \vec{\mathbf{c}} \cdot \left(\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right), \\
&\vec{\mathbf{a}} \times \left(\vec{\mathbf{b}} \times \vec{\mathbf{c}}\right) + \vec{\mathbf{b}} \times (\vec{\mathbf{c}} \times \vec{\mathbf{a}}) + \vec{\mathbf{c}} \times \left(\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right) = \vec{\mathbf{0}}, \\
&\left(\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right) \cdot \left(\vec{\mathbf{c}} \times \vec{\mathbf{d}}\right) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\left(\vec{\mathbf{b}} \cdot \vec{\mathbf{d}}\right) - \left(\vec{\mathbf{a}} \cdot \vec{\mathbf{d}}\right)\left(\vec{\mathbf{b}} \cdot \vec{\mathbf{c}}\right).
\end{aligned}
$$

**Note that the definitions of scalar- and cross-product and all of the associated properties and identities above are** *independent* **of a coordinate system.**

## 1.2 Reference Frames and Physical Vector Coordinates

Up to this point, we have only considered physical vectors, without any mention of a frame of reference. For computational purposes we need to introduce the concept of a reference frame. Reference frames are also needed to describe the orientation of an object, and are needed for the formulation of kinematics and dynamics.

To define a reference frame, say reference frame "1" (which we will label $\mathcal{F}_1$), it is customary to identify three mutually perpendicular unit length (length of one) physical vectors, labeled as $\vec{\mathbf{x}}_1$, $\vec{\mathbf{y}}_1$ and $\vec{\mathbf{z}}_1$ respectively. The notation used here corresponds to the usual *x*-*y*-*z* axes defined for a Cartesian three-dimensional coordinate system. These three vectors then define the reference frame. The unit vectors are chosen according to the right-handed rule, as shown in Figure 1.7. Under the right-handed rule, the unit vectors satisfy

$$
\begin{aligned}
\vec{\mathbf{x}}_1 \times \vec{\mathbf{y}}_1 &= \vec{\mathbf{z}}_1, \\
\vec{\mathbf{y}}_1 \times \vec{\mathbf{z}}_1 &= \vec{\mathbf{x}}_1, \\
\vec{\mathbf{z}}_1 \times \vec{\mathbf{x}}_1 &= \vec{\mathbf{y}}_1.
\end{aligned}
$$

Since they are perpendicular, they also satisfy

$$
\begin{aligned}
\vec{\mathbf{x}}_1 \cdot \vec{\mathbf{x}}_1 = \vec{\mathbf{y}}_1 \cdot \vec{\mathbf{y}}_1 = \vec{\mathbf{z}}_1 \cdot \vec{\mathbf{z}}_1 = 1, \\
\vec{\mathbf{x}}_1 \cdot \vec{\mathbf{y}}_1 = \vec{\mathbf{x}}_1 \cdot \vec{\mathbf{z}}_1 = \vec{\mathbf{y}}_1 \cdot \vec{\mathbf{z}}_1 = 0.
\end{aligned}
\tag{1.10}
$$

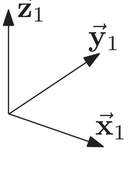


**Figure 1.7** Reference frame basis vectors

Now, since the three unit vectors form a basis for physical three-dimensional space, any physical vector $\vec{\mathbf{r}}$ can be written as a linear combination of the unit vectors, that is

$$\begin{aligned}\vec{\mathbf{r}} &= r_{x,1}\vec{\mathbf{x}}_1 + r_{y,1}\vec{\mathbf{y}}_1 + r_{z,1}\vec{\mathbf{z}}_1 \\ &= \begin{bmatrix}\vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 & \vec{\mathbf{z}}_1\end{bmatrix}\begin{bmatrix} r_{x,1} \\ r_{y,1} \\ r_{z,1}\end{bmatrix} \\ &= \vec{\mathcal{F}}_1^T \mathbf{r}_1.\end{aligned} \tag{1.11}$$

where

$$\mathbf{r}_1 = \begin{bmatrix} r_{x,1} \\ r_{y,1} \\ r_{z,1}\end{bmatrix} \tag{1.12}$$

is a column matrix containing the coordinates of the physical vector $\vec{\mathbf{r}}$ in reference frame $\mathcal{F}_1$, and

$$\vec{\mathcal{F}}_1 = \begin{bmatrix} \vec{\mathbf{x}}_1 \\ \vec{\mathbf{y}}_1 \\ \vec{\mathbf{z}}_1\end{bmatrix} \tag{1.13}$$

is a column matrix containing the unit physical vectors defining reference frame $\mathcal{F}_1$. We shall refer to $\vec{\mathcal{F}}_1$ as a *vectrix* (that is, a matrix of physical vectors).

