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On the concept of stationary Lyapunov basis

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Abstract

We propose the concept of stationary Lyapunov basis – the basis of tangent vectors $\mathbf{e}^{(i)}(x)$ defined at every point x of the attractor of the dynamical system, and show that one can reformulate some algorithms for calculation of Lyapunov exponents λ_i so that each λ_i can be treated as the average of a function $S_i(x)$. This enables one to use measure averaging in theoretical arguments thus proposing the rigorous basis for a number of ideas for calculation of Lyapunov exponents from time series. We also study how the Lyapunov vectors in Benettin's algorithm converge to the stationary basis and show that this convergence rate determines continuity of the field of stationary Lyapunov vectors. © 1998 Elsevier Science B.V.

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1. Introduction

Lyapunov exponents are important for quantification of dynamical systems [1–3]. They characterize whole attractor, not its local properties. To study the latter a number of more detailed characteristics have been proposed, e.g. finite time Lyapunov exponents, local stable and unstable directions, etc.

Calculation of Lyapunov exponents [4] involves the so-called Lyapunov vectors, describing the evolution of infinitesimal phase volumes (or, more precisely, their orientation). In the present paper, we show that in a rather wide class of chaotic systems there exists a field of directions that correspond to distinct *average* stretchings, and that the evolving Lyapunov vectors exponentially converge to them. We hope, that this field may be helpful in understanding the behavior of chaotic systems, as well as in development and substantiation of methods of their numerical treatment.

We shall call this local structure for tangent vectors at every point of the attractor, which arise from the algorithm [4], "the stationary Lyapunov basis". Its vectors at different points of a trajectory are mapped into each other, and the ith one is responsible for calculation of ith exponent λ_i. Within this approach each λ_i can be represented as a time average of a "pointwise" function $\sigma_i(x)$ (uniquely determined by the phase space point x). Usually time average

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coincides with measure (ensemble) average. Whenever this is the case, our ideas help in providing the mathematical background for the numerical algorithms used for calculation of Lyapunov exponents, especially from time series.

We shall focus on iterated maps

$$x_{n+1} = f(x_n), \tag{1.1}$$

i.e., systems with discrete time. Indeed, in the opposite case, i.e. for an ordinary differential equation $\dot{x} = F(x)$, one can easily rewrite the evolution in the form of the "stroboscopic map"

$$x(t+\tau) = F^{\tau}(x(t)),$$

because for an *autonomous* differential equation the value of x at time t uniquely determines all the subsequent evolution, including $x(t+\tau)$. Denoting $x_n \equiv x(n\tau)$ and $f(x) \equiv F^{\tau}(x)$ we obtain nothing but the iterated mapping (1.1).

The sensitivity to initial conditions, i.e. the evolution of *infinitesimal* perturbations δx, is governed by the linearized system

$$\delta x_{n+1} = Df(x_n)\delta x_n,$$

where Df is the Frechet derivative of the nonlinear mapping f:

$$Df(x) \equiv \{\partial f_i/\partial x_j\}.$$

In many systems the growth of the perturbations is exponential:

$$\|\delta x_n\| \sim e^{\lambda_1 n} \|\delta x_0\|,$$

and its increment λ_1 is called the first Lyapunov exponent. A far more detailed information about the behavior of perturbations is given by the complete set of Lyapunov exponents λ_i, of which there are N for an N-dimensional dynamical system. They, in general, characterize divergence/convergence of trajectories along specific directions, which are just the Lyapunov vectors.

The main result of this paper is the proof of the existence of the so-called stationary Lyapunov basis (SLB) – a basis of N vectors $\mathbf{e}^{(1)}(x), \ldots, \mathbf{e}^{(N)}(x)$ at almost all points x of attractor, such that the Lyapunov exponents can be expressed as a sum

$$\lambda_i = \lim_{K\to\infty} \frac{1}{K} \sum_{n=1}^{K} \log |(\mathbf{e}^{(i)}(f(x_n)), Df(x_n)\mathbf{e}^{(i)}(x_n))| \equiv \lim_{K\to\infty} \frac{1}{K} \sum_{n=1}^{K} \sigma_i(x_n).$$

The most important is the fact that λ_i in such a representation can be expressed as an ordinary average along the trajectory. This gives another view on them, enables to apply the usual technique of measure averaging, and gives a rigorous basis for some methods of λ estimating from a time series. Applicability of measure averaging means that $\sigma_i(x_n)$ may be calculated in arbitrary order, and the points x_n even must not belong to the same trajectory. In case of continuous-time dynamical systems $\dot{x} = F(x)$ the corresponding formula is

$$\lambda_i = \lim_{T\to\infty} \frac{1}{T} \int_0^T (\mathbf{e}^{(i)}(x(t)), DF(x(t))\mathbf{e}^{(i)}(x(t)))\, dt,$$

and the use of measure averaging also becomes possible.

In Section 2 we shall consider the definition of Lyapunov exponents and Lyapunov vectors along with some details and notations concerning their calculation. Section 3 contains the proof of stability of the Lyapunov vectors (with respect to their perturbation phase space trajectory assumed fixed). In Section 4 we introduce the concept of the "stationary Lyapunov basis", i.e. the orthonormal basis to which the evolving Lyapunov vectors converge. Some preliminary results concerning its continuity are presented in Section 5. Then we shall discuss some conclusions which follow from its existence.

We should also make several remarks.

1. The very *existence* of the stationary Lyapunov basis (SLB) as the limit of the evolving tangent vectors can be derived from the well-known Oseledec theorem [1] applied to the inverse-time system $x \mapsto f^{-1}(x)$, and the SLB vectors are just the eigenvectors of the corresponding Oseledec matrix Ξ for f^{-1} – see Sections 2.1 and 4. But the approach we present gives more: it enables one to estimate the convergence to this basis along different directions *separately*, which is crucial for studying the *continuity*.
2. Some results on the stability of Lyapunov vectors, including the estimate of the convergence rate, have been also obtained by Goldhirsch et. al. [5], but (i) their method can only been applied to ordinary differential equations, not maps; and (ii) they have considered only infinitesimal perturbations of the Lyapunov vectors, while we present the results for finite perturbations as well. Neither the concept of stationary Lyapunov basis nor its continuity properties, as far as we know, have been considered before.
3. Below we shall assume that the trajectory x_n being analyzed belongs to the system attractor, and therefore may be infinitely continued to $n \to \pm\infty$ (backward continuation may not be unique). The same is true for the points x, for which we shall state the existence of the stationary Lyapunov basis.

2. Definitions of Lyapunov exponents and notations

2.1. Oseledec approach

Let us consider the matrix

$$\Xi(n_0) \equiv ([Df^{n_0}(x)]^* Df^{n_0}(x))^{1/2n_0}$$

and its limit $\Xi \equiv \lim_{n_0\to\infty} \Xi(n_0)$ [1]. The eigenvalues Λ_i of Ξ are called the Lyapunov numbers, and their logarithms $\lambda_i \equiv \log \Lambda_i$ are the Lyapunov exponents. Usually it is assumed that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. There is no common name for the eigenvectors $\{\mathbf{g}^{(m)}\}$ of Ξ. Lyapunov exponents are the same for almost all attractor points, though the vectors $\{\mathbf{g}^{(m)}\}$ are different. There is no good numerical algorithm for $\mathbf{g}^{(m)}$ calculation, and for this reason they usually arise only in theoretical arguments.

Using instead of the limit the finite-time matrix $\Xi(n_0)$, we obtain the so-called "finite-time Lyapunov numbers" and exponents. In contrast to the limiting values, they depend on the point x and n_0.

2.2. The approach of Benettin et al.

The alternative approach is based upon the fact that almost every k-dimensional infinitesimal volume $\mathrm{d}\Omega^{(k)}$ grows as

$$\mathrm{d}\Omega_n^{(k)} \sim \mathrm{d}\Omega_0^{(k)} \exp\left(\sum_{i=1}^{k} n\lambda_i\right).$$

So, let us consider orthonormal vectors $\mathbf{v}_n^{(m)}$ (a basis in tangent space). Their images are $Df(x_n)\mathbf{v}_n^{(m)}$, which usually are neither orthogonal nor normalized, thus to obtain the *basis* $\{\mathbf{v}_{n+1}^{(m)}\}$ at the next time step one must apply the Gram–Schmidt orthonormalization:

$$\begin{aligned} \mathbf{v}_{n+1}^{(1)} &= \frac{Df(x_n)\mathbf{v}_n^{(1)}}{\|Df(x_n)\mathbf{v}_n^{(1)}\|}, \\ \mathbf{v}_{n+1}^{(m)} &= \frac{(1-P_{\mathbf{v}_{n+1}^{(m-1)}})\cdots(1-P_{\mathbf{v}_{n+1}^{(1)}})Df(x_n)\mathbf{v}_n^{(m)}}{\|(1-P_{\mathbf{v}_{n+1}^{(m-1)}})\cdots(1-P_{\mathbf{v}_{n+1}^{(1)}})Df(x_n)\mathbf{v}_n^{(m)}\|}, \quad m=2,\dots,N, \end{aligned} \tag{2.1}$$

where $P_{\mathbf{v}}$ is the projector on $\mathbf{v}$, i.e. $P_{\mathbf{v}}\mathbf{g} \equiv \mathbf{v}\cdot(\mathbf{v},\mathbf{g})$. The action of the projectors is to ensure that the resulting vectors form an orthogonal basis. The Lyapunov exponents are defined as

$$\lambda_m \equiv \lim_{n'\to\infty} \frac{1}{n'} \sum_{n=1}^{n'} \log \|(1-P_{\mathbf{v}_{n+1}^{(m-1)}})\cdots(1-P_{\mathbf{v}_{n+1}^{(1)}})Df(x_n)\mathbf{v}_n^{(m)}\|.$$

One may understand (2.1) like this. First the linearized mapping is applied: $\{\mathbf{v}_n^{(1)},\dots,\mathbf{v}_n^{(N)}\} \mapsto \{Df(x_n)\mathbf{v}_n^{(1)},\dots, Df(x_n)\mathbf{v}_n^{(N)}\}$ and then the resulting basis is reorthonormalized. At the next iteration the procedure is repeated, and so on; thus reorthonormalizing the vectors at *every* time step. It was shown in [8] that the "intermediate" reorthonormalizations can be omitted, so one can obtain $\{\mathbf{v}_{n+n_0}^{(1)},\dots,\mathbf{v}_{n+n_0}^{(N)}\}$ in just *two* steps: first under the linearized n_0th iterate of the map the basis $\{\mathbf{v}_n^{(1)},\dots,\mathbf{v}_n^{(N)}\}$ is transformed into $\{Df^{n_0}(x_n)\mathbf{v}_n^{(1)},\dots, Df^{n_0}(x_n)\mathbf{v}_n^{(N)}\}$, and then reorthonormalized – just once:

$$\mathbf{v}_{n+n_0}^{(m)} = \frac{(1-P_{\mathbf{v}_{n+n_0}^{(m-1)}})\cdots(1-P_{\mathbf{v}_{n+n_0}^{(1)}})Df^{n_0}(x_n)\mathbf{v}_n^{(m)}}{\|(1-P_{\mathbf{v}_{n+n_0}^{(m-1)}})\cdots(1-P_{\mathbf{v}_{n+n_0}^{(1)}})Df^{n_0}(x_n)\mathbf{v}_n^{(m)}\|}, \tag{2.2}$$

and the Lyapunov exponents can be obtained as

$$\lambda_m = \lim_{n_0\to\infty} \frac{1}{n_0} \log \|(1-P_{\mathbf{v}_{n+n_0}^{(m-1)}})\cdots(1-P_{\mathbf{v}_{n+n_0}^{(1)}})Df^{n_0}(x_n)\mathbf{v}_n^{(m)}\|. \tag{2.3}$$

Note that (2.1) defines the mapping in the tangent space

$$\mathbf{v}_{n+n_0}^{(i)} = \mathcal{F}_i^{n_0}(x_n, \mathbf{v}_n^{(1)},\dots,\mathbf{v}_n^{(N)}). \tag{2.4}$$

Below we shall show that for n_0 large enough it is contracting almost everywhere. Therefore, the initially different Lyapunov vectors converge, which appears to be crucial for the proof of existence of SLB.

If we denote $\widehat{\mathbf{v}}_n^{(m)} \equiv Df^{n_0}(x_n)\,\mathbf{v}_n^{(m)}$, the transformation (2.2) can be written in a more explicit form:

$$\begin{aligned} \mathbf{v}_{n+n_0}^{(1)} &= \frac{\widehat{\mathbf{v}}_n^{(1)}}{\|\widehat{\mathbf{v}}_n^{(1)}\|}, \\ \mathbf{v}_{n+n_0}^{(2)} &= \frac{\widehat{\mathbf{v}}_n^{(2)} - (\widehat{\mathbf{v}}_n^{(2)}, \mathbf{v}_{n+n_0}^{(1)})\mathbf{v}_{n+n_0}^{(1)}}{\|\widehat{\mathbf{v}}_n^{(2)} - (\widehat{\mathbf{v}}_n^{(2)}, \mathbf{v}_{n+n_0}^{(1)})\mathbf{v}_{n+n_0}^{(1)}\|}, \\ &\vdots \\ \mathbf{v}_{n+n_0}^{(i)} &= \frac{\widehat{\mathbf{v}}_n^{(i)} - \sum_{j=1}^{i-1}(\widehat{\mathbf{v}}_n^{(i)}, \mathbf{v}_{n+n_0}^{(j)})\mathbf{v}_{n+n_0}^{(j)}}{\left\|\widehat{\mathbf{v}}_n^{(i)} - \sum_{j=1}^{i-1}(\widehat{\mathbf{v}}_n^{(i)}, \mathbf{v}_{n+n_0}^{(j)})\mathbf{v}_{n+n_0}^{(j)}\right\|}, \end{aligned} \tag{2.5}$$

which introducing the upper-triangle matrix $R^{(n_0)}$ such that

$$R_{ij}^{(n_0)} \equiv \begin{cases} (\widehat{\mathbf{v}}_n^{(j)}, \mathbf{v}_{n+n_0}^{(i)}), & i = 1, \dots, j-1, \\ \left\| \widehat{\mathbf{v}}_n^{(i)} - \sum_{j=1}^{i-1} (\widehat{\mathbf{v}}_n^{(i)}, \mathbf{v}_{n+n_0}^{(j)}) \mathbf{v}_{n+n_0}^{(j)} \right\|, & i = j, \\ 0, & i = j, \dots, N, \end{cases} \tag{2.6}$$

becomes

$$\mathbf{v}_{n+n_0}^{(i)} = \frac{1}{R_{ii}^{(n_0)}} \cdot \left(\widehat{\mathbf{v}}_n^{(i)} - \sum_{j=1}^{i-1} R_{ji}^{(n_0)} \mathbf{v}_{n+n_0}^{(j)} \right).$$

So

$$\sum_{j=1}^{i} R_{ji}^{(n_0)} \mathbf{v}_{n+n_0}^{(j)} = \widehat{\mathbf{v}}_n^{(i)} \equiv Df^{n_0}(x_n) \mathbf{v}_n^{(i)}. \tag{2.7}$$

Introducing $N \times N$ matrix V_n whose columns are the vectors $\{\mathbf{v}_n^{(m)}\}$, we can rewrite (2.7) in the matrix form:

$$V_{n+n_0} R^{(n_0)} = Df^{n_0}(x_n) V_n \tag{2.8}$$

or

$$V_{n+n_0} = Df^{n_0}(x_n) V_n (R^{(n_0)})^{-1}. \tag{2.9}$$

By the definition of $R^{(n_0)}$ and (2.3),

$$\lim_{n_0 \to \infty} (R_{ii}^{(n_0)})^{1/n_0} = \exp(\lambda_i), \tag{2.10}$$

or $n_0^{-1} \log R_{ii}^{(n_0)} \xrightarrow{n_0 \to \infty} \lambda_i$. The values

$$l_i^{(n_0)}(x_n, V_n) \equiv \frac{1}{n_0} \ln R_{ii}^{(n_0)}(x_n, V_n) \tag{2.11}$$

can be called the "snapshot Lyapunov exponents".

