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Series 10

1. We have that $W_t \in \mathcal{L}^1(P; |\cdot|)$ for every $t \in [0,T]$ and that $(W_t)_{t\in[0,T]}$ is $(\mathbb{F}_t)_{t\in[0,T]}$-adapted. Furthermore, since the increments of W are independent and stationary with zero mean, we have that for every $0 \le s < t \le T$

$$\mathbb{E}_P[W_t|\mathbb{F}_s] = \mathbb{E}_P[W_t - W_s + W_s|\mathbb{F}_s] = \mathbb{E}_P[W_t - W_s] + \mathbb{E}_P[W_s|\mathbb{F}_s] = W_s,$$

which proves that $(W_t)_{t\in[0,T]}$ is a $(\mathbb{F}_t)_{t\in[0,T]}$-martingale. The Itô formula applied to $W\colon [0,T]\times\Omega \to \mathbb{R}$ and to the function $\mathbb{R} \ni x \mapsto x^2$, yields

$$W_t^2 - t = W_s^2 - s + 2\int_s^t W_r \,\mathrm{d}W_r, \quad s,t \in [0,T],\ s \le t. \tag{1}$$

In particular, for $s = 0$ we have

$$W_t^2 - t = 2\int_0^t W_s \mathrm{d}W_s, \quad t \in [0,T].$$

Then, since $(W_t)_{t\in[0,T]}$ is $(\mathbb{F}_t)_{t\in[0,T]}$-predictable with

$$\int_0^T \mathbb{E}[W_t^2]\mathrm{d}t < +\infty,$$

Proposition 1.6.5 implies that the stochastic integral process $(W_t^2 - t)_{t\in[0,T]}$ is also a $(\mathbb{F}_t)_{t\in[0,T]}$-martingale.

Alternative proof for the second part: We have that $W_t^2 \in \mathcal{L}^1(P; |\cdot|)$ and that $(W_t^2 - t)_{t\in[0,T]}$ is $(\mathbb{F}_t)_{t\in[0,T]}$-adapted. Then, since $\int_s^t W_r\,\mathrm{d}W_r$ is independent of $\mathbb{F}_s$ and of zero mean, Formula (1) yields

$$\begin{aligned}\mathbb{E}_P[W_t^2 - t|\mathbb{F}_s] &= W_s^2 - s + 2\,\mathbb{E}_P[\int_s^t W_r\,\mathrm{d}W_r|\mathbb{F}_s]\\ &= W_s^2 - s + 2\underbrace{\mathbb{E}_P[\int_s^t W_r\,\mathrm{d}W_r]}_{=0} = W_s^2 - s,\end{aligned}$$

which proves the claim.

2. (i) We use the tower property of conditional expectations and the independence of I and W to derive for fixed $t \in [0,T]$ and $x \in \mathbb{R}^m$

$$\begin{aligned}\mathbb{E}_P\left[e^{\mathbf{i}x^\top X_t}\right] &= e^{\mathbf{i}x^\top \gamma_1 t}\,\mathbb{E}_P\left[e^{\mathbf{i}x^\top \gamma_2 I_t + \mathbf{\Sigma} W_{I_t}}\right] \\ &= e^{\mathbf{i}x^\top \gamma_1 t}\,\mathbb{E}_P\left[\mathbb{E}_P\left[e^{\mathbf{i}x^\top \gamma_2 I_t + \mathbf{\Sigma} W_{I_t}} \,|\, I_t\right]\right] \\ &= e^{\mathbf{i}x^\top \gamma_1 t}\,\mathbb{E}_P\left[(e^{\mathbf{i}x^\top \gamma_2 I_t}\,\mathbb{E}_P\left[e^{\mathbf{i}x^\top \mathbf{\Sigma} W_{I_t}} \,|\, I_t\right]\right] \\ &= e^{\mathbf{i}x^\top \gamma_1 t}\,\mathbb{E}_P\left[(e^{\mathbf{i}x^\top \gamma_2 I_t}\exp\left(-\frac{1}{2} I_t x^\top \mathbf{A} x\right)\right].\end{aligned}$$

In the last equality we have used that $\mathbf{\Sigma} W_{I_t} \overset{d}{=} \sqrt{I_t}\mathbf{\Sigma} W_1$ follows a centered normal distribution with covariance matrix $I_t \mathbf{\Sigma}\mathbf{\Sigma}^\top = I_t \mathbf{A}$, conditional on I_t. Regrouping some terms then shows that

$$\mathbb{E}_P\left[e^{\mathbf{i}x^\top X_t}\right] = e^{\mathbf{i}x^\top \gamma_1 t}\,\mathbb{E}_P\left[\exp\left(\mathbf{i}\left(x^\top \gamma_2 + \frac{\mathbf{i}}{2} x^\top \mathbf{A} x\right) I_t\right)\right],$$

which proves the claim.