To determine the coordinates of the vector $\vec{\mathbf{r}}$ in frame $\mathcal{F}_1$, we can simply take the dot product of the physical vector (1.11) with each of the unit vectors. For example,

$$\begin{aligned}\vec{\mathbf{r}} \cdot \vec{\mathbf{x}}_1 &= \left(r_{x,1}\vec{\mathbf{x}}_1 + r_{y,1}\vec{\mathbf{y}}_1 + r_{z,1}\vec{\mathbf{z}}_1\right) \cdot \vec{\mathbf{x}}_1, \\ &= r_{x,1}\vec{\mathbf{x}}_1 \cdot \vec{\mathbf{x}}_1 + r_{y,1}\vec{\mathbf{y}}_1 \cdot \vec{\mathbf{x}}_1 + r_{z,1}\vec{\mathbf{z}}_1 \cdot \vec{\mathbf{x}}_1, \\ &= r_{x,1}.\end{aligned}$$

Here, we have made use of properties (1.1) and (1.4) of the dot product of physical vectors. In fact, these properties allow us to treat the dot product of physical vectors in the same manner as scalar multiplication. Using the vectrix notation, we can take advantage of this fact to concisely determine $\mathbf{r}_1$ by taking the dot product of the vector (1.11) with the vectrix (1.13) as follows

$$\begin{aligned}\vec{\mathcal{F}}_1 \cdot \vec{\mathbf{r}} = \vec{\mathcal{F}}_1 \cdot \left(\vec{\mathcal{F}}_1^T \mathbf{r}_1\right) &= \left(\begin{bmatrix} \vec{\mathbf{x}}_1 \\ \vec{\mathbf{y}}_1 \\ \vec{\mathbf{z}}_1 \end{bmatrix} \cdot \begin{bmatrix} \vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 & \vec{\mathbf{z}}_1 \end{bmatrix}\right) \mathbf{r}_1 \\ &= \begin{bmatrix} \vec{\mathbf{x}}_1 \cdot \vec{\mathbf{x}}_1 & \vec{\mathbf{x}}_1 \cdot \vec{\mathbf{y}}_1 & \vec{\mathbf{x}}_1 \cdot \vec{\mathbf{z}}_1 \\ \vec{\mathbf{x}}_1 \cdot \vec{\mathbf{y}}_1 & \vec{\mathbf{y}}_1 \cdot \vec{\mathbf{y}}_1 & \vec{\mathbf{y}}_1 \cdot \vec{\mathbf{z}}_1 \\ \vec{\mathbf{x}}_1 \cdot \vec{\mathbf{z}}_1 & \vec{\mathbf{y}}_1 \cdot \vec{\mathbf{z}}_1 & \vec{\mathbf{z}}_1 \cdot \vec{\mathbf{z}}_1 \end{bmatrix} \mathbf{r}_1 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{r}_1\end{aligned}$$

Note that properties (1.1) and (1.4) allowed us to treat the dot product in the same manner as scalar multiplication, and apply the associativity rule for matrix multiplication as we did above. Finally, we have

$$\begin{aligned} r_{x,1} &= \vec{\mathbf{r}} \cdot \vec{\mathbf{x}}_1, \\ r_{y,1} &= \vec{\mathbf{r}} \cdot \vec{\mathbf{y}}_1, \\ r_{z,1} &= \vec{\mathbf{r}} \cdot \vec{\mathbf{z}}_1. \end{aligned}$$