Though the equivalence of the approaches of both Oseledec and Benettin et al. is physically obvious, its rigorous proof (which can be found in [7]) is rather tedious.

Below we shall also use the fact that the volume of the p-dimensional parallelepiped, formed by the vectors $\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(p)}$, can be calculated as a square root of the determinant of the $p \times p$ matrix, whose entries are the inner products $(\mathbf{u}^{(i)}, \mathbf{u}^{(j)})$, $i, j = 1, \dots, p$ (see e.g. [9]). We shall denote

$$\Gamma_p(\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(p)}) = \det \begin{pmatrix} (\mathbf{u}^{(1)}, \mathbf{u}^{(1)}) & \cdots & (\mathbf{u}^{(1)}, \mathbf{u}^{(p)}) \\ \cdots & \cdots & \cdots \\ (\mathbf{u}^{(p)}, \mathbf{u}^{(1)}) & \cdots & (\mathbf{u}^{(p)}, \mathbf{u}^{(p)}) \end{pmatrix}, \qquad \Gamma_0 = 1. \tag{2.12}$$

Then, according to the above relations,

$$R_{11}^{(n_0)} R_{22}^{(n_0)} \cdots R_{pp}^{(n_0)} = \sqrt{\Gamma_p(\widehat{\mathbf{v}}_n^{(1)}, \dots, \widehat{\mathbf{v}}_n^{(p)})} \equiv \sqrt{\Gamma_p}$$

or

$$R_{pp}^{(n_0)} = \sqrt{\Gamma_p/\Gamma_{p-1}}. \tag{2.13}$$

We shall use this relation in Section 6 and Appendix A.

3. Decay of perturbations of Lyapunov vectors

Let us return to Eq. (2.9) which governs the evolution of the Lyapunov basis, and assume that at some step they are perturbed: $V_n \mapsto V_n + \delta V_n$. In this section we shall consider how this perturbation decays and how it influences the values of $l_i^{(n_0)} \equiv (1/n_0) \log R_{ii}^{(n_0)}$ used in numerical calculations as the estimates for λ_i (3.24).

Note that changing are only the tangent vectors, while the trajectory $\{x_n\}$ and therefore, the matrices $Df(x_n)$ *remain unperturbed.* In terms of (2.4), we study the behavior of perturbations of Lyapunov vectors, both infinitesimal

$$\delta V_{n+n_0} = D_V \mathcal{F}^{n_0}(x_n, V_n)\delta V_n$$

and finite

$$\Delta V_{n+n_0} \equiv V_{n+n_0} - V'_{n+n_0} = \mathcal{F}^{n_0}(x_n, V_n) - \mathcal{F}^{n_0}(x_n, V'_n).$$

Here D_V means differentiation only by the coordinates of the Lyapunov vectors, not by the position x.

3.1. Estimates for $R_{ij}^{(n_0)}$, finite-time eigenvectors $\mathbf{g}^{(m;n_0)}$ and $\mathbf{e}^{(m;n_0)}$

Whenever one uses matrix elements, a coordinate system is necessary. It is convenient to use the system of eigenvectors $\{\mathbf{g}^{(m;n_0)}\}$ of the matrix $[Df^{n_0}(x)]^* Df^{n_0}(x)$:

$$[Df^{n_0}(x)]^* Df^{n_0}(x)\mathbf{g}^{(m;n_0)} = \gamma_{m;n_0}\mathbf{g}^{(m;n_0)}. \tag{3.1}$$

They certainly depend on x; later we shall write

$$\mathbf{g}_n^{(m;n_0)} \equiv \mathbf{g}^{(m;n_0)}(x_n), \qquad \gamma_{m;n_0} \equiv \gamma_{m;n_0}(x_n)$$

in short if it does not cause a confusion. According to the Oseledec theorem, for almost all starting points x

$$(\gamma_{m;n_0})^{1/2n_0} \overset{n_0\to\infty}{\longrightarrow} \exp(\lambda_m). \tag{3.2}$$

Since the matrix is symmetric, its eigenvectors are orthogonal; besides, it is natural to assume that they are normalized: $\|\mathbf{g}^{(m;n_0)}(x)\| = 1$. We also assume that they are numbered so that $\gamma_{1;n_0} \geq \gamma_{2;n_0} \geq \cdots \geq \gamma_{N;n_0}$.

Besides $\mathbf{g}^{(m;n_0)}$ we shall use another set of vectors

$$\mathbf{e}_n^{(m;n_0)} \equiv \frac{1}{\sqrt{\gamma_{m;n_0}}} Df^{n_0}(x_n)\mathbf{g}_n^{(m;n_0)}, \tag{3.3}$$

which form orthonormal basis in tangent space at the point $f^{n_0}(x_n) = x_{n+n_0}$. Later, in Section 4 we shall show that they are the eigenvectors of the matrix

$$Df^{n_0}(x_n)[Df^{n_0}(x_n)]^*,$$

as well as its inverse

$$[Df^{n_0}(x_n)]^{-*}[Df^{n_0}(x_n)]^{-1} = [Df^{-n_0}(x_{n+n_0})]^* Df^{-n_0}(x_{n+n_0}).$$

These eigenvectors are also assumed to be normalized:

$$\|\mathbf{e}_n^{(m;n_0)}\| = 1. \tag{3.4}$$

In most cases it is assumed that all Lyapunov exponents are distinct, and therefore one can make the ratios $\gamma_{j:n_0}/\gamma_{i:n_0} \approx \mathrm{e}^{n_0(\lambda_j-\lambda_i)}$ for $j > i$ as small as necessary by choosing n_0 large enough (see (3.2)).

All relations including the tangent vectors do not change if one inverts the sign of (some) of them. For this reason we shall regard two bases in tangent space, related to the same x_n, as the same if they can be transformed one into another by reflections $\mathbf{v}_n^{(m)} \mapsto -\mathbf{v}_n^{(m)}$. Therefore, it is always possible to ensure $(\mathbf{v}_n^{(m)}, \mathbf{g}_n^{(m;n_0)}) \geq 0$.

There are two relations, which are important for the arguments presented below, their proof can be found in Appendix A:

1. For a "typical" basis V_n the matrix $R^{(n_0)}$ satisfies

$$R_{ij}^{(n_0)} = \mathrm{O}(1)\sqrt{\gamma_{i:n_0}}, \quad i = 1, \ldots, j,$$

and so can be written as

$$R^{(n_0)} = DR', \tag{3.5}$$

where $D = \mathrm{diag}\{\sqrt{\gamma_{i:n_0}}\}$ is diagonal, and $R' = D^{-1}R^{(n_0)}$ is upper triangle matrix, such that

$$D_{ii} = \sqrt{\gamma_{i:n_0}}, \qquad R'_{ij} = \mathrm{O}(1), \quad i \leq j, \qquad R'_{ij} = 0, \quad i > j. \tag{3.6}$$

The word "typical" means that some generic conditions must be satisfied; namely the vectors $\mathbf{v}_n^{(i)}$, $i = 1, \ldots, k$, should not be orthogonal to the span of $\mathbf{g}_n^{(1;n_0)}, \ldots, \mathbf{g}_n^{(k;n_0)}$ for any $k \leq N$. This can be expressed as

$$\det \begin{vmatrix} (\mathbf{v}_n^{(1)}, \mathbf{g}_n^{(1;n_0)}) & \cdots & (\mathbf{v}_n^{(1)}, \mathbf{g}_n^{(k;n_0)}) \\ \cdots & \cdots & \cdots \\ (\mathbf{v}_n^{(k)}, \mathbf{g}_n^{(1;n_0)}) & \cdots & (\mathbf{v}_n^{(k)}, \mathbf{g}_n^{(k;n_0)}) \end{vmatrix} \neq 0, \quad k = 1, \ldots, N. \tag{3.7}$$

2. There is the relation between the vectors $\mathbf{v}_{n+n_0}^{(i)}$ and $\mathbf{e}_n^{(j;n_0)}$:

$$(\mathbf{v}_{n+n_0}^{(i)}, \mathbf{e}_n^{(j;n_0)}) = \begin{cases} \mathrm{O}(1)\sqrt{\gamma_{j:n_0}/\gamma_{i:n_0}} & \text{for } j > i, \\ 1 - \mathrm{O}(1)\left(\sum\limits_{k<i} \dfrac{\gamma_{i:n_0}}{\gamma_{k:n_0}} + \sum\limits_{k>i} \dfrac{\gamma_{k:n_0}}{\gamma_{i:n_0}}\right) & \text{for } j = i, \\ \mathrm{O}(1)\sqrt{\gamma_{i:n_0}/\gamma_{j:n_0}} & \text{for } j < i. \end{cases} \tag{3.8}$$

This relation plays the key role in the proof of existence of SLB.

3.2. *Infinitesimal perturbations* δV

Perturbing the Lyapunov vectors (and therefore, the matrix $R^{(n_0)}$) in (2.8), one obtains

$$\delta V_{n+n_0} R^{(n_0)} + V_{n+n_0}\delta R^{(n_0)} = Df^{n_0}(x_n)\delta V_n$$

or

$$\delta V_{n+n_0} = Df^{n_0}(x_n)\delta V_n (R^{(n_0)})^{-1} - V_{n+n_0}\delta R^{(n_0)}(R^{(n_0)})^{-1}. \tag{3.9}$$

To calculate the variation δR we can use the relation

$$(R^{(n_0)})^* R^{(n_0)} = V_n^*([Df^{n_0}(x_n)]^* Df^{n_0}(x_n))V_n, \tag{3.10}$$

which can be derived if one multiplies the relation $V_{n+n_0}R^{(n_0)} = Df^{n_0}(x_n)V_n$ by its conjugate and takes into account that the matrices V_n represent *orthonormal* bases and so $V_n^* V_n = 1$. Taking the variation of (3.10) one obtains

$$\begin{aligned}&\delta(R^{(n_0)})^* R^{(n_0)} + (R^{(n_0)})^* \delta R^{(n_0)} \\ &\quad = \delta V_n^*([Df^{n_0}(x_n)]^* Df^{n_0}(x_n))V_n + V_n^*([Df^{n_0}(x_n)]^* Df^{n_0}(x_n))\delta V_n.\end{aligned} \tag{3.11}$$

Below it is convenient to introduce two infinitesimal matrices ρ and ϵ:

$$\rho \equiv \delta R^{(n_0)}(R^{(n_0)})^{-1}, \qquad \epsilon \equiv V_n^* \delta V_n. \tag{3.12}$$

in which notations (recall that $V_n^* V_n = 1$)

$$\delta R^{(n_0)} = \rho R^{(n_0)}, \qquad \delta V_n = V_n \epsilon. \tag{3.13}$$

Since the perturbed matrix $V_n + \delta V_n$ is also orthogonal, $[V_n + \delta V_n]^*[V_n + \delta V_n] = 1$. This immediately implies that for infinitesimal perturbations

$$V_n^* \delta V_n + V_n \delta V_n^* = 0,$$

i.e.

$$\epsilon^* + \epsilon = 0,$$

which means that ϵ is an antisymmetric matrix and so $\epsilon_{ii} = 0$.

Substituting (3.13) into (3.11) we get

$$(R^{(n_0)})^*(\rho + \rho^*)R^{(n_0)} = \epsilon^* V_n^*([Df^{n_0}(x_n)]^* Df^{n_0}(x_n))V_n + V_n^*([Df^{n_0}(x_n)]^* Df^{n_0}(x_n))V_n \epsilon,$$

which in view of (3.10) becomes

$$(R^{(n_0)})^*(\rho + \rho^*)R^{(n_0)} = \epsilon^*(R^{(n_0)})^* R^{(n_0)} + (R^{(n_0)})^* R^{(n_0)} \epsilon,$$

from what it follows that

$$\begin{aligned}\rho + \rho^* &= [(R^{(n_0)})^*]^{-1}\epsilon^*(R^{(n_0)})^* + R^{(n_0)}\epsilon(R^{(n_0)})^{-1} \\ &= (R^{(n_0)}\epsilon(R^{(n_0)})^{-1})^* + R^{(n_0)}\epsilon(R^{(n_0)})^{-1}.\end{aligned}$$

Denoting $Z \equiv R^{(n_0)}\epsilon(R^{(n_0)})^{-1}$ we can write this equation for matrix elements:

$$\rho_{ij} + \rho_{ji} = Z_{ij} + Z_{ji}. \tag{3.14}$$

By definition, R is upper triangle, and so are $(R^{(n_0)})^{-1}$, $\delta R^{(n_0)}$ and ρ. Therefore, for $i < j$ ρ_{ji} vanishes and (3.14) becomes $\rho_{ij} = Z_{ij} + Z_{ji}$, while for $i = j$ it gives $\rho_{ii} = Z_{ii}$. Combining these cases, one has

$$\rho_{ij} = \begin{cases} 0, & i > j, \\ Z_{ii}, & i = j, \\ Z_{ij} + Z_{ji}, & i < j. \end{cases} \tag{3.15}$$

Now let us rewrite (3.9) in terms of ϵ and ρ. Substituting (3.13) in (3.9), one obtains (it is easy to check the result by direct calculation) that

$$\delta V_{n+n_0} = V_{n+n_0}(Z - \rho).$$

Introducing

$$\widehat{\epsilon} = V^*_{n+n_0}\delta V_{n+n_0}, \tag{3.16}$$

which is the analog of the matrix ϵ, corresponding to time $n + n_0$, we rewrite this equation in the form

$$\widehat{\epsilon} = Z - \rho,$$

and applying (3.15) one arrives at

$$\widehat{\epsilon}_{ij} = \begin{cases} Z_{ij}, & i > j, \\ 0, & i = j, \\ -Z_{ji}, & i < j. \end{cases} \tag{3.17}$$

In view of (3.5) we can write $Z \equiv R^{(n_0)}\epsilon(R^{(n_0)})^{-1} \equiv DR'\epsilon(R')^{-1}D^{-1}$, or

$$Z = DZ'D^{-1}, \quad \text{where } Z' \equiv R'\epsilon(R')^{-1}.$$

An upper triangle matrix with bounded elements is invertible, and the estimate holds

$$\|(R')^{-1}\| \le N \cdot \left(\max_i \frac{\max_j |R'_{ij}|}{|R'_{ii}|}\right)^N,$$

which can be easily derived from the usual Gauss successive elimination procedure when applied to an upper triangle matrix. In view of (A.16) this gives

$$\|(R')^{-1}\| \le \mathrm{O}(1)$$

and therefore

$$\|Z'\| \le \mathrm{O}(1) \cdot \|\epsilon\|.$$

Then, by definition

$$Z_{ij} \equiv (Z')_{ij}\frac{D_{ii}}{D_{jj}} = (Z')_{ij}\sqrt{\frac{\gamma_{i;n_0}}{\gamma_{j;n_0}}},$$

so

$$|Z_{ij}| \le \mathrm{O}(1) \cdot \|\epsilon\| \cdot \sqrt{\frac{\gamma_{i;n_0}}{\gamma_{j;n_0}}}. \tag{3.18}$$

Substituting this in (3.17) we have

$$|\widehat{\epsilon}_{ij}| \le \|\epsilon\| \cdot \begin{cases} \mathrm{O}(1)\sqrt{\gamma_{j;n_0}/\gamma_{i;n_0}} & \text{for } j > i, \\ 0 & \text{for } j = i, \\ \mathrm{O}(1)\sqrt{\gamma_{i;n_0}/\gamma_{j;n_0}} & \text{for } j < i. \end{cases}$$

According to the definition of ϵ and $\widehat{\epsilon}$ Eqs. (3.12) and (3.16),

$$\epsilon_{ij} = (\mathbf{v}_n^{(j)}, \delta\mathbf{v}_n^{(i)}), \qquad \widehat{\epsilon}_{ij} = (\mathbf{v}_{n+n_0}^{(j)}, \delta\mathbf{v}_{n+n_0}^{(i)}),$$

so

$$|(\mathbf{v}^{(j)}_{n+n_0},\ \delta\mathbf{v}^{(i)}_{n+n_0})| \le \max_m \|\delta\mathbf{v}^{(m)}_n\| \cdot \begin{cases} \mathrm{O}(1)\sqrt{\gamma_{j;n_0}/\gamma_{i;n_0}} & \text{for } j > i, \\ 0 & \text{for } j = i, \\ \mathrm{O}(1)\sqrt{\gamma_{i;n_0}/\gamma_{j;n_0}} & \text{for } j < i. \end{cases} \tag{3.19}$$

For large n_0, using the asymptotics (3.2), it becomes

$$|(\mathbf{v}^{(j)}_{n+n_0},\ \delta\mathbf{v}^{(i)}_{n+n_0})| \le \mathrm{O}(1) \max_m \|\delta\mathbf{v}^{(m)}_n\| \cdot \begin{cases} \exp(-n_0|\lambda_i - \lambda_j|) & \text{for } j \ne i, \\ 0 & \text{for } j = i. \end{cases} \tag{3.20}$$

3.3. *Finite perturbations*

Now let us derive the estimate for *finite* perturbations. Unfortunately, we cannot base upon that for *infinitesimal* perturbations – mainly because (3.19) and (3.20) expand them in the *varying* basis, and one should also take into account its "rotation" if one wants to treat finite deviations. Fortunately, the necessary estimate can be derived, and rather simply, from our earlier auxiliary relation (3.8).

