Motivation and connection between some special methods for L.P.P.s (linear programming problems)

Richard (RUICHENG) Lu
Bachelor of science
cchmantle@gmail.com

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1. Abstract

The history of L.P.P. is not long, but during the last century, various ways of solving it raise rapidly. In this article, the author will focus more on revealing the inner connections between several famous methods for solving L.P.P.s but also includes some rigorous proof of small additional formulas in some of those methods.

2. Introduction:

In this paper, I will introduce some special methods for solving the L.P.P(linear programming problems) and analyze their intrinsic motivations and connections. The methods include 2-dimension graphical methods, Basic simplex method, Revised Simplex methods, quick simplex methods, dual simplex method and Gomory's cutting plane algorithm. We will start from a simple example and connect its idea to other methods, meanwhile build some motivation for each method.

3. Statement of example:

Example 1:

Consider a L.P.P such as:

$$\text{Max } Z = 20\, x_1 + 10\, x_2$$

Subject to:

$$x_1 + 2x_2 \leq 40$$

$$3x_1 + 2x_2 \leq 60$$

$$(x_i \geq 0,\ i \in N)$$

Example 2 (only for method 5):

Consider a L.P.P such as:

$$\text{Min } M = 1\,x_1 - 2\,x_2$$

Subject to:

$$2x_1 + x_2 \leq 5$$

$$-4x_1 + 4x_2 \leq 5$$

$$(x_i \geq 0,\ i,\ x_i, \in N)$$

We start from example 1 and analysis it by using method 1 to 4:

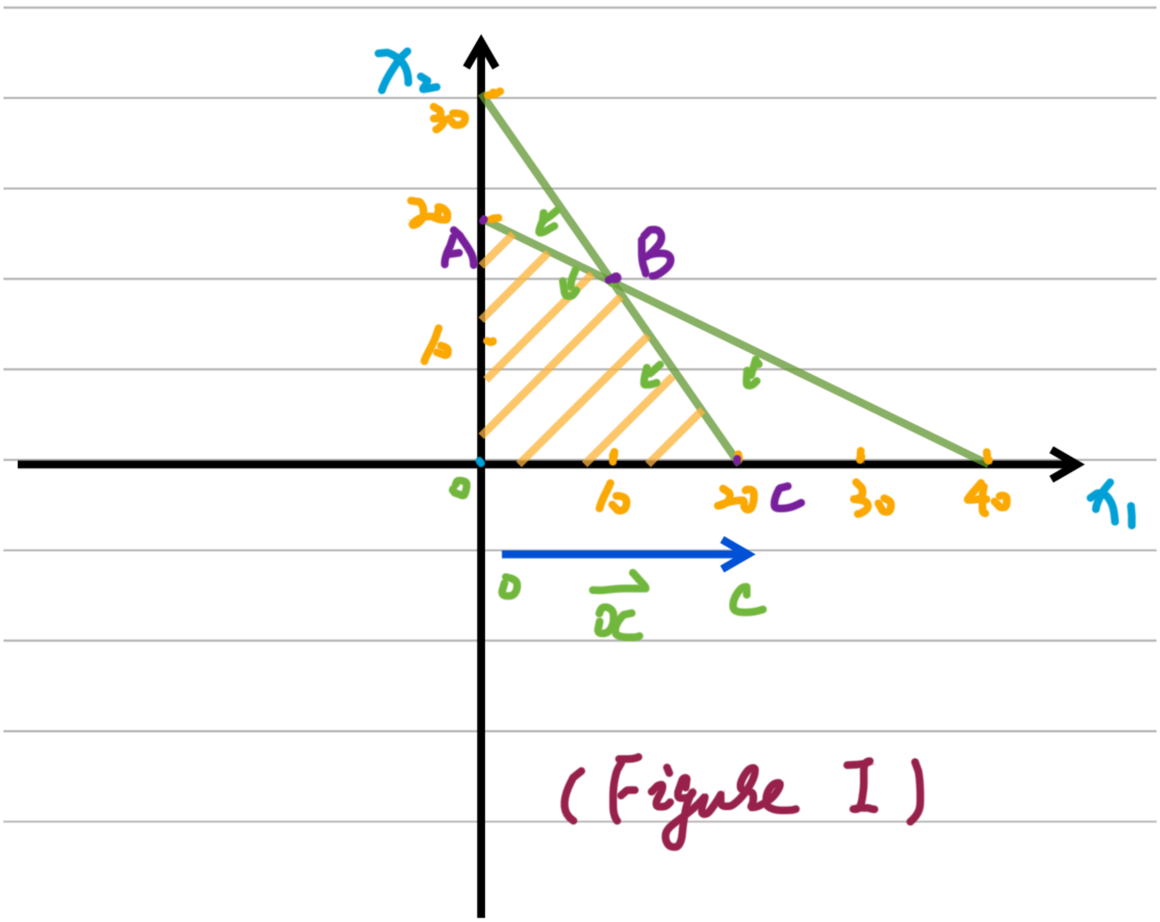
4. 2-dimension graphical method (1):

