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Series 4

1. The price process S is a geometric Brownian motion with closed form solution

$$S_t = S_0 \exp\left(\left(r - \sigma^2/2\right) t + \sigma W_t\right), \quad t \in [0, T].$$

In particular,

$$S_T = S_0 \exp\left(\left(r - \sigma^2/2\right) T + \sigma W_T\right) = \exp\left(\ln(S_0) + \left(r - \sigma^2/2\right) T + \sigma W_T\right).$$

Furthermore, $W_T \sim \sqrt{T} Z$ with $Z \sim \mathcal{N}_{0,1}$, and consequently

$$S_T = \exp\left(\ln(S_0) + \left(r - \sigma^2/2\right) T + \sigma \sqrt{T} Z\right).$$

We denote

$$\alpha_T = \ln(S_0) + \left(r - \sigma^2/2\right) T, \quad \beta_T = \sigma \sqrt{T}.$$

Then, for every $K \in (0, +\infty)$ we have that

$$\mathbb{E}_P[f(S_T)] = \mathbb{E}_P\left[\max\left\{e^{\alpha_T + \beta_T Z} - K, 0\right\}\right] = \int_{\mathbb{R}} \max\left\{e^{\alpha_T + \beta_T y} - K, 0\right\} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} \, \mathrm{d}y.$$

We observe that

$$e^{\alpha_T + \beta_T y} - K \geq 0 \Longleftrightarrow \alpha_T + \beta_T y \geq \ln(K) \Longleftrightarrow y \geq \frac{\ln(K) - \alpha_T}{\beta_T},$$

and consequently

$$\begin{aligned}
\mathbb{E}_P[f(S_T)] &= \int_{\frac{\ln(K) - \alpha_T}{\beta_T}}^{\infty} \left(e^{\alpha_T + \beta_T y} - K\right) \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} \, \mathrm{d}y \\
&= e^{\alpha_T} \int_{\frac{\ln(K) - \alpha_T}{\beta_T}}^{\infty} \tfrac{1}{\sqrt{2\pi}} e^{\left(\beta_T y - \frac{1}{2} y^2\right)} \, \mathrm{d}y - K \int_{\frac{\ln(K) - \alpha_T}{\beta_T}}^{\infty} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} \, \mathrm{d}y \\
&= e^{\alpha_T} \int_{\frac{\ln(K) - \alpha_T}{\beta_T}}^{\infty} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y^2 - 2\beta_T y + \beta_T^2 - \beta_T^2\right)} \, \mathrm{d}y - K \, \Phi\left(\tfrac{\alpha_T - \ln(K)}{\beta_T}\right) \\
&= e^{\alpha_T + \frac{1}{2}\beta_T^2} \int_{\frac{\ln(K) - \alpha_T}{\beta_T}}^{\infty} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y - \beta_T)^2} \, \mathrm{d}y - K \, \Phi\left(\tfrac{\alpha_T - \ln(K)}{\beta_T}\right).
\end{aligned}$$

Finally, we make the change of variable $z = y - \beta_T$ and we obtain

$$\begin{aligned}
\mathbb{E}_P[f(S_T)] &= e^{\alpha_T + \frac{1}{2}\beta_T^2} \int_{\frac{\ln(K) - \alpha_T}{\beta_T} - \beta_T}^{\infty} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} \, \mathrm{d}z - K \, \Phi\left(\tfrac{\alpha_T - \ln(K)}{\beta_T}\right) \\
&= e^{\alpha_T + \frac{1}{2}\beta_T^2} \, \Phi\left(\tfrac{\alpha_T - \ln(K)}{\beta_T} + \beta_T\right) - K \, \Phi\left(\tfrac{\alpha_T - \ln(K)}{\beta_T}\right).
\end{aligned}$$