### *1.2.1 Vector Addition and Scalar Multiplication*

We can now determine how to perform vector addition and scalar multiplication operations in terms of the coordinates of a vector in a given reference frame. To this end, let us consider two physical vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ expressed in the same reference frame $\mathcal{F}_1$, and a scalar, $c$:

$$\vec{\mathbf{a}} = \begin{bmatrix} \vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 & \vec{\mathbf{z}}_1 \end{bmatrix} \begin{bmatrix} a_{x,1} \\ a_{y,1} \\ a_{z,1} \end{bmatrix} = \vec{\mathcal{F}}_1^T \mathbf{a}, \quad \vec{\mathbf{b}} = \begin{bmatrix} \vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 & \vec{\mathbf{z}}_1 \end{bmatrix} \begin{bmatrix} b_{x,1} \\ b_{y,1} \\ b_{z,1} \end{bmatrix} \vec{\mathcal{F}}_1^T \mathbf{b},$$

It is obvious from the rules for physical vector addition and scalar multiplication that

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = \begin{bmatrix} \vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 & \vec{\mathbf{z}}_1 \end{bmatrix} \begin{bmatrix} a_{x,1} + b_{x,1} \\ a_{y,1} + b_{y,1} \\ a_{z,1} + b_{z,1} \end{bmatrix} = \vec{\mathcal{F}}_1^T \left(\mathbf{a} + \mathbf{b}\right),$$

and

$$c\vec{\mathbf{a}} = \begin{bmatrix} \vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 & \vec{\mathbf{z}}_1 \end{bmatrix} \begin{bmatrix} c\, a_{x,1} \\ c\, a_{y,1} \\ c\, a_{z,1} \end{bmatrix} = \vec{\mathcal{F}}_1^T\,(c\mathbf{a})\,.$$

That is, vector addition and scalar multiplication operations can be directly applied to the coordinates of the vectors.

## 1.2.2 Scalar Product

Let us now examine how to compute the scalar (or dot) product in terms of the coordinates of the vectors in a given reference frame. To this end, let us consider two physical vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ expressed in the same reference frame $\mathcal{F}_1$:

$$\vec{\mathbf{a}} = \begin{bmatrix} \vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 & \vec{\mathbf{z}}_1 \end{bmatrix} \begin{bmatrix} a_{x,1} \\ a_{y,1} \\ a_{z,1} \end{bmatrix}, \quad \vec{\mathbf{b}} = \begin{bmatrix} \vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 & \vec{\mathbf{z}}_1 \end{bmatrix} \begin{bmatrix} b_{x,1} \\ b_{y,1} \\ b_{z,1} \end{bmatrix}$$

The dot product is now given by

$$\begin{aligned}
\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} &= \begin{bmatrix} a_{x,1} & a_{y,1} & a_{z,1} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{x}}_1 \\ \vec{\mathbf{y}}_1 \\ \vec{\mathbf{z}}_1 \end{bmatrix} \cdot \begin{bmatrix} \vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 & \vec{\mathbf{z}}_1 \end{bmatrix} \begin{bmatrix} b_{x,1} \\ b_{y,1} \\ b_{z,1} \end{bmatrix} \\
&= \begin{bmatrix} a_{x,1} & a_{y,1} & a_{z,1} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{x}}_1 \cdot \vec{\mathbf{x}}_1 & \vec{\mathbf{x}}_1 \cdot \vec{\mathbf{y}}_1 & \vec{\mathbf{x}}_1 \cdot \vec{\mathbf{z}}_1 \\ \vec{\mathbf{x}}_1 \cdot \vec{\mathbf{y}}_1 & \vec{\mathbf{y}}_1 \cdot \vec{\mathbf{y}}_1 & \vec{\mathbf{y}}_1 \cdot \vec{\mathbf{z}}_1 \\ \vec{\mathbf{x}}_1 \cdot \vec{\mathbf{z}}_1 & \vec{\mathbf{y}}_1 \cdot \vec{\mathbf{z}}_1 & \vec{\mathbf{z}}_1 \cdot \vec{\mathbf{z}}_1 \end{bmatrix} \begin{bmatrix} b_{x,1} \\ b_{y,1} \\ b_{z,1} \end{bmatrix} \\
&= \begin{bmatrix} a_{x,1} & a_{y,1} & a_{z,1} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{x,1} \\ b_{y,1} \\ b_{z,1} \end{bmatrix} \\
&= \mathbf{a}_1^T \mathbf{b}_1
\end{aligned}$$

Again, properties (1.1) and (1.4) allowed us to treat the dot product in the same manner as scalar multiplication, and apply the associativity rule for matrix multiplication as we did above. Making use of identity (1.2), we can relate the length of the physical vector to the length of its coordinate representation, that is:

$$||\mathbf{a}_1|| = \sqrt{\mathbf{a}_1^T \mathbf{a}_1} = |\vec{\mathbf{a}}|,$$

where $||\mathbf{a}_1||$ is the standard Euclidean length of a column matrix.