We first regard the variables in these two constraints as x-y coordinate use x_1 -> x, x_2 -> y, and change it into slope-intercept form:

$$x_2 = -\,1/2x_1 + 20\ (\leq)$$

$$x_2 = -3/2x_1 + 30\ (\leq)$$

we can easily draw the following picture:



(Figure I)

We can see from the graph, in the feasible region (Winston.W. ,1987), find all the extreme points (Winston.W. ,1987): A(0,20); B(10, 15); C(20,0), then test all the extreme points in Z, find $Z_{max} = Z_c = 200$, feasible solution: C(20,0).

5. Basic Simplex method (2):

First change constraints into standard form (add two slack variables x_3 and x_4):

$$\text{Max } Z = 20\,x_1 + 10\,x_2$$

Subject to:

$$x_1 + 2x_2 + x_3 = 40$$

$$3x_1 + 2x_2 + x_4 = 60$$

$(x_i \geq 0,\ i \in N)$

① initial tableau:

	Z	X_1	X_2	X_3	X_4	RHS
C_i	1	-20	-10	0	0	0
X_3	0	1	**2**	1	0	40
X_4	0	**3**	2	0	1	60

② first pivot: (also optimal tableau)

	Z	X_1	X_2	X_3	X_4	RHS
C_i	1	0	$\frac{10}{3}$	0	$\frac{20}{3}$	400
X_3	0	0	$\frac{4}{3}$	**1**	$-\frac{1}{3}$	20
X_1	0	**1**	$\frac{3}{2}$	0	$\frac{1}{3}$	20

Notice: here optimal solution is judged by $C_i \geq 0$, the coefficient of OF (Objective Function) (Winston.W. ,1987) since now $C_i \geq 0$ stands, the optimal solution is RHS = 400 with $x_1 = 20$, $x_2 = 0$.

Connections

A. Method 1 is pretty straight forward and easy, but it has its limitation for solving problems containing x_i $(0 < i \leq 2,\ I \in N)$.

B. Method 2 can be applied to even higher dimension problems but lose its simplicity of visualization.

C. Every pivot step in method 2 corresponds to the move of a feasible solution. Eg: pivot step ① in method 2 is also a vector $\overrightarrow{OC}$ in method 1.

D. If the system is non-degenerated (Winston.W.,1987), and has no (≥) symbol in the initial system, the initial point of feasible region in method two will be 0 vectors$\vec{0}$.

E. Every pivot moves in method two can only move along the edge (Munkers.J. 2015) from one vertex (Munkers.J. 2015) to another adjacent vertex in the convex (Munkers.J. 2015) feasible region.

F. The feasible region of L.P.P is a convex set.

G. The moving direction of the feasible solution is decided by pivoting variable, for example, here since the most negative coefficient in OF is $C_{1,}$ so we make x_1 to be BV (basic variable); also we can see now x_1 has a value, but x_2 is still zero, which means that the feasible solution moves along the x-axis to another feasible solution C, also if we ignore the optimal solution and keep pivoting without going back, we will obtain point B in method one and keeps moving in CCW(counter-clockwise) direction . on the other hand, there are no constraints for us to choose between C_1 or C_2 since they are all negative, hence if we decide to make a change in column C_2, it will make x_2 be BV, this time x_2 will have a positive value. At the same time, x_1 stays 0, so the feasible solution moves along the y-axis. Then if we keep pivoting the tableau with the negative C_1, we shall enter x_1 into BV, and we could see this time we have x_1, x_2 both with a positive value, which corresponds to point B in method 1. From here, if we keep pivoting, we will return to point C in method 1, but the direction never changes