Since

$$e^{-rT}e^{\alpha_T+\frac{1}{2}\beta_T^2} = e^{\ln(S_0)} = S_0,$$

we arrive at the desired formula

$$\begin{aligned}
e^{-rT}\mathbb{E}_P[f(S_T)] &= S_0\Phi\Big(\tfrac{\alpha_T-\ln(K)}{\beta_T}+\beta_T\Big) - e^{-rT}K\,\Phi\Big(\tfrac{\alpha_T-\ln(K)}{\beta_T}\Big)\\
&= S_0\Phi\left(\frac{\left(r-\frac{\sigma^2}{2}\right)T+\ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}}+\sigma\sqrt{T}\right) - e^{-rT}K\,\Phi\left(\frac{\left(r-\frac{\sigma^2}{2}\right)T+\ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}}\right)\\
&= S_0\Phi\left(\frac{\left(r+\frac{\sigma^2}{2}\right)T+\ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}}\right) - e^{-rT}K\,\Phi\left(\frac{\left(r-\frac{\sigma^2}{2}\right)T+\ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}}\right).
\end{aligned}$$

2. (i) We define the mappings

$$\mu\colon \mathbb{R}\to\mathbb{R},\quad x\mapsto \log(1+x^2)$$

and

$$\sigma\colon \mathbb{R}\to\mathbb{R},\quad x\mapsto \mathbf{1}_{\{x>0\}}x.$$

In order to show existence and uniqueness of the solution process $X\colon[0,T]\times\Omega\to\mathbb{R}$, we prove that the conditions of Theorem 2.5.1 are fulfilled.
Since $\xi\in\mathcal{L}^p(P|_{\mathbb{F}_0};|\cdot|)$ for some $p\geq 2$ by assumption, it remains to show that μ and σ are globally Lipschitz continuous. We start considering the mapping μ. By the mean value theorem, we obtain that for every $x,y\in\mathbb{R}$

$$|\mu(x)-\mu(y)| = |\log(1+x^2)-\log(1+y^2)| \leq \sup_{\xi\in\mathbb{R}}\left|\mu'(\xi)\right||x-y| = \sup_{\xi\in\mathbb{R}}\left|\frac{2\xi}{1+\xi^2}\right||x-y|.$$

Moreover, we note that for every $\xi\in\mathbb{R}$

$$(1+\xi)^2\geq 0 \quad\Longleftrightarrow\quad 1+\xi^2\geq|2\xi| \quad\Longleftrightarrow\quad \frac{|2\xi|}{1+\xi^2}\leq 1.$$

Hence, we have that

$$|\mu(x)-\mu(y)| \leq \sup_{\xi\in\mathbb{R}}\frac{|2\xi|}{1+\xi^2}|x-y| \leq |x-y|.$$

In addition, it is immediate to check that σ satisfies for every $x,y\in\mathbb{R}$

$$|\sigma(x)-\sigma(y)| \leq |x\mathbf{1}_{\{x>0\}} - y\mathbf{1}_{\{y>0\}}| \leq |x-y|.$$

The above computations show that μ and σ are globally Lipschitz continuous. Therefore, the conditions of Theorem 2.5.1 are fulfilled and the SDE

$$\mathrm{d}X_t = \log(1+X_t^2)\mathrm{d}t + \mathbf{1}_{\{X_t>0\}}X_t\mathrm{d}W_t,\quad t\in[0,T],\quad X_0=\xi,$$

admits a unique (up to indistinguishability) solution process $X\colon[0,T]\times\Omega\to\mathbb{R}$.

(ii) See file `EulerMaruyama.m`.

Siehe nächstes Blatt!

(iii) See file `ErrorEM.m`. We know by item (i) that the coefficients of the SDE are globally Lipschitz continuous and $\xi = 1$. Hence, all assumptions of Theorem 3.3.10 are satisfied and consequently the Euler-Maruyama scheme converges in the strong L^2-sense with order $\alpha = 1/2$. The expected convergence rate is realized in the experiment.

3. (i) See file `EulerMaruyamaGL.m`.

(ii) See file `EulerMaruyamaGLEstimator.m`. The drift coefficient of X is not globally Lipschitz continuous and grows superlinearly. A a consequence, the the Euler–Maruyama scheme does not converge in the strong L^2-sense

(iii) See files `TamedEulerMaruyamaGL.m` and `TamedEulerMaruyamaGLEstimator.m`.

Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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