Indeed, let us take some different initial bases $\{\mathbf{v}^{(i)}_n\}$ and $\{\widetilde{\mathbf{v}}^{(i)}_n\}$. Quite obviously, for their images after n_0 iterations we can write

$$(\widetilde{\mathbf{v}}^{(i)}_{n+n_0},\ \mathbf{v}^{(j)}_{n+n_0}) = \sum_{k=1}^{N} (\widetilde{\mathbf{v}}^{(i)}_{n+n_0},\ \mathbf{e}^{(k;n_0)}_n)(\mathbf{v}^{(j)}_{n+n_0},\ \mathbf{e}^{(k;n_0)}_n).$$

Now let $j < i$; in this case, splitting the summation domain into five parts:

$$\begin{aligned}(\widetilde{\mathbf{v}}^{(i)}_{n+n_0},\ \mathbf{v}^{(j)}_{n+n_0}) = {} & \sum_{k<j} (\widetilde{\mathbf{v}}^{(i)}_{n+n_0},\ \mathbf{e}^{(k;n_0)}_n)(\mathbf{v}^{(j)}_{n+n_0},\ \mathbf{e}^{(k;n_0)}_n) + (\widetilde{\mathbf{v}}^{(i)}_{n+n_0},\ \mathbf{e}^{(j;n_0)}_n)(\mathbf{v}^{(j)}_{n+n_0},\ \mathbf{e}^{(j;n_0)}_n) \\ & + \sum_{j<k<i} (\widetilde{\mathbf{v}}^{(i)}_{n+n_0},\ \mathbf{e}^{(k;n_0)}_n)(\mathbf{v}^{(j)}_{n+n_0},\ \mathbf{e}^{(k;n_0)}_n) + (\widetilde{\mathbf{v}}^{(i)}_{n+n_0},\ \mathbf{e}^{(i;n_0)}_n)(\mathbf{v}^{(j)}_{n+n_0},\ \mathbf{e}^{(i;n_0)}_n) \\ & + \sum_{k>i} (\widetilde{\mathbf{v}}^{(i)}_{n+n_0},\ \mathbf{e}^{(k;n_0)}_n)(\mathbf{v}^{(j)}_{n+n_0},\ \mathbf{e}^{(k;n_0)}_n)\end{aligned}$$

and estimating the inner products by means of (3.8) which holds for both bases, we have

$$\begin{aligned}|(\widetilde{\mathbf{v}}^{(i)}_{n+n_0},\ \mathbf{v}^{(j)}_{n+n_0})| & \le \mathrm{O}(1)\sqrt{\frac{\gamma_{i;n_0}}{\gamma_{j;n_0}}} + \sum_{k<j} \mathrm{O}(1)\sqrt{\frac{\gamma_{i;n_0}}{\gamma_{k;n_0}}}\sqrt{\frac{\gamma_{j;n_0}}{\gamma_{k;n_0}}} + \sum_{k>i} \mathrm{O}(1)\sqrt{\frac{\gamma_{k;n_0}}{\gamma_{i;n_0}}}\sqrt{\frac{\gamma_{k;n_0}}{\gamma_{j;n_0}}} \\ & = \mathrm{O}(1)\sqrt{\frac{\gamma_{i;n_0}}{\gamma_{j;n_0}}} \cdot \left\{ 1 + \sum_{k<j} \frac{\gamma_{j;n_0}}{\gamma_{k;n_0}} + \sum_{k>i} \frac{\gamma_{k;n_0}}{\gamma_{i;n_0}} \right\} \\ & = \mathrm{O}(1)\sqrt{\gamma_{i;n_0}/\gamma_{j;n_0}}.\end{aligned}$$

In quite a similar way (suffice it to swap i and j) one finds that for $j > i$

$$(\widetilde{\mathbf{v}}^{(i)}_{n+n_0},\ \mathbf{v}^{(j)}_{n+n_0}) \le \mathrm{O}(1)\sqrt{\gamma_{j;n_0}/\gamma_{i;n_0}},$$

thus combining these two cases we arrive at

$$(\widetilde{\mathbf{v}}^{(i)}_{n+n_0},\ \mathbf{v}^{(j)}_{n+n_0}) \le \begin{cases} \mathrm{O}(1)\sqrt{\gamma_{j;n_0}/\gamma_{i;n_0}} & \text{for } j > i, \\ \mathrm{O}(1)\sqrt{\gamma_{i;n_0}/\gamma_{j;n_0}} & \text{for } j < i. \end{cases} \tag{3.21}$$

The case $j = i$ is now obvious: orthonormality of the bases $\{\tilde{\mathbf{v}}^{(i)}_{n+n_0}\}$ and $\{\mathbf{v}^{(i)}_{n+n_0}\}$ implies that

$$\sum_{j=1}^{N} (\tilde{\mathbf{v}}^{(i)}_{n+n_0}, \mathbf{v}^{(j)}_{n+n_0})^2 = 1,$$

so

$$|(\tilde{\mathbf{v}}^{(i)}_{n+n_0}, \mathbf{v}^{(i)}_{n+n_0})| = \sqrt{1 - \sum_{j \neq i} (\tilde{\mathbf{v}}^{(i)}_{n+n_0}, \mathbf{v}^{(j)}_{n+n_0})^2}.$$

Estimating $(\tilde{\mathbf{v}}^{(i)}_{n+n_0}, \mathbf{v}^{(j)}_{n+n_0})$ by means of (3.21), one obtains that for large n_0, when the fractions in (3.21) are very small,

$$1 \geq |(\tilde{\mathbf{v}}^{(i)}_{n+n_0}, \mathbf{v}^{(i)}_{n+n_0})| \geq 1 - \mathrm{O}(1) \left(\sum_{k<i} \frac{\gamma_{i;n_0}}{\gamma_{k;n_0}} + \sum_{k>i} \frac{\gamma_{k;n_0}}{\gamma_{i;n_0}} \right).$$

Combining all these three possible cases, and choosing the bases so that $(\tilde{\mathbf{v}}^{(i)}_n, \mathbf{v}^{(i)}_n) > 0$ for any n, we finally arrive at

$$|(\tilde{\mathbf{v}}^{(i)}_{n+n_0}, \mathbf{v}^{(j)}_{n+n_0})| \leq \begin{cases} \mathrm{O}(1)\sqrt{\gamma_{j;n_0}/\gamma_{i;n_0}} & \text{for } j > i, \\ 1 - \mathrm{O}(1) \left(\sum_{k<i} \dfrac{\gamma_{i;n_0}}{\gamma_{k;n_0}} + \sum_{k>i} \dfrac{\gamma_{k;n_0}}{\gamma_{i;n_0}} \right) & \text{for } j = i, \\ \mathrm{O}(1)\sqrt{\gamma_{i;n_0}/\gamma_{j;n_0}} & \text{for } j < i. \end{cases}$$

Obviously, the deviation $\Delta\mathbf{v}^{(i)}_{n+n_0} \equiv \tilde{\mathbf{v}}^{(i)}_{n+n_0} - \mathbf{v}^{(i)}_{n+n_0}$ satisfies

$$(\Delta\mathbf{v}^{(i)}_{n+n_0}, \mathbf{v}^{(j)}_{n+n_0}) \equiv (\tilde{\mathbf{v}}^{(i)}_{n+n_0} - \mathbf{v}^{(i)}_{n+n_0}, \mathbf{v}^{(j)}_{n+n_0}) = (\tilde{\mathbf{v}}^{(i)}_{n+n_0}, \mathbf{v}^{(j)}_{n+n_0}) - \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

and thus

$$|(\Delta\mathbf{v}^{(i)}_{n+n_0}, \mathbf{v}^{(j)}_{n+n_0})| \leq \begin{cases} \mathrm{O}(1)\sqrt{\gamma_{j;n_0}/\gamma_{i;n_0}} & \text{for } j > i, \\ \mathrm{O}(1) \left(\sum_{k<i} \dfrac{\gamma_{i;n_0}}{\gamma_{k;n_0}} + \sum_{k>i} \dfrac{\gamma_{k;n_0}}{\gamma_{i;n_0}} \right) & \text{for } j = i, \\ \mathrm{O}(1)\sqrt{\gamma_{i;n_0}/\gamma_{j;n_0}} & \text{for } j < i. \end{cases} \tag{3.22}$$

Using the asymptotics (3.2) $\gamma_{i,n_0} \overset{n_0 \to \infty}{\longrightarrow} \exp(n_0 \lambda_i)$, one can write this as

$$|(\Delta\mathbf{v}^{(i)}_{n+n_0}, \mathbf{v}^{(j)}_{n+n_0})| \leq \begin{cases} \mathrm{O}(1)\exp(-n_0|\lambda_i - \lambda_j|) & \text{for } j \neq i, \\ \mathrm{O}(1) \left(\sum_{k \neq i} \exp(-n_0|\lambda_i - \lambda_k|) \right) & \text{for } j = i. \end{cases} \tag{3.23}$$

Therefore, if all Lyapunov exponents are different, both infinitesimal and finite perturbations of Lyapunov basis decay exponentially.

3.4. The estimate for variations of "snapshot" Lyapunov exponents

The "snapshot" Lyapunov exponents were defined as (see (2.11))

$$l_i^{(n_0)}(x_n, V_n) \equiv \frac{1}{n_0} \log R_{ii}^{(n_0)}(x_n, V_n).$$

So, their deviations due to the deviation of the Lyapunov vectors $\mathbf{v}_n^{(m)}$ are

$$\delta l_i^{(n_0)}(x_n, V_n) = \frac{1}{n_0} \delta(\log R_{ii}^{(n_0)}(x_n, V_n) = \frac{1}{n_0} \delta R_{ii}^{(n_0)} / R_{ii}^{(n_0)}.$$

Recalling that $\rho \equiv \delta R^{(n_0)} \cdot (R^{(n_0)})^{-1}$ while $\delta R^{(n_0)}$ and $(R^{(n_0)})^{-1}$ are upper triangle matrices, we see that $\rho_{ii} = \delta R_{ii}^{(n_0)} / R_{ii}^{(n_0)}$, so

$$\delta l_i^{(n_0)}(x_n) = \frac{1}{n_0} \rho_{ii}.$$

Meanwhile, due to (3.15) $\rho_{ii} = Z_{ii}$, and thus $\delta l_i^{(n_0)} = Z_{ii}$ which using (3.18) becomes

$$|\delta l_i^{(n_0)}| \leq \frac{1}{n_0} \mathrm{O}(\|\epsilon\|), \tag{3.24}$$

where $\epsilon_{ij} \equiv (\delta \mathbf{v}_n^{(i)}, \mathbf{v}_n^{(j)})$ and so $\|\epsilon\|$ is a measure of the deviation of the Lyapunov vectors $\mathbf{v}_n^{(m)}$ which form the basis V_n.

4. Stationary Lyapunov basis (SLB)

According to (3.8), if started *more* than n_0 iterations ago, the Lyapunov vectors at the given point x_n are very close to $\mathbf{e}_{n-n_0}^{(m;n_0)}$:

$$\|\mathbf{v}_n^{(m)} - \mathbf{e}_{n-n_0}^{(m;n_0)}\| \overset{n_0 \to \infty}{\longrightarrow} 0,$$

where

$$\mathbf{e}_{n-n_0}^{(m;n_0)} \equiv \frac{1}{\sqrt{\gamma_{m;n_0}}} Df^{n_0}(x_{n-n_0}) \mathbf{g}_{n-n_0}^{(m;n_0)},$$

see (3.3). Therefore, if the latter *converge* as $n_0 \to \infty$, the evolving Lyapunov vectors (if started long before) will *only* depend on the point x where they are observed. In other words, in such a case the "extended" phase space, introduced in [8], reduces to the original phase space. The limiting vectors are naturally called "stationary Lyapunov vectors".

