Analytical Calculation of Bulk Velocity Inside a Sphere given a Tangential Surface Velocity

Shashank kumar Roy

August 13, 2018

1 Calculation of the bulk velocity

A velocity field can be expanded in vector spherical harmonics as follows :

$$\mathbf{V} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(V_{lm}^{Y}(r)\mathbf{Y}_{lm} + V_{lm}^{\psi}(r)\mathbf{\Psi}_{lm} + V_{lm}^{\phi}(r)\mathbf{\Phi}_{lm} \right) \tag{1}$$

In more compact form,we can write them as:

$$\mathbf{V} = \mathbf{V}^{Y} + \mathbf{V}^{\psi} + \mathbf{V}^{\phi} \tag{2}$$

Where,

$$\mathbf{V}^{Y} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} V_{lm}^{Y}(r)\mathbf{Y}_{lm} \tag{3}$$

$$\mathbf{V}^{\psi} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} V_{lm}^{\psi}(r)\mathbf{\Psi}_{lm} \tag{4}$$

$$\mathbf{V}^{\phi} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} V_{lm}^{\phi}(r)\mathbf{\Phi}_{lm} \tag{5}$$

We take vector Laplacian of this form of the velocity with the rules that ,

$$\Delta\left(v(r)\mathbf{Z}_{lm}\right) = \left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial V}{\partial r}\right)\mathbf{Z}_{lm} + v(r)\Delta\mathbf{Z}_{lm} \tag{6}$$

$$\mathbf{Z}_{lm} = \mathbf{Y}_{lm}, \mathbf{\Phi}_{lm}, \mathbf{\Psi}_{lm} \tag{7}$$

where Z can take the values

$$\mathbf{Z}_{lm} = \mathbf{Y}_{lm}, \mathbf{\Psi}_{lm}, \mathbf{\Phi}_{lm} \tag{8}$$

$$\Delta\mathbf{Y}_{lm} = -\frac{1}{r^2}(2 + l(l+1))\mathbf{Y}_{lm} + \frac{2}{r^2}\mathbf{\Psi}_{lm} \tag{9}$$

$$\Delta\mathbf{\Psi}_{lm} = \frac{2}{r^2}l(l+1)\mathbf{Y}_{lm} - \frac{1}{r^2}l(l+1)\mathbf{\Psi}_{lm} \tag{10}$$

$$\Delta\mathbf{\Phi}_{lm} = -\frac{1}{r^2}l(l+1)\mathbf{\Phi}_{lm} \tag{11}$$

Thus operating the vector laplacian on the velocity,we get

$$\Delta \mathbf{V} = \sum_{z=r,\psi,\phi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial v^z}{\partial r} \right) \mathbf{Z}_{lm} + v(r) \Delta \mathbf{Z}_{lm} \tag{12}$$

We can use compact notation to simplify further as follows: Let

$$F^z = \left(\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial V^z}{\partial r} \right) \tag{13}$$

Using the identities (5),(6) and (7) in equation ,we seperate the cofficients of the three VSH as,

$$(\boldsymbol{\Delta}\mathbf{V})^{Y}_{lm} = F^{Y} - \frac{2 + (l(l+1)}{r^2} V^{Y}_{lm} + \frac{2(l(l+1))}{r^2} V^{\psi}_{lm} = P_{lm} \tag{14}$$

$$(\boldsymbol{\Delta}\mathbf{V})^{\psi}_{lm} = F^{\psi} + \frac{2}{r^2} V^{Y}_{lm} - \frac{(l(l+1))}{r^2} V^{\psi}_{lm} = Q_{lm} \tag{15}$$

$$(\boldsymbol{\Delta}\mathbf{V})^{\phi}_{lm} = F^{\phi} - \frac{(l(l+1))}{r^2} V^{\phi}_{lm} = R_{lm} \tag{16}$$

Expressing the laplacian of the velocity in compact form as:

$$(\Delta \mathbf{V})_{lm} = P_{lm} \mathbf{Y}_{lm} + Q_{lm} \boldsymbol{\Psi}_{lm} + R_{lm} \boldsymbol{\Phi}_{lm} \tag{17}$$

