Dr. Andreas Stein
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Series 1

1. (i) Write a MATLAB function `Normal(v1,v2,c11,c12,c22)` with input $v_1, v_2, c_{11}, c_{12}, c_{22} \in \mathbb{R}$ and output a realization of a $\mathcal{N}_{v,Q}$-distributed random variable, where

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}.$$

Hint: Use Proposition 0.4.15. in Chapter 0 of the lecture notes. You may also use the MATLAB function `chol()`.

(ii) Write a MATLAB function `Poisson(N,lambda)` with input $N \in \mathbb{N}$, $\lambda \in (0, +\infty)$, which returns as output N independent realizations of a Poisson distributed random variable with parameter λ. Then, write a MATLAB function `PoissonPlot()` which plots 10^5 realizations of a Poisson distributed random variable with $\lambda = 10$ generated with your Matlab function `Poisson()` in a histogram with 50 bins.

2. Let $\mathbb{T} \subseteq \mathbb{R}$ be a set, let $(\Omega, \mathcal{F}, P)$ be a probability space, let $(S, \mathcal{S})$ be a measurable space, and let $X\colon \mathbb{T} \times \Omega \to S$ be a stochastic process. Consider the filtration $\mathbb{F}^X$ generated by X and defined by

$$\mathbb{F}_t^X = \sigma_\Omega((X_s)_{s \in \mathbb{T} \cap (-\infty, t]}), \quad t \in \mathbb{T}.$$

(i) Show that X is $\mathbb{F}^X/\mathcal{S}$-adapted.

(ii) Let $\mathbb{F}\colon \mathbb{T} \to \mathcal{P}(\mathcal{P}(\Omega))$ be an other filtration such that X is $\mathbb{F}/\mathcal{S}$-adapted. Show that $\mathbb{F}_t^X \subseteq \mathbb{F}_t$ for every $t \in \mathbb{T}$.

(iii) Prove or disprove the following statement: if X is a stochastic process with continuous sample paths, then the filtration $\mathbb{F}^X$ is right-continuous.

3. Let $T \in (0, \infty)$, $m \in \mathbb{N}$, let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $W\colon [0, T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion.

(i) Show that for every $t \in [0, T]$ and $s \in (0, T]$ it holds that

$$W_t = \frac{\sqrt{t}}{\sqrt{s}} W_s$$

in distribution on $\mathcal{B}(\mathbb{R}^m)$.

(ii) Prove that W has P-independent and stationary increments.

(iii) Let $m = 1$. Show that for every $s, t \in [0, T]$ it holds that

$$\mathrm{Cov}_P(W_s, W_t) = \mathbb{E}[W_s W_t] = \min\{s, t\}.$$

Due: 16:00 o'clock, Monday, 3rd October 2022
Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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