As well as $\mathbf{g}_n^{(m;n_0)}$ and $\gamma_{m;n_0}$, these $\mathbf{e}_n^{(m;n_0)}$ depend on n only implicitly, via x_n:

$$\mathbf{e}_{n-n_0}^{(m;n_0)} \equiv \mathbf{e}^{(m;n_0)}(x_n).$$

To study this dependence we shall base upon the fact that $\mathbf{e}^{(m;n_0)}(x)$ are the eigenvectors of the matrix

$$\begin{aligned} A_{n_0}(x) &\equiv ([Df^{n_0}(f^{-n_0}(x))]^{-1})^* [Df^{n_0}(f^{-n_0}(x))]^{-1} \\ &= [Df^{-n_0}(x_n)]^* Df^{-n_0}(x_n). \end{aligned}$$

Indeed, denoting $x_{-n_0} \equiv f^{-n_0}(x)$ and using the definition of $\mathbf{e}_n^{(m;n_0)}$ in the form

$$\mathbf{e}^{(m;n_0)}(x) \equiv \frac{1}{\sqrt{\gamma_{m;n_0}(x_{-n_0})}} Df^{n_0}(x_{-n_0})\mathbf{g}^{(m;n_0)}(x_{-n_0}),$$

one easily obtains

$$\begin{aligned} A_{n_0}(x)\mathbf{e}^{(m;n_0)}(x) &= \frac{1}{\sqrt{\gamma_{m;n_0}(x_{-n_0})}}([Df^{n_0}(x_{-n_0})]^{-1})^*\mathbf{g}^{(m;n_0)}(x_{-n_0}) \\ &= \frac{1}{\sqrt{\gamma_{m;n_0}(x_{-n_0})}} Df^{n_0}(x_{-n_0})([Df^{n_0}(x_{-n_0})]^* Df^{n_0}(x_{-n_0}))^{-1}\mathbf{g}^{(m;n_0)}(x_{-n_0}), \end{aligned}$$

and since by definition

$$[Df^{n_0}(x)]^* Df^{n_0}(x)\mathbf{g}^{(m;n_0)}(x) = \gamma_{m;n_0}(x) \cdot \mathbf{g}^{(m;n_0)}(x),$$

we have

$$\begin{aligned} A_{n_0}(x)\mathbf{e}^{(m;n_0)}(x) &= \frac{1}{\gamma_{m;n_0}(x_{-n_0})\sqrt{\gamma_{m;n_0}(x_{-n_0})}} \cdot Df^{n_0}(x_{-n_0})\mathbf{g}^{(m;n_0)}(x_{-n_0}) \\ &= \frac{1}{\gamma_{m;n_0}(x_{-n_0})} \cdot \mathbf{e}^{(m;n_0)}(x). \end{aligned} \tag{4.1}$$

Also, we shall use the following obvious factorization of the matrix $A_n(x)$:

$$A_n(x) \equiv ([Df^n(x_{-n})]^{-1})^*[Df^n(x_{-n})]^{-1} = (J_0^* J_1^* \cdots J_n^*)(J_n J_{n-1} \cdots J_0), \tag{4.2}$$

where $x_{-n} \equiv f^{-n}(x)$ and $J_n \equiv Df^{-1}(x_{-n})$ is the Jacobian of the inverse transformation at the point x_{-n}. Here the absence of a subscript means the moment when the trajectory hits the point x, and the subscript "$-n$" means "n iterations before it had hit that point".

So we conclude that if the evolution starts from the point x_{-n}, and the set of Lyapunov vectors $\mathbf{v}_{-n}^{(m;n)} = \mathbf{g}^{(m;n_0)}(x_{-n})$, then after n iterations they will be mapped into $x = f^n(x_{-n})$ and $\mathbf{v}^{(m;n)} = \mathbf{e}^{(m;n)}(x)$, respectively, which using notation (2.4) can be written as

$$\mathbf{e}^{(m;n)}(x) = \mathcal{F}_m^n(x_{-n},\ \mathbf{g}^{(1;n_0)}(x_{-n}), \ldots, \mathbf{g}^{(N;n_0)}(x_{-n})). \tag{4.3}$$

So the basis $\{\mathbf{e}^{(1;n)}(x), \ldots, \mathbf{e}^{(N;n)}(x)\}$ is the set of Lyapunov vectors (at this point) corresponding to some particular choice of their "initial" orientation n iterations before.

Now let us take some greater $\bar{n} > n$. Again we have

$$\mathbf{e}^{(m;\bar{n})}(x) = \mathcal{F}_m^{\bar{n}}(x_{-\bar{n}},\ \mathbf{g}^{(1;\bar{n})}(x_{-\bar{n}}), \ldots, \mathbf{g}^{(N;\bar{n})}(x_{-\bar{n}})),$$

which, according to [8] (see also the beginning of Section 3), can be as well achieved in two steps: first we pass from $x_{-\bar{n}}$ to $f^{\bar{n}-n}(x_{-\bar{n}}) = x_{-n}$ while the Lyapunov vectors are transformed into

$$\mathcal{E}^{(m)} = \mathcal{F}_m^{\bar{n}-n}(x_{-\bar{n}},\ \mathbf{g}^{(1;\bar{n})}(x_{-\bar{n}}), \ldots, \mathbf{g}^{(N;\bar{n})}(x_{-\bar{n}})),$$

and then from x_{-n} to $f^n(x_{-n}) = x$, this time the Lyapunov vectors being transformed into the "final" $\{\mathbf{v}^{(m;n)}\}$:

$$\mathbf{e}^{(m;\bar{n})}(x) = \mathcal{F}_m^n(x_{-n},\ \mathcal{E}^{(1)}, \ldots, \mathcal{E}^{(N)}). \tag{4.4}$$

In other words, in n iterations the basis $\{\mathcal{E}^{(m)}\}$ is transformed into $\{\mathbf{e}^{(m;\bar{n})}(x)\}$.

Comparing this with (4.3) we see that the difference between $\mathbf{e}^{(m;\bar{n})}$ and $\mathbf{e}^{(m;n)}$ can be thought of as being caused by the difference between their predecessors ($\{\mathbf{g}^{(m;\bar{n})}(x_{-\bar{n}})\}$ and $\{\mathcal{E}^{(m)}\}$, respectively) n iterations before. Therefore, it can be estimated using (3.22), only the number of iterations done is now denoted n instead of n_0:

$$|(\mathbf{e}^{(i;\bar{n})} - \mathbf{e}^{(i;n)}, \mathbf{e}^{(j;n)})| \leq \begin{cases} \mathrm{O}(1)\sqrt{\gamma_{j;n}/\gamma_{i;n}} & \text{for } j > i, \\ \mathrm{O}(1)\left(\sum\limits_{k<i} \dfrac{\gamma_{i;n}}{\gamma_{k;n}} + \sum\limits_{k>i} \dfrac{\gamma_{k;n}}{\gamma_{i;n}}\right) & \text{for } j = i, \\ \mathrm{O}(1)\sqrt{\gamma_{i;n}/\gamma_{j;n}} & \text{for } j < i. \end{cases} \tag{4.5}$$

Applying the asymptotics (3.2) $\gamma_{i,n_0} \overset{n_0\to\infty}{\longrightarrow} \exp(n_0\lambda_i)$, one can rewrite this a bit more graphically:

$$|(\mathbf{e}^{(i;\bar{n})} - \mathbf{e}^{(i;n)}, \mathbf{e}^{(j;n)})| \leq \begin{cases} \mathrm{O}(1)\sum\limits_{k\neq i} \exp(-n\,|\lambda_k - \lambda_i|) & \text{if } j = i, \\ \mathrm{O}(1)\exp(-n\,|\lambda_j - \lambda_i|) & \text{if } j \neq i. \end{cases} \tag{4.6}$$

Remark 1. Note that as well as (3.22), these estimates hold not for all orientations of Lyapunov vectors. Namely, they fail if $(\mathbf{g}^{(m;\bar{n})}(x_{-\bar{n}}), \mathcal{E}^{(m)}) = 0$ for some m. Since both $\{\mathbf{g}^{(m;\bar{n})}(x_{-\bar{n}})\}$ and $\{\mathcal{E}^{(m)}\}$ are uniquely determined by x, we conclude that, though (4.5) and (4.6) are valid for most points x, there can exist some exceptional points at which they fail.

Therefore, there is a convergence as $n \to \infty$, and one may define the *limiting* vectors which are just the stationary Lyapunov basis at the point x as

$$\mathbf{e}^{(m)}(x) = \lim_{n\to\infty} \mathbf{e}^{(m;n)}(x),$$

where $\mathbf{e}^{(m;n)}(x)$ are the eigenvalues of the matrix $A_n(x)$ (4.2). According to (4.1), its eigenvalues are $\alpha_{m;n}(x) = 1/\gamma_{m;n}$, and thus $\alpha_{1;n}(x) \leq \alpha_{2;n}(x) \leq \cdots \leq \alpha_{N;n}(x)$. They obviously have the following asymptotics as $n \to \infty$:

$$\alpha_{k;n}^{1/n}(x) \overset{n\to\infty}{\longrightarrow} \exp(-2\lambda_k),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ are the Lyapunov exponents of the mapping f.

Remark 2. Other stationary bases may exist, but converging to them are only very specific initial vectors (those which *do not* satisfy (3.7)). In other words, these *stationary* bases are *unstable*, and tiny deviation of the initial basis results in its convergence to the stable $\{\mathbf{e}^{(m)}\}$. Further, the latter is *unique* in view of the results of Section 3 (up to changing the directions of the vectors or reflection transformations of the basis).

Remark 3. The very existence of the limiting basis $\mathbf{e}^{(m)}(x) = \lim_{n_0\to\infty} \mathbf{e}^{(m;n_0)}(x)$ can be derived from the Oseledec theorem applied to the inverse dynamical system $f^{-1}(x)$, provided the latter exists, but our approach gives *separate* estimates for the convergence along different directions. That proves to be crucial when studying continuity of the stationary Lyapunov vectors.

Remark 4. For non-invertible systems there are several trajectories through the same point (merging at it), and each of them can have its own stationary Lyapunov basis. But if we consider only one of the trajectories, then again the stationary Lyapunov basis becomes unique. In particular, in time series analysis we usually deal with a single trajectory, and therefore it is possible to apply the concept of SLB for both invertible and non-invertible dynamical systems.

5. Continuity of the field of stationary Lyapunov vectors

5.1. Why we need equicontinuity. Infinitesimal perturbations δx

Since the matrix $A_n(x)$ depends on the point x, so do its eigenvectors, and therefore the stationary Lyapunov basis, their limit, also changes from point to point. It is interesting to find out, whether the vectors of SLB change continuously with x or not, i.e. to study the continuity with respect to the perturbations of basic trajectory. Let us consider two infinitely close points x and $x + \delta x$, such that (1) both of these points belong to the same attractor, (2) there exists the SLB in both of them, (3) the trajectories through these points are "typical", i.e. the sets of $\{\lambda_i\}$ for them coincide and (4) all exponents λ_i are different.

According to the theory of perturbations of symmetric operators, for *any finite n* the eigenvectors of $A_n(x)$ change continuously with x. This does not guarantee that *their limit* is continuous, though. Indeed, *if* $\mathbf{e}^{(m:n)}$, as a function of x, becomes more and more ragged, though remaining continuous, its limit may be discontinuous. To ensure the smoothness of the limiting vectors, they should be *equicontinuous*, i.e. there must exist some function $q(r) \overset{r\to\infty}{\longrightarrow} 0$ such that

$$|\mathbf{e}^{(m:n)}(x + \delta x) - \mathbf{e}^{(m:n)}(x)| \le q(\|\delta x\|)$$

for any n large enough. To find if this inequality holds, and under which conditions, we must have an estimate for deviation of the eigenvectors for an arbitrary n. Note that the matrix $A_n(x)$ itself has *no limit* as $n \to \infty$.

In the remainder of this section we will write $\mathbf{e}^{(m:n)}$ and $\alpha_{m:n}$ instead of $\mathbf{e}^{(m:n)}(x)$ and $\alpha_{m:n}(x)$ wherever this does not cause a confusion.

Denoting the perturbed point $x' \equiv x + \delta x$, we can write the perturbation $\delta A_n \equiv A_n(x') - A_n(x)$ of the matrix

$$A_n = (J_0^* J_1^* \cdots J_n^*)(J_n J_{n-1} \cdots J_0)$$

as

$$\delta A_n = \sum_{k=0}^{n} J_0^* \cdots J_{k-1}^* \cdot \delta J_k^* \cdot J_{k+1}^* \cdots J_n^* \cdot J_n \cdots J_0 + \sum_{k=0}^{n} J_0^* \cdots J_n^* \cdot J_n \cdots J_{k+1} \cdot \delta J_k \cdot J_{k-1} \cdots J_0, \tag{5.1}$$

where

$$\delta J_k \equiv Df^{-1}(x_{-k}) - Df^{-1}(x'_{-k}) \equiv Df^{-1}(f^{-k}(x)) - Df^{-1}(f^{-k}(x')).$$

The corresponding perturbations of the eigenvectors are (see, e.g. Landau and Lifshitz, "Quantum Mechanics" or application of this technique in the context of Lyapunov exponents in [5]):

$$\delta \mathbf{e}^{(i:n)} = \sum_{j\neq i} \frac{(\mathbf{e}^{(j:n)}, \delta A_n \mathbf{e}^{(i:n)})}{\alpha_{i:n} - \alpha_{j:n}} \mathbf{e}^{(j:n)}. \tag{5.2}$$

The main question is: when the deviation $\|\delta \mathbf{e}^{(i:n)}\|$ remains bounded for $n \to \infty$?

We must note that applicability of this relation requires more slightly stronger condition than the degeneracy of of Lyapunov specturm. We shall assume that no quasidegeneracy occurs, i.e. the shift in the specturm of $\alpha_{i:n}$ is less than the spectral difference. In Appendix B it is shown that with the help of the above formulas and the estimate $\exp[n \cdot (\lambda_i + \varphi_{1n})] \leq \alpha_{i;n} \leq \exp[n \cdot (\lambda_i + \varphi_{2n})]$, where $\varphi_{jn} \overset{n\to\infty}{\longrightarrow} +0$, it is possible to obtain the relation

$$\|\delta \mathbf{e}^{(i;n)}\| \leq \mathrm{O}(1) \cdot \|\delta x\| \left\{ \sum_{j<i} \sum_{k=0}^{n} \exp[k(\lambda_i - \lambda_j - \lambda_N + \psi_k)] + \sum_{j>i} \sum_{k=0}^{n} \exp[k(\lambda_j - \lambda_i - \lambda_N + \psi_k)] \right\}. \tag{5.3}$$

Now let us assume that there exists such $q > 0$ that $\lambda_i - \lambda_j - \lambda_N \leq -q$ for $j < i$, and $\lambda_j - \lambda_i - \lambda_N \leq -q$ for $j > i$. In this case, it follows from (5.3) that

$$\|\delta \mathbf{e}^{(i;n)}\| \leq \mathrm{O}(1) \cdot \|\delta x\| \sum_{k=0}^{n} \exp[-k(q - \psi_k)].$$

Beginning with such k_* that $\psi_{k_*} < q$, its terms decay exponentially, so the series converges, and $\delta \mathbf{e}^{(i;n)}$ is *bounded* for $n \to \infty$.

Due to the arbitrariness of the q, we can say that the deviations $\|\delta \mathbf{e}^{(i;n)}\|$ are bounded for $n \to \infty$ if

$$\lambda_j > \lambda_i - \lambda_N \quad \text{for } j < i, \qquad \lambda_j < \lambda_i + \lambda_N \quad \text{for } j > i. \tag{5.4}$$

In other words, in this case the eigenvector $\mathbf{e}^{(i;n)}(x)$ as a function of x is equicontinuous, and so its limit, the static Lyapunov vector $\mathbf{e}^{(i)}(x) = \lim_{n\to\infty} \mathbf{e}^{(i;n)}(x)$, continuously depends on x.

Remark 5. It should be emphasized that the above consideration, based on the estimates from above, may only provide *sufficient*, but *not necessary* conditions for the continuity of the stationary Lyapunov vector field. In other words, there may be continuity even when inequalities (5.4) cease to hold.

In the framework of our approach, *all* of the stationary Lyapunov vectors are continuous if (5.4) holds for every i. In this case it implies

$$\lambda_{i-1} > \lambda_i - \lambda_N, \qquad \lambda_{i+1} < \lambda_i + \lambda_N,$$

which means that

$$\lambda_{N-1} > 0, \quad \lambda_{N-2} > -\lambda_N, \ldots, \quad \lambda_{N-k} > (-k+1)\lambda_N.$$

and therefore, *unless* $N = 2$,

$$\sum_{m=1}^{N} \lambda_m > \lambda_N + \lambda_{N-1} + \lambda_{N-2} > 0,$$

which is *impossible* if the dynamics is *invertible*. So, our approach *never* states continuity of *all* vectors if the dimension is greater than 2.

Now let us discuss two simple examples and consider the conditions, under which some of vectors of SLB of dissipative systems for which $\sum_i \lambda_i < 0$ are continuous.

