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Series 5

1. (i) See file `Milstein1D.m`

(ii) See file `Milstein_SDE1.m`.

```
>> Milstein_SDE1
Strong rate of convergence in L^1: 1.016
Strong rate of convergence in L^2: 1.0051
```

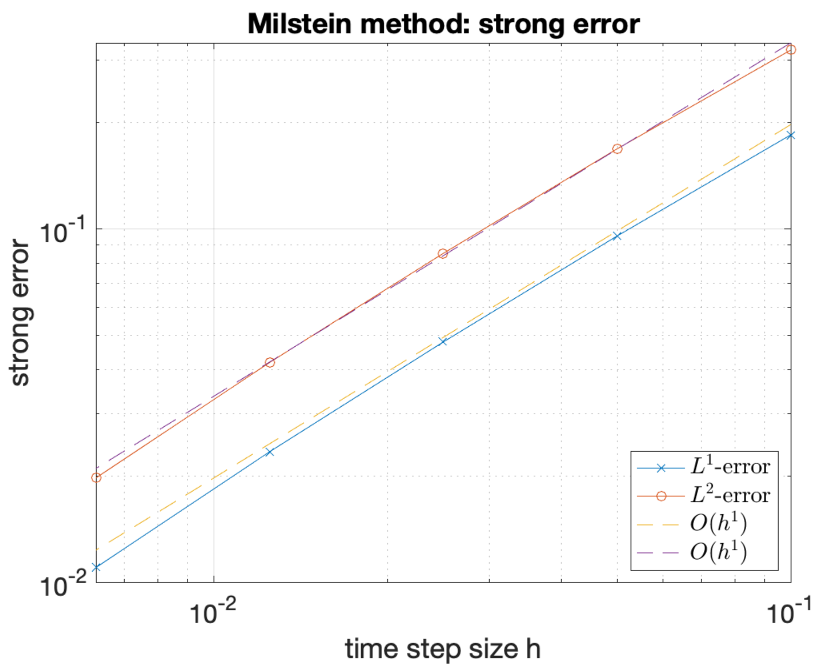

Abbildung 1: Since all assumptions of Theorem 3.6.5 are satisfied, we see strong convergence of order one in both the L^1- and the L^2-norm.

(iii) See file `Milstein_SDE2.m`.

```
>> Milstein_SDE2
Strong rate of convergence in L^1: 0.87553
Strong rate of convergence in L^2: 0.60607
```

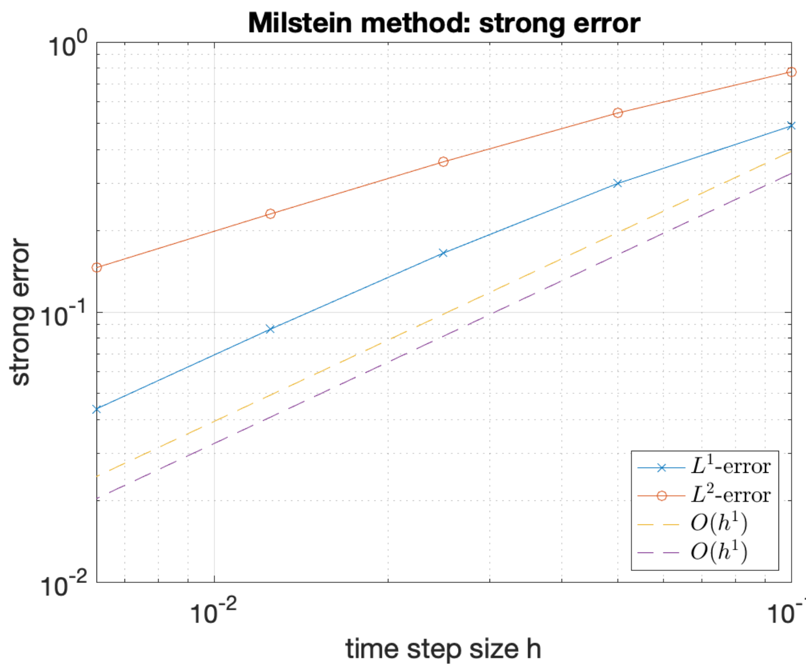

Abbildung 2: We do not recover convergence of order one for the strong error, but we observe a slower decay of roughly $\mathcal{O}(\Delta t^{0.88})$ for the L^1-error and of $\mathcal{O}(\Delta t^{0.61})$ for the L^2-error. This is due to the fact that the diffusion coefficient σ has an unbounded derivative $\sigma'(x) = 2x\cos(1+x^2)$.