Now , taking curl of the laplacian of the velocity using the standard identitys , we have :

$$\nabla \times (P \mathbf{Y}_{lm}) = -\frac{1}{r} P \boldsymbol{\Phi}_{lm} \tag{18}$$

$$\nabla \times (Q \boldsymbol{\Psi}_{lm}) = \left(\frac{\mathrm{d}Q}{\mathrm{d}r} + \frac{1}{r} Q \right) \boldsymbol{\Phi}_{lm} \tag{19}$$

$$\nabla \times (R \boldsymbol{\Phi}_{lm}) = -\frac{l(l+1)}{r} R \mathbf{Y}_{lm} - \left(\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{1}{r} R \right) \boldsymbol{\Psi}_{lm} \tag{20}$$

Thus we have the curl of the laplacian as:

$$\nabla \times (\nabla^2 \mathbf{V})_{lm} = -\frac{l(l+1)}{r} R \mathbf{Y}_{lm} - \left(\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{1}{r} R \right) \boldsymbol{\Psi}_{lm} + \left(-\frac{1}{r} P + \frac{\mathrm{d}Q}{\mathrm{d}r} + \frac{1}{r} Q \right) \boldsymbol{\Phi}_{lm} \tag{21}$$

From the Orthogonality and linear Independence of VSH's,each term in the beackets must be identicaly zero . Thus we get the differential equations as :

$$\frac{l(l+1)}{r} R = 0 \tag{22}$$

$$\left(\frac{\mathrm{d}R}{\mathrm{d}r} + \frac{1}{r} R \right) = 0 \tag{23}$$

$$\left(-\frac{1}{r} P + \frac{\mathrm{d}Q}{\mathrm{d}r} + \frac{1}{r} Q \right) = 0 \tag{24}$$

We can now expand the two equations and removing the redundant equations , we get

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial V_{lm}^{\phi}}{\partial r}\right)-\frac{(\ell(\ell+1))}{r^2}V_{lm}^{\phi}=0 \tag{25}$$

$$-\frac{1}{r}P+\frac{\mathrm{d}Q}{\mathrm{d}r}+\frac{1}{r}Q=0 \tag{26}$$

The general Solution to equation(23) is:

$$V_{lm}^{\phi}=A_m^l r^{\ell}+B_m^l r^{-(\ell+1)} \tag{27}$$

With the condition that at $r=0$, $V_{lm}^{\phi}=0$,$B_m^l=0$.Hence the general solution to $\mathbf{V}^{\phi}$ is :

$$\mathbf{V}^{\phi}=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}A_m^l r^{\ell}\mathbf{\Phi}_{lm} \tag{28}$$

Equation (26) can be expanded as ;

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r}\left(F^{\psi}+\frac{2}{r^2}V^{Y}-\frac{\ell(\ell+1)}{r^2}V^{\psi}\right)\\&+\frac{1}{r}\left(F^{\psi}+\frac{2}{r^2}V^{Y}-\frac{\ell(\ell+1)}{r^2}V^{\psi}\right)-\frac{1}{r}\left(F^{r}-\frac{2+(\ell(\ell+1)}{r^2}V^{Y}+\frac{2\ell(\ell+1)}{r^2}V^{\psi}\right)=0\end{aligned} \tag{29}$$

We have simplified with the assumption that the flow is incompressible

$$\nabla\cdot\mathbf{V}=0 \tag{30}$$

$$\nabla\cdot\mathbf{V}=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{\mathrm{d}V_{lm}^{Y}}{\mathrm{d}r}+\frac{2}{r}V_{lm}^{Y}-\frac{l(l+1)}{r}V_{lm}^{\psi}\right)\mathbf{Y}_{lm} \tag{31}$$

$$\Rightarrow\quad\left(\frac{\mathrm{d}V_{lm}^{Y}}{\mathrm{d}r}+\frac{2}{r}V_{lm}^{Y}-\frac{l(l+1)}{r}V_{lm}^{\psi}\right)=0 \tag{32}$$

The equation (28) can be simplified as follows:

$$\frac{r^3\partial^3}{\partial r^3}V_{lm}^{\psi}+\frac{3r^2\partial^2}{\partial r^2}V_{lm}^{\psi}-\frac{l(l+1)r\partial}{\partial r}V_{lm}^{\psi}-\frac{r^2\partial^2}{\partial r^2}V_{lm}^{Y}+\ell(\ell+1)\left(V_{lm}^{Y}-V_{lm}^{\psi}\right)=0 \tag{33}$$

$$\tag{34}$$

For the velocity component V_{lm}^{ϕ} and V_{lm}^{r}.Equation (31) and (32) are coupled differential equations and reduce to euler form of differentail equation.The simplified equations are solvable. Using the incompressibility condition , we get the ordinary differential equation below.

$$\frac{r^4\partial^4}{\partial r^4}V_{lm}^{Y}+\frac{8r^3\partial^3}{\partial r^3}V_{lm}^{Y}-\frac{12-2\ell(\ell+1)r^2\partial^2}{\partial r^2}V_{lm}^{Y}-\frac{4\ell(\ell+1)r\partial}{\partial r}V_{lm}^{Y}+((\ell(\ell+1))^2-2\ell(\ell+1))V_{lm}^{Y}=0 \tag{35}$$

The above is an Euler-Lagrange type differential equation and can be solved by substituting $r=e^{z}$ and then transforming the ode into the following form:

$$\frac{\partial^4}{\partial z^4}V_{lm}^{Y}+\frac{2\partial^3}{\partial z^3}V_{lm}^{Y}+\frac{-2-\ell(\ell+1)\partial^2}{\partial z^2}V_{lm}^{Y}+\frac{(-1-3\ell(\ell+1))\partial}{\partial z}V_{lm}^{Y}+((\ell(\ell+1))^2-2\ell(\ell+1))V_{lm}^{Y}=0 \tag{36}$$

The solution to this has the form:

$$V_{lm}^{Y} = C_1^{\ell} r^{\ell-1} + C_2^{\ell} r^{\ell+1} + C_3^{\ell} r^{-\ell} + C_4^{\ell} r^{-(\ell+2)} \tag{37}$$

The Boundary condition inside the cell i.e finite at centre and zero at the surface does not permit solutions which diverge at the centre. The cofficients C_1^0 and C_4^0 must vanish and C_3^{ℓ} and C_4^{ℓ} must vanish for all $\ell > 1$. The solution thus retained must be of the form

$$V_{lm}^{Y} = \begin{cases} C_2^0 r + C_3^0, \ell = 0 \\ C_1^{\ell} r^{\ell-1} + C_2^{\ell} r^{\ell+1}, \ell > 0 \end{cases}$$

At the surface of the cell $r = R$,the radial velocity must be zero.This condition must be satisfied individually for every mode present i.e. for each value of ℓ. Using this condition,we get for $\ell = 0$ mode $C_3^0 = -C_2^0 R$ and for $\ell > 0$ modes ,$C_1^l = -C_2^l R^2$

$$\mathbf{V}^{Y} = C_2^0 (r - R) \mathbf{Y}_{00} + \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} C_2^l \left(r^{\ell+1} - R^2 r^{\ell-1} \right) \mathbf{Y}_{lm} \tag{38}$$

Now from the incompressibility condition,we have :

$$\frac{\mathrm{d}V_{lm}^{Y}}{\mathrm{d}r} + \frac{2}{r} V_{lm}^{Y} - \frac{\ell(\ell+1)}{r} V_{lm}^{\psi} = 0 \tag{39}$$

This condition can be used to dtermine V_{lm}^{r} along with the condition that at r= 0, $V_{lm}^{\psi} = 0$,we find that only $\ell >= 1$ and higher modes exist.

$$\mathbf{V}^{\psi} = \sum_{l=1}^{\infty} \sum_{m=-\ell}^{\ell} C_m^{\ell} \left(-\frac{R^2 r^{\ell-1}}{\ell} + \frac{r^{\ell+1}(\ell+3)}{\ell(\ell+1)} \right) \mathbf{\Psi}_{lm} \tag{40}$$

Also the equation(42) changes as

$$\mathbf{V}^{Y} = \sum_{l=1}^{\infty} \sum_{m=l}^{m=-l} C_2^l \left(r^{\ell+1} - R^2 r^{\ell-1} \right) \mathbf{Y}_{lm} \tag{41}$$

Given any cortical flow of the form:

$$\mathbf{V}_s = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left(v_{lm}^{\psi} \mathbf{\Psi}_{lm} + v_{lm}^{\phi} \mathbf{\Phi}_{lm} \right) \tag{42}$$

Now,given that we have the cortical Velocity we can determine the cofficients C_m^l and A_m^l by matching the general velocity at $r = R$ for $\ell >= 1$

$$A_m^l = \frac{v_{lm}^{\phi}}{R^{\ell}} \tag{43}$$

$$C_m^l = \frac{\ell(\ell+1) v_{lm}^{\psi}}{2R^{\ell+1}} \tag{44}$$

Once we have determined these cofficients,we can build the bulk flow inside the sphere. Comparing equation (3),(4),(5) with (28),(40),(41),we get

$$V^Y_{lm} = C^\ell_m \left(r^{\ell+1} - R^2 r^{\ell-1}\right) \tag{45}$$

$$V^\psi_{lm} = C^\ell_m \left(-\frac{R^2 r^{\ell-1}}{\ell} + \frac{r^{\ell+1}(\ell+3)}{\ell(\ell+1)}\right) \tag{46}$$

$$V^\phi_{lm} = A^\ell_m r^\ell \tag{47}$$

Also, We can expand $\mathbf{\Psi}_{lm}$ and $\mathbf{\Phi}_{lm}$ in terms of scalar Y_{lm} as

$$\mathbf{\Psi}_{lm} = \frac{\partial Y_{lm}}{\partial \theta}\hat{\theta} + \frac{\partial Y_{lm}}{\sin\theta \partial\phi}\hat{\phi} \tag{48}$$

$$\mathbf{\Phi}_{lm} = -\frac{\partial Y_{lm}}{\partial \theta}\hat{\phi} + \frac{\partial Y_{lm}}{\sin\theta \partial\phi}\hat{\theta} \tag{49}$$

Thus we can write the velocity back in terms of r, θ, ϕ basis as:

$$\mathbf{V} = \sum_{lm} \left(V^Y_{lm} Y_{lm}\right)\hat{r} + \left(V^\psi_{lm}\frac{\partial Y_{lm}}{\partial\theta} + \frac{V^\phi_{lm}\partial Y_{lm}}{\sin\theta\partial\phi}\right)\hat{\theta} + \left(V^\psi_{lm}\frac{\partial Y_{lm}}{\sin\theta\partial\phi} - \frac{V^\phi_{lm}\partial Y_{lm}}{\partial\phi}\right)\hat{\phi} \tag{50}$$

2 Calculation of Stress Tensor

We can compute the Stress generated by the flows inside the sphere by the expression :

$$[\sigma] = \nabla\mathbf{V} + (\nabla\mathbf{V})^{\mathbf{T}} \tag{51}$$

where

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{\partial}{r\theta} + \hat{\phi}\frac{\partial}{r\sin\theta\partial\phi} \tag{52}$$

The two simple cases where we can easily compute the independent components of the stress tensor are given below:

2.1 Case I : All $\mathbf{C}_{lm}$ are zero

In this case we have only V^ϕ_{lm} non-zeros. The Velocity field reduces to the form:

$$\mathbf{V} = \sum_{lm} \left(V^\phi_{lm}\frac{\partial Y_{lm}}{\sin\theta\partial\phi}\right)\hat{\theta} + \left(-\frac{V^\phi_{lm}\partial Y_{lm}}{\partial\theta}\right)\hat{\phi} \tag{53}$$

For this case now,we six independent components of the stress tensor for a particular mode (ℓm) of the velocity are

$$\sigma_{rr} = 0 \tag{54}$$

$$\sigma_{r\theta} = im\csc\theta Y_{lm}\left(\frac{-V^\phi_{lm}}{r} + \frac{\partial V^\phi_{lm}}{\partial r}\right) \tag{55}$$

$$\sigma_{r\phi} = m\cot^2\theta\left(Y_{lm} + e^{i\phi}\sqrt{(\ell-m)(\ell+m+1)}Y_{lm+1}\right)\left(\frac{V^\phi_{lm}}{r} - \frac{\partial V^\phi_{lm}}{\partial r}\right) \tag{56}$$

$$\sigma_{\theta\phi} = \left(\frac{V_{lm}^{\phi}}{r}\right)\Big(m\left(\cot^2\theta + \csc^2\theta\right)Y_{lm} + 2e^{-i\phi}m\cot\theta\sqrt{(\ell-m)(l+m+1)}Y_{lm+1} +$$
$$e^{-2i\phi}\sqrt{(\ell-m)(\ell-m-1)(\ell+m+1)(\ell+m+2)}Y_{lm+2} \quad (57)$$

$$\sigma_{\theta\theta} = -2im\csc\theta\left(-(-1+m)\cot\theta Y_{lm} - e^{-i\phi}(l-m)(l+m+1)Y_{lm+1}\right)\frac{V_{lm}^{\phi}}{r} \quad (58)$$
$$\sigma_{\phi\phi} = -2im\csc\theta\left(-(-1+m)\cot\theta Y_{lm} + e^{-i\phi}(l-m)(l+m+1)Y_{lm+1}\right)\frac{V_{lm}^{\phi}}{r} \quad (59)$$

2.2 Case II : All $\mathbf{A}_{lm}$ are zeros

In this case we have V_{lm}^{Y} and V_{lm}^{ψ} as non-zeros. The Velocity field reduces to the form:

$$\mathbf{V} = \sum_{lm}\left(V_{lm}^{Y}Y_{lm}\right)\hat{r} + \left(V_{lm}^{\psi}\frac{\partial Y_{lm}}{\partial\theta}\right)\hat{\theta} + \left(V_{lm}^{\psi}\frac{\partial Y_{lm}}{\sin\theta\partial\phi}\right)\hat{\phi} \quad (60)$$

For this case,the six independent components of the stress tensora for particular mode (ℓm) of the velocity are

$$\sigma_{rr} = 2Y_{lm}\frac{\partial V_{lm}^{Y}}{\partial r} \quad (61)$$
$$\sigma_{r\theta} = \left(m\cot\theta Y_{lm} + e^{-i\phi}\sqrt{(\ell-m)(l+m+1)}Y_{lm+1}\right)\left(\frac{V_{lm}^{Y} - V_{lm}^{\psi} + r\frac{V_{lm}^{\psi}}{\partial r}}{r}\right) \quad (62)$$
$$\sigma_{r\phi} = im\csc\theta Y_{lm}\left(\frac{V_{lm}^{Y} - V_{lm}^{\psi} + r\frac{\partial V_{lm}^{\psi}}{\partial r}}{r}\right) \quad (63)$$
$$\sigma_{\theta\phi} = \frac{2im\csc\theta}{r}\left((m-1)\cot\theta Y_{lm} + e^{-i\phi}\sqrt{(\ell-m)(l+m+1)}Y_{lm+1}\right)\frac{V_{lm}^{\psi}}{r} \quad (64)$$

$$\sigma_{\theta\theta} = \Big(Y_{lm}V_{lm}^{Y} + \Big(m(m\cot^2\theta - \csc^2\theta Y_{lm} + e^{-i\phi}(1+2m)\cot\theta\sqrt{(\ell-m)(\ell+m+1)}Y_{lm+1}$$
$$+ e^{-i2\phi}\sqrt{(\ell-m)(\ell-m-1)(\ell+m+3)(\ell+m+2)}Y_{lm+2}V_{lm}^{\psi} \quad (65)$$

$$\sigma_{\phi\phi} = \frac{2\csc\theta}{r}\left(\sin\theta Y_{lm}V_{lm}^{Y} + (m(\cos\theta\cot\theta - m\csc\theta)Y_{lm}\right.$$
$$+ e^{-i\phi}\cos\theta\sqrt{(\ell-m)(\ell+m+1)}Y_{lm+1}V_{lm}^{\psi} \quad (66)$$

In general,the stress tensor is a sum over all modes when both $\mathbf{C}_{lm}$ and $\mathbf{A}_{lm}$ are non-zero, stress tensor components would be the sum of the respective components as described in the above cases.