Dr. Andreas Stein
Dr. Francesca Bartolucci

# Series 1

**1.** (i) See the MATLAB file `Normal.m`

(ii) See the MATLAB files `Poisson.m` and `PoissonPlot.m`.

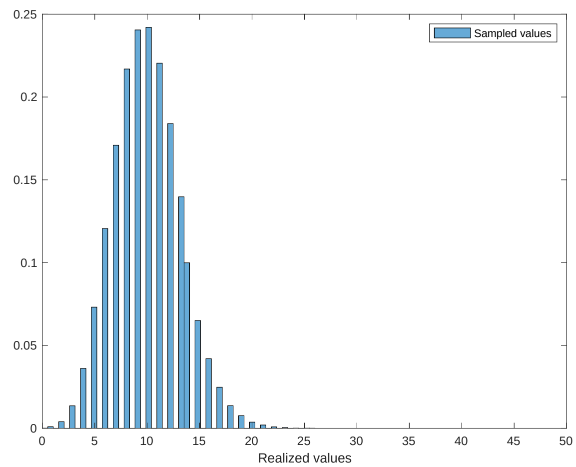


Abbildung 1: Result of a call of the MATLAB function `PoissonPlot()`.

**2.** (i) Let $t \in \mathbb{T}$. We need to show that $X_t \in \mathcal{M}(\mathbb{F}_t^X, \mathcal{S})$. By definition of $\mathbb{F}^X$, we have that

$$\mathbb{F}_t^X = \sigma_\Omega((X_s)_{s \in \mathbb{T} \cap (-\infty, t]}) = \sigma_\Omega\big(\{X_s^{-1}(A) \in \mathcal{P}(\Omega) : A \in \mathcal{S}, s \in \mathbb{T} \cap (-\infty, t]\}\big). \quad (1)$$

In particular,

$$\mathbb{F}_t^X \supseteq \{X_t^{-1}(A) \in \mathcal{P}(\Omega) : A \in \mathcal{S}\}.$$

Therefore, for every $A \in \mathcal{S}$ it holds that

$$X_t^{-1}(A) \in \mathbb{F}_t^X,$$

which concludes the proof.

(ii) Let $t \in \mathbb{T}$. We want to show that $\mathbb{F}_t^X \subseteq \mathbb{F}_t$. By hypothesis, $X_s \in \mathcal{M}(\mathbb{F}_s, \mathcal{S})$ for every $s \in \mathbb{T}$. Then, for every $s \in \mathbb{T}$ and for every $A \in \mathcal{S}$ it holds that

$$X_s^{-1}(A) \in \mathbb{F}_s.$$

Furthermore, by definition of filtration we know that $\mathbb{F}_s \subseteq \mathbb{F}_t$ for every $s \in \mathbb{T} \cap (-\infty, t]$. Therefore,

$$\{X_s^{-1}(A) \in \mathcal{P}(\Omega) : A \in \mathcal{S}, s \in \mathbb{T} \cap (-\infty, t]\} \subseteq \mathbb{F}_t,$$

which by equation (1) implies that $\mathbb{F}_t^X \subseteq \mathbb{F}_t$.

(iii) Let $T \in (0, +\infty)$. We consider the stochastic process $X\colon [0,T] \times \Omega \to \mathbb{R}$ defined by $X(t,\omega) = tZ(\omega)$, where $Z\colon \Omega \to \mathbb{R}$ is a standard normal random variable. Clearly, $X$ is a stochastic process with continuous sample paths. By definition of $\mathbb{F}^X$, it holds that

$$\mathbb{F}_0^X = \sigma_\Omega(X_0) = \sigma_\Omega\big(\{X_0^{-1}(A) \in \mathcal{P}(\Omega) : A \in \mathcal{B}(\mathbb{R})\}\big) = \sigma_\Omega\big(\{\Omega, \emptyset\}\big) = \{\Omega, \emptyset\}.$$

On the other hand, for every $t \in (0, T]$ it holds that

$$\begin{aligned}
\mathbb{F}_t^X &= \sigma_\Omega\big(\{X_s^{-1}(A) \in \mathcal{P}(\Omega) : A \in \mathcal{B}(\mathbb{R}), s \in \mathbb{T} \cap (-\infty, t]\}\big) \\
&= \sigma_\Omega\big(\{X_s^{-1}(sB) \in \mathcal{P}(\Omega) : B \in \mathcal{B}(\mathbb{R}), s \in \mathbb{T} \cap (-\infty, t]\}\big) \\
&= \sigma_\Omega\big(\{Z^{-1}(B) \in \mathcal{P}(\Omega) : B \in \mathcal{B}(\mathbb{R})\}\big) = \sigma_\Omega\big(Z\big),
\end{aligned}$$

and consequently

$$\mathbb{F}_0^{X+} = \bigcap_{t \in (0,T]} \mathbb{F}_t^X = \sigma_\Omega\big(Z\big).$$

Therefore, $\mathbb{F}_0^X \neq \mathbb{F}_0^{X+}$ and the filtration $\mathbb{F}^X$ generated by $X$ is not right-continuous.