2. (i) We keep the notation as in the lecture slides and define the mappings

$$\mu\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto a(b-x)$$

and

$$\sigma\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sigma_{\mathrm{v}}\sqrt{|x|}.$$

In order to show the uniqueness of the solution process $V\colon [0,T]\times\Omega \to \mathbb{R}$, we prove that the conditions of the Yamada-Watanabe theorem are satisfied (for the case $d = m = 1$). We start showing that the mapping $\mu\colon [0,T]\times\mathbb{R} \to \mathbb{R}$ satisfies condition (2) with $\kappa\colon [0,+\infty) \to [0,+\infty)$ given by

$$\kappa(x) = ax, \quad x \in \mathbb{R}.$$

Indeed, we have that for every $t \in [0,T]$ and for every $x, y \in \mathbb{R}$

$$|a(b-x) - a(b-y)| = a|b-x-b+y| = a|x-y| = \kappa(|x-y|).$$

Moreover, the mapping $\sigma\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ satisfies condition (2) with $\varrho\colon [0,+\infty)\to[0,+\infty)$ given by

$$\varrho(x)=\sigma_{\mathrm{v}}\sqrt{|x|},\quad x\in\mathbb{R}.$$

Precisely, we have that for every $t\in[0,T]$ and for every $x,y\in\mathbb{R}$

$$\begin{aligned}|\sigma_{\mathrm{v}}\sqrt{|x|}-\sigma_{\mathrm{v}}\sqrt{|y|}|=\sigma_{\mathrm{v}}|\sqrt{|x|}-\sqrt{|y|}|=\sigma_{\mathrm{v}}\frac{||x|-|y||}{\sqrt{|x|}+\sqrt{|y|}}&=\sigma_{\mathrm{v}}\sqrt{||x|-|y||}\frac{\sqrt{||x|-|y||}}{\sqrt{|x|}+\sqrt{|y|}}\\&\leq\sigma_{\mathrm{v}}\sqrt{|x-y|}=\varrho(|x-y|),\end{aligned}$$

where the last inequality follows by the trivial inequality $\sqrt{||x|-|y||}/\sqrt{|x|}+\sqrt{|y|}\leq 1$. Now, since $d=m=1$, we can conclude the exercise showing that conditions (3') and (4') are satisfied. The function $\kappa(x)=ax$ is concave and satisfies condition (3'). Finally, for every $\gamma>0$ we have that

$$\int_0^\gamma\kappa(z)^{-1}\mathrm{d}z=a\int_0^\gamma\frac{1}{z}\mathrm{d}z=+\infty,\quad\int_0^\gamma\varrho(z)^{-2}\mathrm{d}z=\sigma_{\mathrm{v}}\int_0^\gamma\frac{1}{z}\mathrm{d}z=+\infty,$$

which shows that condition (4') is also satisfied. Therefore, we can apply the Yamada-Watanabe theorem to conclude that the Cox-Ingersoll-Ross process is the unique (up to indistinguishability) solution to the SDE

$$\mathrm{d}V_t=a(b-V_t)\mathrm{d}t+\sigma_{\mathrm{v}}\sqrt{V_t}\mathrm{d}W_t,\quad t\in[0,T],\quad V_0=v_0.$$