(CW- clockwise), all the tableaus and solution with CW direction are listed below:

① initial tableau: (pivot with entry 2: $X_2 \rightarrow X_3$)

	Z	X_1	X_2	X_3	X_4	RHS
C_i	1	-20	-10	0	0	0
X_3	0	1	**2**	1	0	40
X_4	0	**3**	2	0	1	60

② After one pivot ($\overrightarrow{OA}$ in method 1)

	Z	X_1	X_2	X_3	X_4	RHS
C_i	1	-15	0	5	0	200
X_2	0	$\frac{1}{2}$	**1**	$\frac{1}{2}$	0	20
X_4	0	**2**	0	-1	1	20

③ keep pivoting ($\overrightarrow{AB}$)

	Z	X_1	X_2	X_3	X_4	RHS
C_i	1	0	0	$-\frac{5}{2}$	$\frac{15}{2}$	350
X_2	0	0	**1**	$\mathbf{\frac{3}{4}}$	$-\frac{1}{4}$	15
X_1	0	**1**	0	$-\frac{1}{2}$	$\frac{1}{2}$	10

④ pivot again, back to C ($\overrightarrow{BC}$)

	Z	X_1	X_2	X_3	X_4	RHS
C_i	1	0	$\frac{10}{3}$	0	$\frac{20}{3}$	400
X_3	0	0	$\frac{4}{3}$	**1**	$-\frac{1}{3}$	20
X_1	0	**1**	$\frac{3}{2}$	0	$\frac{1}{3}$	20

6. Revised Simplex method[1] (3):

We've already seen that there are so many connections between methods 1 and 2, but the visual information contained in method 1 is much less than that in method 2. Actually for an L.P.P. if there have m constraints and n variables, for one single tableau, it contains (n+2)(m+1) numbers ($n \geq m$), but for us, we need (n - m) non-basic variables to decide to enter variable (denote as x_k) (Winston.W.,1987), and m numbers in a column of entering variable (P_k) also we need m numbers of RHS(Right-hand side of tableau) ($\bar{b}$), which means we only need (n - m + m + m) = (n + m) numbers in one tableau, hence now we introduce the Revised Simplex method (method 3) to sharply reduce the numbers appearing in our calculation.

For a max L.P.P.

$$\text{Max } Z = C^T X$$

Subject to:

[1] The concrete algorithm steps can be found in Winston.W(1987) Operations Research: Applications and Algorithms.

$$AX = b$$

$$\text{With: } X \geq 0$$

C, X is a column vector with n row, b is a column vector with m row, A is a m x n matrix. The following set-ups are all getting from the standard form with the initial order, and BV, NBV is relative to the objective tableau.

Denote C_{BV} as the coefficient of BV, C_{NBV} as the coefficient of NBV in OF, B as the matrix containing all the BV columns, N as the matrix containing all the Non-BV columns, X_{BV} as the column vector for BV, X_{NBV} as the column vector for Non-BV.

$$\text{Max } Z = C_{BV}X_{BV} + C_{NBV}X_{NBV}$$

$$BX_{BV} + NX_{NBV} = b$$

Times $B^{-1}$by both side:

$$B^{-1}BX_{BV} + B^{-1}NX_{NBV} = B^{-1}b$$

$$\text{So:} \quad X_{BV} + B^{-1}NX_{NBV} = B^{-1}b \quad \ldots\ldots\ldots\ldots\ldots \text{(I)}$$

Since in the objective tableau, the coefficients of BV are all equal to 1, then this is the objective tableau formula we want. Hence for NBV, we have:

$$B^{-1}P_i = \bar{P}_i$$

For i^th NBV, the column in objective tableau($\bar{P}_i$) is equal to B^{-1} times the column in initial tableau (P_i); also, the RHS of the constraints is the vector: $B^{-1}b$.