Example 1. $N = 2$. Directly from (5.4) we see that for both vectors to be continuous it suffices that λ_1 be positive. That is, *any* invertible 2D chaotic map (e.g. the Hénon map) possesses a continuous Lyapunov basis.

Example 2. $N = 3$. Again directly from (5.4) we see that:
– $\mathbf{e}^{(1)}$ is continuous if $\lambda_1 > 0$, $2\lambda_2 < \lambda_1 + \lambda_2 + \lambda_3 < 0$, $\lambda_3 < 0$;

– continuity of $\mathbf{e}^{(2)}$ can neither be proved nor rejected using the approach presented;
– $\mathbf{e}^{(3)}$ is continuous if $\lambda_1, \lambda_2 > 0$, $\lambda_3 < 0$.

That is, there are three cases: continuity can be proved for (a) $\mathbf{e}^{(1)}$; (b) $\mathbf{e}^{(3)}$; (c) none.

For the Lorenz system ($\sigma = 10$, $r = 28$, $b = 8/3$, $\lambda_1 = 0.9$, $\lambda_2 = 0$, $\lambda_3 = -14.8$) our experiments show that $\mathbf{e}^{(3)}$ is continuous even despite $\lambda_2 = 0$.

We should emphasize again that all these are *sufficient*, not *necessary* conditions, and our approach *never* enables one to prove *discontinuity* of the stationary Lyapunov vectors.

5.2. *Continuity and finite perturbations*

Unfortunately, it is impossible to extend our approach directly to *finite* perturbations.

Indeed, at first glance we could have proceeded as follows. Take two points x and x' and connect them by a smooth curve $\xi(t)$, $0 \le t \le 1$ of the length l, $\xi(0) = x$, $\xi(1) = x'$. Then, assuming that $\mathbf{e}^{(i;n)}(x)$ is differentiable, we can write

$$\mathbf{e}^{(i;n)}(x') - \mathbf{e}^{(i;n)}(x) = \int_0^1 \mathrm{d}\xi \frac{\partial \mathbf{e}^{(i;n)}(\xi)}{\partial \xi}$$

and therefore

$$\|\mathbf{e}^{(i;n)}(x') - \mathbf{e}^{(i;n)}(x)\| \le \int_0^1 \mathrm{d}\xi \left\| \frac{\partial \mathbf{e}^{(i;n)}(\xi)}{\partial \xi} \right\| \le l \sup_{x\in\xi(t)} \|\nabla \mathbf{e}^{(i;n)}(x)\|.$$

However, this obviously requires that (1) $\xi(t)$ also belong to attractor and (2) in all its points there exists the stationary Lyapunov basis. Moreover since chaotic attractors usually have very complex geometry, this curve may prove very long even for very close points x and x'. In other words, some points may be close, even arbitrary close, in euclidean space, i.e. $\|x - x'\|$ small, but remote in the "attractor's metric", i.e. l may be large, e.g. because of fractal structure of the attractor. In such situations our approach cannot guarantee the closeness of $\mathbf{e}^{(i)}(x)$ and $\mathbf{e}^{(i)}(x')$.

Indeed, our calculations for the Hénon mapping confirmed that at most points x $\|\mathbf{e}^{(i)}(x') - \mathbf{e}^{(i)}(x)\| \to 0$ as $\|x - x'\| \to 0$. But at some points the deviations of $\mathbf{e}^{(i)}$ are $O(1)$ for $\|\Delta x\|$ as small as 10^{-5}. The origin of this effect is not completely clear, since the estimates of $\|\nabla \mathbf{e}^{(i)}(x)\|$ show that usually it does not exceed 10, but sometimes, at rare points, may reach the values of 10^{10}, and even greater.

On the contrary, for the Lorenz system the continuity of $\mathbf{e}^{(3)}(x)$ can be easily detected: $\max_{x'} \|\mathbf{e}^{(3)}(x') - \mathbf{e}^{(3)}(x)\|$ goes to 0 approximately as $\|x - x'\|$.

Therefore, the numerical verification of the continuity of SLB can be a very difficult problem.

6. Stationary Lyapunov basis and calculation of Lyapunov exponents

6.1. *Lyapunov exponents as measure averages*

Now let us consider the consequences which follow from the existence of stationary Lyapunov basis and the estimates of the rate of convergence towards it. But first let us return to the algorithm of Benettin et al. of calculation of Lyapunov exponents and introduce the "one-step contribution" terms for them.

Let us denote

$$S_i^{(n_0)}(x_n, V_n) = \log R_{ii}^{(n_0)}(x_n, V_n)$$

(see (2.6)). As mentioned in Section 2.2, orthonormalization of the evolving vectors can be performed after every iteration, which in matrix notation means that the matrix $R^{(n_0)}(x_n, V_n)$ is the product of single-step matrices $R^{(1)}(x_k, V_k)$:

$$R^{(n_0)}(x_n, V_n) = R^{(1)}(x_n, V_n)R^{(1)}(x_{n+1}, V_{n+1}) \cdots R^{(1)}(x_{n+n_0-1}, V_{n+n_0-1}),$$

which, in its turn, means that $S_i^{(n_0)}$ is the sum of stretchings per one iteration $S_i^{(1)}(x_k, V_k)$ which later we shall denote simply as $S_i(x_k, V_k)$. By definition,

$$S_i \equiv \log R_{ii}^{(1)} = \log |(\mathbf{v}_{n+1}^{(i)}, Df(x_n)\mathbf{v}_n^{(i)})|.$$

With these notations, the Lyapunov exponents are

$$\lambda_i = \lim_{n_0\to\infty} \frac{1}{n_0} \sum_{k=0}^{n_0} S_i(x_k, V_k) \equiv \lim_{n_0\to\infty} \frac{1}{n_0} \sum_{k=0}^{n_0} \log |(\mathbf{v}_{k+1}^{(i)}, Df(x_k)\mathbf{v}_k^{(i)})|. \tag{6.1}$$

This means that λ_i is the time-average of a function which depends both on the phase space point x and the orientation of the tangent vectors $\mathbf{v}_k^{(i)}$. It is the latter dependence that makes it impossible to replace the time averaging with the measure averaging without passing to the "extended phase space" which incorporates both usual and tangent coordinates x and V [8].

But if the SLB exists, the situation is quite different. Indeed, now the evolving Lyapunov vectors converge to their stationary directions $\mathbf{e}^{(m)}(x)$, thus after some transient period $V_k = E(x_k)$, i.e. their orientation is uniquely determined by the phase space point. Therefore, (6.1) becomes

$$\lambda_i = \lim_{n_0\to\infty} \frac{1}{n_0} \sum_{k=0}^{n_0} \sigma_i(x_k), \tag{6.2}$$

where (x_{k+1} is replaced by $f(x_k)$)

$$\sigma_i(x_k) = S_i(x_k, E(x_k)) = \log |(\mathbf{e}^{(i)}(f(x_k)), Df(x_k)\mathbf{e}^{(i)}(x_k))|.$$

Now the averaged function depends only on the phase space point x, and one can replace the time averaging with the measure averaging in the original phase space:

$$\lambda_i = \int \sigma_i(x)\mu(\mathrm{d}x). \tag{6.3}$$

Note that in the sum (6.2), which can be considered as a numerical approximation to this integral, the terms can be calculated *independently* from each other and in an *arbitrary* order. Sometimes this may be very important for numerical implementation.

For continuous time dynamical systems $\dot{x} = F(x)$, when the time step τ, corresponding to one iteration, may be arbitrarily small, we can obtain the relation for $\mathrm{d}S/\mathrm{d}t$ instead of S. In continuous time systems the Lyapunov exponents are related to *time*, not to number of iterations – thus (6.1) becomes

$$\lambda_i = \lim_{n_0\to\infty} \frac{1}{\tau n_0} \sum_{k=0}^{n_0} S_i(x_k, V_k),$$

which can be written as

$$\lambda_i = \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{S_i(x(t), V(t))}{\tau}\, dt.$$

For $\tau \ll 1$ we can use approximations $f(x) \approx 1 + \tau F(x)$, $\mathbf{v}_{n+1}^{(i)} \approx \mathbf{v}_n^{(i)}$ which leads to

$$\frac{S_i(x(t), V(t))}{\tau} \approx (\mathbf{v}^{(i)}, DF(x)\mathbf{v}^{(i)}),$$

thus

$$\lambda_i = \lim_{T\to\infty} \frac{1}{T} \int_0^T (\mathbf{v}^{(i)}(t), DF(x(t))\mathbf{v}^{(i)}(t))\, dt.$$

This relation, but without stationary basis context, has been obtained in [5,6]. Again the expression for Lyapunov exponents can be rewritten as a measure average

$$\lambda_i = \int (\mathbf{e}^{(i)}(x), DF(x)\mathbf{e}^{(i)}(x))\mu(dx).$$

6.2. *Some consequences of SLB existence*

Relations (6.1)–(6.3) are convenient for showing the role of SLB $E(x)$.

(1) Since during the evolution the basis V_k converges to $E(x_k)$, the first iterations, when V_k has not yet converged, produce an error in "snapshot" Lyapunov exponents. Approximately, it can be estimated using (3.24). This error decays very slowly as

$$\Delta\lambda_i \sim \frac{1}{n} \|V_0 - E\|,$$

where $\|V_0 - E\|$ denotes the deviation of the starting Lyapunov vectors from their stationary directions. Since this deviation decays rapidly, the accuracy will be substantially improved if one starts averaging a bit later, thus discarding the very first iterations. This means that several first terms must be excluded from the sum (6.1); their optimal number depends on the convergence rate. Everyone who has calculated Lyapunov exponents numerically usually knows this fact from one's own experience, though.

(2) The relation (6.3) enables one to propose another approach to the calculation of Lyapunov exponents. It is known that usually most initial probability distributions converge to the invariant measure. Let us take M points x_{0j}, $j = 1, \ldots, M$, on the attractor of the dynamical system, and let them evolve during some time T. The latter should be large enough for convergence of distribution to invariant and for convergence of Lyapunov vectors to the stationary Lyapunov basis. After that has occurred, at the end of each trajectory we calculate the single value $\sigma_i(f^T(x_{0j}))$. Then, according to (6.3)

$$\frac{1}{M} \sum_{j=1}^{M} \sigma_i(f^T(x_{0j})) \to \lambda_i \quad \text{as } M \to \infty,\ T \to \infty. \tag{6.4}$$

This method is of little practical importance for calculation of Lyapunov exponents from equations of motion, while it has been successfully applied for time-series analysis [11,12], when trajectories (which are actually just different parts of the same long trajectory) remain close only for some limited time.

(3) Now let us briefly show, how this idea can be used in time-series analysis. Let there be a vector time series $x_1, \dots, x_M$, it may be either points in original phase space or delay reconstructions of a scalar time series. Let us consider a point x_i and its neighborhood, the points x_{j_k}, $\|x_{j_k} - x_i\| < \epsilon$, $k = 1, \dots, K$. If ϵ is small, the vectors $\mathbf{v}_{0,k}(x_i) = x_{j_k} - x_i$ can be considered as approximately tangent vectors. The same usually will be true for n_f forward and n_b backward iterates: $\mathbf{v}_{n,k}(x_i) = x_{j_k+n} - x_{i+n}$, $-n_b \le n \le n_f$. The old idea is to use the vectors $\mathbf{v}_{n,k}$ to obtain the estimates of Lyapunov exponents.

Suppose that n_b iterates are enough for a tangent vector to converge with good accuracy to the direction of $\mathbf{e}^{(1)}(x_i)$. Then the ratio $\|\mathbf{v}_{n,k}\|/\|\mathbf{v}_{0,k}\|$ can be considered as an approximation for $R_{11}^{(n)}$ (see (2.6)). For better performance of the method it is possible to average this ratio over all K neighbors, and therefore,

$$\sigma_1^{(n)}(x_i) \simeq \log\left(\frac{1}{K}\sum_{k=1}^{K}\frac{\|\mathbf{v}_{n,k}(x_i)\|}{\|\mathbf{v}_{0,k}(x_i)\|}\right)$$

(the upper index (n) means that we estimate the σ_1 for the nth iterate of our map, i.e. $f^n(x)$). Then, according to (6.2),

$$n\lambda_1 \simeq \frac{1}{M}\sum_{i=1}^{M}\sigma_1^{(n)}(x_i) = \frac{1}{M}\sum_{i=1}^{M}\log\left(\frac{1}{K}\sum_{k=1}^{K}\frac{\|\mathbf{v}_{n,k}(x_i)\|}{\|\mathbf{v}_{0,k}(x_i)\|}\right). \tag{6.5}$$

Analogs of this formula with some differences, which are inessential for us now, has been proposed in [11,12] for estimating λ_1 from a time series. Our results form the rigorous mathematical background for this approach. The important feature of this approach is that it does not require approximation of the matrix Df and it is insensitive to the dimension of the phase space, which is essential for attractor reconstruction from a scalar time series.

Note that without SLB one has to follow the usual definitions of Lyapunov exponents, and to ensure consistency of the "tangent" vectors $\mathbf{v}$ at successive points. This creates additional problems with vectors selection, accuracy etc. – see [10] for the details.

In a similar way it is possible to propose methods for estimating λ_2. We assume that during n_b iterates the pairs $\{\mathbf{v}_{n,k_1}(x_i), \mathbf{v}_{n,k_2}(x_i)\}$ converge with good accuracy to span$\{\mathbf{e}^{(1)}(x_i), \mathbf{e}^{(2)}(x_i)\}$. Then one should calculate 2D volumes formed by these pairs. This can be done for example by using the Gram determinant – see (2.12):

$$\Gamma_{2,k_1,k_2}^{(n)}(x_i) = \det\begin{pmatrix} (\mathbf{v}_{n,k_1}(x_i), \mathbf{v}_{n,k_1}(x_i)) & (\mathbf{v}_{n,k_1}(x_i), \mathbf{v}_{n,k_2}(x_i)) \\ (\mathbf{v}_{n,k_2}(x_i), \mathbf{v}_{n,k_1}(x_i)) & (\mathbf{v}_{n,k_2}(x_i), \mathbf{v}_{n,k_2}(x_i)) \end{pmatrix}.$$

According to (2.13),

$$R_{11}^{(n)}R_{22}^{(n)} \simeq \sqrt{\Gamma_2^{(n)}/\Gamma_2^{(0)}},$$

and, as in the case of (6.5), we obtain

$$n(\lambda_1 + \lambda_2) \simeq \frac{1}{M}\sum_{i=1}^{M}\log\left(\frac{1}{N(\text{pairs})}\sum_{\{k_1,k_2\}}\sqrt{\frac{\Gamma_{2,k_1,k_2}^{(n)}(x_i)}{\Gamma_{2,k_1,k_2}^{(0)}(x_i)}}\right). \tag{6.6}$$

In [10] the attempts to estimate λ_2 have been done by other methods also using 2D phase volumes, and it was pointed out that estimates for it are worse than for λ_1. It is natural to expect similar problems for (6.6) as well, though averaging over many pairs may improve situation.

In principle, it is possible to propose generalization of this approach for other λ_i.

Therefore, the SLB concept gives new and interesting viewpoint on the Lyapunov exponents and the problem of their computation. Though it does not give practical outcome at once, it may serve as a basis for a number of algorithms.

7. Conclusion

In this paper we have studied the mathematical background of some algorithms for calculation of Lyapunov exponents.

(i) We have shown that the evolving basis of Lyapunov vectors, used in the algorithm of Benettin et al. [4], converges to the stationary Lyapunov basis, which is unique at every point x of the attractor.

