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THE EXACT SOLUTION OF ELASTICITY MIXED PLAIN BOUNDARY VALUE PROBLEM IN A RECTANGULAR DOMAIN

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Abstract: *The plane mixed boundary value problem of elasticity on a rectangular domain is solved exactly. With the help of Fourier transformation the one-dimensional vector boundary problem in the transformation's domain is obtained. The components of the unknown vector are the displacement transformations. The problem is solved exactly with the methods of the matrix differential calculations. The constructed vector is inversed by the corresponding formulas of inverse Fourier transformation, so the displacement expressions are found in the form of Fourier series. The numerical investigation of the stress in dependence of the external loading value and domain's size is presented.*

Keywords: Rectangular domain, Mixed plain boundary value problem, Exact solution.

1. Introduction

Various engineering problems are reduced to the mixed plain boundary value problem in a rectangular domain. The history of this problem and its solution is very long and interesting (Sailendra N. Chatterjee amd Shyam N. Prasad, 1973; Meleshko, 1998; Tokovyi and Vigak, 2002; Golovchan, 2006). The novelty of the presented paper is in the new approach (Popov and Vaysfeld, 2011) to the problem solution based on the reduction of the initial boundary value problem to the one-dimensional vector boundary value problem in the transformed domain. This new method of constructing the solution, with the help of the matrix differential calculations, allows to obtain the exact solution of the elasticity problem in a rectangular domain, when the smooth contact conditions are fulfilled on the domain faces.

2. Problem Statement

The elasticity plane deformation problem is assumed. The rectangular problem domain is defined as

$$0 < x < a, \quad 0 < y < b \tag{1}$$

The boundary conditions on the edges $x = 0, \quad x = a$ are given by the following expressions

$$U(0, y) = 0, \quad \tau_{xy}(0, y) = 0\,,\ 0 < y < b \tag{2}$$

$$U(a, y) = 0, \quad \tau_{xy}(a, y) = 0\,,\ 0 < y < b \tag{3}$$

On the edge $y = b$ the normal stress are given

$$\sigma_y(x,b) = -p(x), \quad \tau_{yx}(x,b) = 0\,,\ 0 < x < a \tag{4}$$

The face $y = 0$ is in the smooth contact with the solid base

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$$V(x,0)=0, \quad \tau_{xy}(x,0)=0, \ 0<x<a \tag{5}$$

where $U(x,y)=u_x(x,y), \quad V(x,y)=u_y(x,y)$.

The displacements $U(x,y), \quad V(x,y)$ satisfy the equilibrium equations (Nowacki, 1970)

$$U''(x,y)+U^{\bullet\bullet}(x,y)+\mu_0(U''(x,y)+V^{/\bullet}(x,y))=0 \tag{6}$$

$$V''(x,y)+V^{\bullet\bullet}(x,y)+\mu_0(U^{/\bullet}(x,y)+V^{\bullet\bullet}(x,y))=0 \tag{7}$$

in the domain (1), where $\mu_0 = (1-2\mu)^{-1}$, μ is the Poison's coefficient, and G is the shear modulus. The stroke denotes the derivative with respect to the variable r, the dot denotes the derivative with respect to the variable θ. The goal is to solve the boundary problem (2-7) in the domain (1).

3. The Solution of the Problem

Let's apply the sin- and cos- integral Fourier transformations (Sneddon, 1951) to the differential equations (6, 7)

$$U_n(y)=\int_0^a U(x,y)\sin(\alpha_n x)dx, \ V_n(y)=\int_0^a V(x,y)\cos(\alpha_n x)dx, \ \alpha_n=\frac{\pi n}{a} \tag{8}$$