On the other hand, since Max Z - $C_{BV}X_{BV}$ - $C_{NBV}X_{NBV}$ = 0, from formula I, we multiply both sides by C_{BV} to creates $C_{BV}X_{BV}$:

$$C_{BV}X_{BV} + C_{BV}B^{-1}NX_{NBV} = C_{BV}B^{-1}b \quad \ldots\ldots\ldots\ldots\ldots \text{(II)}$$

Then add it to max Z equation, we obtain:

$$\text{Max } Z + X_{NBV}(C_{BV}B^{-1}N - C_{NBV}) = C_{BV}B^{-1}b$$

If it's a Min problem, then Min $M + C_{BV}X_{BV} + C_{NBV}X_{NBV} = 0$, minus formula II, we get:

$$\text{Min } M + X_{NBV}(C_{NBV} - C_{BV}B^{-1}N) = -C_{BV}B^{-1}b$$

From these two formulas:

①The value of OF is:

$C_{BV}B^{-1}b$ for Max problem

$-C_{BV}B^{-1}b$ for Min problem

②the coefficients of NBV(for i^{th} NBV):

Max: $C_{BV}\bar{P}_i - C_i$

Min: $C_i - C_{BV}\bar{P}_i$

Until now, we've finished most of our work, but remember we cannot get optimal tableau right after one pivot step(unless it's a relatively simple problem), because we don't know the BV in optimal tableau, so we regard our initial standard form (initial tableau) as stage one, then use all the same idea in method 2 to decide how to pivot, in other words, we can decide the next tableau's BV. We can use our formulas to substitute and simplify tableaus, but we have one little obstacle, in order to use our formula, we have to recalculate B^{-1} every time once we change the BV, we try to find a relationship between B_{new}^{-1} and B^{-1}:

Claim:

$$B_{new}^{-1} = EB^{-1}$$

Where $E = [e_1, e_2 \dots e_{j-1}, \eta, e_{j+1}, \dots, e_n]$

$$e_i = \begin{bmatrix} \delta_{i1} \\ \vdots \\ \delta_{in} \end{bmatrix}, \ \delta_{ij} = \begin{cases} 0, & (i \neq j) \\ 1, & (i = j) \end{cases}$$

$$\eta = \begin{bmatrix} \frac{-N_1}{N_j} \\ \vdots \\ \frac{1}{N_j} \\ \vdots \\ \frac{-N_n}{N_j} \end{bmatrix}$$

Proof:

WLOG, we set up three important stages in solving L.P.P. using Revised Simplex method:

Stage 1:

BV matrix in initial tableau $B_0 = [S_1, S_2 \dots S_n]$

Stage 2:

After some pivoting steps from stage 1

BV matrix $B = [B_1, B_2 \dots B_j \dots B_n]$ (B_i represent the column vector of i^{th} BV in Stage 1, B_j is the column vector of whom is going to leaving BV) with some non-BV including a non-BV with column N (stage 2) for next pivoting step

Stage 3:

Right after one step pivoting step from stage 2

$$B_{new} = [B_1, B_2 \ldots \widetilde{N} \ldots B_n]$$

$\widetilde{N}$ is the original column in stage 1 tableau, which will also be transformed to N in stage 2 and ready to enter BV in stage 3.

From the above set-up:

In stage 2: $B^{-1}\widetilde{N} = N$

Now define:

$$\eta = \begin{bmatrix} \frac{-N_1}{N_j} \\ \vdots \\ \frac{1}{N_j} \\ \vdots \\ \frac{-N_n}{N_j} \end{bmatrix}$$

$$\text{Where } N = \begin{bmatrix} N_1 \\ \vdots \\ N_j \\ \vdots \\ N_n \end{bmatrix}$$

First, we notice $[e_1, e_2 \ldots e_{j-1}, N, e_{j+1}, \ldots, e_n]\ \eta = e_j$