(ii) We estimated the rate of convergence and stability of this basis against trajectory perturbation

(iii) In contrast with the paper [5], our results on convergence rate are applicable for dynamical systems with both continuous and discrete time. This is also important for applications in time-series analysis, since usually the reconstructed system is a mapping.

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Appendix A. Proof of (3.5) and (3.8)

A.1. Estimation of the diagonal elements $R_{ii}^{(n_0)}$

Since the eigenvectors $\{\mathbf{g}_n^{(m;n_0)}\}$ form an orthonormal basis in the tangent space at the point x_n, we can expand another element of this space, the Lyapunov vector $\mathbf{v}_n^{(i)}$ as

$$\mathbf{v}_n^{(i)} = \sum_{j=1}^{N} c_{ij}\mathbf{g}_n^{(j;n_0)}. \tag{A.1}$$

Obviously,

$$\hat{\mathbf{v}}_n^{(i)} = Df^{n_0}(x_n)\mathbf{v}_n^{(i)} = \sum_{k=1}^{N} c_{ik}Df^{n_0}(x_n)\mathbf{g}_n^{(k;n_0)} = \sum_{k=1}^{N} c_{ik}\sqrt{\gamma_{k;n_0}}\mathbf{e}_n^{(k;n_0)}$$

and

$$(\hat{\mathbf{v}}_n^{(i)}, \hat{\mathbf{v}}_n^{(j)}) = \sum_{k=1}^{N} c_{ik}c_{jk}\gamma_{k;n_0}.$$

The coefficients c_{ik} form an orthogonal matrix – the transition matrix from one orthogonal basis to another, i.e. $\sum_{k=1}^{N} c_{ik}c_{jk} = \delta_{ij}$.

Using the Gram matrix for $\hat{\mathbf{v}}_n^{(i)}$ and the definition of det(·), we can write

$$\begin{aligned}\Gamma_p &= \det \|(\hat{\mathbf{v}}_n^{(i)}, \hat{\mathbf{v}}_n^{(j)})\| \\ &= \det\left|\begin{pmatrix} c_{11} & \cdots & c_{1N} \\ \cdots & \cdots & \cdots \\ c_{p1} & \cdots & c_{pN}\end{pmatrix}\begin{pmatrix} \gamma_{1;n_0} & & \\ & \cdots & \\ & & \gamma_{N;n_0}\end{pmatrix}\begin{pmatrix} c_{11} & \cdots & c_{p1} \\ \cdots & \cdots & \cdots \\ c_{1N} & \cdots & c_{pN}\end{pmatrix}\right| \\ &= \sum_{1\le i_1<i_2\cdots<i_p\le N} (\gamma_{i_1;n_0}\gamma_{i_2;n_0}\cdots\gamma_{i_p;n_0})\cdot(J_{i_1i_2\cdots i_p})^2,\end{aligned}$$

where

$$J_{i_1 i_2 \cdots i_p} = \det\begin{pmatrix} c_{1i_1} & \cdots & c_{1i_p} \\ \cdots & \cdots & \cdots \\ c_{pi_1} & \cdots & c_{pi_p} \end{pmatrix}.$$

Therefore,

$$\frac{\Gamma_p}{\gamma_{1;n_0}\gamma_{2;n_0}\cdots\gamma_{p;n_0}} = \sum_{1\le i_1<i_2\cdots<i_p\le N} \left(\frac{\gamma_{i_1;n_0}\gamma_{i_2;n_0}\cdots\gamma_{i_p;n_0}}{\gamma_{1;n_0}\gamma_{2;n_0}\cdots\gamma_{p;n_0}}\right)\cdot (J_{i_1 i_2\cdots i_p})^2 \ge (J_{12\cdots p})^2.$$

By choosing n_0 large enough it is possible to make all ratios in the first brackets ≤ 1. Then, for this n_0

$$\frac{\Gamma_p}{\gamma_{1;n_0}\gamma_{2;n_0}\cdots\gamma_{p;n_0}} \le \sum_{1\le i_1<i_2\cdots<i_p\le N} (J_{i_1 i_2\cdots i_p})^2$$
$$= \det\left\|\begin{pmatrix} c_{11} & \cdots & c_{1N} \\ \cdots & \cdots & \cdots \\ c_{p1} & \cdots & c_{pN} \end{pmatrix}\begin{pmatrix} 1 & & \\ & \cdots & \\ & & 1 \end{pmatrix}\begin{pmatrix} c_{11} & \cdots & c_{p1} \\ \cdots & \cdots & \cdots \\ c_{1N} & \cdots & c_{pN} \end{pmatrix}\right\| = 1.$$

This means that for n_0 large enough and all $p = 1, \ldots, N$,

$$(J_{12\ldots p})^2 \gamma_{1;n_0}\gamma_{2;n_0}\cdots\gamma_{p;n_0} \le \Gamma_p \le \gamma_{1;n_0}\gamma_{2;n_0}\cdots\gamma_{p;n_0},$$

and, using the relation $R_{pp} = \sqrt{\Gamma_p/\Gamma_{p-1}}$,

$$|J_{12\cdots p}|\sqrt{\gamma_{p;n_0}} \le R_{pp} \le \frac{1}{|J_{12\cdots p-1}|}\sqrt{\gamma_{p;n_0}}.$$

Therefore, one obtains that

$$O(1)\sqrt{\gamma_{p;n_0}} \le R_{pp}^{(n_0)} \le O(1)\sqrt{\gamma_{p;n_0}} \tag{A.2}$$

as long as

$$J_{12\ldots k} = \det\begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ c_{k1} & c_{k2} & \cdots & c_{kk} \end{vmatrix} \ne 0, \qquad k = 1, \ldots, p, \qquad c_{ij} = (\mathbf{v}_n^{(i)}, \mathbf{g}_n^{(j;n_0)}),$$

i.e. as long as the projections of the vectors $\mathbf{v}_n^{(1)}, \ldots, \mathbf{v}_n^{(p)}$ onto the span of $\mathbf{g}_n^{(1;n_0)}, \ldots, \mathbf{g}_n^{(p;n_0)}$ are linearly independent.

A.2. Estimation of the off-diagonal elements $R_{ij}^{(n_0)}$

Now assume that conditions (3.7) hold. Using expansion (A.1) one obtains

$$\hat{\mathbf{v}}_n^{(i)} \equiv Df^{n_0}(x_n)\mathbf{v}_n^{(i)} = \sum_{j=1}^{N} c_{ij} Df^{n_0}(x_n)\mathbf{g}_n^{(j;n_0)} \equiv \sum_{j=1}^{N} \sqrt{\gamma_{j;n_0}}c_{ij}\mathbf{e}_n^{(j;n_0)}, \tag{A.3}$$

where the vectors $\mathbf{e}_n^{(m;n_0)}$ have been defined in (3.3), and like $\mathbf{g}_n^{(m;n_0)}$, they form orthonormal basis, but in tangent space at x_{n+n_0}.

Let us also expand in $\{\mathbf{e}_n^{(m;n_0)}\}$ the vector $\mathbf{v}_{n+n_0}^{(i)}$:

$$\mathbf{v}_{n+n_0}^{(i)} = \sum_{j=1}^{N} C_{ij}\, \mathbf{e}_n^{(j;n_0)} \tag{A.4}$$

and calculate the coefficients C_{ij}. Substituting (A.3) in (2.7) and comparing with (A.4), we have

$$C_{ij} = \frac{\sqrt{\gamma_j} c_{ij} - \sum_{m=1}^{i-1} C_{mj} \sum_{k=1}^{N} \sqrt{\gamma_{k;n_0}} c_{ik} C_{mk}}{R_{ii}^{(n_0)}}. \tag{A.5}$$

Now take $n_0 \gg 1$. In this case, according to (3.2) $\sqrt{\gamma_{m;n_0}} \approx \mathrm{e}^{n_0 \lambda_m}$, so $\sqrt{\gamma_{k;n_0}} \gg \sqrt{\gamma_{k';n_0}}$ for $k' > k$. For $i = 1$, using the estimate for $R_{11}^{(n_0)}$ (A.2) one obtains

$$C_{1j} \le \mathrm{O}(1) \cdot \sqrt{\gamma_{j;n_0}/\gamma_{1;n_0}}.$$

Let us show, by induction, that it implies

$$C_{ij} \le \mathrm{O}(1) \cdot \begin{cases} \sqrt{\gamma_{j;n_0}/\gamma_{i;n_0}} & \text{for } j \ge i, \\ \sqrt{\gamma_{i;n_0}/\gamma_{j;n_0}} & \text{for } j \le i. \end{cases} \tag{A.6}$$

Assume that it holds for $i = 1, \ldots, I-1$. Will it then be true for $i = I$?
Denoting

$$C'_{ij} \equiv C_{ij} \cdot \begin{cases} \sqrt{\gamma_{i;n_0}/\gamma_{j;n_0}} & \text{for } j \ge i, \\ \sqrt{\gamma_{j;n_0}/\gamma_{i;n_0}} & \text{for } j \le i, \end{cases}$$

we can write our supposition as

$$|C'_{ii'}| \le \mathrm{O}(1) \quad \text{for } i < I \text{ and any } i'. \tag{A.7}$$

In terms of C'_{ij}, (A.5) becomes for $j \ge i$

$$C'_{ij} = \frac{\sqrt{\gamma_{i;n_0}}}{R_{ii}^{(n_0)}} \left\{ c_{ij} - \sum_{m=1}^{i-1} C'_{mj} \left(\sum_{k=1}^{m} c_{ik} C'_{mk} + \sum_{k=m+1}^{N} \frac{\gamma_{k;n_0}}{\gamma_{m;n_0}} c_{ik} C'_{mk} \right) \right\},$$

and substituting in it the estimate for $R_{ii}^{(n_0)}$ (A.2), one obtains

$$C'_{ij} = \mathrm{O}(1) - \mathrm{O}(1) \sum_{m=1}^{i-1} C'_{mj} \sum_{k=1}^{m} c_{ik} C'_{mk} - \mathrm{O}(1) \sum_{m=1}^{i-1} C'_{mj} \sum_{k=m+1}^{N} \frac{\gamma_{k;n_0}}{\gamma_{m;n_0}} c_{ik} C'_{mk} \quad \text{for } j \ge i.$$

Now let $j \ge I$ and $i = I$. Since $m < i = I$, (A.7) implies that $C'_{mj} \le \mathrm{O}(1)$ and $C'_{mk} \le \mathrm{O}(1)$, so

$$|C'_{Ij}| \le \mathrm{O}(1) \quad \text{for } j \ge I. \tag{A.8}$$

Now let us estimate C'_{Ij} for $j < I$.

The basis $\{\mathbf{v}_{n+n_0}^{(m)}\}$ is orthogonal: $(\mathbf{v}_{n+n_0}^{(i)}, \mathbf{v}_{n+n_0}^{(j)}) = 0$, which using expansion (A.4) and the orthogonality of the eigenvectors $\{\mathbf{e}_n^{(m;n_0)}\}$ can be written as

$$\sum_{k=1}^{N} C_{i,k} C_{j,k} = 0 \quad \text{for } j \ne i.$$

Splitting the summation domain $1 \le k \le N$ into three parts and using our notation of C'_{ij}, one rewrites the above equality as

$$\sum_{k=1}^{j-1} C'_{i,k}C'_{j,k}\sqrt{\frac{\gamma_{i;n_0}}{\gamma_{k;n_0}}\frac{\gamma_{j;n_0}}{\gamma_{k;n_0}}} + \sqrt{\frac{\gamma_{i;n_0}}{\gamma_{j;n_0}}}\sum_{k=j}^{i-1} C'_{i,k}C'_{j,k} + \sum_{k=i}^{N} C'_{i,k}C'_{j,k}\sqrt{\frac{\gamma_{k;n_0}}{\gamma_{i;n_0}}\frac{\gamma_{k;n_0}}{\gamma_{j;n_0}}} = 0 \quad \text{for } j < i,$$

which after multiplying by $\sqrt{\gamma_{j;n_0}/\gamma i}$ becomes

$$\sum_{k=1}^{j-1} C'_{i,k}C'_{j,k}\frac{\gamma_{j;n_0}}{\gamma_{k;n_0}} + \sum_{k=j-1}^{i-1} C'_{i,k}C'_{j,k} + C'_{i,i}C'_{j,i} + \sum_{k=i+1}^{N} C'_{i,k}C'_{j,k}\frac{\gamma_{k;n_0}}{\gamma_{i;n_0}} = 0 \quad \text{for } j < i. \tag{A.9}$$

Now let us take $i = I$. Denoting

$$\xi_j \equiv C'_{ij}/C'_{ii}$$

we can rewrite (A.9) in the form

$$\sum_{k=1}^{j-1} \frac{\gamma_{j;n_0}}{\gamma_{k;n_0}} C'_{j,k}\xi_k + \sum_{k=j}^{I-1} C'_{j,k}\xi_k = -C'_{j,I} - \Delta_j, \quad j = 1, \ldots, I-1, \tag{A.10}$$

where

$$\Delta_j \equiv \frac{1}{C'_{II}} \sum_{k=I+1}^{N} \frac{\gamma_{k;n_0}}{\gamma_{I;n_0}} C'_{j,k}C'_{I,k}.$$

In our case, i.e. for $j < I$ and large n_0, it is negligible. Indeed, now according to (A.8) $C'_{I,k} \le \mathrm{O}(1)$; and also $C'_{j,k} \le \mathrm{O}(1)$ because of (A.7). Therefore,

$$|\Delta_j| = \left| \frac{1}{C'_{II}} \sum_{k=i+1}^{N} C'_{i,k}C'_{j,k}\frac{\gamma_{k;n_0}}{\gamma_{i;n_0}} \right| \le \mathrm{O}(1) \sum_{k=i+1}^{N} \frac{\gamma_{k;n_0}}{\gamma_{i;n_0}} \le \mathrm{O}(1)\frac{\gamma_{i+1:n_0}}{\gamma_{i;n_0}} \ll 1 \quad \text{for } j < I.$$

If not for this tiny term Δ_j in the right-hand side, (A.10) would be an $(I-1) \times (I-1)$ linear system with respect to $I-1$ variables $\xi_1, \ldots, \xi_{I-1}$. Moreover, it would be nearly an upper-triangle system, because the lower-triangle matrix elements, $(\gamma_{j;n_0}/\gamma_{k;n_0})C'_{j,k}$, are very small: here $C'_{j,k} = \mathrm{O}(1)$ according to (A.7), while for $n_0 \gg 1$

$$\frac{\gamma_{j;n_0}}{\gamma_{k;n_0}} \le \frac{\gamma_{j;n_0}}{\gamma_{j-1;n_0}} \ll 1.$$

Therefore, the first approximation to the solution, $\xi^{(1)}$, can be found from the upper triangle system

$$\sum_{k=j-1}^{i-1} C'_{j,k}\xi_k^{(1)} = -C'_{j,I}, \quad j = 1, \ldots, I-1, \tag{A.11}$$

which matrix elements and right-hand side are all $\le \mathrm{O}(1)$. Moreover if $n_0 \gg 1$, for its diagonal elements we have the estimate $C'_{jj} \approx 1$. Indeed, the condition

$$\|\mathbf{v}_{n+n_0}^{(j)}\| = 1$$

implies $\sum_{k=1}^{N} C_{jk}^2 = 1$ or

$$|C_{jj}| = \sqrt{1 - \sum_{k=1}^{j-1} C_{jk}^2 - \sum_{k=j+1}^{N} C_{jk}^2}. \tag{A.12}$$