(ii) The drift-implicit Milstein scheme of the stochastic process V with stepsize $\Delta t=T/N$ and initial value $V_0^N=V_0>0$ is given for $n=0,\ldots,N-1$ by

$$V_{n+1}^N=V_n^N+a(b-V_{n+1}^N)\Delta t+\sigma_{\mathrm{v}}\sqrt{V_n^N}(W_{t_{n+1}}-W_{t_n})+\frac{\sigma_{\mathrm{v}}^2}{4}\left((W_{t_{n+1}}-W_{t_n})^2-\Delta t\right). \tag{1}$$

Denoting the increments of W by $\Delta W_n:=W_{t_{n+1}}-W_{t_n}$ for $n=0,\ldots,N-1$ and resolving equation (1) for V_{n+1}^N yields

$$\begin{aligned}V_{n+1}^N&=\frac{1}{1+a\Delta t}\left(V_n^N+ab\Delta t+\sigma_{\mathrm{v}}\sqrt{V_n^N}\Delta W_n+\frac{\sigma_{\mathrm{v}}^2}{4}\left((\Delta W_n)^2-\Delta t\right)\right)\\&=\frac{1}{1+a\Delta t}\left(V_n^N+\left(ab-\frac{\sigma_{\mathrm{v}}^2}{4}\right)\Delta t+\sigma_{\mathrm{v}}\sqrt{V_n^N}\Delta W_n+\frac{\sigma_{\mathrm{v}}^2}{4}(\Delta W_n)^2\right)\\&\geq\frac{1}{1+a\Delta t}\left(V_n^N+\sigma_{\mathrm{v}}\sqrt{V_n^N}\Delta W_n+\frac{\sigma_{\mathrm{v}}^2}{4}(\Delta W_n)^2\right)\\&=\frac{1}{1+a\Delta t}\left(\sqrt{V_n^N}+\frac{\sigma_{\mathrm{v}}}{2}\Delta W_n\right)^2,\end{aligned}$$

where in the third step we have used the hypothesis $4ab\geq\sigma_{\mathrm{v}}^2$. Since $V_0^N=V_0>0$ and $P(\sigma_{\mathrm{v}}\Delta W_0/2=-\sqrt{V}_0)=0$, we have that $P(V_1^N>0)=1$. Hence, by induction it follows that $P(V_n^N>0)=1$ for all $n=0,\ldots,N$.

(iii) See the files `DriftImplicitMilstein.m` and `DriftImplicitMilsteinPlot.m`.

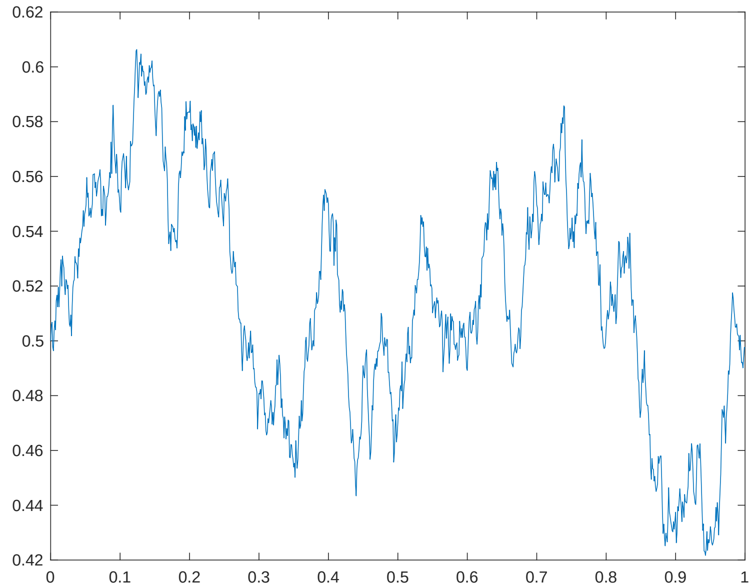

Abbildung 3: By item (ii) we know that $P(V_n^N > 0) = 1$ for all $n \in \{0, \dots, N\}$ whenever the parameters $a, b, \sigma_{\mathrm{v}}$ satisfy $4ab \geq \sigma_{\mathrm{v}}^2$. The condition $4ab \geq \sigma_{\mathrm{v}}^2$ is fulfilled by the choice of the parameters $a = 2, b = 0.5$ and $\sigma_{\mathrm{v}} = 0.25$. The expected behaviuor of the drift-implicit Milstein approximation is realized in the experiment.