**3.** (i) Let $t \in [0,T]$ and $s \in (0,T]$. By item (iii) in the definition of standard Brownian motion and by Proposition 0.4.15. in Chapter 0, it holds that

$$W_t \sim \mathcal{N}_{0, tI_{\mathbb{R}^m}} \sim \frac{\sqrt{t}}{\sqrt{s}} \mathcal{N}_{0, sI_{\mathbb{R}^m}} \sim \frac{\sqrt{t}}{\sqrt{s}} W_s.$$

(ii) We first show that $W$ has $P$-independent increments. Let $n \in \mathbb{N}$ and let $t_0, \ldots, t_n \in [0,T]$ with $t_0 < t_1 < \ldots < t_n$. We have to show that the random variables

$$(W_{t_1} - W_{t_0}), (W_{t_2} - W_{t_1}), \ldots, (W_{t_n} - W_{t_{n-1}})$$

are $P$-independent. First, by item (i) in Exercise 2, we know that $W$ is $\mathbb{F}^W/\mathcal{B}(\mathbb{R}^m)$-adapted. This fact together with the inclusions $\mathbb{F}_{t_0}^W \subseteq \ldots \subseteq \mathbb{F}_{t_{n-1}}^W$ implies that

$$\sigma_\Omega\big((W_{t_1} - W_{t_0}), (W_{t_2} - W_{t_1}), \ldots, (W_{t_{n-1}} - W_{t_{n-2}})\big) \subseteq \mathbb{F}_{t_{n-1}}^W.$$

Consequently, by item (iv) in the definition of standard Brownian motion,

$$\sigma_\Omega\big((W_{t_1} - W_{t_0}), (W_{t_2} - W_{t_1}), \ldots, (W_{t_{n-1}} - W_{t_{n-2}})\big) \quad \text{and} \quad \sigma_\Omega\big(W_{t_n} - W_{t_{n-1}}\big)$$

are $P$-independent $\sigma$-algebras. We can then repeat the above reasoning $t_{n-2}$ times to obtain that the $\sigma$-algebras

$$\sigma_\Omega\big(W_{t_1} - W_{t_0}\big), \sigma_\Omega\big(W_{t_2} - W_{t_1}\big) \ldots, \sigma_\Omega\big(W_{t_n} - W_{t_{n-1}}\big)$$

are $P$-independent. Therefore, for every choice of $A_0, \ldots, A_{n-1} \in \mathcal{B}(\mathbb{R}^m)$, it holds that

$$\begin{aligned} &P(\{W_{t_1} - W_{t_0} \in A_0\} \cap \cdots \cap \{W_{t_n} - W_{t_{n-1}} \in A_{n-1}\}) \\ &= P(W_{t_1} - W_{t_0} \in A_0) \cdot \ldots \cdot P(W_{t_n} - W_{t_{n-1}} \in A_{n-1}), \end{aligned}$$

which proves the first part of item (ii). Furthermore, by item (iii) in the definition of standard Brownian motion, for every $t \in [0, T]$ and $h \in [0, T-t]$ it holds that $W_{t+h} - W_t$ is $\mathcal{N}_{0,hI_{\mathbb{R}^m}}$-distributed, which shows that $W$ has stationary increments and concludes the proof.

(iii) Let $s, t \in [0, T]$. Without loss of generality, we suppose that $s = \min\{s, t\}$. Since $\mathbb{E}[W_s] = \mathbb{E}[W_t] = 0$ and $W_0 = 0$, we have that

$$\begin{aligned} \mathrm{Cov}_P(W_s, W_t) = \mathbb{E}[W_s W_t] &= \mathbb{E}[(W_t - W_s)(W_s - W_0)] + \mathbb{E}[W_s^2] \\ &= \mathbb{E}[(W_t - W_s)]\mathbb{E}[(W_s - W_0)] + \mathbb{E}[W_s^2] \\ &= \mathbb{E}[W_s^2], \end{aligned}$$

where the last two equalities follow by the fact that $W$ has $P$-independent increments and increments are normally distributed random variables with zero mean. Finally, since $W_s$ is $\mathcal{N}_{0,s}$-distributed and $\mathrm{Var}[W_s] = \mathbb{E}[W_s^2] - \mathbb{E}[W_s]^2 = \mathbb{E}[W_s^2]$, it holds that

$$\mathrm{Cov}_P(W_s, W_t) = s = \min\{s, t\},$$

which proves item (iii).

**Webpage:** `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
**Organisation:** Francesca Bartolucci, HG G 53.2