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Series 6

Throughout this exercise sheet, let $T \in (0, +\infty)$, let $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$ be a stochastic basis, $m \in \mathbb{N}$, and let $W\colon [0,T] \times \Omega \to \mathbb{R}^m$ be a m-dimensional standard $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$-Brownian motion.

1. Let $d \in \mathbb{N}$, $\xi \in \mathcal{M}(\mathbb{F}_0, \mathcal{B}(\mathbb{R}^d))$ be a random variable, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be measurable mappings, and consider the SDE

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad t \in [0,T], \qquad X_0 = \xi. \tag{1}$$

(i) Let $M, N \in \mathbb{N}$. Write a MATLAB function `EMMultiDim(`T`,`m`,`ξ`,`μ`,`σ`,`W`)` with inputs $T \in (0, +\infty)$, $m \in \mathbb{N}$, $\xi \in \mathbb{R}^{Md}$, $\mu\colon \mathbb{R}^{d\times M} \to \mathbb{R}^{d\times M}$, $\sigma\colon \mathbb{R}^{d\times M} \to \mathbb{R}^{d\times Mm}$, and $W \in \mathbb{R}^{(N+1)\times mM}$, which returns M realizations $Y_T^N(\omega_i)$, $i = 1, 2, \ldots, M$, of the Euler–Maruyama approximation Y_T^N of X_T.

Note that μ and σ are extended versions of the SDE coefficients from $\mathbb{R}^{d\times M}$ into $\mathbb{R}^{d\times M}$ and $\mathbb{R}^{d\times Mm}$, respectively, and that $\xi \in \mathbb{R}^{Md}$ holds M realizations of the initial condition. Furthermore, the input parameter $W \in \mathbb{R}^{(N+1)\times Mm}$ is a realization of M independent m-dimensional Brownian motions at the equally spaced time points $\{n\Delta t : n = 0, \ldots, N\}$. Hence, $W \in \mathbb{R}^{N+1,Mm}$ should be of the form

$$W = \begin{pmatrix} W_0^\top(\omega_1) & \ldots & W_0^\top(\omega_M) \\ W_{\Delta t}^\top(\omega_1) & \ldots & W_{\Delta t}^\top(\omega_M) \\ \vdots & \ddots & \vdots \\ W_{N\Delta t}^\top(\omega_1) & \ldots & W_{N\Delta t}^\top(\omega_M) \end{pmatrix}$$

where for $n = 0, \ldots, N$ and $i = 1, \ldots, M$ we have

$$W_{n\Delta t}^\top(\omega_i) := \left(W_{n\Delta T}^{(1)}(\omega_i), W_{n\Delta T}^{(2)}(\omega_i), \ldots, W_{n\Delta T}^{(m)}(\omega_i)\right) \in \mathbb{R}^{1\times m}.$$

You may use the template `EMMultiDim.m`.

(ii) Investigate the strong error of the Euler–Maruyama scheme by fixing the parameters $T = 1$, $m = 2$,

$$\xi = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \mu(x_1, x_2) = \begin{pmatrix} 0.5x_1 \\ 2x_2 \end{pmatrix}, \qquad \sigma(x_1, x_2) = \begin{pmatrix} x_1 & 0 \\ 0 & 2x_2 \end{pmatrix},$$

and using $M = 10^5$ and $N = N_\ell = 10\cdot 2^\ell$ for $\ell \in \{0, 1, \ldots, 4\}$. To do so, generate M realizations for every $\ell \in \{0, 1, \ldots, 4\}$ of the Euler–Maruyama approximation $Y_T^{N_\ell}$ of X_T. Then, for every $\ell \in \{0, 1, \ldots, 4\}$ compute a Monte Carlo approximation

$$E_M^\ell := \frac{1}{M}\sum_{i=1}^{M} \|Y_T^{N_\ell}(\omega_i) - X_T(\omega_i)\|_{\mathbb{R}^2} \approx \mathbb{E}[\|Y_T^{N_\ell} - X_T\|_{\mathbb{R}^2}]$$

based on M samples, and determine the "experimental strong convergence rate" with respect to N_ℓ^{-1}. You may use the template `ErrorEM2dGBM.m`.

