Module II

Transportation & Assignment Problems

Part 1 – Initial Solution by Northwest Corner Rule, Least Cost Method and VAM

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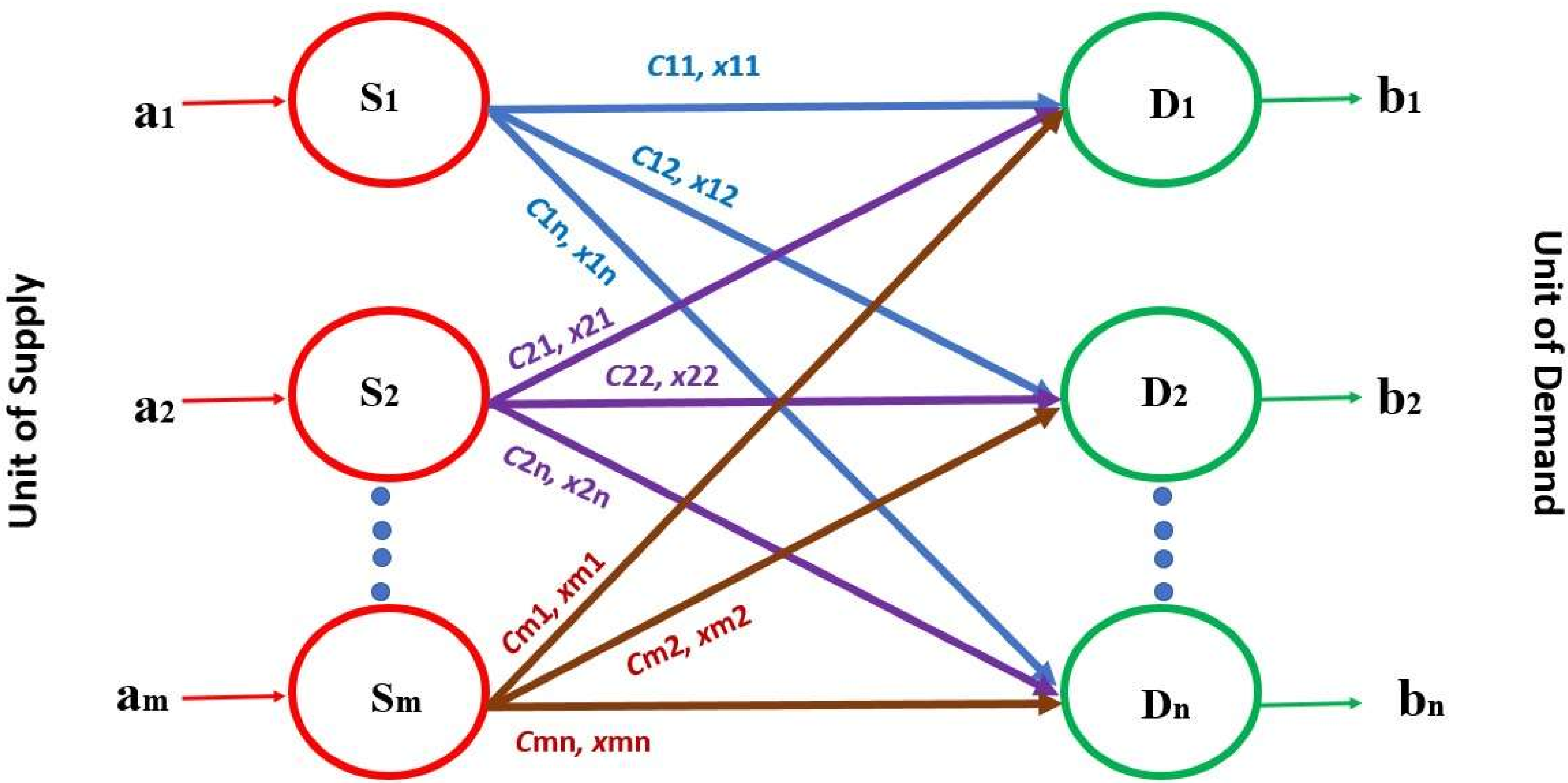
Transportation problem
- a special class of LPP dealing with the transportation of a commodity from m sources (supply) to n destinations (demand).



TRANSPORTATION PROBLEM

Here,

$a_i -$ the amount of supply at source S_i

$b_j -$ the amount of demand at destination D_j

$x_{ij} -$ the amount shipped from S_i to D_j

$c_{ij} -$ the transportation cost from i to j per unit

Transportation Problem as LPP:

$$\text{Minimize } Z = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1,2, \ldots m \, (supply)$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1,2, \ldots n \, (demand)$$

$$x_{ij} \geq 0.$$

Balanced Transportation Problem:

Total supply = Total demand

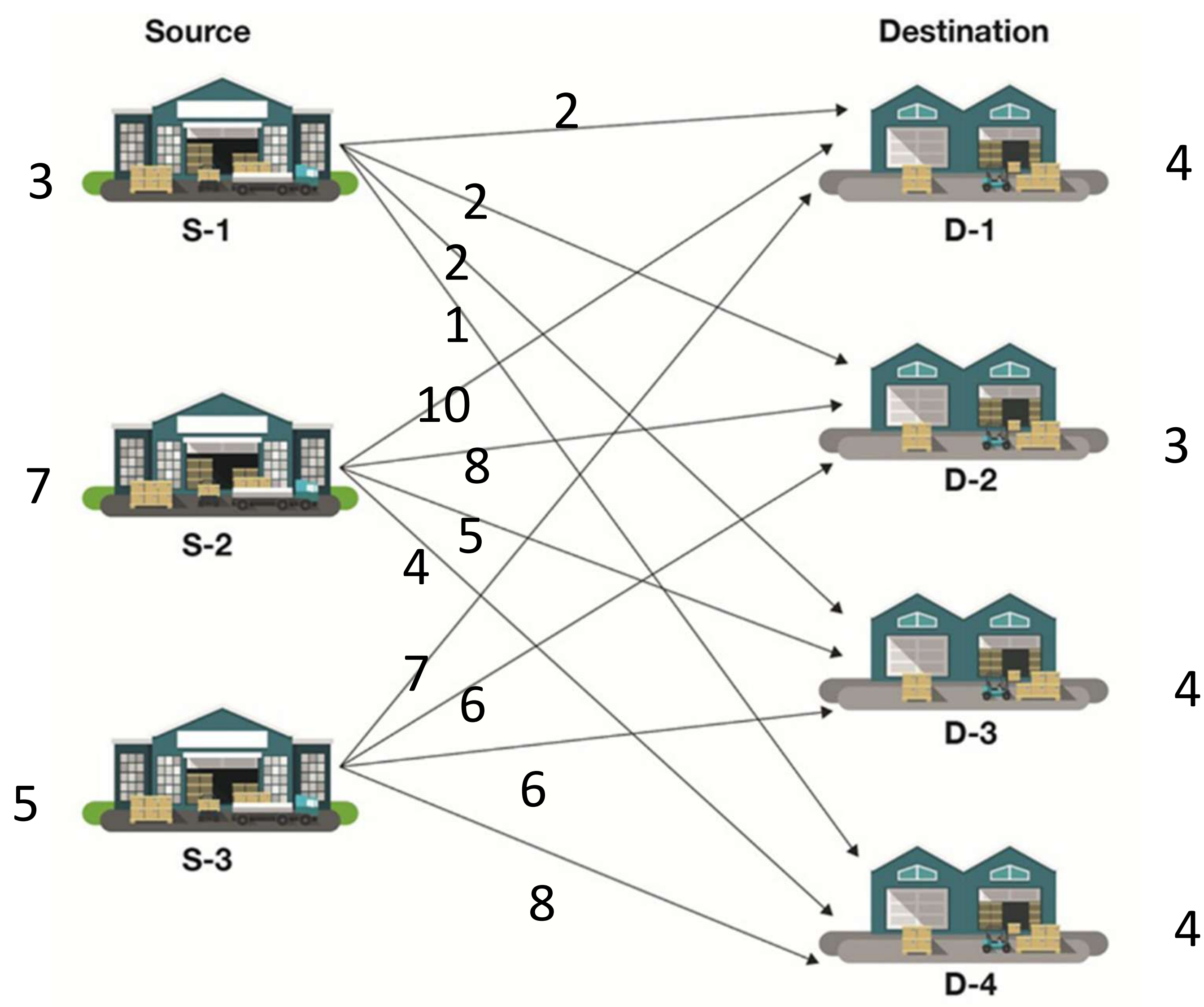
$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Unbalanced Transportation Problem:

Total supply ≠ Total demand

$$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

Assume that you are the owner of a sports equipment sales chain. Your products are manufactured at three factories, and you have to deliver them to four customers. After a survey, you found that the production capacity at each factory, the transportation cost to customers, and the demand amount at each customer are as shown in figure.



So, which of the transport routes would you choose to minimize the cost?

How to solve transportation problem?

Step 1: Identify the Initial Basic Feasible Solution

- North-west corner rule
- Least cost method
- Vogel’s approximation method (VAM)

Step 2: Find the optimal solution

- UV method
- Modified Distribution method (MODI)

Feasible solution

- A set of non-negative allocations x_{ij} which satisfies the row and column restrictions.

Basic Feasible Solution

- A feasible solution is said to be basic feasible solution if it contains no more than $(m + n - 1)$ non-negative allocations. No. of allocations $\leq (m+n-1)$

Optimum solution

- A feasible solution which minimizes the transportation cost is called the optimal solution.

Degenerate Basic Feasible Solution

- A feasible solution containing less than $(m + n - 1)$ allocations.

Non-degenerate Basic Feasible Solution

- A feasible solution containing exactly $(m + n - 1)$ allocations.

What is the significance of $(\mathbf{m} + \mathbf{n} - \mathbf{1})$ allocations?

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1,2, \dots m \ (supply)$$

$x_{11} + x_{12} = a_1$

$x_{21} + x_{22} = a_2$

$m=2, n=2$

		supply
x_{11}	x_{12}	a_1
x_{21}	x_{22}	a_2
demand b_1	b_2	

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1,2, \dots n \ (demand)$$

$x_{11} + x_{21} = b_1$

$x_{12} + x_{22} = b_2$

$m+n$ constraints

$m \times n$ variables

For the balanced TP,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Total supply = Total demand = λ (say)

$x_{11}+x_{12}+x_{21}+x_{22} = a_1+a_2$ (or) $= b_1+b_2$

② $\Rightarrow (x_{11}+x_{12}) + a_2 = a_1+a_2$

$\Rightarrow x_{11}+x_{12} = a_1$

Any one of the constraints can be obtained from the balanced TP condition.

m+n-1 remaining equations.

This makes one of the constraints redundant.
$\Rightarrow$ the maximum rank of the matrix is $m + n - 1$.
$\Rightarrow$ the no. of positive allocations = the no. of basic variables of a TP is atmost $m + n - 1$.

Notes:

(i) The necessary and sufficient condition for the existence of a feasible solution to a TP is

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

(ii) The solution of a TP is never unbounded.

Example 1:

Find the IBFS to the following transportation model:

	D_1	D_2	D_3	Supply
S_1	0	2	1	$a_1 =$ 6
S_2	2	1	5	a_2 7
S_3	2	4	3	$a_3 =$ 7
demand	$b_1 =$ 5	$b_2 =$ 5	$b_3 =$ 10	20

Compare the solutions of

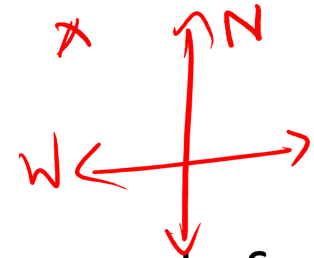
(i) North-west corner rule

(ii) Least cost method

(iii) Vogel's approximation method

$\sum a_i = \sum b_j$

Balanced Transportation Prob.



North-West Corner Rule:
Step 1:

The first assignment is made in the upper left corner (North-West corner) of the table. The maximum possible amount is allocated there, that is, $x_{11} = \min\{a_1, b_1\}$

(i) If $\min\{a_1, b_1\} = a_1$, delete the corresponding row and decrease b_1 by a_1.

(ii) If $\min\{a_1, b_1\} = b_1$, delete the corresponding column and decrease a_1 by b_1.

(iii) If $\min\{a_1, b_1\} = a_1 = b_1$, delete both row and column.

Step 2:

Move to the upper left corner cell in the reduced table and continue with the process followed in Step 1, until the basic feasible solution is reached.

North-West Corner rule

	D_1	D_2	D_3	supply
S_1	$\boxed{5}$ 0	$\boxed{1}$ 2	1	~~6~~ ~~1~~ 0
S_2	2	$\boxed{4}$ 1	$\boxed{3}$ 5	~~7~~ ~~3~~ 0
S_3	2	4	$\boxed{7}$ 3	~~7~~ 0
demand	~~5~~ 0	~~5~~ ~~4~~ 0	~~10~~ ~~7~~ 0	

Total cost $= 0\times5 + 2\times1 + 1\times4 + 5\times3 + 3\times7$

$= 42;$

All the allocations are $\geq 0 \Rightarrow$ Feasible Solution.

The no. of allocations : 5

$m+n-1 = 3+3-1 = 5$ $\Big]$ equal $\Rightarrow$ Basic Feasible Sol.

Non-Degenerate BFS.

Least Cost Method:

Step 1:

Allocate $\min\{a_i, b_j\}$ to the cell with the least cost value and delete the row or column accordingly.

Step 2:

Choose the next minimum cell in the reduced table and continue with the process followed in Step 1, until the basic feasible solution is reached.

Least Cost Method

	D1	D2	D3	Supply
S1	0 $\boxed{5}$	2	1 $\boxed{1}$	~~6~~ ~~1~~ 0
S2	2	1 $\boxed{5}$	5 $\boxed{2}$	~~7~~ ~~2~~ 0
S3	2	4	3 $\boxed{7}$	~~7~~ 0
demand	~~5~~ 0	~~5~~ 0	~~10~~ ~~9~~ ~~2~~ 0	

Total cost $= 0 \times 5 + 5 \times 1 + 1 \times 1 + 5 \times 2 + 7 \times 3$

$= 37$

No. of allocations $= 5 = m+n-1$

$\Rightarrow$ Non-degenerate BFS.

Vogel’s Approximation Method:

Step 1:

Calculate the difference between the smallest and the next smallest value in each row and column.

Step 2:

Identify the largest penalty row or column.

Step 3:

Identify the cell with the least cost in the selected row or column.

Step 4:

Allocate $\min\{a_i, b_j\}$ and delete the row or column accordingly.

Step 5:

Repeat the above steps for the reduced table until the basic feasible solution is reached.

					Row diff.			VAM
	0	2	1 (6)	~~6~~ 0	1-0=1	—	—	—
	2 (2)	1 (5)	5	~~7~~ ~~2~~ 0	2-1=1	2-1=1	5-2=3	—
	2 (3)	4	3 (4)	~~7~~ ~~4~~ 0	3-2=1	3-2=1	3-2=1	3-2=1
	~~5~~ ~~3~~ 0	~~5~~ 0	~~10~~ ~~4~~ 0					
col.diff	2-0=2	2-1=1	3-1=2					
	2-2=0	4-1=3	5-3=2					
	2-2=0	—	5-3=2					

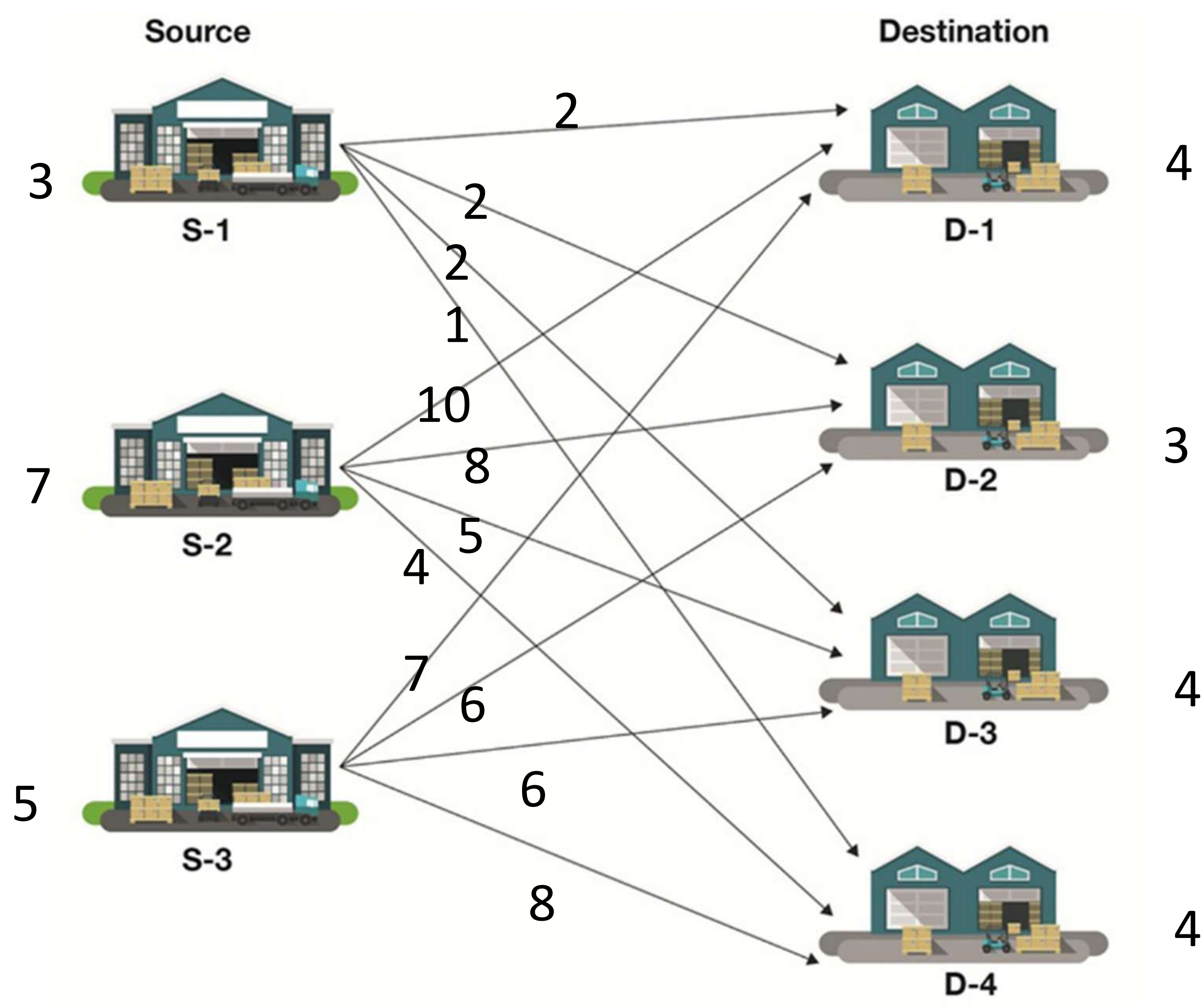
Total cost $= 1\times 6 + 2\times 2 + 1\times 5 + 2\times 3 + 3\times 4$

$= 33$

VAM gives better cost (least) among the three methods for IBFS.

No. of allocations $= 5 = m+n-1$

$\Rightarrow$ Non-degenerate BFS.



Example 2:

Find the IBFS for the transportation problem using

(i) North-West corner rule

(ii) Least cost method

(iii) Vogel's approximation method

	D_1	D_2	D_3	D_4	supply
S_1	2	2	2	1	3
S_2	10	8	5	4	7
S_3	7	6	6	8	5
demand	4	3	4	4	15

Total supply = Total demand $\Rightarrow$ Balanced TP.

North West Corner

	D_1	D_2	D_3	D_4	supply
S_1	2 $\boxed{3}$	2	2	1	$\cancel{3}$ 0
S_2	10 $\boxed{1}$	8 $\boxed{3}$	5 $\boxed{3}$	4	$\cancel{7}$ $\cancel{6}$ $\cancel{3}$ 0
S_3	7	6	6 $\boxed{1}$	8 $\boxed{4}$	$\cancel{5}$ $\cancel{4}$ 0
demand	$\cancel{4}$ $\cancel{1}$ 0	$\cancel{3}$ 0	$\cancel{4}$ $\cancel{1}$ 0	$\cancel{4}$ 0	

$$\text{Total cost} = 2\times3 + 10\times1 + 8\times3 + 5\times3 + 6\times1 + 8\times4 = 93$$

$$\left.\begin{array}{l} \text{No. of allocations} = 6 \\ m+n-1 = 3+4-1 = 6 \end{array}\right\} \text{equal}$$

Non-degenerate BFS.

Least Cost Method

2	2	2	1 $\boxed{3}$	~~3~~ 0
10 $\boxed{2}$	8	5 $\boxed{4}$	4 $\boxed{1}$	~~7~~ ~~6~~ ~~2~~ 0
7 $\boxed{2}$	6 $\boxed{3}$	6	8	~~5~~ ~~2~~ 0
~~4~~ ~~2~~ 0	~~3~~ 0	~~4~~ 0	~~4~~ ~~1~~ 0	

Total cost $= 1 \times 3 + 10 \times 2 + 5 \times 4 + 4 \times 1$
$+ 7 \times 2 + 6 \times 3 = 79$

No. of allocations $= 6 = m+n-1$

$\Rightarrow$ Non-Degenerate BFS.

VAM

2 [3]	2	2	1	~~3~~ 0	2-1=1	—	—	—
10	8	5 [3]	4 [4]	~~7~~ ~~3~~ 0	5-4=1	5-4=1	8-5=3	—
7 [1]	6 [3]	6 [1]	8	~~5~~ ~~4~~ ~~1~~	7-6=1	7-6=1	7-6=1	7-6=1
~~4~~ ~~1~~ 0	~~3~~ 0	~~4~~ ~~1~~ 0	~~4~~ 0					
7-2=5	6-2=4	5-2=3	4-1=3					
10-7=3	8-6=2	6-5=1	8-4=4					
10-7=3	8-6=2	6-5=1	—					
—	—	—	—					

Total cost $= 2\times 3 + 5\times 3 + 4\times 4$
$+ 7\times 1 + 6\times 3 + 6\times 1 = 68$

No. of allocations $= 6 = m+n-1$

$\Rightarrow$ Non-degenerate BFS.