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Series 3

Throughout this exercise sheet, let $T \in (0, +\infty)$, let $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$ be a stochastic basis, and let $W\colon [0,T] \times \Omega \to \mathbb{R}$ be a one-dimensional standard $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$-Brownian motion.

1. (i) Let $\beta \in \mathbb{R}$, and let $X\colon \Omega \to \mathbb{R}$ be an $\mathcal{N}_{0,1}$-distributed random variable. Show that

$$\mathbb{E}_P\big[e^{\beta X}\big] = e^{\frac{1}{2}\beta^2}.$$

(ii) Let $T = 1$ and use Itô's formula to show that the process X given for $t \in [0, 1)$ by

$$X_t := (t-1)\int_0^t \frac{1}{s-1}\mathrm{d}W_s$$

satisfies for all $t \in [0, 1)$ P-a.s. the equation

$$X_t = \int_0^t \frac{1}{s-1} X_s \mathrm{d}s + W_t.$$

2. (i) Let X be an Itô process given by

$$X_t = X_0 + \int_0^t Y_s \mathrm{d}s + \int_0^t Z_s \mathrm{d}W_s$$

for $\mathbb{F}/\mathcal{B}(\mathbb{R})$-predictable stochastic processes $Y\colon [0,T]\times\Omega \to \mathbb{R}$ and $Z\colon [0,T]\times\Omega \to \mathbb{R}$ such that $P(\int_0^T |Y_s| + |Z_s|^2 \mathrm{d}s < +\infty) = 1$. Define the *stochastic exponential* of X via

$$\mathcal{E}(X)_t := \exp\left(X_t - X_0 - \frac{1}{2}\int_0^t Z_s^2 \mathrm{d}s\right), \quad t \in [0,T].$$

Use the Itô formula to show that P-a.s. for all $t \in [0,T]$ it holds

$$\mathcal{E}(X)_t = 1 + \int_0^t \mathcal{E}(X)_s Y_s \mathrm{d}s + \int_0^t \mathcal{E}(X)_s Z_s \mathrm{d}W_s.$$

(ii) Let $X\colon [0,T] \times \Omega \to \mathbb{R}$ be the solution process to the SDE

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad t \in [0,T], \quad X_0 = 1,$$

Bitte wenden!

where $\mu, \sigma \colon [0,T] \times \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous mappings, and assume that X has P-a.s. strictly positive sample paths. Moreover, assume that

$$P\left(\int_0^t \frac{1}{X_s} |\mu(X_s)| \mathrm{d}s + \int_0^t \left(\frac{1}{X_s} \sigma(X_s) \right)^2 \mathrm{d}s < +\infty \right) = 1.$$

The *stochastic logarithm* $\mathcal{L}(X)$ of X is defined as the process

$$\mathcal{L}(X)_t := \int_0^t \frac{1}{X_s} \mu(X_s) \mathrm{d}s + \int_0^t \frac{1}{X_s} \sigma(X_s) \mathrm{d}W_t, \quad t \in [0,T].$$

Show that

$$\mathcal{E}(\mathcal{L}(X))_t = X_t$$

for all $t \in [0,T]$, where $\mathcal{E}$ denotes the stochastic exponential.

3. Consider the non-homogeneous linear SDE

$$\mathrm{d}X_t = (\mu_1(t)X_t + \mu_2(t))\mathrm{d}t + (\sigma_1(t)X_t + \sigma_2(t))\mathrm{d}W_t, \quad t \in [0,T], \quad X_0 = \xi. \tag{1}$$

Assume that $\mu_1, \mu_2, \sigma_1, \sigma_2 \colon [0,T] \to \mathbb{R}$ are globally Lipschitz continuous functions and that $\xi \in \mathcal{L}^p(P|_{\mathbb{F}_0}; |\cdot|)$ for some $p \geq 2$. In this exercise, we will first derive the solution of a homogeneous linear SDE and then solve Equation (1) by a stochastic analogue of the *variations-of-constants formula*.

(i) Consider the homogeneous linear SDE with $\mu_2 = \sigma_2 = 0$ and $\xi = 1$ in (1), i.e.

$$\mathrm{d}Y(t) = \mu_1(t)Y_t\mathrm{d}t + \sigma_1(t)Y_t\mathrm{d}W_t, \quad t \in [0,T], \quad Y_0 = 1. \tag{2}$$

Show that the unique (up to indistinguishability) solution process of the SDE (2) is given by

$$Y_t = \exp\left(\int_0^t \mu_1(s) - \frac{\sigma_1(s)^2}{2} \mathrm{d}s + \int_0^t \sigma_1(s)\mathrm{d}W_s \right), \quad t \in [0,T]. \tag{3}$$

Hint: Use the stochastic exponential from Exercise 2. Concerning uniqueness, it is sufficient to show that the functions $\mu \colon [0,T] \times \mathbb{R} \to \mathbb{R}$, $\mu(t,x) = x \cdot \mu_1(t)$, and $\sigma \colon [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma(t,x) = x \cdot \sigma_1(t)$, are globally Lipschitz continuous in the second variable, uniformly in t, and that there exists a positive constant C such that for all $t \in [0,T]$

$$|\mu(t,x)| + |\sigma(t,x)| \leq C|x|, \quad \forall x \in \mathbb{R}.$$

(ii) Show that the process

$$X_t = Y_t \left(X_0 + \int_0^t \frac{\mu_2(s) - \sigma_1(s)\sigma_2(s)}{Y_s} \mathrm{d}s + \int_0^t \frac{\sigma_2(s)}{Y_s} \mathrm{d}W_s \right), \quad t \in [0,T],$$

with $Y \colon [0,T] \times \Omega \to \mathbb{R}$ as in Equation (3), is a solution of the SDE (1).
Hint: Use the ansatz $X = YZ$, where $Z \colon [0,T] \times \Omega \to \mathbb{R}$ is a suitable Itô-process with the same driving noise W as X and Y. Apply the two-dimensional Itô formula to the product YZ to derive the result.

Siehe nächstes Blatt!

Remark for item (ii). The uniqueness (up to indistinguishability) of the solution for the SDE (1) can also be proved by showing that the functions $\mu\colon [0,T] \times \mathbb{R} \to \mathbb{R}, \mu(t,x) = x \cdot \mu_1(t) + \mu_2(t)$, and $\sigma\colon [0,T] \times \mathbb{R} \to \mathbb{R}, \sigma(t,x) = x \cdot \sigma_1(t) + \sigma_2(t)$, are globally Lipschitz continuous in the second variable, uniformly in t, together with the linear growth condition

$$|\mu(t,x)| + |\sigma(t,x)| \leq C(1 + |x|), \quad \forall t \in [0,T], \forall x \in \mathbb{R},$$

for some positive constant C independent of t. The proof follows by analogous computations as in item (i) and it is therefore **not** required.

Due: 16:00 o'clock, Monday, 17th October 2022
Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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