Hints:

- To construct the matrix $W \in \mathbb{R}^{(N+1)\times 2M}$, first generate the $\mathbb{R}^{(N+1)\times M}$-matrix that holds the realizations of $W^{(1)}$ on the discrete grid. Then repeat this procedure for $W^{(2)}$ and concatenate both matrices horizontally.
- You can derive the exact value X_T for the SDE (1) by Exercise 3 on Sheet 3.
- Estimate the convergence rate by a linear regression of $\log(E_M^\ell)$ on the log-stepsizes $\log(N_\ell^{-1})$. For this you may use the MATLAB function `polyfit`.

2. Consider the setting from Exercise 1 with $d = m = 1$ and let $f : \mathbb{R} \to \mathbb{R}$ be a given test function.

(i) Let the SDE (1) be given by

$$\mathrm{d}X_t = \mu_0 X_t \,\mathrm{d}t + \sigma_0 X_t \,\mathrm{d}W_t, \qquad t \in [0, T], \qquad X_0 = \xi, \tag{2}$$

where $\mu_0, \sigma_0 \in \mathbb{R}$ are constant and $\xi \in \mathcal{M}(\mathbb{F}_0, \mathcal{B}(\mathbb{R}^d))$ is such that $\xi \sim \mathcal{N}_{0,1}$. Show that (2) admits a unique strong solution X such that for all $p \in (0, \infty)$ there holds

$$\mathbb{E}_P[X_t^p] = \mathbb{E}_P[\xi^p]\exp\left(p\mu_0 t + \frac{\sigma_0^2}{2}(p^2 - p)t\right) \in \mathbb{R}.$$

(ii) Let Y^N the Euler-Maruyama approximation of X with $N \in \mathbb{N}$ time steps and let $f(x) := x^n$ for $x \in \mathbb{R}$ and a fixed $n \in \mathbb{N}$. Prove that the sequence $(Y^N, N \in \mathbb{N})$ converges weakly to X for given f, i.e. show that $|\mathbb{E}(f(Y_N^N) - f(X_T))| \le CN^{-1}$ for a $C > 0$ and any $N \in \mathbb{N}$.

3. (i) Investigate the weak convergence of the Euler scheme for the SDE (2) with parameters $T = 1$ and $\mu_0 = 0.5, \sigma_0 = 1$ for the test function $f(x) = x^2$. Generate $M = 10^6$ samples of $f(Y_N^N)$ for each $N = N_\ell = 10 \cdot 2^\ell$ with $\ell \in \{0, 1, \ldots, 4\}$ and use the Monte Carlo approximations

$$E_M^\ell := \frac{1}{M}\sum_{i=1}^{M} f(Y_{N_\ell}^{N_\ell}(\omega_i)) \approx \mathbb{E}(f(Y_{N_\ell}^{N_\ell}))$$

and Exercise 2(i) to calculate the weak error $|\mathbb{E}(f(Y_N^N) - f(X_T))|$ for given N. You may use the template `ErrorEMWeak.m`.

Hints:

- You may modify the solution `EulerMaruyama.m` from Series 4, Exercise 1 to implement the Euler-Maruyama method in one dimension. Alternatively, you could also use your implementation of `EMMulti` from exercise 1(i).
- You may use the built-in function `random(...)` to sample the initial condition ξ.
- Estimate the convergence rate by a linear regression of $\log(E_M^\ell)$ on the log-stepsizes $\log(N_\ell^{-1})$. For this you may use the MATLAB function `polyfit`.

(ii) Repeat part (i) with the same parameters, but using the Milstein scheme instead of the Euler-Maruyama scheme. Comment on the results.

Hint: You may modify the solution `Milstein1D.m` from Series 5, Exercise 1 to implement the Milstein method in one dimension and modify the template/solution `ErrorEMWeak.m` from part (i).

(iii) Repeat part (i) (again with the Euler scheme), but now with the deterministic initial value $\xi \equiv 1$ and for the test function $f(x) := \mathbf{1}_{\{x>5\}}$. Comment on the results.

Hint: Use the built-in function `normcdf` to calculate the exact mean $\mathbb{E}_P[f(X_1)]$.

Due: 16:00 o'clock, Monday, 31st October 2022
Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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