Recalling the definition of C'_{ij}, we derive from this

$$|C'_{jj}| = \sqrt{1 - \sum_{k=1}^{j-1} (C'_{jk})^2 \frac{\gamma_{j;n_0}}{\gamma_{k;n_0}} - \sum_{k=j+1}^{N} (C'_{jk})^2 \frac{\gamma_{k;n_0}}{\gamma_{j;n_0}}} \geq \sqrt{1 - \left(\frac{\gamma_{j;n_0}}{\gamma_{j-1;n_0}} + \frac{\gamma_{j+1;n_0}}{\gamma_{j;n_0}}\right) \sum_{k \neq j}^{N} (C'_{jk})^2}. \tag{A.13}$$

For $j < I$ we have $C'_{jk} \leq \mathrm{O}(1)$ according to (A.7). Therefore,

$$|C'_{jj}| \overset{n_0 \to \infty}{\longrightarrow} 1. \tag{A.14}$$

Thus we conclude that the matrix of the linear system (A.11) is well-defined and invertible; and so is its *small* perturbation, the matrix of (A.10). Therefore, its solution is of the same order of magnitude as the right-hand side:

$$|\xi_k| \leq \mathrm{O}(1).$$

By our definition, $\xi_j \equiv C'_{Ij}/C'_{II}$ while C'_{II} is not estimated as yet. Though, similarly to (A.13) we can write

$$(C'_{II})^2 \geq 1 - \left(\frac{\gamma_{I;n_0}}{\gamma_{I-1;n_0}} + \frac{\gamma_{I+1;n_0}}{\gamma_{I;n_0}}\right) \left(\sum_{k=1}^{I-1} (C'_{Ik})^2 + \sum_{k=I+1}^{N} (C'_{Ik})^2\right)$$

$$= 1 - \left(\frac{\gamma_{I;n_0}}{\gamma_{I-1;n_0}} + \frac{\gamma_{I+1;n_0}}{\gamma_{I;n_0}}\right) \left((C'_{II})^2 \sum_{k=1}^{I-1} (\xi_k)^2 + \sum_{k=I+1}^{N} (C'_{Ik})^2\right).$$

The values of ξ_k for $k < I$ were estimated just above; and C'_{Ik} for $k > I$ satisfy (A.8). Thus,

$$(C'_{II})^2 \geq 1 - \mathrm{O}(1) \cdot \left(\frac{\gamma_{I;n_0}}{\gamma_{I-1;n_0}} + \frac{\gamma_{I+1;n_0}}{\gamma_{I;n_0}}\right) - \mathrm{O}(1) \cdot \left(\frac{\gamma_{I;n_0}}{\gamma_{I-1;n_0}} + \frac{\gamma_{I+1;n_0}}{\gamma_{I;n_0}}\right) (C'_{II})^2,$$

so

$$|C'_{II}| \geq \sqrt{\frac{1 - \mathrm{O}(1) \cdot (\gamma_{I;n_0}/\gamma_{I-1;n_0} + \gamma_{I+1;n_0}/\gamma_{I;n_0})}{1 + \mathrm{O}(1) \cdot (\gamma_{I;n_0}/\gamma_{I-1;n_0} + \gamma_{I+1;n_0}/\gamma_{I;n_0})}}.$$

On the other hand, by definition $C'_{ii} = C_{ii}$ while, according to (A.12), $|C_{ii}| \leq 1$. Therefore, $|C'_{II}| \leq 1$ and so

$$\sqrt{\frac{1 - \mathrm{O}(1) \cdot (\gamma_{I;n_0}/\gamma_{I-1;n_0} + \gamma_{I+1;n_0}/\gamma_{I;n_0})}{1 + \mathrm{O}(1) \cdot (\gamma_{I;n_0}/\gamma_{I-1;n_0} + \gamma_{I+1;n_0}/\gamma_{I;n_0})}} \leq |C'_{II}| \leq 1$$

from what it follows that

$$C'_{II} \overset{n_0 \to \infty}{\longrightarrow} 1,$$

and thus

$$C'_{Ij} \equiv \xi_j C'_{II} = \mathrm{O}(1) \quad \text{for } j < I.$$

Combining this with the estimate (A.8), we conclude that for large n_0

$$|C'_{Ij}| \le \mathrm{O}(1) \quad \text{for any } j,$$

which means that

$$C_{Ij} \le \mathrm{O}(1) \cdot \begin{cases} \sqrt{\gamma_{j;n_0}/\gamma_{I;n_0}} & \text{for } j \ge I, \\ \sqrt{\gamma_{i;n_0}/\gamma_{I;n_0}} & \text{for } j \le I. \end{cases}$$

Therefore, we have shown that if the estimate (A.6) holds for $i = I - 1$, then it is also true for $i = I$. Repeating the process, we conclude that it is valid for all *any* i.

The diagonal elements can now be estimated substituting (A.6) in (A.12) which gives

$$|C_{jj}| \ge \sqrt{1 - \mathrm{O}(1)\left(\sum_{k<j} \frac{\gamma_{j;n_0}}{\gamma_{k;n_0}} + \sum_{k>j} \frac{\gamma_{k;n_0}}{\gamma_{j;n_0}}\right)}.$$

For large n_0 when the fractions are very small, it reduces to

$$|C_{jj}| \ge 1 - \mathrm{O}(1)\left(\sum_{k<j} \frac{\gamma_{j;n_0}}{\gamma_{k;n_0}} + \sum_{k>j} \frac{\gamma_{k;n_0}}{\gamma_{j;n_0}}\right),$$

and recalling that $(\mathbf{v}^{(i)}_{n+n_0}, \mathbf{e}^{(j;n_0)}_n) \equiv C_{ij}$ we finally come to (3.8):

$$(\mathbf{v}^{(i)}_{n+n_0}, \mathbf{e}^{(j;n_0)}_n) = \begin{cases} \mathrm{O}(1)\sqrt{\gamma_{j;n_0}/\gamma_{i;n_0}} & \text{for } j > i, \\ 1 - \mathrm{O}(1)\left(\sum_{k<i} \frac{\gamma_{i;n_0}}{\gamma_{k;n_0}} + \sum_{k>i} \frac{\gamma_{k;n_0}}{\gamma_{i;n_0}}\right) & \text{for } j = i, \\ \mathrm{O}(1)\sqrt{\gamma_{i;n_0}/\gamma_{j;n_0}} & \text{for } j < i. \end{cases}$$

Now writing $R^{(n_0)}_{ij}$ as

$$\begin{aligned} R^{(n_0)}_{ij} &\equiv (\widehat{\mathbf{v}}^{(j)}_n, \mathbf{v}^{(i)}_{n+n_0}) \\ &= \sum_{k=1}^{N} (\widehat{\mathbf{v}}^{(j)}_n, \mathbf{e}^{(k;n_0)}_n)(\mathbf{v}^{(i)}_{n+n_0}, \mathbf{e}^{(k;n_0)}_n) \\ &= \sum_{k=1}^{N} c_{jk}\sqrt{\gamma_{k;n_0}}(\mathbf{v}^{(i)}_{n+n_0}, \mathbf{e}^{(k;n_0)}_n) \end{aligned}$$

and using the estimate (3.8), one obtains for $i = 1, \dots, j-1$ and large n_0:

$$|R^{(n_0)}_{ij}| \le \sqrt{\gamma_{i;n_0}} \cdot \left[\sum_{k=1}^{i} \mathrm{O}(1) \cdot c_{jk} + \sum_{k=i+1}^{N} \mathrm{O}(1) \cdot c_{jk} \frac{\gamma_{k;n_0}}{\gamma_{i;n_0}}\right] = \mathrm{O}(1) \cdot \sqrt{\gamma_{i;n_0}}. \tag{A.15}$$

By definition, $R'_{ij} \equiv R^{(n_0)}_{ij}/\sqrt{\gamma_{i;n_0}}$. So combining (A.15) with (A.2), one obtains that

$$R'_{ij} = \begin{cases} \mathrm{O}(1) & \text{for } j \ge i, \\ 0 & \text{for } j < i. \end{cases} \tag{A.16}$$

so its upper triangle (which only is distinct from zero) contains neither too large nor *too small* elements. The latter fact, which follows from (A.2), is crucial for obtaining the estimate of the inverse matrix, $(R')^{-1}$.

Appendix B. Estimates for perturbations of stationary Lyapunov vectors

Using our expansion of δA_n (5.1) we see that

$$(\mathbf{e}^{(j;n)}, \delta A_n \mathbf{e}^{(i;n)}) = V_{ij} + V_{ji},$$

where

$$\begin{aligned} V_{ij} &= \sum_{k=0}^{n} (\delta J_k J_{k-1} \cdots J_0 \mathbf{e}^{(j;n)},\ J_{k+1}^* \cdots J_n^* J_n \cdots J_0 \mathbf{e}^{(i;n)}) \\ &= \sum_{k=0}^{n} (\delta J_k J_{k-1} \cdots J_0 \mathbf{e}^{(j;n)},\ [J_0^* \cdots J_k^*]^{-1} A_n \mathbf{e}^{(i;n)}) \\ &= \alpha_{i;n} \sum_{k=0}^{n} ([J_k \cdots J_0]^{-1} \delta J_k J_{k-1} \cdots J_0 \mathbf{e}^{(j;n)},\ \mathbf{e}^{(i;n)}) \\ &= \alpha_{i;n} \sum_{k=0}^{n} (Df^k(x_{-k}) \delta J_k J_{k-1} \cdots J_0 \mathbf{e}^{(j;n)},\ \mathbf{e}^{(i;n)}), \end{aligned}$$

and so

$$\begin{aligned} (\mathbf{e}^{(j;n)}, \delta A_n \mathbf{e}^{(i;n)}) = {} & \alpha_{i;n} \sum_{k=0}^{n} (Df^k(x_{-k}) \delta J_k J_{k-1} \cdots J_0 \mathbf{e}^{(j;n)},\ \mathbf{e}^{(i;n)}) \\ & + \alpha_{j;n} \sum_{k=0}^{n} (Df^k(x_{-k}) \delta J_k J_{k-1} \cdots J_0 \mathbf{e}^{(i;n)},\ \mathbf{e}^{(j;n)}). \end{aligned} \tag{B.1}$$

B.1. Estimation of the inner product $(Df^k(x_{-k}) \delta J_k J_{k-1} \cdots J_0 \mathbf{e}^{(j;n)},\ \mathbf{e}^{(i;n)})$ for large k

To estimate the magnitude of $Df^k(x_{-k}) \delta J_k J_{k-1} \cdots J_0 \mathbf{e}^{(j;n)}$ we shall use the eigenvectors of A_k, i.e. $\mathbf{e}^{(m;k)}$, and the vectors

$$\mathbf{e}_{-k}^{(m;k)} \equiv \frac{J_k J_{k-1} \cdots J_0 \mathbf{e}^{(m;k)}}{\| J_k J_{k-1} \cdots J_0 \mathbf{e}^{(m;k)} \|} \equiv \frac{Df^{-k}(x) \mathbf{e}^{(m;k)}}{\| Df^{-k}(x) \mathbf{e}^{(m;k)} \|},$$

which are defined at the point x_{-k}. They are the eigenvectors of the matrix $A_k \equiv [Df^k(x_{-k})]^* Df^k(x_{-k})$ (see Section 4) and are therefore orthogonal.

Let us expand $\delta J_k J_{k-1} \cdots J_0 \mathbf{e}^{(j;n)}$ in the basis $\{\mathbf{e}_{-k}^{(m;k)}\}$:

$$\delta J_k J_{k-1} \cdots J_0 \mathbf{e}^{(j;n)} = \sum_{m=1}^{N} b_{jm} \mathbf{e}_{-k}^{(m;k)}.$$

which gives

$$
\begin{aligned}
&(Df^k(x_{-k})\delta J_k J_{k-1}\cdots J_0 \mathbf{e}^{(j;n)},\ \mathbf{e}^{(i;n)}) \\
&= \left(Df^k(x_{-k})\sum_{m=1}^{N} b_{jm}\mathbf{e}_{-k}^{(m;k)},\ \mathbf{e}^{(i;n)}\right) = \sum_{m=1}^{N} b_{jm}(Df^k(x_{-k})\mathbf{e}_{-k}^{(m;k)},\ \mathbf{e}^{(i;n)}) \\
&= \sum_{m=1}^{N} \frac{b_{jm}}{(\mathbf{e}^{(m;n)},\ A_k\mathbf{e}^{(m;n)})^{1/2}}\cdot(\mathbf{e}^{(m;k)},\ \mathbf{e}^{(i;n)}).
\end{aligned}
$$

At the same time, $b_{jm} = (\mathbf{e}_{-k}^{(m;k)},\ \delta J_k J_{k-1}\cdots J_0\mathbf{e}^{(j;n)})$, so

$$|b_{jm}| \le \|\delta J_k J_{k-1}\cdots J_0\mathbf{e}^{(j;n)}\|$$

which writing $\delta J_k J_{k-1}\cdots J_0\mathbf{e}^{(j;n)} = (\delta J_k J_k^{-1})J_k\cdots J_0\mathbf{e}^{(j;n)}$ can be estimated as

$$
\begin{aligned}
|b_{jm}| &\le \|\delta J_k\|\cdot\|J_k^{-1}\|\cdot\|J_k J_{k-1}\cdots J_0\mathbf{e}^{(j;n)}\| \\
&= \|\delta J_k\|\cdot\|J_k^{-1}\|\cdot(\mathbf{e}^{(j;n)},\ A_k\mathbf{e}^{(j;n)})^{1/2}
\end{aligned}
$$

and thus, assuming that the map has finite derivative so that $J_k^{-1} \equiv Df(x_{-k})$ is bounded, one arrives at

$$
\begin{aligned}
&|(Df^k(x_{-k})\delta J_k J_{k-1}\cdots J_0\mathbf{e}^{(j;n)},\ \mathbf{e}^{(i;n)})| \\
&\le \mathrm{O}(1)\cdot\|\delta J_k\|\sum_{m=1}^{N}\frac{(\mathbf{e}^{(j;n)},\ A_k\mathbf{e}^{(j;n)})^{1/2}}{(\mathbf{e}^{(m;n)},\ A_k\mathbf{e}^{(m;n)})^{1/2}}\cdot(\mathbf{e}^{(m;k)},\ \mathbf{e}^{(i;n)}) \\
&\equiv \mathrm{O}(1)\cdot\|\delta J_k\|\sum_{m=1}^{N}\frac{(\mathbf{e}^{(j;n)},\ A_k\mathbf{e}^{(j;n)})^{1/2}}{(\mathbf{e}^{(m;n)},\ A_k\mathbf{e}^{(m;n)})^{1/2}}\cdot\beta_{im},
\end{aligned}
\tag{B.2}
$$

where

$$\beta_{im} \equiv (\mathbf{e}^{(i;n)}, \mathbf{e}^{(m;k)}). \tag{B.3}$$

Expanding $\mathbf{e}^{(j;n)}$ in the series of the eigenvectors of A_k:

$$\mathbf{e}^{(j;n)} = \sum_{m=1}^{N}(\mathbf{e}^{(j;n)}, \mathbf{e}^{(m;k)})\mathbf{e}^{(m;k)} \equiv \sum_{m=1}^{N}\beta_{jm}\mathbf{e}^{(m;k)}$$

and using orthogonality of the latter, we obtain

$$(\mathbf{e}^{(j;n)},\ A_k\mathbf{e}^{(j;n)}) = \sum_{m=1}^{N}\beta_{jm}^2\alpha_{m;k}. \tag{B.4}$$