And: $B^{-1}B_{new} = [B^{-1}B_1, B^{-1}B_2, \ldots B^{-1}\widetilde{N}, \ldots B^{-1}B_n]$

$$= [e_1, e_2 \ldots e_{j-1}, N, e_{j+1}, \ldots, e_n]$$

$$\text{So: } B^{-1}B_{new}\eta = e_j$$

$$B_{new}\eta = Be_j = B_j$$

$$\text{Since } B_{new}\, e_i = B_i \ (i \neq j)$$

$$B_{new}\,[e_1, e_2 \ldots e_{j-1}, \eta, e_{j+1}, \ldots, e_n] = B = [B_1, B_2 \ldots B_j \ldots B_n]$$

Hence:

$$\text{If we set } E = [e_1, e_2 \dots e_{j-1}, \eta, e_{j+1}, \dots, e_n]$$

$$B_{new}E = B$$

$$B_{new}^{-1} = EB^{-1}$$

□

After getting all the formulas, we can put it together and test the power of Revised Simplex method, again, we consider the statement problem:

Like before, we first change it to standard form:

$$\text{Max } Z = 20\, x_1 + 10\, x_2$$

Subject to:

$$x_1 + 2x_2 + x_3 = 40$$

$$3x_1 + 2x_2 + x_4 = 60$$

$$(x_i \geq 0,\ i \in N)$$

① initiate the problem:

$$BV = \{x_3, x_4\},\ NBV = \{x_1, x_2\}$$

$$C_{BV} = [0, 0],\ C_{NBV} = [10, 10],\ b = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

② pivot 1:

$$\overline{C_{NBV}} = C_{BV}B^{-1}N - C_{NBV} = [-20, -10]$$

Find the entering variable (E.V.): x_1

$$\overline{P_1} = B^{-1}P_1 = I \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\bar{b} = B^{-1}b = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$$

$$\theta = \min\{\frac{40}{1}, \frac{60}{3}\} = 20$$

Find the departing variable (D.V.): x_4

$$\eta = \begin{bmatrix} \frac{-1}{3} \\ \frac{1}{3} \end{bmatrix}, E = \begin{bmatrix} 1 & \frac{-1}{3} \\ 0 & \frac{1}{3} \end{bmatrix}, B_1^{-1} = EB^{-1} = \begin{bmatrix} 1 & \frac{-1}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$BV = \{x_3, x_1\}, NBV = \{x_2, x_4\}$$

$$C_{BV} = [0, 20], C_{NBV} = [10, 0], b = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$$

③pivot 2:

$$C_{NBV} = C_{BV}B^{-1}N - C_{NBV} = [0, \frac{20}{3}] \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix} - [10, 0] = [\frac{10}{3}, \frac{20}{3}]$$

Since all the coefficient of C_{NBV} is positive in Max form, it obtains the optimal solution with:

$$\bar{b} = B_1^{-1}b = \begin{bmatrix} 20 \\ 20 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Optimal solution: $C_{BV}B_1^{-1}b = 400$.

We could clearly see that by using method 3, we can still get the exact same solution as before.

Connections

A. Since the method 3 is just a simplified version of method 2, it still has the capability to restore all the information containing in the tableau, but beyond that, it's also an efficient way to get selected information from selected tableau with corresponded BVs.

B. At the same time, just like method 2, it still keeps all the connections to method 1.

C. We can also use the powerful formulas in method 3 to execute <u>sensitivity analysis</u> to the system, even more, <u>we can sometimes restore all the information in tableau</u> (method 2) only by drawing the graph of method 1.

For instance, we suppose that we have a graph like a Figure 1 in method 1, in the feasible region, if we want to get a tableau which corresponds to point A, we first need to get the BVs corresponds to this tableau. Since the initial $X_{BV} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$, the moving direction is x_2-axis, which means x_2 enter the BV, also by testing $\theta = \min \{\frac{40}{2}, \frac{60}{2}\} = 20$, D.V. will be x_3, so the BV in tableau point A will be $X_{BV} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$, then using the formulas:

$$\text{Max } \overline{C_i} = C_{BV}\bar{P}_i - C_i$$

$$B^{-1}P_i = \bar{P}_i$$

$$\bar{b} = B^{-1}b$$

$$Z_{max} = C_{BV}B^{-1}b$$

We can quickly get the corresponding tableau to point A:

	Z	X_1	X_2	X_3	X_4	RHS
C_i	1	-15	0	5	0	200
X_2	0	$\frac{1}{2}$	**<u>1</u>**	$\frac{1}{2}$	0	20
X_4	0	**<u>2</u>**	0	-1	1	20

7. Quick Simplex method (4):

The basic idea of Quick Simplex method comes from the pivoting steps from tableau computation (Gauss elimination) when you do one pivot at a time, this will be the method 2 (or method three if you tend to simplify the tableau). Still, if you do two or more pivot steps at the same time, this is the crucial idea of the quick simplex method, now we shall apply this new concept to our statement problem. First, I will claim the formulas[2] for the quick simplex method: (2-pivot form):

Suppose we have the following tableau to pivot (Gauss elimination) with pivot point a_1& b_2 (regard X_1 & X_2 as the entering variable):

Z	X_1	X_2	X_3
0	$\mathbf{a_1}$	b_1	c_1
0	a_2	$\mathbf{b_2}$	c_2
0	a_3	b_3	c_3
0	a_4	b_4	c_4

Now, after pivot steps, we get:

Z	X_1	X_2	X_3

[2] The rigorous proof of the formulas can be found in Vaidya.N (2014) Quick Simplex Algorithm for Optimal Solution to the Linear Programming Problem along with Theoretical Proof of Formulae.

0	**1**	0	$\tilde{c}_1 = \frac{(-1)^1 \det\left(\begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right)}$
0	0	**1**	$\tilde{c}_2 = \frac{(-1)^2 \det\left(\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right)}$
0	0	0	$\tilde{c}_3 = \frac{\det\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\right)}{det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right)}$
0	0	0	$\tilde{c}_4 = \frac{det\left(\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{bmatrix}\right)}{det\left(\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\right)}$

Now, we can try to use this to solve our statement problem, first, again change it to standard form:

$$\text{Max } Z = 20\, x_1 + 10\, x_2$$

Subject to:

$$x_1 + 2x_2 + x_3 = 40$$

$$3x_1 + 2x_2 + x_4 = 60$$

$$(x_i \geq 0,\ i \in N)$$

If we put it into the initial tableau:

	Z	X_1	X_2	X_3	X_4	RHS
C_i	1	-20	-10	0	0	0
X_3	0	1	**<u>2</u>**	1	0	40
X_4	0	**<u>3</u>**	2	0	1	60

Now, since for x_1 and x_2 column, we both have negative value, the normal way would be pivoting the most negative column's variable first.

Still, here, since we're using the quick simplex method, we should judge the leaving variable first, for x_1:

$$\theta = \min\{\frac{40}{1}, \frac{60}{3}\} = 20$$

So, the first pivot point $a_1 = 3$, the corresponding row will be the row of $\tilde{c}_1$ in formula's tableau, same, for x_2:

$$\theta = \min\{\frac{40}{2}, \frac{60}{2}\} = 20$$

The second pivot point is $b_2 = 2$, and the corresponding row will be the row of $\tilde{c}_2$ in formula's table, now we apply the formulas to the whole table using the pivot points as state above, get:

	Z	X_1	X_2	X_3	X_4	RHS
C_i	1	0	0	$-\frac{5}{2}$	$\frac{15}{2}$	350
X_2	0	0	**1**	$\boldsymbol{\frac{3}{4}}$	$-\frac{1}{4}$	15
X_1	0	**1**	0	$-\frac{1}{2}$	$\frac{1}{2}$	10

After using the quick simplex method, we should first check the RHS, which should all be non-negative beside the first row, then we find out that C_3 is still negative. Hence we should pivot $\frac{3}{4}$ in the X_3 column (since $-\frac{1}{2} < 0$, we can only use $\frac{3}{4}$ as pivot point which means that X_2 will be the leaving variable, X_3 will be the entering variable), after one step of normal simplex pivot:

	Z	X_1	X_2	X_3	X_4	RHS

C_i	1	0	$\frac{10}{3}$	0	$\frac{20}{3}$	400
X_3	0	0	$\frac{4}{3}$	**1**	$-\frac{1}{3}$	20
X_1	0	**1**	$\frac{3}{2}$	0	$\frac{1}{3}$	20

Now we obtain the optimal solution tableau just like before in method 2.