In the transformations domain (8) system can be written in the following form (Nowacki, 1970)

$$U_n''(y)-\alpha_n^2(1+\mu_0)U_n(y)-\mu_0\alpha_n V_n'(y)=0 \tag{9}$$

$$(1+\mu_0)V_n''(y)-\alpha_n^2 V_n(y)+\mu_0\alpha_n(1+\mu_0)U_n'(y)=0 \tag{10}$$

One should take into consideration that conditions of the smooth contact (2), (3) were satisfied during the transformation. The boundary conditions (4), (5) were reformulated in the terms of the displacements

$$V_n(0)=0, \quad U_n'(0)=0 \tag{11}$$

$$\alpha_n V_n(b)-U_n'(b)=0, \ \alpha_n\mu U_n(b)-(1-\mu)V_n'(b)=0 \tag{12}$$

The one-dimensional boundary problem (9-12) was formulated as the vector one. The following vector and matrix terms are introduced

$$\mathbf{E}=\begin{pmatrix}1 & 0\\ 0 & 1+\mu_0\end{pmatrix} \ \mathbf{P}=\begin{pmatrix}0 & -\mu_0\alpha_n\\ 0 & 1+\mu_0\end{pmatrix} \ \mathbf{Q}=\begin{pmatrix}-\mu_0\alpha_n^2(1+\mu_0) & 0\\ 0 & \alpha_n^2\end{pmatrix} \ \mathbf{Z}(y)=\begin{pmatrix}U_n(y)\\ V_n(y)\end{pmatrix}$$

$$\mathbf{A_1}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \ \mathbf{B_1}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \ \mathbf{A_2}=\begin{pmatrix}-1 & 0\\ 0 & \mu-1\end{pmatrix} \ \mathbf{B_2}=\begin{pmatrix}0 & \alpha_n\\ \alpha_n\mu & 0\end{pmatrix} \tag{13}$$

Using this notation, the problem (9-12) is rewritten in the form

$$\begin{cases} L_2(\mathbf{Z}(y))=0, \\ U_i(\mathbf{Z}(y))=0, \ i=1,2 \end{cases} \tag{14}$$

where $L_2(\mathbf{Z}(y))=\mathbf{E}\mathbf{Z}''(y)+\mathbf{P}\mathbf{Z}'(y)+\mathbf{Q}\mathbf{Z}(y)$ is the differential operator, $U_i(\mathbf{Z}(y))=\mathbf{A_i}\mathbf{Z}'(b_i)+\mathbf{B_i}\mathbf{Z}(b_i)$ are the boundary functionals, $i=1$ corresponds to $b_1=0$, $i=2$ corresponds to $b_2=b$. For the solution of vector equation one needs to construct the homogenous equation $L_2(\mathbf{Y}(y))=0$. One can assure that the correspondence $L_2(e^{sy}\mathbf{I})=\mathbf{M}(s)e^{sy}$ is correct ($\mathbf{I}$ is an unitary matrix). The construction of the fundamental matrix was done using the formula (Gantmakher, 1998)

$$\mathbf{Y}(y)=\frac{1}{2\pi i}\oint_C e^{sy}\mathbf{M}^{-1}(s)ds=\frac{1}{2\pi i}\oint_C \frac{e^{sy}\tilde{\mathbf{M}}(s)}{\det\mathbf{M}(s)}ds \tag{15}$$

where $\mathbf{M}^{-1}(s)$ is the inverse matrix to the matrix $\mathbf{M}(s)=\begin{pmatrix} s^2-\alpha_n^2(1+\mu_0) & -\mu_0\alpha_n s \\ (1+\mu_0)s^2+\alpha_n s & -\alpha_n^2 \end{pmatrix}$, C is the contour including the poles of the under the integral function, $\det\mathbf{M}(s)=(s^2-\alpha_n^2)^2$. After the calculation of integral (15) using the residual theorem, one obtains two linearly independent solutions of the matrix equation

$$\mathbf{Y}_0(y)=\frac{e^{\alpha_n y}}{4(1+\mu_0)}\begin{pmatrix} \dfrac{\mu_0\alpha_n y+\mu_0+2}{\alpha_n} & \mu_0 y \\ -\mu_0 y & -\dfrac{\mu_0\alpha_n y-\mu_0-2}{\alpha_n} \end{pmatrix}$$

$$\mathbf{Y}_1(y)=\frac{e^{-\alpha_n y}}{4(1+\mu_0)}\begin{pmatrix} \dfrac{\mu_0\alpha_n y-\mu_0-2}{\alpha_n} & -\mu_0 y \\ -\mu_0 y & -\dfrac{\mu_0\alpha_n y+\mu_0+2}{\alpha_n} \end{pmatrix} \tag{16}$$