### 1.2.3 Vector Cross Product

We can also determine the cross-product of two vectors in terms of the coordinates with respect to a given reference frame. Consider again the same two vectors as in Section 1.2.2. Since the vector cross product satisfies the same distributive and scalar multiplication properties (1.7) and (1.9) as the vector dot product (compare to (1.1) and (1.4)), we can concisely determine the vector cross-product in terms of the coordinates in the same manner as we determined the dot product in Section 1.2.2 (provided we respect the order in which each individual vector cross-product is taken). We have

$$
\begin{aligned}
\vec{\mathbf{a}} \times \vec{\mathbf{b}} &= \begin{bmatrix} a_{x,1} & a_{y,1} & a_{z,1} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{x}}_1 \\ \vec{\mathbf{y}}_1 \\ \vec{\mathbf{z}}_1 \end{bmatrix} \times \begin{bmatrix} \vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 & \vec{\mathbf{z}}_1 \end{bmatrix} \begin{bmatrix} b_{x,1} \\ b_{y,1} \\ b_{z,1} \end{bmatrix} \\
&= \begin{bmatrix} a_{x,1} & a_{y,1} & a_{z,1} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{x}}_1 \times \vec{\mathbf{x}}_1 & \vec{\mathbf{x}}_1 \times \vec{\mathbf{y}}_1 & \vec{\mathbf{x}}_1 \times \vec{\mathbf{z}}_1 \\ \vec{\mathbf{y}}_1 \times \vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 \times \vec{\mathbf{y}}_1 & \vec{\mathbf{y}}_1 \times \vec{\mathbf{z}}_1 \\ \vec{\mathbf{z}}_1 \times \vec{\mathbf{x}}_1 & \vec{\mathbf{z}}_1 \times \vec{\mathbf{y}}_1 & \vec{\mathbf{z}}_1 \times \vec{\mathbf{z}}_1 \end{bmatrix} \begin{bmatrix} b_{x,1} \\ b_{y,1} \\ b_{z,1} \end{bmatrix} \\
&= \begin{bmatrix} a_{x,1} & a_{y,1} & a_{z,1} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{0}} & \vec{\mathbf{z}}_1 & -\vec{\mathbf{y}}_1 \\ -\vec{\mathbf{z}}_1 & \vec{\mathbf{0}} & \vec{\mathbf{x}}_1 \\ \vec{\mathbf{y}}_1 & -\vec{\mathbf{x}}_1 & \vec{\mathbf{0}} \end{bmatrix} \begin{bmatrix} b_{x,1} \\ b_{y,1} \\ b_{z,1} \end{bmatrix} \\
&= \begin{bmatrix} \vec{\mathbf{x}}_1 & \vec{\mathbf{y}}_1 & \vec{\mathbf{z}}_1 \end{bmatrix} \begin{bmatrix} 0 & -a_{z,1} & a_{y,1} \\ a_{z,1} & 0 & -a_{x,1} \\ -a_{y,1} & a_{x,1} & 0 \end{bmatrix} \begin{bmatrix} b_{x,1} \\ b_{y,1} \\ b_{z,1} \end{bmatrix} \\
&= \vec{\mathcal{F}}_1^T \mathbf{a}_1^\times \mathbf{b}_1,
\end{aligned}
$$

where the $3 \times 3$ matrix

$$
\mathbf{a}_1^\times \triangleq \begin{bmatrix} 0 & -a_{z,1} & a_{y,1} \\ a_{z,1} & 0 & -a_{x,1} \\ -a_{y,1} & a_{x,1} & 0 \end{bmatrix}
$$

is the cross-product operator matrix corresponding to the vector $\vec{\mathbf{a}}$ in reference frame $\mathcal{F}_1$ coordinates.

### 1.2.4 Column Matrix Identities

The vector identities presented in Sections 1.1.2 and 1.1.3 can all be rewritten in terms of column matrices. To this end, let us consider four physical vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, $\vec{\mathbf{c}}$ and $\vec{\mathbf{d}}$ expressed in the same reference frame $\mathcal{F}$, with corresponding coordinates $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively.