PRICING PERPETUAL OPTIONS FOR JUMP PROCESSES

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ABSTRACT

We consider two models in which the logarithm of the price of an asset is a shifted compound Poisson process. Explicit results are obtained for prices and optimal exercise strategies of certain perpetual American options on the asset, in particular for the perpetual put option. In the first model in which the jumps of the asset price are upwards, the results are obtained by the martingale approach and the smooth junction condition. In the second model in which the jumps are downwards, we show that the value of the strategy corresponding to a constant option-exercise boundary satisfies a certain renewal equation. Then the optimal exercise strategy is obtained from the continuous junction condition. Furthermore, the same model can be used to price certain reset options. Finally, we show how the classical model of geometric Brownian motion can be obtained as a limit and also how it can be integrated in the two models.

1. THE PROBLEM

Let $S(t)$ be the price of an asset, for example, a stock, at time t, $t \geq 0$. We assume that the market is risk neutral; hence, the price of a security is the expectation of the discounted future payments. Under the assumption that the asset does not pay any dividends and that the risk-free instantaneous rate of interest r is a positive constant, the discounted asset price process $\{e^{-rt}S(t)\}$ is a martingale.

Let $U(t) = \ln S(t)$, $t \geq 0$. We suppose that $\{U(t), t \geq 0\}$ is a process with independent and stationary increments, starting with initial value $U(0) = u$. The classical stock price model is the special case in which $\{U(t)\}$ is a Wiener process, with drift $\mu = r - \frac{1}{2}\sigma^2$ and infinitesimal variance σ^2. Here we examine two models:

Model I: $\quad U(t) = u - ct + Z(t),$

Model II: $\quad U(t) = u + ct - Z(t).$

In both cases, $c > 0$ is constant, and $\{Z(t)\}$ is a compound Poisson process, specified by the Poisson parameter $\lambda > 0$ and the distribution of the jump amounts. To simplify notation, we suppose the latter is continuous with probability density $p(x)$, $x \geq 0$. In Model II, we make the additional assumption that $c > r$; without it there would be a possibility for arbitrage. (If $c \leq r$, we would have $S(t) \leq S(0)e^{rt}$, with strict inequality after the first jump. Hence by short-selling the stock and investing the proceeds $S(0)$ at the risk-free interest rate r, we could obtain a riskless profit.) Note that Model II resembles the model that is used for the surplus process in classical risk theory.

Our main goal is to evaluate a perpetual American option with payoff function $\Pi(s)$. Although the qualifier "American" is not necessary, it is added to emphasize that the option can be exercised at any time. If such an option is exercised at time T, the payoff is $\Pi(S(T))$. We restrict ourselves to options with payoff functions for which it is a priori clear that the optimal option-exercise strategies are *stopping times* of the form

$$T_L = \inf\{t{:}S(t) < L\}, \tag{1}$$

where the option-exercise boundary L is a positive constant. This is in particular the case for a *perpetual put option,* where

$$\Pi(s) = \max(K - s, 0) = (K - s)_+,$$

with K denoting the exercise price. The problem is first to find the value of the strategy T_L,

$$V(s; L) = E[e^{-rT_L}\, \Pi(S(T_L)) \mid S(0) = s], \quad s \geq L, \tag{2}$$

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and then to determine $\tilde{L}$, the optimal value of L, which maximizes $V(s; L)$. Hence $\tilde{L}$ is the *optimal option-exercise boundary*. The *price* of the option is

$$\begin{cases} V(S(0); \tilde{L}) & \text{if } S(0) \geq \tilde{L} \\ \Pi(S(0)) & \text{if } S(0) < \tilde{L} \end{cases}.$$

2. SOLUTION FOR MODEL I

Because the sample paths of the process $\{S(t)\}$ are jump-free downwards, we have $S(T_L) = L$, and hence (2) simplifies to

$$V(s; L) = E[e^{-rT_L} \mid S(0) = s]\ \Pi(L), \qquad s \geq L. \quad (3)$$

It remains to determine the expectation. For this purpose we consider a martingale of the form $\{e^{-rt}S(t)^{\xi}\}$. Because $e^{-rt}S(t)^{\xi} = e^{-rt+\xi U(t)}$ and $\{U(t),\ t \geq 0\}$ is a process with independent and stationary increments, the martingale condition for ξ is that

$$e^{-r}\ \mathrm{E}[e^{\xi U(1)} \mid U(0) = u] = e^{\xi u}, \quad (4)$$

or that

$$e^{-r-c\xi}\ \mathrm{E}[e^{\xi Z(1)}] = 1,$$

which leads to the condition

$$\lambda \left[\int_0^{\infty} e^{\xi x}\ p(x)dx - 1 \right] - r - c\xi = 0. \quad (5)$$

Because the expression on the left-hand side is a strictly convex function of ξ, this equation has at most two solutions. In fact, it has exactly two solutions: one is $\xi_1 = 1$, since $\{e^{-rt}S(t),\ t \geq 0\}$ is supposed to be a martingale; the other solution is negative, $\xi_2 = -R < 0$, since the expression on the left-hand side is a continuous function of ξ, with value $-r$ for $\xi = 0$ and tending to ∞ for $\xi \to -\infty$.

We note that the positive martingale $\{e^{-rt}S(t)^{-R}\}$ is bounded from above by the constant L^{-R} for $t < T_L$. Hence we can apply the *optional sampling theorem* to conclude that

$$\begin{aligned} s^{-R} &= \mathrm{E}[e^{-rT_L}\ S(T_L)^{-R} \mid S(0) = s] \\ &= \mathrm{E}[e^{-rT_L} \mid S(0) = s]L^{-R}. \end{aligned} \quad (6)$$

By combining (6) and (3), we obtain

$$V(s; L) = \left(\frac{L}{s}\right)^R \Pi(L), \qquad s \geq L. \quad (7)$$

Finally $\tilde{L}$ is the value of L that maximizes $L^R\Pi(L)$. The first-order condition is that

$$\frac{R}{\tilde{L}}\ \Pi(\tilde{L}) + \Pi'(\tilde{L}) = 0. \quad (8)$$

Moreover, by using (7) and (8) we see that

$$\left. \frac{\partial V(s; \tilde{L})}{\partial s} \right|_{s=\tilde{L}} = \Pi'(\tilde{L}). \quad (9)$$

This means that the functions $\Pi(s)$, $s \leq \tilde{L}$, and $V(s; \tilde{L})$, $s \geq \tilde{L}$, have a *smooth junction* at the point $s = \tilde{L}$. This type of optimality condition is also called a *smooth pasting* or *high contact condition* (Samuelson 1965).

The preceding analysis and the generalization of the martingale approach to price perpetual American options on two assets is contained in Gerber and Shiu (1994, 1996a, 1996b). The next section presents the main contribution of this paper, pricing perpetual options on assets whose prices can jump across the optimal option-exercise boundary.

3. SOLUTION FOR MODEL II

3.1 The Martingale Approach

As before, we are looking for a coefficient ξ such that the process $\{e^{-rt+\xi U(t)}\}$ is a martingale. In Model II this leads to the condition

$$\lambda \left[\int_0^{\infty} e^{-\xi x}\ p(x)dx - 1 \right] - r + c\xi = 0. \quad (10)$$

Similar to (5), this equation has at most two solutions. One is $\xi_1 = 1$ (since $\{e^{-rt}S(t)\}$ is a martingale). Under some mild regularity conditions for $p(x)$, it has another, negative solution, $\xi_2 = -R < 0$. Hence the positive martingale $\{e^{-rt}S(t)^{-R}\}$ is bounded from above by the constant L^{-R} for $t < T_L$, and the optional sampling theorem can be applied and yields

$$s^{-R} = \mathrm{E}[e^{-rT_L}\ S(T_L)^{-R} \mid S(0) = s].$$

But now the problem is that $S(T_L) < L$ and that the distribution of $S(T_L)$ is not known in general. A noteworthy exception is the case in which the jump amounts have an exponential distribution, that is, $p(x) = \beta e^{-\beta x}$ for $x > 0$. This distribution is *memoryless:* given the value of T_L and the price of the stock immediately before the jump occurring at time T_L, the conditional distribution of $\ln L - \ln S(T_L)$ is the same exponential distribution. It follows that

$$s^{-R} = \mathrm{E}[e^{-rT_L} \mid S(0) = s]\ \beta \int_0^\infty (Le^{-x})^{-R}\, e^{-\beta x}\, dx$$

$$= \mathrm{E}[e^{-rT_L} \mid S(0) = s]\ L^{-R} \frac{\beta}{\beta - R}$$

and

$$V(s; L)$$

$$= \mathrm{E}[e^{-rT_L} \mid S(0) = s]\ \beta \int_0^\infty \Pi(Le^{-x})e^{-\beta x}\, dx, \quad s \geq L.$$

Finally, we combine the last two equations to obtain

$$V(s; L) = (\beta - R)\left(\frac{L}{s}\right)^R \int_0^\infty \Pi(Le^{-x})e^{-\beta x}\, dx. \quad (11)$$

To determine the optimal option-exercise boundary $\tilde{L}$, we can use two ideas. One is that $\tilde{L}$ must maximize the expression

$$L^R \int_0^\infty \Pi(Le^{-x})e^{-\beta x}\, dx.$$

This leads to the first-order condition,

$$\frac{R}{\tilde{L}} \int_0^\infty \Pi(\tilde{L}e^{-x})e^{-\beta x}\, dx + \int_0^\infty \Pi'(\tilde{L}e^{-x})e^{-x}\, e^{-\beta x}\, dx = 0. \quad (12)$$

The other idea is that we must have

$$V(\tilde{L}; \tilde{L}) = \Pi(\tilde{L}). \quad (13)$$

This means that the functions $\Pi(s)$, $s \leq \tilde{L}$, and $V(s; \tilde{L})$, $s \geq \tilde{L}$, have a *continuous junction* at the point $s = \tilde{L}$. Equation (13) leads to the condition

$$(\beta - R) \int_0^\infty \Pi(\tilde{L}e^{-x})e^{-\beta x}\, dx = \Pi(\tilde{L}). \quad (14)$$

By integrating the second integral of (12) by part, we can verify the equivalence of Conditions (12) and (14).

The continuous junction condition can be obtained by the following reasoning. If L is a number such that $V(L; L) < \Pi(L)$, we conclude that $\tilde{L} > L$; if, on the other hand, L is a number such that $V(L; L) > \Pi(L)$, we gather that $\tilde{L} < L$. This shows that $\tilde{L}$ must satisfy Equation (13). We note that this reasoning is not limited to the case of exponential jump amounts distributions. Hence, $\tilde{L}$ must satisfy the continuous junction condition in the general case, which is discussed in the next section.

3.2 The General Case

In the general case the martingale approach fails. But there is another method for solving the problem. Let

$$T = \inf\{t: U(t) < u\}$$

be the first time when the process $\{U(t)\}$ falls below the initial level u, and

$$Y = u - U(T)$$

the amount by which it falls below the initial level. We denote by $f(y, t)$ the joint probability density of Y and T. By conditioning on T and Y and remembering that the strategy is to exercise the option when the asset price drops below the level L, we see that

$$V(s; L) = \int_0^{\ln(s/L)} \int_0^\infty e^{-rt}\, V(se^{-y}; L) f(y, t)dtdy$$

$$+ \int_{\ln(s/L)}^\infty \int_0^\infty e^{-rt}\, \Pi(se^{-y}) f(y, t)dtdy, \quad s \geq L.$$

Let

$$g(y) = \int_0^\infty e^{-rt} f(y, t)dt, \qquad y > 0.$$

Thus $g(y)dy$ is the "discounted probability" that the process $\{U(t)\}$ will ever fall below its initial level u and will be between $u - y$ and $u - y + dy$ when it happens for the first time. Then we can write (15) more concisely as follows:

$$V(s; L) = \int_0^{\ln(s/L)} V(se^{-y}; L)g(y)dy$$

$$+ \int_{\ln(s/L)}^\infty \Pi(se^{-y})g(y)dy, \qquad s \geq L. \quad (16)$$

Luckily, there is an explicit expression for $g(y)$. Gerber and Shiu (1997, 1998) show that

$$g(y) = \frac{\lambda}{c} e^y \int_y^\infty e^{-x}\, p(x)dx. \quad (17)$$

Setting $s = L$ in (16), we get

$$V(L; L) = \int_0^\infty \Pi(Le^{-y})g(y)dy$$

$$= \frac{\lambda}{c} \int_0^\infty e^y\, \Pi(Le^{-y}) \int_y^\infty e^{-x}\, p(x)dxdy.$$

Now $\tilde{L}$ is obtained from the continuous junction condition, that is, from the equation

$$\frac{\lambda}{c} \int_0^\infty e^y\, \Pi(\tilde{L}e^{-y}) \int_y^\infty e^{-x}\, p(x)dxdy = \Pi(\tilde{L}). \quad (18)$$

In general, this equation cannot be solved explicitly, and $\tilde{L}$ has to be found by numerical methods.

To further discuss $V(s; L)$ for $s > L$, it is judicious to make a change of variable, replacing s by

$$x = \ln \frac{s}{L}, \qquad x > 0,$$

and to consider the function $W(x; L)$, defined by

$$W(x; L) = V(s; L).$$

Then Equation (16) can be rewritten as follows:

$$W(x: L) = \int_0^x W(x - y; L)g(y)dy + \int_x^\infty \Pi(Le^{x-y})g(y)dy, \qquad x \geq 0. \quad (19)$$

In the terminology of Feller (1966), this is a *defective renewal equation* for the function $W(x; L)$. Hence we can analyze this function by using standard techniques of renewal theory. One idea is to take Laplace transforms in Equation (19), which leads to a linear equation for the Laplace transform of the function W. The analytical inversion of the Laplace transform of W may be possible in some cases. Another idea is to discretize Equation (19) and to obtain an approximative solution by a recursive algorithm. A third idea is to examine the asymptotic behavior. Observe that

$$\int_0^\infty e^{Ry} g(y)dy = \frac{\lambda}{c}\int_0^\infty e^{(R+1)y}\int_y^\infty e^{-x} p(x)dxdy.$$

By changing the order of integration, we see that this expression is equal to

$$\frac{\lambda}{c}\int_0^\infty e^{-x} p(x)\int_0^x e^{(R+1)y}\, dydx = \frac{\lambda}{c}\frac{1}{R+1}\left[\int_0^\infty e^{Rx} p(x)dx - \int_0^\infty e^{-x} p(x)dx\right].$$

Because $\xi_1 = 1$ and $\xi_2 = -R$ are solutions of Equation (10), the last expression is equal to 1. Hence we have shown that

$$\int_0^\infty e^{Ry} g(y)dy = 1.$$

From this, (19), and Theorem 2 of section XI.6 of Feller (1966), an asymptotic formula is obtained:

$$W(x; L) \sim Ce^{-Rx}, \qquad \text{for } x \to \infty,$$

or

$$V(s; L) \sim C\left(\frac{L}{s}\right)^R, \qquad \text{for } s \to \infty,$$

where C is a constant.

Remark

In Model I and Model II the coefficient $R > 0$ was defined by the condition that the processes $\{e^{-rt-RU(t)}\}$ are martingales. This construction is reminiscent of the *adjustment coefficient* R in risk theory, which is defined by the condition that the process $\{e^{-RU(t)}\}$ is a martingale. In the context of risk theory, Equations (5) and (10) are attributable to Lundberg (1932).

4. THE PERPETUAL PUT OPTION

In this section, we consider the particular case in which $\Pi(s) = (K - s)_+$. In Model I, it follows from (7) that

$$V(s; L) = \left(\frac{L}{s}\right)^R (K - L), \qquad \text{for } s \geq L. \quad (20)$$

This expression is maximized for

$$\tilde{L} = K\frac{R}{R+1}. \quad (21)$$

Hence the price of the perpetual put option is

$$\begin{cases} [\tilde{L}/S(0)]^R(K - \tilde{L}) & \text{if } S(0) \geq \tilde{L} \\ K - S(0) & \text{if } S(0) < \tilde{L}. \end{cases} \quad (22)$$

In Model II, $\tilde{L}$ is determined from Equation (18). Obviously $\tilde{L} < K$. Hence,

$$\Pi(\tilde{L}e^{-y}) = K - \tilde{L}e^{-y}, \qquad \text{for } y \geq 0.$$

Upon substitution in (18), we obtain a linear equation for the optimal option-exercise boundary $\tilde{L}$. Its solution is

$$\tilde{L} = K\frac{c - \lambda\int_0^\infty e^y \int_y^\infty e^{-x} p(x)dxdy}{c - \lambda\int_0^\infty\int_y^\infty e^{-x} p(x)dxdy}. \quad (23)$$

By changing the order of integration, we can simplify both double integrals. First, we have

$$\lambda\int_0^\infty e^y\int_y^\infty e^{-x} p(x)dxdy = \lambda\int_0^\infty e^{-x} p(x)\int_0^x e^y\, dydx$$

$$= \lambda\int_0^\infty p(x)dx - \lambda\int_0^\infty e^{-x} p(x)dx$$

$$= -r + c$$

since $\xi_1 = 1$ is a solution of (10). Hence the numerator of (23) reduces to r. Second, the double integral of the denominator becomes

$$\int_0^\infty \int_y^\infty e^{-x}\, p(x)dxdy = \int_0^\infty xe^{-x}\, p(x)dx. \quad (24)$$

Hence

$$\tilde{L} = K \frac{r}{c - \lambda \int_0^\infty xe^{-x}\, p(x)dx}. \quad (25)$$

By comparing (21) and (25), we note that Model I and Model II lead to quite different expressions for $\tilde{L}$.

We remark that the particular case of a constant jump size has been treated by Michaud (1997). He obtains a series expansion for the price of a perpetual put option, and his formula for the optimal option-exercise boundary is a special case of (25).

As an illustration, let us consider the case in which $p(x) = \beta e^{-\beta x}$, $x > 0$. In Model I, we must assume that $\beta > 1$ (otherwise $E[S(t)]$ would be infinite). Then Equation (5) for ξ_1 and ξ_2 can be written as

$$\lambda \frac{\xi}{\beta - \xi} - r \;\; - c\xi = 0,$$

which leads to a quadratic equation:

$$c\xi^2 + (\lambda + r - \beta c)\xi - \beta r = 0.$$

Because $\xi_1 = 1$ must be a solution, it follows from the formula of Vieta (concerning the product of the roots of a quadratic equation) that $\xi_2 = -\beta r/c$, or

$$R = \frac{\beta r}{c}. \quad (26)$$

By substituting this in (21), we obtain

$$\tilde{L} = K \frac{r}{r + c/\beta}. \quad (27)$$

In Model II, we first observe that

$$\int_0^\infty xe^{-x}\, p(x)dx = \frac{\beta}{(\beta + 1)^2}.$$

Because $\xi_1 = 1$ is a solution of (10), we have

$$-\frac{\lambda}{\beta + 1} = r - c.$$

Hence

$$-\lambda \int_0^\infty xe^{-x}\, p(x)dx = \frac{\beta}{\beta + 1}\,(r - c).$$

Substitution in (25) leads to

$$\tilde{L} = K \frac{r}{r + \dfrac{c - r}{\beta + 1}}. \quad (28)$$

As before, we note that Model I and Model II lead to substantially different expressions for $\tilde{L}$; see (27) and (28).

5. RESET OPTIONS

Motivated by certain product designs of the currently popular equity-indexed annuities, we consider the pricing of a reset option. These ideas were inspired by Pafumi's discussion of Gerber and Shiu (1998).

We consider a mutual fund or an equity-indexed annuity whose value tracks that of a stock index. We assume that the stock index $I(t)$ incorporates the reinvestments of all dividends. Hence the discounted stock index is a martingale. We assume that $I(t) = \exp(U(t))$, $t \geq 0$, where $U(t)$ is given by Model II. Of course it is possible that $I(t)$ falls below $I(0)$ at some future time, which results in a loss.

A *reset option* provides the following protection against losses. Let $F(t)$ denote the *modified* fund value at time t, $t \geq 0$. Then $F(t) = I(t)$ until time T_1, the time when the fund value drops below the initial value $F(0)$ for the first time. At this time the fund value is reset to $F(0)$, so that

$$F(t) = F(0) \frac{I(t)}{I(T_1)}$$

for t between T_1 and time T_2, when the fund value drops below $F(0)$ for the second time and is reset to $F(0)$, and so on. Thus a reset option provides implicitly payments of

$$D_i = F(0) - F(T_i)$$

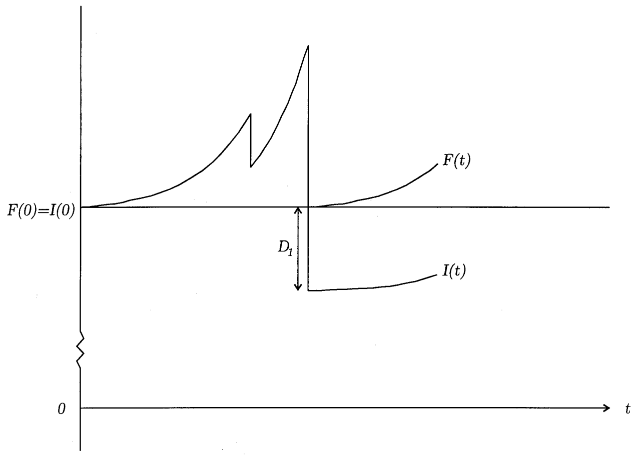
at times T_i, $i = 1, 2, \ldots$. This is illustrated in Figure 1.

A minimal type of reset option covers only the first time when the fund value falls below the initial level; at the other end of the spectrum, a maximal reset option provides a perpetual guarantee. We look at a family of reset options that contains both extremes as special cases.

We consider a reset option that provides up to n resets. Let A_n denote the price at time 0 of this guarantee; that is, A_n is the expectation of the sum of the discounted values of $D_1, D_2, \ldots, D_n$. We have

$$A_1 = F(0) \int_0^\infty (1 - e^{-y})g(y)dy.$$

Figure 1
The Definition of D_1



When the fund value falls below $F(0)$ for the first time, the amount D_1 is due and the amount A_{n-1} is the value of the remaining guarantee. Hence

$$A_n = \int_0^\infty (F(0)(1 - e^{-y}) + A_{n-1})g(y)dy, \qquad n = 2, 3, \ldots . \quad (29)$$

Recalling that

$$\int_0^\infty g(y)dy = 1 - \frac{r}{c}, \quad (30)$$

we obtain from (29) the recursive relation

$$A_n = A_1 + A_{n-1}\left(1 - \frac{r}{c}\right).$$

Its solution is

$$A_n = A_1 \sum_{j=0}^{n-1} \left(1 - \frac{r}{c}\right)^j = \frac{c}{r} A_1 \left[1 - \left(1 - \frac{r}{c}\right)^n\right].$$

In the limit $n \to \infty$, we see that the price A_∞ of a perpetual reset coverage is

$$A_\infty = \frac{c}{r} A_1.$$

It remains to calculate

$$A_1 = F(0) \int_0^\infty g(y)dy - F(0) \int_0^\infty e^{-y} g(y)dy.$$

Substituting from (30), (17), and (24), we find that

$$A_1 = F(0)\left[1 - \frac{r}{c} - \frac{\lambda}{c}\int_0^\infty xe^{-x}\, p(x)dx\right].$$

Then, for example

$$A_\infty = F(0)\left[\frac{c}{r} - 1 - \frac{\lambda}{r}\int_0^\infty xe^{-x}\, p(x)dx\right]. \quad (31)$$

6. THE CLASSICAL MODEL AS A LIMIT

The classical model of *geometric Brownian motion* can be viewed as a limiting case, from both Model I and Model II.

Consider Model I with an exponential jump amount distribution. This is a family of models with three parameters: $\beta > 1$, λ, c. The idea is to vary the parameters in such a way that the martingale condition is preserved,

$$\frac{\lambda}{\beta - 1} - r - c \equiv 0,$$

and that the variance per unit time of the process $\{Z(t)\}$ does not vary,

$$\frac{2\lambda}{\beta^2} \equiv \sigma^2.$$

Thus λ and c can be expressed by β as follows:

$$\lambda = \beta^2 \sigma^2/2,$$

and

$$c = \frac{1}{2}\frac{\beta^2 \sigma^2}{\beta - 1} - r. \quad (32)$$

In the limit $\beta \to \infty$, the classical model is obtained. From (32), see that

$$\frac{c}{\beta} = \frac{1}{2}\frac{\beta\sigma^2}{\beta - 1} - \frac{r}{\beta}.$$

From this and (27), it follows that the optimal option-exercise boundary for a perpetual put option is

$$\tilde{L} = K\frac{r}{r + \sigma^2/2} \quad (33)$$

in the classical model. Alternatively, see from (26) that $R \to 2r/\sigma^2$ for $\beta \to \infty$. By substituting this value in (21), we can confirm (33). Formulas (22) and (33) can be found in Merton (1973, sec. 8), and in textbooks such as Lamberton and Lapeyre (1996, Prop. 4.4.5).

If we use Model II with an exponential jump amount distribution, the three parameters must be varied such that

$$-\frac{\lambda}{\beta+1} - r + c \equiv 0 \tag{34}$$

and

$$\frac{2\lambda}{\beta^2} \equiv \sigma^2. \tag{35}$$

Then

$$\frac{c-r}{\beta+1} = \frac{1}{2}\sigma^2\left(\frac{\beta}{\beta+1}\right)^2.$$

Bu substituting this in (28) and taking the limit $\beta \to \infty$, we obtain again (33). Furthermore, we note that (11) becomes (7) in the limit.

Let us finally discuss the price of a perpetual reset option in the classical model. For an exponential jump amount distribution with parameter β, we see from (31) that the price is

$$A_\infty = F(0)\left[\frac{c}{r} - 1 - \frac{\lambda}{r}\frac{\beta}{(\beta+1)^2}\right]. \tag{36}$$

Substituting according to (34) and (35), we obtain

$$A_\infty = F(0)\frac{\sigma^2}{2r}\left(\frac{\beta}{\beta+1}\right)^2.$$

Taking the limit $\beta \to \infty$, we see that the price of a perpetual reset option in the classical model is

$$A = F(0)\frac{\sigma^2}{2r}.$$

Note that this expression can also be written as $F(0)/R$.

7. INTEGRATION OF THE CLASSICAL MODEL

It is possible to embed the classical and the compound Poisson models in a single, richer model. Let $\{W(t), t \geq 0\}$ be a standard Wiener process that is independent of the compound Poisson process $\{Z(t), t \geq 0\}$. Then we define the following models:

Model I*: $U(t) = u - ct + Z(t) + \sigma W(t)$,

Model II*: $U(t) = u + ct - Z(t) + \sigma W(t)$,

with $\sigma > 0$. The analysis of Model I carries over to Model I* (Gerber and Shiu 1994). Now the process $\{e^{-rt}S(t)^\xi, t \geq 0\}$ is a martingale provided that

$$\lambda\left[\int_0^\infty e^{\xi x} p(x)dx - 1\right] - r - c\xi + \frac{1}{2}\sigma^2\xi^2 = 0.$$

This equation has two solutions, $\xi_1 = 1$ (because $\{e^{-rt}S(t), t \geq 0\}$ is supposed to be a martingale) and $\xi_2 = -R < 0$. With this value for R, Formulas (7), (8), (9), (20), and (21) remain valid. The analysis of Model II* necessitates a different approach. Gerber and Landry (1998) show that in this case (25) is replaced by

$$\tilde{L} = K\frac{r}{c + \sigma^2 - \lambda\int_0^\infty xe^{-x}p(x)dx}. \tag{37}$$

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DISCUSSIONS

X. SHELDON LIN*

This paper considers valuation of perpetual options when the underlying stock follows a shifted compound Poisson process. The paper contains some elegant results. Here I comment on one aspect of the paper: the martingale approach.

In option-pricing theory, the price of a stock or a stock index is often assumed to follow a geometric Brownian motion. Valuation of an option written on the stock or stock index involves analysis of Brownian motion, which requires a good knowledge of stochastic process theory and stochastic calculus. On the other hand, we actuaries have applied distribution theory to model insurance risks over a long time. As a result, we are familiar with distributions and transforms. However, we are not so familiar with the general theory of stochastic processes and stochastic calculus. The martingale approach by Gerber and Shiu enables us to solve some valuation problems in option pricing without knowing deep results in stochastic calculus such as the reflection principle and Girsanov theorem. Furthermore, it is a powerful tool for computing various barrier hitting time distributions of a geometric Brownian motion. In this discussion I show how to use Gerber and Shiu's approach to compute single-barrier hitting time distributions. Single-barrier hitting time distributions play a key role in the valuation of barrier options such as cash-or-nothing options and lookback options. The latter is often used to price certain guarantees of equity-indexed annuities. As we will see, the martingale approach provides a simple analytical method without detailed knowledge of stochastic calculus as opposed to the classical method.

Let us assume that the price of a stock follows a geometric Brownian motion, namely,

$$S(t) = S(0)e^{\mu t+\sigma W(t)}, \tag{1}$$

where $W(t)$ is a Brownian motion with drift 0 and diffusion coefficient 1. Parameters μ and σ are the drift and the volatility of the stock price, respectively. The term μ may represent the actual rate of return of the stock or $\mu = r - \frac{1}{2}\sigma^2$, where r is the constant interest rate. The latter corresponds to the so-called risk-neutral probability measure in finance literature. Which one to choose depends on the problems, and we need to proceed with caution. In general, to calculate the price of an option, the drift $\mu = r - \frac{1}{2}\sigma^2$ should be used, while to calculate distributional properties for prediction purposes, the actual rate of return should be used.

Let U be an upper barrier, that is, $S(0) < U$, and

$$\tau = \inf\{t; S(t) \geq U\}$$

be the first hitting time to the barrier. We now apply the martingale approach to compute the distribution of τ. The idea is to calculate the Laplace transform of the distribution and then identify the distribution. I refer interested readers to *Financial Economics* (Panjer et al. 1998) for more details.

For a fixed real value $z \geq 0$, consider the process $Z(t) = e^{-zt}[S(t)]^{\xi}$. Following the line of the derivation in Sections 2 and 3, we look for a value of ξ such that $Z(t)$ is a martingale. Independent increment property yields that ξ will satisfy

$$E\{e^{-z+\mu\xi+\sigma\xi W(1)}\} = 1.$$

Equivalently, we have

$$\frac{1}{2}\sigma^2\xi^2 + \mu\xi - z = 0.$$

Thus,

$$\xi_1 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 z}}{\sigma^2},\ \xi_2 = \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 z}}{\sigma^2}.$$

Since $\xi_1 \geq 0$, the positive martingale

$$Z_1(t) = e^{-zt}[S(t)]^{\xi_1}$$

is bounded from above for $t \leq \tau$. The optional sampling theorem yields

$$S(0)^{\xi_1} = E\{e^{-z\tau}[S(\tau)]^{\xi_1}\} = E\{e^{-z\tau}\}U^{\xi_1}.$$

Hence the Laplace transform of τ is

$$E\{e^{-z\tau}\} = \left[\frac{S(0)}{U}\right]^{\xi_1} = \exp\left\{\left(-\frac{\mu}{\sigma^2}\ln\frac{S(0)}{U}\right)\left[1 - \sqrt{1 + \frac{2\sigma^2 z}{\mu^2}}\right]\right\}. \tag{2}$$

We now identify the distribution of τ. Recall that the density function of an inverse Gaussian distribution (see Bowers et al. 1997, p. 39) with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ is given by

$$f_{IG}(t) = \frac{\alpha}{\sqrt{2\pi\beta t^3}}e^{-(\beta t-\alpha)^2/2\beta t},\ t > 0, \tag{3}$$

and its Laplace transform is given by

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$$\int_0^\infty e^{-zt} f_{IG}(t)dt = e^{\alpha[1-\sqrt{1+2z/\beta}]},\ z > 0. \qquad (4)$$

Comparing (2) with (4), we conclude that τ has an inverse Gaussian distribution with shape parameter $\alpha = -(\mu/\sigma^2)\ \ln\ (S(0)/U)$ and scale parameter $\beta = (\mu/\sigma)^2$.

The lower barrier case can be dealt with similarly. Instead of choosing ξ_1, we choose ξ_2 to satisfy the condition of the optional sampling theorem. The resulting distribution will be a defective inverse Gaussian distribution. Furthermore, this approach can be used for a more complicated case in which there are two barriers. In that case, the corresponding Laplace transform is solved and reexpressed as an infinite series. Each term of the series is then identified. See Lin (1998) for details.

The advantage of this approach is now evident. The classical approach requires the use of the reflection principle to obtain the corresponding hitting time distribution for the standard Brownian motion and the use of the Girsanov theorem to derive the one for general Brownian motion. What we need here is the optional sampling theorem, which is relatively easier to understand and check. The rest of the derivation depends on solving a Laplace transform and identifying it, with which we are quite familiar.

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XIAOLAN ZHANG*

This paper, based on probabilistic methods, presents useful and important results on the pricing of perpetual options in models with jumps. The techniques used to treat the case in which the jumps are downwards are very interesting; usually, this is a delicate case for this kind of problem. Finally, the authors present some applications of these models.

Pricing perpetual options for jumps, diffusion, or jump-diffusion models has been studied by various authors with different methods. In this paper, by using the martingale approach, the authors obtain some explicit results in jump models when the jumps are positive. Then, they introduce a so-called ''discounted probability,'' which has an explicit expression (Gerber and Shiu 1997, 1998), to derive a certain renewal equation satisfied by the value function when jump sizes are negative, and also obtain some explicit results in this case. In addition to these, applying the variational inequalities techniques (Zhang 1995) yields a quasi-explicit formula that is similar to results in Section 2 and Formulas (20), (21), and (22) for perpetual put options in the jump-diffusion model with positive jumps. Based on the computation of the Laplace transform, by using the stopping theorem and the Levy exponent, Chesney (1996) obtains some formulas in the jump-diffusion model with deterministic jump amounts.

One common feature of all these methods is the use of the smooth junction condition to conclude the results. I think that some comments about methods will be welcome, in particular those used in Gerber and Shiu's papers and Chesney's paper, since both provide techniques that enable us to deal with the problem regardless of the jump's sign.

For me the most important point in this paper is that its technique (Section 3.2) is so general that it can open the door to other interesting problems, such as pricing American options with finite maturity. It is well known that there is no explicit formula for this problem: usually, we have to use numerical methods, such as finite difference methods or binomial trees. These methods are universal, but would be expensive with regard to the computation time. Fortunately, for some particular cases such as put and call options, some approximation procedures can help us to find explicit or quasi-explicit formulas [see Barone-Adesi (1987) and MacMillan (1986) for diffusion models, and Chesney (1996) and Zhang (1995) for jump-diffusion models with positive jumps]. These techniques lead us to deal with a problem that is very similar to pricing perpetual American options. Let us explain more precisely the techniques used in MacMillan (1986) and Zhang (1995). If we denote the current stock price by x, the payoff function by $f(x)$, the price of the American option by $u_a(t, x)$, and the price of the European option by $u_e(t, x)$, we can show that $w(t, x) = u_a(t, x) - u_e(t, x)$ satisfies a variational inequality (see Zhang 1995, 1997). By discretizing it only one step in time, we derive that an approximate value of

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$u(0, x)$ can be interpreted as the price of a perpetual American option with new payoff function $\bar{f}(x) = f(x) - u_e(0, x)$, where the underlying asset price is governed by the same SDE but with different parameters. For instance, see Model II, $\bar{c} = cT$, $\bar{r} = rT + 1$, $\bar{\lambda} = \lambda T$, where T is the maturity of the option (see Zhang 1995 and the appendix following this discussion for details). Usually, it turns out that $u_e(0, x)$ can be calculated easily; furthermore, in some particular cases it has some explicit form (see Merton 1976). So, if we can find the value of $w(0, x)$, then $u_a(0, x)$ can be easily obtained by $u_a(0, x) = w(0, x) + u_e(0, x)$.

Hereafter, we will focus on the computation of the "perpetual price" $w(0, x)$ by using the techniques of Section 3.2. Let us limit our discussion to Model II, since techniques in Chesney (1996) and Zhang (1995) fail in this case. It is worth mentioning that the results in Section 3.2 are still valid, since they have been obtained for a general payoff function $\Pi(\cdot)$. This means that we have the computation formulas for $w(0, x)$ if (18) has a unique solution. The latter is not given in the paper.

So, at first, it suffices for us to show that the system (18) has one and only one solution belonging to $[0, K]$ under the following conditions:

(a) For $x \in [0, K]$, $\Pi'(x) \le 0$, $\Pi''(x) \le 0$; and $\Pi(0) > 0$, $\Pi(K) \le 0$

(b) Let us write

$$I = \int_0^\infty \int_y^\infty e^{-x} p(x) dx dy = \int_0^\infty x e^{-x} p(x) dx,$$

and suppose that $\lambda I \le c$.

Then, we will see that the new payoff function $(K - x)^+ - u_e(0, x)$ satisfies the above conditions.

In addition, it is worth pointing out that conditions (a) and (b) are reasonable. We see that condition (b) is necessary even for perpetual put option $\Pi(x) = (K - x)^+$ to ensure the positivity of $\tilde{L}$ [see Formula (25)]. The authors do not explicitly point this out in their paper. However, it is easy to see that $(K - x)^+$ satisfies condition (a).

For the following discussion, we assume that $r > 0$. Now, let us write:

$$\phi(L) = \frac{\lambda}{c} \int_0^\infty e^y \Pi(Le^{-y}) \int_y^\infty e^{-x} p(x) dx dy - \Pi(L).$$

By using the formula

$$\int_0^\infty e^y \int_y^\infty e^{-x} p(x) dx dy = (c - r)/\lambda$$

[see Section 4, the formula between (23) and (24)], we derive

$$\begin{aligned} \phi(0) &= \frac{\lambda}{c} \int_0^\infty e^y \Pi(0) \int_y^\infty e^{-x} p(x) dx dy - \Pi(0) \\ &= \Pi(0) \left[\frac{c - r}{c} - 1 \right] \\ &= -\Pi(0) \frac{r}{c} < 0 \end{aligned}$$

since $\Pi(0)$, r, c are > 0, and

$$\begin{aligned} \phi(K) &= \frac{\lambda}{c} \int_0^\infty e^y \Pi(Ke^{-y}) \int_y^\infty e^{-x} p(x) dx dy - \Pi(K) \\ &= \frac{\lambda}{c} \int_0^\infty e^y \Pi(Ke^{-y}) \int_y^\infty e^{-x} p(x) dx dy \\ &\quad - \frac{\lambda}{c - r} \int_0^\infty e^y \Pi(K) \int_y^\infty e^{-x} p(x) dx dy. \end{aligned}$$

Here, we distinguish two cases:

- If $\Pi(K) < 0$, since $r > 0$ and $\Pi'(\cdot) \le 0$, then we have

$$\begin{aligned} \phi(K) &> \frac{\lambda}{c} \int_0^\infty e^y [\Pi(Ke^{-y}) - \Pi(K)] \int_y^\infty e^{-x} p(x) dx dy \\ &= \frac{\lambda}{c} \int_0^\infty e^y \Pi'(\theta)(e^{-y} - 1)K \int_y^\infty e^{-x} p(x) dx dy \ge 0, \end{aligned}$$

  with $\theta \in [e^{-y}K, K]$.

- If $\Pi(K) = 0$, taking into account the fact that $\Pi(\cdot)$ is a decreasing function, we have $\Pi(Ke^{-y}) > 0 \ \forall\, y \ge M_0$ except in the trivial case where $\Pi(\cdot)$ equals the constant zero. Here M_0 is some constant. That yields $\phi(K) > 0$.

Finally, let us show that $\phi(\cdot)$ is an increasing function. In fact, recalling

$$I = \int_0^\infty \int_y^\infty e^{-x} p(x) dx dy = \int_0^\infty x e^{-x} p(x) dx,$$

$$\begin{aligned} \phi'(L) &= \frac{\lambda}{c} \int_0^\infty \Pi'(Le^{-y}) \int_y^\infty e^{-x} p(x) dx dy \\ &\quad - \frac{1}{I} \int_0^\infty \Pi'(L) \int_y^\infty e^{-x} p(x) dx dy \\ &= \int_0^\infty \left[\frac{\lambda}{c} \Pi'(Le^{-y}) - \frac{1}{I} \Pi'(L) \right] \int_y^\infty e^{-x} p(x) dx dy. \end{aligned}$$

Since $\lambda l \leq c$, $\Pi'(\cdot) \leq 0$ and $\Pi''(\cdot) \leq 0$, we easily derive

$$\phi'(L) \geq \frac{\lambda}{c}\int_0^{\infty} [\Pi'(Le^{-y}) - \Pi'(L)] \int_y^{\infty} e^{-x} p(x)dxdy \geq 0.$$

It yields that $\phi(\cdot)$ is an increasing function with $\phi(0) < 0$ and $\phi(K) > 0$. This completes the proof.

Now let us come back to our discussion about pricing American options with finite maturity. In our case $\Pi(x) = (K - x)^+ - u_e(0, x)$, since $u_e''(0, x) \geq 0$ and $1 + u_e'(0, x) \geq 0$ (see Zhang 1995); we derive $\Pi'(x) \leq 0$, $\Pi''(x) \leq 0$ for $x \leq K$, and $\Pi(0) = K(1 - e^{-rT}) > 0$, $\Pi(K) \leq 0$. So under the condition (b)[1], we can say that $\tilde{L}$ is the unique solution belonging to $[0, K]$ of the following system

$$\frac{\lambda}{c}\int_0^{\infty} e^y[(K - Le^{-y}) - u_e(0, Le^{-y})] \int_y^{\infty} e^{-x} p(x)dxdy$$
$$= (K - L) - u_e(0, L).$$

As mentioned in the paper, generally $\tilde{L}$ has to be found by numerical methods (for instance, Newton's algorithm), since it has no explicit form. In this way, we obtain a quasi-explicit approach formula for American put options with finite maturity.

Let us end our discussion with two minor comments about the paper:

It is more and more popular to treat the mutual fund or even derivative product about the fund. In this paper, the authors model the stock index by Model II; that is, jumps are downwards and they obtain a pricing formula for a so-called reset option. Just from a computational point of view, can the techniques given in the paper be easily extended to the jump-diffusion model?

At the end of this paper, the authors mention a possibility to embed the classical and compound Poisson models in a single, rich model (that is, a jump-diffusion model); I agree that the analysis of Model II* requires a different approach because of the diffusion part. Normally, the pure jump model is only a particular case of the jump-diffusion model. Thus, since we see that (25) is just a particular case of (37) with $\sigma = 0$, I wonder if the techniques used to deal with Model II are also a particular case for Model II*.

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APPENDIX

ABOUT THE MACMILLAN TECHNIQUE

Recalling

$$U(t) = u + ct - Z(t),$$

the infinitesimal generator is given by

$$c\frac{\partial f}{\partial x} + \lambda \int_0^{\infty} [f(x - y) - f(x)]p(y)dy.$$

As above, we note that $u_a(t, x)$ and $u_e(t, x)$ are the price of American options and European options with the payoff function $\Pi(x)$ and maturity T. If $w(t, x) = u_a(t, x) - u_e(t, x)$, then w satisfies the following system (see Zhang 1997 for details):

$$\begin{cases} \left(\dfrac{\partial w}{\partial t} + Bw\right)(t, x) \leq 0 \\ w(t, x) \geq \Pi(x) - u_e(t, x) \\ \left(\dfrac{\partial w(t, x)}{\partial t} + Bw(t, x)\right)(w(t, x) - \Pi(x) + u_e(t, x)) = 0 \\ w(T, x) = 0 \end{cases}$$

with

$$Bw = c\frac{\partial w}{\partial x} + \lambda \int_0^{\infty} [w(x - y) - w(x)]p(y)dy - rw.$$

By discretizing only one step in time, and noting that $v(x) = w(0, x)$ and $\overline{\Pi}(x) = \Pi(x) - u_e(0, x)$, we derive

[1] By taking into account the definitions of $\bar{\lambda}$, $\bar{c}$, we see that $\bar{\lambda} \cdot l \leq \bar{c} \Longleftrightarrow \lambda \cdot l \leq c$.

$$\begin{cases} \bar{B}v \le 0 \\ v(x) \ge \bar{\Pi}(x) \\ (\bar{B}v) \cdot (v - \bar{\Pi}) = 0 \end{cases}$$

with

$$\bar{B}v = \bar{c}\,\frac{\partial v}{\partial x} + \bar{\lambda}\int_0^\infty (v(t, x - y) - v(t, x))p(y)dy - \bar{r}v,$$

and

$$\bar{c} = cT, \quad \bar{\lambda} = \lambda T, \; \bar{r} = rT + 1.$$

We can then interpret v as the price of a perpetual American option with payoff $\Pi(x) - u_e(0, x)$, and the dynamic of underlying is given as above, but with $\bar{c}$, $\bar{\lambda}$, $\bar{r}$ instead of c, λ, r.

AUTHORS' REPLY

HANS U. GERBER AND ELIAS S.W. SHIU

We thank the two discussants for their valuable and stimulating contributions. Dr. Lin shows that for a geometric Brownian motion, the first passage time at an upper level has an inverse Gaussian distribution. More generally, the martingale approach can be applied to determine the Laplace transform of the first passage time at an upper level, if the logarithm of the stock price behaves according to Model II or Model II*; see Formula (5.7) of Gerber and Shiu (1998). In the case of Model I, there is an explicit expression for the Laplace transform of the first passage time of jumping across the initial level; see Formula (3.9) of Gerber and Shiu (1998). For higher levels, we can determine the Laplace transform under special assumptions, for example, that the jump amount distribution is exponential or a combination of exponentials. Related questions for Model I* are discussed by Gerber and Landry (1998). Lin (1998) discusses the time of absorption in the case of two horizontal absorbing barriers. Gerber, Goovaerts, and De Pril (1981) obtain a series expansion for the absorption time density, if the other barrier is reflecting and linearly increasing.

Dr. Zhang describes computational procedures to price American options with a finite maturity date. The last equation before the References caught our particular attention. The reader should be aware that the optimal exercise boundary is now a function of time. Dr. Zhang's equation shows how its value can be determined for any given point in time. Furthermore, in the case of a perpetual option, the price of the corresponding European option is zero; so in that case her equation reduces to Equation (18) of our paper. At the end of the discussion, Dr. Zhang asks whether the techniques of our paper can be adapted to the jump-diffusion model. A partial answer is given in Section 7 of our paper and in Gerber and Landry (1998). In a sense, Model I* is the limit of a family of models of type I, and Model II* is the limit of models of type II. To see this, we represent the diffusion term in Model I* and Model II* as a limit of a sequence of independent shifted compound Poisson processes.

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Additional discussions on this paper can be submitted until January 1, 1999. The authors reserve the right to reply to any discussion. See the "Submission Guidelines for Authors" for detailed instructions on the submission of discussions.