The solution of the vector equation (14) was written in the form

$$\mathbf{Z}(y)=\mathbf{Y}_0(y)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}+\mathbf{Y}_1(y)\begin{pmatrix} C_3 \\ C_4 \end{pmatrix} \tag{17}$$

where $C_j,\ j=\overline{1,4}$ - are the unknown constants, which are found from the boundary conditions (14). Hence, the exact solution of the one-dimensional vector boundary problem (9-12) is constructed in the form

$$U_n(y)=\Delta_n(e^{\alpha_n(b+y)}(2\mu-\alpha_n(y-b)-1)+e^{\alpha_n(-b+y)}\left(-2\mu+\alpha_n(y+b)+1\right)+ \tag{18}$$
$$+e^{\alpha_n(b-y)}(2\mu+\alpha_n(y+b)-1)-e^{\alpha_n(-b-y)}(2\mu+\alpha_n(y-b)-1))$$

$$V_n(y)=\Delta_n(e^{\alpha_n(b+y)}(2\mu+\alpha_n(y+b)-2)-e^{\alpha_n(-b+y)}\left(2\mu+\alpha_n(y+b)-2\right)+ \tag{19}$$
$$+e^{\alpha_n(b-y)}(-2\mu+\alpha_n(y+b)+2)+e^{\alpha_n(-b-y)}(2\mu-\alpha_n(y-b)-2))$$

where $\Delta_n=P_n\left[2(4\alpha_n b+e^{2\alpha_n b}-e^{-2\alpha_n b})G\alpha_n\right]^{-1}$, $P_n=\int_0^a p(x)\cos\alpha_n x dx$. The formulas (18, 19) are correct when $n=1,2,...$ The particular case is $n=0$, when the equilibrium equations take the form

$$V_0''(y)=0 \tag{20}$$

The solution of the equation (20), taking into consideration the boundary conditions, is obtained

$$V_0'''(y) = -\frac{P_0(1-2\mu)}{2G(1-\mu)} y \tag{21}$$

The application of the inverse formulas of the integral transformations (8) to the expressions (18, 19) finishes the construction of the exact solution

$$U(x,y) = \frac{2}{a}\sum_{n=1}^{\infty} U_n(y)\sin\left(\alpha_n x\right), \quad V(x,y) = \frac{V_0(y)}{a} + \frac{2}{a}\sum_{n=1}^{\infty} V_n(y)\cos\left(\alpha_n x\right) \tag{22}$$

4. Numerical Results

With the help of the formulas (22), the stress is calculated exactly. The different forms of the loading $p(x) \equiv P, \quad p(x) = Ax^2 + Bx + C$ in the rectangular domain $a = 2, b = 1$ were considered. The calculations were done on the face $x = 0$. The main interests are to estimate the values of the normal stress and to investigate the condition for the creation of stretching stress. The results of the calculations are shown in the Tab. 1 for different values of Poison's coefficient (here the normal stress' values are multiplied by the 10).

Tab. 1: Values of normal stress on the face $x = 0$ for the different values of Poison's coefficient.

μ	y										
	0	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1.0**
0.1	***-5.99***	***-6.0***	***-6.1***	***-6.1***	***-6.1***	***-5.9***	***-5.2***	***-3.9***	***-0.96***	***5.11***	***22.76***
0.2	***-9.7***	***-9.7***	***-9.8***	***-9.8***	***-9.8***	***-9.6***	***-8.9***	***-7.6***	***-4.67***	***1.40***	***19.03***
0.3	***-14.5***	***-14.5***	***-14.5***	***-14.6***	***-14.5***	***-14.3***	***-13.7***	***-12.3***	***-9.43***	***-3.36***	***14.21***
0.4	***-20.8***	***-20.8***	***-20.9***	***-20.9***	***-20.9***	***-20.7***	***-20.1***	***-18.7***	***-15.8***	***-9.71***	***7.75***

The results illustrate that the stretching stress appears as one approach to the upper face of the rectangle. The zones of its occurrence are investigated, depending on the geometrical parameters and elastic properties of the domain.

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