Connections

A. The basic idea of the simplex method (method 2) never change in quick simplex method, which means every tableau in the quick simplex method can also be obtained by some combinations of pivot steps in method 2, on the other hand, since method two can be modified to be the form in method 3, the quick simplex method can also be changed into a more simple way, but whenever you try to pivot more than one variable, you should calculate more than one non-BV's column and recalculate B^{-1} each time.

B. We can see the connections between this method and the graphical method (method 1), since, after the first quick simplex, the tableau corresponds to point B in Figure I, hence, the first step will correspond to $\overrightarrow{OB}$, then we pivot once more, corresponds to $\overrightarrow{BC}$. However, this problem needs more pivot steps than the basic simplex method, but sometimes, the optimal solution tend to be gotten in a more central area of the first quadrant of Figure I, in other word, more pivot steps will be needed, since the quick simplex method introduces all the Non-BVs directly into the table.

In some problem whose optimal solution can be found or near a special solution which contains all the variables in the OF (before turning into the standard form) as BV, solving the L.P.P. using the quick simplex method would be much quicker relative to the basic simplex method. For example, here, the tableau corresponds to B containing all the BV: X_1 & X_2 in the original OF. Overall, the quick simplex method changes the direction of the pivot from the start. It always pivots to the diagonal direction($\overrightarrow{OB}$) of the feasible region then go CW($\overrightarrow{BC}$)or CCW($\overrightarrow{BA}$)direction.

From now on, we will turn our focus on example 2:

8. Dual simplex method and Gomory's cutting plane algorithm (5):

In example 2, we need to solve an integer L.P.P. problem since all x_i are required to be non-negative. I will concisely illustrate the motivation and idea behind the method and focus more on their inner connections with other methods, but I won't state the **detail of the algorithm**[3] itself. First of all, to solve an integer L.P.P. the normal way would be finding the optimal solution without the integer constraints, then find an integer solution around our non-integer optimal solution, if you're using method 1 to show what we've done,

[3] The detail of dual simplex method and gomory's cutting plane algorithm should be found in Winston.W. (1987) Operations Research: Applications and Algorithms.

it's just like cutting the original feasible region into several pieces according to the different algorithms, now observe the example:

As usual, change it to standard form by adding slack variables:

$$\text{Max } Z = -1\, x_1 + 2\, x_2$$

Subject to:

$$2x_1 + x_2 + x_3 = 5$$

$$-4x_1 + 4x_2 + x_4 = 5$$

$$(x_i \geq 0,\ i,\ x_i, \in N)$$

Now, we try to solve this problem using either method 2,3 or 4 without the integer constraints. We get straight into optimal tableau:

	Z	X_1	X_2	X_3	X_4	RHS
C_i	1	0	0	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{15}{4}$
X_1	0	**1**	0	$\frac{1}{3}$	$-\frac{1}{12}$	$\frac{5}{4}$
X_2	0	0	**1**	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{5}{2}$

From here, we find the optimal solution is non-integer, then pick one BV row in optimal tableau randomly, like Row (X_2):

$$x_2 + \frac{1}{3}\, x_3 + \frac{1}{6}\, x_4 = \frac{5}{2}$$

now separating the fraction:

$$x_2 + (0 + \frac{1}{3})x_3 + (0 + \frac{1}{6})x_4 = 2 + \frac{1}{2}$$

rearrange:

$$\frac{1}{3}\, x_3 + \frac{1}{6}\, x_4 - \frac{1}{2} = 2 - x_2$$

Notice RHS has to be an integer number since x_2 is integer, also LHS (=RHS) can only be non-negative since x_3 & x_4 are integers and non-negative, so we find out that RHS is an integer & non-negative value, it just like other variable in the problem, so we make it a new variable x_5

Set $x_5 = x_2 - 2$

$$-\frac{1}{3}x_3 - \frac{1}{6}x_4 + x_5 = -\frac{1}{2} \text{ .... (III)}$$

Now we get a whole new constraint, but since it has a negative value on RHS, we shall use the **dual simplex method,** the optimal tableau after pivot is:

	Z	X_1	X_2	X_3	X_4	X_5	RHS
C_i	1	0	0	0	$\frac{1}{4}$	1	$\frac{13}{4}$
X_1	0	1	0	0	$-\frac{1}{4}$	1	$\frac{3}{4}$
X_2	0	0	1	0	0	1	2
X_3	0	0	0	1	$\frac{1}{2}$	-3	$\frac{3}{2}$

Now, we get a new optimal solution, but it's still not an integer, so we take Row (X_1) to make a new constraint and repeat the steps:

$$\frac{3}{4}X_4 - \frac{3}{4} = -X_1 - X_5 + X_4$$

Set: $-X_1 - X_5 + X_4$ = new variable X_6

$$-\frac{3}{4}X_4 + X_6 = -\frac{3}{4} \text{ ....(IV)}$$

put it back into our tableau and pivot it again using the dual simplex method:

	Z	X_1	X_2	X_3	X_4	X_5	RHS
C_i	1	0	0	0	1	$\frac{1}{3}$	3
X_1	0	1	0	0	1	$-\frac{1}{3}$	1

X_2	0	0	1	0	1	0	2
X_3	0	0	1	0	-3	$\frac{2}{3}$	1
X_4	0	0	0	1	0	$-\frac{4}{3}$	1

Now, we finally get our optimal integer solution:

$$Z_{min} = -3 \text{ iff } X_1 = 1 \ \& \ X_2 = 2.$$

Connections

A. For dual simplex method itself, the basic idea still never change compare to the basic simplex method, only it changes the problem into dual form than do the algorithm, in this way, it can avoid the negative value in RHS, because since the dual simplex algorithm starts from semi-optimal solution (with $C_i \geq 0$ but some $b_i < 0$), in the dual form C_i will become b_i, on the contrary, b_i will become C_I, in this way we can continue our simplex method, at the same time we have dual theorem, yield that $Z_{max} = M_{min}$ for the dual and primal problem which means the solution we get in dual form can also be the solution to our real problem. Moreover, the dual simplex method plays an important role in the sensitivity analysis. If we add a new constraint or new variable into our original question, we can use the dual form to check it if it's still optimal, if not, when the $b_{new} < 0$, we can use dual simplex to continue our calculation. An even better application will be using it in ($\geq$) included problems, instead of using big M method/ two phases method[4], we can turn the excess variable into a slack variable by multiply (-1) then using dual simplex (since now $b_i < 0$).

B. Gomory's cutting plane algorithm can be visualized in method 1. The result will be cutting feasible region into pieces, for example here, as for formula (III) and (IV), if we eliminate all the slack and new variable by plug in the Rows in tableau, we get:

$$X_2 \leq 2$$

$$X_1 - X_2 \geq -1$$

Which means the two steps of adding new variable are just adding these two constraints to the original question respectively. Since formula (III) and (IV) are derived by randomly chosen Row in non-integer optimal tableau, the way of cutting the plan may not be only one. If we show these constraints in graphical form, whenever it adds a new variable, it cuts the graph into two pieces, but notice it can not only cut it parallel to the X_1 or X_2 axis like B&B algorithm [5](branch and bound algorithm), but also can cut it obliquely, which gives more possibility to find an optimal solution rapidly.

[4] Methods can be found in Winston.W. (1987) Operations Research: Applications and Algorithms.
[5] Detail explained in Winston.W. (1987) Operations Research: Applications and Algorithms.

9. Conclusion:

From all the analysis and connections listed above, the graphical method almost connects to all other methods, we could say in some sense, the method 1 and 2 are virtually identical, because every point in method 1 of its feasible region has its own. Only simplex tableau, reversely, every feasible simplex tableau has a feasible point in method one, which corresponds to it. We can also restore the feasible simplex tableau, which corresponds to its feasible point in the graph directly by using method 3. Other than this, the direction of the pivot of the basic simplex method depends on the choice of C_i. If we choose to pivot multiple C_i at the same time, the direction will be diagonal (quick simplex method). At last, if we tend to get an integer solution of L.P.P.s, cutting the feasible region into pieces in a different way by choosing other Rows in non-integer optimal tableau around the optimal solution.

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