(ii) For every $s, t \in [0,T]$, $s < t$, we have that

$$\begin{aligned}L_t - L_s &= \gamma_1(t-s) + \gamma_2(I_t - I_s) + \mathbf{\Sigma}(W_{I_t} - W_{I_s}) \\ &\overset{d}{=} \gamma_1(t-s) + \gamma_2 I_{t-s} + \mathbf{\Sigma} W_{I_{t-s}} \\ &\overset{d}{=} \gamma_1(t-s) + \gamma_2 I_{t-s} + \sqrt{I_{t-s}}\,\mathbf{\Sigma}\, Z_m,\end{aligned}$$

where $Z_m \sim \mathcal{N}_{0,I_m}$ is m-dimensional standard normal. The last identity follows from the scaling property of Brownian motions together with the independence of I and W.

3. (i) Let $n \in \mathbb{N}$. By the given hint, we have that for all $x \in \mathbb{R}$

$$\phi_X(x) = \left(1 - i\frac{x}{\beta}\right)^{-\alpha} = \prod_{k=1}^n \left(1 - i\frac{x}{\beta}\right)^{-\frac{\alpha}{n}} = \prod_{k=1}^n \phi_{Y_k}(x) = \phi_{\sum_{k=1}^n Y_k}(x),$$

where $Y_1, \ldots, Y_n$ are i.i.d. $\Gamma_{\frac{\alpha}{n},\beta}$-random variables. Consequently, by Proposition 0.5.2, $X \overset{d}{=} \sum_{k=1}^n Y_k$, which proves the claim.

(ii) By the given hint, the Lévy-Khintchine formula reads

$$\psi_X(x) = i\gamma x - \frac{Ax^2}{2} + \int_{\mathbb{R}} (e^{ixz} - 1 - ixz\mathbb{1}_{\{|z|\le 1\}})\alpha e^{-\beta z} z^{-1}\mathbb{1}_{\{z \ge 0\}} \mathrm{d}z.$$

We have to determine $\gamma, A \in \mathbb{R}$ such that

$$\phi_X(x) = e^{\psi_X(x)},$$

which implies

$$\log(\phi_X(x)) = \psi_X(x) + 2ki\pi,$$

Siehe nächstes Blatt!

for some $k \in \mathbb{Z}$. But, since $\log(\phi_X(0)) = \psi_X(0) = 0$, then necessarily $k = 0$ and

$$\phi_X(x) = e^{\psi_X(x)} \Longleftrightarrow \log(\phi_X(x)) = \psi_X(x).$$

Then, taking derivatives we obtain

$$\log(\phi_X)'(x) = i\gamma - Ax + \frac{\alpha i}{\beta - ix} + \frac{\alpha}{\beta} i \left(e^{-\beta} - 1\right)$$

and

$$\psi_X'(x) = \frac{\alpha i}{\beta - ix}.$$

Hence, we can choose $A = 0$ and $\gamma = \frac{\alpha}{\beta}\left(1 - e^{-\beta}\right)$ to get

$$\log(\phi_X)'(x) = \psi_X'(x), \quad x \in \mathbb{R}.$$

As a consequence, for every $x \in \mathbb{R}$ we have that

$$\log(\phi_X(x)) = \int_0^x \log(\phi_X)'(s)\mathrm{d}s = \int_0^x \psi_X'(s)\mathrm{d}s = \psi_X(x),$$

where the second and fourth equalities follow by $\log(\phi_X(0)) = \psi_X(0) = 0$. Therefore, the characteristic triplet (γ, A, ν) of $I^{\Gamma_{\alpha,\beta}}$ is given by

$$\gamma = \frac{\alpha}{\beta}\left(1 - e^{-\beta}\right), \; A = 0, \; \nu = \alpha e^{-\beta z} z^{-1} \mathbb{1}_{\{z \geq 0\}} \mathrm{d}z.$$

(iii) See file `VarianceGamma.m`.

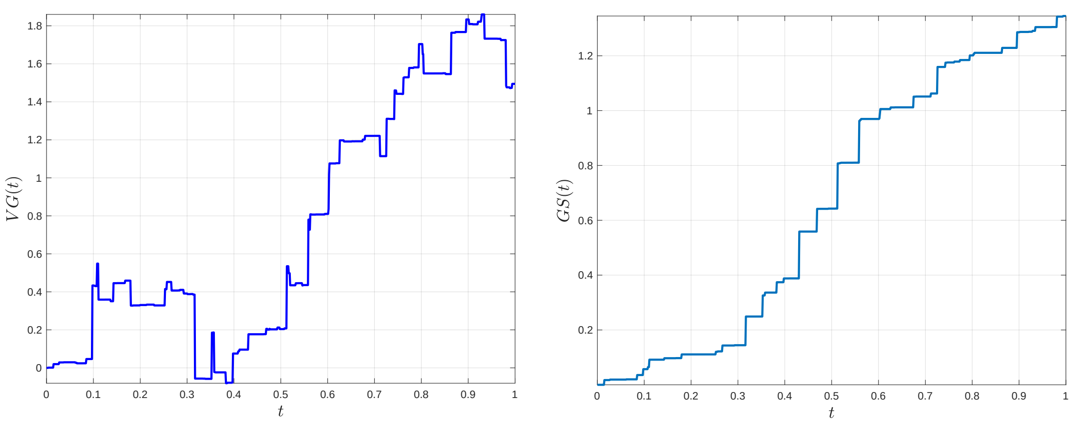

Abbildung 1: A realization of a sample path of a subordinated Brownian motion L (on the left) with Gamma subordinator (on the right).

Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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