These values of β_{jm} can be estimated by means of (4.5) (note that in this expression we should use the eigenvalues, corresponding to the *least* between n and k, i.e. $\gamma_{j;\min(n,k)} = \gamma_{j;k}$):

$$
\beta_{jm} = \begin{cases}
\mathrm{O}(1)\sqrt{\gamma_{j;k}/\gamma_{m;k}} & \text{for } j > m, \\
\mathrm{O}(1)\left(\sum_{\ell<i}\dfrac{\gamma_{m;k}}{\gamma_{\ell;k}} + \sum_{\ell>m}\dfrac{\gamma_{\ell;k}}{\gamma_{m;k}}\right) & \text{for } j = m, \\
\mathrm{O}(1)\sqrt{\gamma_{m;k}/\gamma_{j;k}} & \text{for } j < m,
\end{cases}
$$

or, recalling that $\alpha_{m;k} = 1/\gamma_{m;k}$,

$$\beta_{jm} = \begin{cases} O(1)\sqrt{\alpha_{m;k}/\alpha_{j;k}} & \text{for } j > m, \\ O(1)\left(\sum_{\ell<m} \frac{\alpha_{\ell;k}}{\alpha_{m;k}} + \sum_{\ell>m} \frac{\alpha_{m;k}}{\alpha_{\ell;k}}\right) & \text{for } j = m, \\ O(1)\sqrt{\alpha_{j;k}/\alpha_{m;k}} & \text{for } j < m. \end{cases} \tag{B.5}$$

Therefore,

$$\begin{aligned}(\mathbf{e}^{(j;n)}, A_k\mathbf{e}^{(j;n)}) &= \sum_{m=1}^{N} \beta_{jm}^2 \alpha_{m;k} \\ &= \sum_{m=1}^{j-1} \beta_{jm}^2 \alpha_{m;k} + \beta_{jj}^2 \alpha_{j;k} + \sum_{m=j+1}^{N} \beta_{jm}^2 \alpha_{m;k} \\ &= \alpha_{j;k} \cdot \left(\sum_{m=1}^{j-1} O(1) \left(\frac{\alpha_{m;k}}{\alpha_{j;k}}\right)^2 + O(1) \left(\sum_{\ell<j} \frac{\alpha_{\ell;k}}{\alpha_{j;k}} + \sum_{\ell>j} \frac{\alpha_{j;k}}{\alpha_{\ell;k}} \right)^2 + \sum_{m=j+1}^{N} O(1) \right).\end{aligned}$$

Since $\alpha_{i;k} \leq \alpha_{i';k}$ for $i \leq i'$, all the fractions here cannot exceed 1, and so

$$(\mathbf{e}^{(j;n)}, A_k\mathbf{e}^{(j;n)}) \leq O(1) \cdot \alpha_{j;k}.$$

Similarly we can derive the estimate from below:

$$(\mathbf{e}^{(j;n)}, A_k\mathbf{e}^{(j;n)}) = \sum_{m=1}^{N} \beta_{jm}^2 \alpha_{m;k} \geq \sum_{m=j+1}^{N} \beta_{jm}^2 \alpha_{m;k} = \alpha_{j;k} \sum_{m=j+1}^{N} O(1),$$

so finally

$$O(1) \cdot \alpha_{j;k} \leq (\mathbf{e}^{(j;n)}, A_k\mathbf{e}^{(j;n)}) \leq O(1) \cdot \alpha_{j;k}.$$

Substituting this inequality together with (B.5) in (B.2), one obtains

$$\begin{aligned}&|(Df^k(x_{-k})\delta J_k J_{k-1} \cdots J_0 \mathbf{e}^{(j;n)}, \mathbf{e}^{(i;n)})| \\ &\leq O(1) \cdot \|\delta J_k\| \sum_{m=1}^{N} \beta_{im} \sqrt{\frac{\alpha_{j;k}}{\alpha_{m;k}}} \\ &= O(1) \cdot \|\delta J_k\| \cdot \sqrt{\frac{\alpha_{j;k}}{\alpha_{i;k}}} \left(\sum_{m=1}^{i-1} O(1) + O(1) \left(\sum_{\ell<i} \frac{\alpha_{\ell;k}}{\alpha_{i;k}} + \sum_{\ell>i} \frac{\alpha_{i;k}}{\alpha_{\ell;k}} \right) + \sum_{m=i+1}^{N} O(1) \frac{\alpha_{i;k}}{\alpha_{m;k}} \right).\end{aligned}$$

Again we note that all the fractions here cannot exceed 1, so

$$|(Df^k(x_{-k})\delta J_k J_{k-1} \cdots J_0 \mathbf{e}^{(j;n)}, \mathbf{e}^{(i;n)})| \leq O(1) \cdot \|\delta J_k\| \cdot \sqrt{\frac{\alpha_{j;k}}{\alpha_{i;k}}}.$$

The deviation of the Jacobian satisfies

$$\delta J_k \equiv \partial^2 f^{-1}(f^{-k}(x)) Df^{-k}(x)\delta x, \tag{B.6}$$

where $\partial^2 f^{-1}$ is the second derivative, and if it is bounded,

$$\|\delta J_k\| \sim \|Df^{-k}(x)\| \cdot \|\delta x\| \leq O(1) \cdot \|\delta x\| \cdot \sqrt{\alpha_{N;k}},$$

because the growth of the norm of the matrix is determined by its largest eigenvalue $\alpha_{N;k}$ (associated with the *smallest* Lyapunov exponent λ_N). So finally

$$|(Df^k(x_{-k})\delta J_k J_{k-1}\cdots J_0\mathbf{e}^{(j;n)},\ \mathbf{e}^{(i;n)})| \le \mathrm{O}(1)\cdot\|\delta x\|\sqrt{\frac{\alpha_{N;k}\alpha_{j;k}}{\alpha_{i;k}}} \tag{B.7}$$

Remark B.1. In case when the system in N_1-dimensional phase space possesses an N_2-dimensional stable invariant manifold containing its attractor, in the above estimate we should take $N = N_2$, because the dynamics on the attractor is completely determined by N_2 first Lyapunov exponents; the others only control convergence towards the attractor and so are unessential in the stationary regime.

That can be easily understood from the following example. Consider the three-dimensional dynamical system (i.e., $N_1 = 3$):

$$x_{n+1} = 1 - a\,|x_n| + y_n, \qquad y_{n+1} = bx_n, \qquad z_{n+1} = \kappa z_n.$$

The first two equations form the Lozy map Φ; and the third one is "self-consistent". The matrix $D\mathbf{f}$ is

$$D\mathbf{f}(x,y,z) = \begin{pmatrix} D\Phi(x,y) & 0 \\ 0 & \kappa \end{pmatrix},$$

so

$$[D\mathbf{f}^{-n}]^* D\mathbf{f}^{-n} = \begin{pmatrix} [D\Phi^{-n}]^* D\Phi^{-n} & 0 \\ 0 & \kappa^{-2n} \end{pmatrix}.$$

Obviously, *one* of its eigenvalues is κ^{-2n}; for $\kappa \ll 1$ it is the *greatest* one and so $\alpha_{3;n} = \kappa^{-2n}$. The attractor is the invariant manifold $z = 0$, so the displacement $\delta\mathbf{x} \equiv \{\delta x, \delta y, \delta z\} = \{\delta x, \delta y, 0\}$ and $D\mathbf{f}^{-k}(\mathbf{x})\delta\mathbf{x}$ in (B.6) is

$$D\mathbf{f}^{-k}(\mathbf{x})\delta\mathbf{x} = \left\{ D\Phi^k\begin{pmatrix}\delta x \\ \delta y\end{pmatrix}, 0\right\}.$$

Evidently, it grows not like $\sqrt{\alpha_{3;k}}$, but like $\sqrt{\alpha_{2;k}}$, where $\alpha_{2;k}$ corresponds to the smallest Lyapunov exponent of the Lozy map Φ; i.e. the smallest Lyapunov exponent among those describing the growth of displacements *along the attractor*. So in this example we should take $N = 2$, not $N = 3$.

Quite similarly (suffice it to exchange i with j),

$$|(Df^k(x_{-k})\delta J_k J_{k-1}\cdots J_0\mathbf{e}^{(i;n)},\ \mathbf{e}^{(j;n)})| \le \mathrm{O}(1)\cdot\|\delta x\|\sqrt{\frac{\alpha_{N;k}\alpha_{i;k}}{\alpha_{j;k}}}. \tag{B.8}$$

B.2. *Estimation of the inner product* $(Df^k(x_{-k})\delta J_k J_{k-1}\cdots J_0\mathbf{e}^{(j;n)},\ \mathbf{e}^{(i;n)})$ *for small* k

For small k the asymptotics $\alpha_{m;k} \approx \exp(-2k\lambda_m)$ that Section B.1 was based upon is no longer valid. But the natural assumption that both Df and Df^{-1} are bounded immediately gives that for $k = \mathrm{O}(1)$

$$|(Df^k(x_{-k})\delta J_k J_{k-1}\cdots J_0\mathbf{e}^{(j;n)},\ \mathbf{e}^{(i;n)})| \le \mathrm{O}(1)\cdot\|\delta x\|,$$

and similarly

$$|(Df^k(x_{-k})\delta J_k J_{k-1}\cdots J_0\mathbf{e}^{(i;n)},\ \mathbf{e}^{(j;n)})| \le \mathrm{O}(1)\cdot\|\delta x\|.$$

This coincides with what the estimates (B.7) and (B.8) would have become *if* they had been true for such small k. Therefore, we can use them *for any* value of k.

B.3. Deviations of the stationary Lyapunov vectors $\delta \mathbf{e}^{(i:n)}$

Substituting (B.7) and (B.8) in (B.1) one obtains

$$|(\mathbf{e}^{(j:n)}, \delta A_n \mathbf{e}^{(i:n)})| \leq \mathrm{O}(1) \cdot \|\delta x\| \cdot \sum_{k=0}^{n} \left[\alpha_{i:n} \sqrt{\frac{\alpha_{N:k}\alpha_{j:k}}{\alpha_{i:k}}} + \alpha_{j:n} \sqrt{\frac{\alpha_{N:k}\alpha_{i:k}}{\alpha_{j:k}}} \right],$$

which together with expansion (5.2) gives

$$\begin{aligned} \|\delta \mathbf{e}^{(i:n)}\| &\leq \mathrm{O}(1) \cdot \|\delta x\| \cdot \left(\sum_{j \neq i} \frac{\alpha_{i:n}}{|\alpha_{i:n} - \alpha_{j:n}|} \sum_{k=0}^{n} \sqrt{\frac{\alpha_{N:k}\alpha_{j:k}}{\alpha_{i:k}}} + \sum_{j \neq i} \frac{\alpha_{j:n}}{|\alpha_{i:n} - \alpha_{j:n}|} \sum_{k=0}^{n} \sqrt{\frac{\alpha_{N:k}\alpha_{i:k}}{\alpha_{j:k}}} \right) \\ &= \mathrm{O}(1) \cdot \|\delta x\| \cdot \left\{ \sum_{j<i} \frac{1}{|1 - \alpha_{j:n}/\alpha_{i:n}|} \sum_{k=0}^{n} \sqrt{\frac{\alpha_{N:k}\alpha_{j:k}}{\alpha_{i:k}}} \left(1 + \frac{\alpha_{j:n}}{\alpha_{i:n}} \frac{\alpha_{i:k}}{\alpha_{j:k}} \right) \right. \\ &\quad \left. + \sum_{j>i} \frac{1}{|1 - \alpha_{i:n}/\alpha_{j:n}|} \sum_{k=0}^{n} \sqrt{\frac{\alpha_{N:k}\alpha_{i:k}}{\alpha_{j:k}}} \left(1 + \frac{\alpha_{i:n}}{\alpha_{j:n}} \frac{\alpha_{j:k}}{\alpha_{i:k}} \right) \right\}. \end{aligned} \tag{B.9}$$

For any *finite* n the deviation of the stationary vectors is obviously finite; so actually we are only interested in its growth when $n \to \infty$. In this case the eigenvalues differ drastically, and so for n large enough

$$\frac{1}{|1 - \alpha_{j:n}/\alpha_{i:n}|} \leq 2, \quad j < i, \qquad \frac{1}{|1 - \alpha_{i:n}/\alpha_{j:n}|} \leq 2, \quad j > i.$$

Thus

$$\|\delta \mathbf{e}^{(i:n)}\| \leq \mathrm{O}(1) \cdot \|\delta x\| \cdot \left\{ \sum_{j<i} \sum_{k=0}^{n} \sqrt{\frac{\alpha_{N:k}\alpha_{j:k}}{\alpha_{i:k}}} \left(1 + \frac{\alpha_{j:n}}{\alpha_{i:n}} \frac{\alpha_{i:k}}{\alpha_{j:k}} \right) + \sum_{j>i} \sum_{k=0}^{n} \sqrt{\frac{\alpha_{N:k}\alpha_{i:k}}{\alpha_{j:k}}} \left(1 + \frac{\alpha_{i:n}}{\alpha_{j:n}} \frac{\alpha_{j:k}}{\alpha_{i:k}} \right) \right\}.$$

Moreover, for large n

$$1 + \frac{\alpha_{j:n}}{\alpha_{i:n}} \frac{\alpha_{i:k}}{\alpha_{j:k}} \approx 1 + \exp(2(n-k)(\lambda_i - \lambda_j)), \qquad 1 + \frac{\alpha_{i:n}}{\alpha_{j:n}} \frac{\alpha_{j:k}}{\alpha_{i:k}} \approx 1 + \exp(2(n-k)(\lambda_j - \lambda_i)),$$

and since $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$, for n large enough, one has

$$1 + \frac{\alpha_{j:n}}{\alpha_{i:n}} \frac{\alpha_{i:k}}{\alpha_{j:k}} \leq 2, \quad j < i, \qquad 1 + \frac{\alpha_{i:n}}{\alpha_{j:n}} \frac{\alpha_{j:k}}{\alpha_{i:k}} \leq 2, \quad j > i.$$

Substituting these estimates in (B.9), we arrive at

$$\|\delta \mathbf{e}^{(i:n)}\| \leq \mathrm{O}(1) \cdot \|\delta x\| \cdot \left\{ \sum_{j<i} \sum_{k=0}^{n} \sqrt{\frac{\alpha_{N:k}\alpha_{j:k}}{\alpha_{i:k}}} + \sum_{j>i} \sum_{k=0}^{n} \sqrt{\frac{\alpha_{N:k}\alpha_{i:k}}{\alpha_{j:k}}} \right\}. \tag{B.10}$$

The eigenvalues α have the exponential asymptotics $\alpha_{i:k}^{-1/k} \xrightarrow{k \to \infty} \exp(2\lambda_i)$, which readily implies that

$$\sqrt{\frac{\alpha_{N:k}\alpha_{j:k}}{\alpha_{i:k}}} \leq \exp[k \cdot (\lambda_i - \lambda_j - \lambda_N + \psi_k)],$$
$$\sqrt{\frac{\alpha_{N:k}\alpha_{i:k}}{\alpha_{j:k}}} \leq \exp[k \cdot (\lambda_j - \lambda_i - \lambda_N + \psi_k)]$$

for some function $\psi_k \xrightarrow{k\to\infty} +0$. Applying these estimates to (B.10), one obtains (5.3)

$$\|\delta \mathbf{e}^{(i;n)}\| \le \mathrm{O}(1)\cdot\|\delta x\| \left\{ \sum_{j<i}\sum_{k=0}^{n} \exp[k(\lambda_i - \lambda_j - \lambda_N + \psi_k)] + \sum_{j>i}\sum_{k=0}^{n} \exp[k(\lambda_j - \lambda_i - \lambda_N + \psi_k)] \right\}.$$

This formula is only valid for "typical" points x. It may cease to hold at exceptional points, see Remark 1.

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