Dr. Andreas Stein
Dr. Francesca Bartolucci

Series 11

1. (i) See file `EulerMaruyamaLevySDE.m`.

(ii) See file `EulerMaruyamaGammaSDE.m`.

2. (i) Let $n \in \mathbb{N}$. By the given hint, we have that for all $x \in \mathbb{R}$

$$\phi_X(x) = e^{i\mu x - \lambda|x|} = \prod_{k=1}^{n} e^{\frac{1}{n}(i\mu x - \lambda|x|)} = \prod_{k=1}^{n} e^{i\frac{\mu}{n}x - \frac{\lambda}{n}|x|} = \prod_{k=1}^{n} \phi_{Y_k}(x) = \phi_{\sum_{k=1}^{n} Y_k}(x),$$

where $Y_1, \dots, Y_n$ are i.i.d. $C_{\frac{\mu}{n},\frac{\lambda}{n}}$-random variables. Consequently, by Propositi-on 0.5.2, $X \stackrel{d}{=} \sum_{k=1}^{n} Y_k$, which proves the claim.

(ii) By definition of mean, we have to compute the improper integral

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f_{\mu,\lambda}(x)\mathrm{d}x = \frac{1}{\pi\lambda}\int_{\mathbb{R}} \frac{x\lambda^2}{(x-\mu)^2+\lambda^2}\mathrm{d}x. \tag{1}$$

Since the integrand is an odd function, if the integral converges it must equal zero. However, the integrand goes to zero at infinity with infinitesimal order 1 and consequently the integral diverges. On the other hand, by symmetry, the integral cannot diverge to either $+\infty$ or $-\infty$. As a result, the improper integral (1) is undefined. Alternatively, we can compute

$$\lim_{a\to+\infty} \int_{-a}^{a} x f_{\mu,\lambda}(x)\mathrm{d}x = 0,$$

together with the improper integral

$$\begin{aligned}
\lim_{a\to+\infty} \int_{-a}^{2a} x f_{\mu,\lambda}(x)\mathrm{d}x &= \lim_{a\to+\infty} \int_{a}^{2a} x f_{\mu,\lambda}(x)\mathrm{d}x \\
&= \lim_{a\to+\infty} \frac{\lambda}{\pi}\left[\frac{1}{2}\int_{a}^{2a} \frac{2(x-\mu)}{(x-\mu)^2+\lambda^2}\mathrm{d}x + \int_{a}^{2a} \frac{\mu}{(x-\mu)^2+\lambda^2}\mathrm{d}x\right] \\
&= \lim_{a\to+\infty} \frac{\lambda}{2\pi}\log[(x-\mu)^2+\lambda^2]_a^{2a} + 0 \\
&= \lim_{a\to+\infty} \frac{\lambda}{2\pi}\log\left[\frac{(2a-\mu)^2+\lambda^2}{(a-\mu)^2+\lambda^2}\right] = \frac{\lambda\log(2)}{\pi}.
\end{aligned}$$

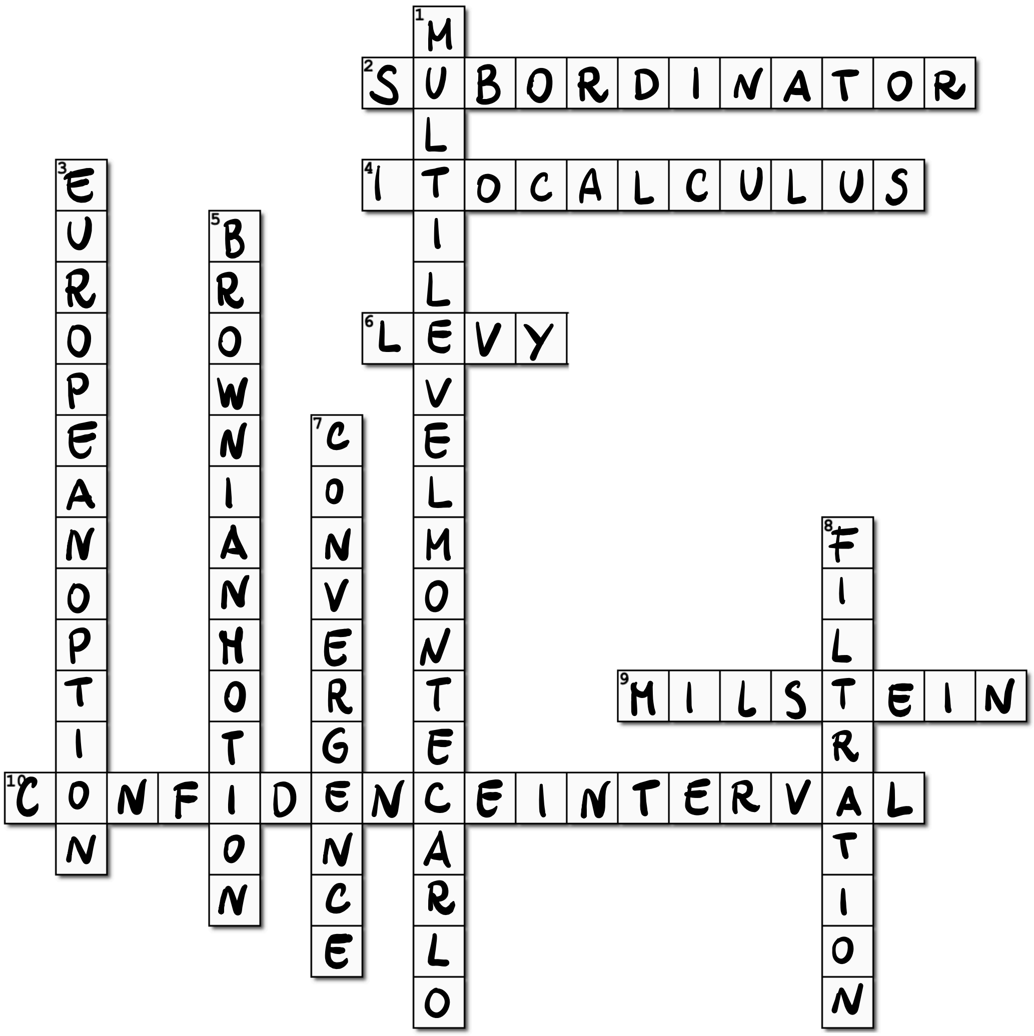
Then, we have that

$$\lim_{a\to+\infty} \int_{-a}^{a} x f_{\mu,\lambda}(x)\mathrm{d}x \neq \lim_{a\to+\infty} \int_{-a}^{2a} x f_{\mu,\lambda}(x)\mathrm{d}x,$$

and we conclude that $\mathbb{E}[X]$ is undefined.

(iii) See file `EulerMaruyamaCauchySDE.m`.

3. Solution of the crossword puzzle:



Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
Organisation: Francesca Bartolucci, HG G 53.2