3. *Proof of Proposition 3.6.6.* We start computing the iterated integrals (4) given in the hint. Let $(W^{(1)}, W^{(2)})\colon [0,T] \times \Omega \to \mathbb{R}^2$ be a 2-dimensional standard Brownian motion.

The Itô's formula applied to $W^{(1)}\colon [0,T] \times \Omega \to \mathbb{R}$ and to the function $\mathbb{R} \ni x \mapsto x^2$, yields

$$(W_t^{(1)})^2 = (W_{t_0}^{(1)})^2 + 2\int_{t_0}^{t} W_s^{(1)} \, \mathrm{d}W_s^{(1)} + t - t_0. \tag{2}$$

Furthermore, we may compute the second expression in (4) by applying the multi-dimensional Itô formula to $(W^{(1)}, W^{(2)})\colon [0,T] \times \Omega \to \mathbb{R}^2$ together with the function $\mathbb{R} \ni (x,y) \mapsto xy$, i.e.

$$W_t^{(1)} W_t^{(2)} = W_{t_0}^{(1)} W_{t_0}^{(2)} + \int_{t_0}^{t} W_s^{(1)} \, \mathrm{d}W_s^{(2)} + \int_{t_0}^{t} W_s^{(2)} \, \mathrm{d}W_s^{(1)}. \tag{3}$$

We are now ready to prove Proposition 3.6.6. By the commutative noise assumption, we have that for all $n \in \{0, 1, \ldots, N-1\}$ it holds P-a.s. that

$$
\begin{aligned}
&\sum_{i,j=1}^{m} (\sigma_i)'(Y_n^N)\,\sigma_j(Y_n^N) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(j)}\,\mathrm{d}W_s^{(i)} \\
&= \sum_{i=1}^{m} (\sigma_i)'(Y_n^N)\,\sigma_i(Y_n^N) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(i)}\,\mathrm{d}W_s^{(i)} \\
&\quad + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \Bigg((\sigma_i)'(Y_n^N)\,\sigma_j(Y_n^N) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(j)}\,\mathrm{d}W_s^{(i)} \\
&\quad + (\sigma_j)'(Y_n^N)\,\sigma_i(Y_n^N) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(i)}\,\mathrm{d}W_s^{(j)} \Bigg) \\
&= \sum_{i=1}^{m} (\sigma_i)'(Y_n^N)\,\sigma_i(Y_n^N) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(i)}\,\mathrm{d}W_s^{(i)} \qquad (4) \\
&\quad + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} (\sigma_i)'(Y_n^N)\,\sigma_j(Y_n^N) \left(\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(j)}\,\mathrm{d}W_s^{(i)} + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(i)}\,\mathrm{d}W_s^{(j)} \right).
\end{aligned}
$$

We apply Equation (2) to compute the iterated integral in (4)

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(i)}\,\mathrm{d}W_s^{(i)} = \int_{t_n}^{t_{n+1}} \int_{0}^{s} \mathrm{d}W_u^{(i)}\,\mathrm{d}W_s^{(i)} - \int_{t_n}^{t_{n+1}} \int_{0}^{t_n} \mathrm{d}W_u^{(i)}\,\mathrm{d}W_s^{(i)} \\
&= \int_{t_n}^{t_{n+1}} W_s^{(i)}\,\mathrm{d}W_s^{(i)} - \int_{t_n}^{t_{n+1}} \int_{0}^{t_n} \mathrm{d}W_u^{(i)}\,\mathrm{d}W_s^{(i)} \\
&= \frac{(W_{t_{n+1}}^{(i)})^2 - (W_{t_n}^{(i)})^2 - \Delta t}{2} - W_{t_n}^{(i)} \left(W_{t_{n+1}}^{(i)} - W_{t_n}^{(i)} \right) \\
&= \frac{1}{2} \left(W_{t_{n+1}}^{(i)} - W_{t_n}^{(i)} \right)^2 - \frac{\Delta t}{2}. \qquad (5)
\end{aligned}
$$

By (5) we have that

$$
\begin{aligned}
&\sum_{i,j=1}^{m} (\sigma_i)'(Y_n^N)\,\sigma_j(Y_n^N) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(j)}\,\mathrm{d}W_s^{(i)} = \\
&= \frac{1}{2} \sum_{i=1}^{m} (\sigma_i)'(Y_n^N)\,\sigma_i(Y_n^N) \left(\left(W_{t_{n+1}}^{(i)} - W_{t_n}^{(i)} \right)^2 - \Delta t \right) \\
&\quad + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} (\sigma_i)'(Y_n^N)\,\sigma_j(Y_n^N) \left(\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(j)}\,\mathrm{d}W_s^{(i)} + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(i)}\,\mathrm{d}W_s^{(j)} \right).
\end{aligned}
\qquad (6)
$$

Then, we exploit Equation (3) to compute the iterated integrals in (6) and we obtain

$$
\begin{aligned}
&\int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(j)}\,\mathrm{d}W_s^{(i)} + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(i)}\,\mathrm{d}W_s^{(j)} \\
&= \int_{t_n}^{t_{n+1}} W_s^{(j)}\,\mathrm{d}W_s^{(i)} - W_{t_n}^{(j)} \left(W_{t_{n+1}}^{(i)} - W_{t_n}^{(i)} \right) \\
&\quad + \int_{t_n}^{t_{n+1}} W_s^{(i)}\,\mathrm{d}W_s^{(j)} - W_{t_n}^{(i)} \left(W_{t_{n+1}}^{(j)} - W_{t_n}^{(j)} \right) \\
&= W_{t_{n+1}}^{(i)} W_{t_{n+1}}^{(j)} - W_{t_n}^{(i)} W_{t_n}^{(j)} + 2W_{t_n}^{(i)} W_{t_n}^{(j)} - W_{t_n}^{(j)} W_{t_{n+1}}^{(i)} - W_{t_n}^{(i)} W_{t_{n+1}}^{(j)} \\
&= \left(W_{t_{n+1}}^{(i)} - W_{t_n}^{(i)} \right) \left(W_{t_{n+1}}^{(j)} - W_{t_n}^{(j)} \right).
\end{aligned}
$$

Therefore, we obtain the desired result

$$
\begin{aligned}
&\sum_{i,j=1}^{m} (\sigma_i)'(Y_n^N)\,\sigma_j(Y_n^N) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathrm{d}W_u^{(j)}\,\mathrm{d}W_s^{(i)} \\
&= \frac{1}{2} \sum_{i=1}^{m} (\sigma_i)'(Y_n^N)\,\sigma_i(Y_n^N) \left(\left(W_{t_{n+1}}^{(i)} - W_{t_n}^{(i)} \right)^2 - \frac{T}{N} \right) \\
&\quad + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{m} (\sigma_i)'(Y_n^N)\,\sigma_j(Y_n^N) \left(W_{t_{n+1}}^{(i)} - W_{t_n}^{(i)} \right) \left(W_{t_{n+1}}^{(j)} - W_{t_n}^{(j)} \right) \\
&= \frac{1}{2} \sum_{i,j=1}^{m} (\sigma_i)'(Y_n^N)\sigma_j(Y_n^N)\big(W_{t_{n+1}}^{(i)} - W_{t_n}^{(i)} \big)\big(W_{t_{n+1}}^{(j)} - W_{t_n}^{(j)} \big) \\
&\quad - \frac{\Delta t}{2} \sum_{i=1}^{m} (\sigma_i)'(Y_n^N)\,\sigma_i(Y_n^N).
\end{aligned}
$$

□

Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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