D-MATH Numerical Analysis of Stochastic ODEs Autumn 2022

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Series 8

Throughout this sheet let $T \in (0, +\infty)$, $d, m, N, K \in \mathbb{N}$, let $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]})$ be a stochastic basis, and let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be an m-dimensional standard $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]})$-Brownian motion.

1. Let $\xi \in \mathbb{R}^d$, $\mu \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, $\sigma \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^{d\times m}))$, let $X\colon [0,T]\times\Omega \to \mathbb{R}^d$ be a solution process of the SDE

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad t \in [0,T], \qquad X_0 = \xi,$$

and let $f \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$ satisfy $\mathbb{E}_P\big[|f(X_T)|\big] < +\infty$.

(i) Write a MATLAB function `MonteCarloEuler(`T`,`m`,`N`,`K`,`ξ`,`μ`,`σ`,`f`)` with inputs $T, m, N, K, \xi, \mu, \sigma, f$ as above and output a realization of a Monte Carlo Euler approximation of $\mathbb{E}_P\big[f(X_T)\big]$ based on K samples and time step size $\Delta t = T/N$. Hint: You may use the file `EMMultiDim.m` from Sheet 6.

(ii) Consider the Black-Scholes model, where the price process of an underlying S is modeled by the SDE

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \quad t \in [0,T], \quad S_0 = s_0,$$

for a fixed interest rate $r \in \mathbb{R}$, a volatility parameter $\sigma > 0$ and an initial price $s_0 > 0$. Test your MATLAB function `MonteCarloEuler()` from item (i) to evaluate a European call option with strike price $K_{\text{strike}} > 0$ and payoff at T given by $f(S_T) = \max(S_T - K_{\text{strike}}, 0)$. To this end, run the Monte Carlo-Euler scheme with $K \in \{2^{10}, 2^{12}, \ldots, 2^{18}\}$ Monte Carlo samples and adjust the step size $\Delta t = T/N$ for each K so the that statistical error and the discretization bias in the the root mean squared error (RMSE)

$$\left\| e^{-rT}\mathbb{E}_P\big[f(S_T)\big] - e^{-rT}\frac{1}{K}\sum_{k=1}^{K} f\big(Y_N^{N,k}\big) \right\|_{L^2(P;|\cdot|_{\mathbb{R}})} \tag{1}$$

are balanced. In Equation (1), $Y_N^{N,k}$ denotes the k-th sample of the Euler-Maruyama approximation of S_T with stepsize $\Delta t = T/N$. Use the Black-Scholes parameters $T = 1, S_0 = 100, r = 0.01, \sigma = 0.1$ and $K_{\text{strike}} = 90$. Estimate the RMSE in Equation (1) by generating 10 realizations of the weak error

$$e^{-rT}\mathbb{E}_P\big[f(S_T)\big] - e^{-rT}\frac{1}{K}\sum_{k=1}^{K} f\big(Y_N^{N,k}\big)$$

for each K and Δt and average the squared realizations (see the template). Plot the estimated RMSE against Δt in a logarithmic diagram and estimate the convergence rates of the weak error with respect to Δt and K.

Hints: You may use template `MonteCarloEulerBSCall.m`. The exact value of the call price $e^{-rT}\mathbb{E}_P\big[f(S_T)\big]$ is given by the Black Scholes formula in Series 4. You may also use the MATLAB function `blsprice()`. Depending on your workstation this simulation might take a minute or two. Set the parameter `K_RMSE` in the template to `K_RMSE = 1` first to make sure everything works, and then rerun the experiment with `K_RMSE = 10`.

2. Consider the Heston model with stochastic volatility for the underlying S, that is for $t \in [0,T]$ the dynamics are given by the system of SDEs

$$\begin{aligned} \mathrm{d}S_t =& rS_t\,\mathrm{d}t + \sqrt{V_t}S_t\,\mathrm{d}W_t^{(1)}, \\ \mathrm{d}V_t =& a(b - V_t)\,\mathrm{d}t + \sigma_{\mathrm{v}}\sqrt{V_t}\,\left(\rho\,\mathrm{d}W_t^{(1)} + \sqrt{1-\rho^2}\,\mathrm{d}W_t^{(2)}\right), \end{aligned} \tag{2}$$

with initial values $S_0 > 0$ and $V_0 > 0$. Here, $\rho \in [-1,1]$ and $r, a, b, \sigma_{\mathrm{v}} > 0$ are constants. For simplicity, we assume uncorrelatedness, i.e., $\rho = 0$.

a) Repeat item (ii) from Exercise 1 in case that the price process S is given by the Heston model. To this end, implement the *truncated Euler scheme* for the Heston model (see Exercise 1 in Sheet 7). Set the Heston parameters to $T = 1$, $S_0 = 100$, $V_0 = 0.5$, $r = 0.01$, $a = 2$, $b = 0.5$, $\sigma_{\mathrm{v}} = 0.25$ and choose $K_{\text{strike}} = 90$. Use the simulation parameters $K \in \{2^{10}, 2^{12}, \ldots, 2^{18}\}$ and $\Delta t = K^{-1/2}$ for each K.

Hints: You may use the template `MonteCarloEulerHestonCall.m`. The true price of the call option for the given parameters is $e^{-rT}\mathbb{E}_P[f(S_T)] \approx 31.8991$. This simulation might take up to ten minutes.

b) Repeat part a) for the new model parameters $V_0 = 0.1$, $r = 0.01$, $a = 0.25$, $b = 0.1$, and $\sigma_{\mathrm{v}} = 0.5$. Vary the coupling of K and Δt by running the test for

1. $K \in \{2^{10}, 2^{12}, \ldots, 2^{18}\}$ and $\Delta t = K^{-1/2}$,
2. $K \in \{2^{7}, 2^{8}, \ldots, 2^{11}\}$ and $\Delta t = K^{-1}$.

Compare the results with part a) and explain the different behaviors of the RMSE.

Hint: The true price of the call option for the given parameters is $e^{-rT}\mathbb{E}_P[f(S_T)] \approx 17.5817$.

3. Consider the Black-Scholes model, where the price process of an underlying S is modeled by the SDE

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \quad t \in [0,T], \quad S_0 = s_0,$$

for a fixed interest rate $r \in \mathbb{R}$, a volatility parameter $\sigma > 0$ and an initial price $s_0 > 0$. Edit the template `MCERichardsonExtrapolation.m` to valuate a European call option with strike price $K_{\text{strike}} > 0$ and payoff at T given by $f(S_T) = \max(S_T - K_{\text{strike}}, 0)$. To this end, combine the Monte Carlo-Euler scheme with the Richardson extrapolation method using a uniform temporal mesh with $N = 10 : 2 : 20$ and K Monte Carlo samples. Adjust the parameter K for each given N such that the statistical error and the discretization bias in the root mean squared error (RMSE)

$$\left\| e^{-rT}\mathbb{E}_P\big[f(S_T)\big] - e^{-rT}\frac{1}{K}\sum_{k=1}^{K}\big(2\cdot f\big(Y_{2N}^{2N,k}\big) - f\big(Y_N^{N,k}\big)\big)\right\|_{L^2(P;|\cdot|_{\mathbb{R}})} \tag{3}$$

are balanced. In Equation (3), $Y_N^{N,k}$ denotes the k-th sample of the Euler-Maruyama approximation of S_T with stepsize $\Delta t = T/N$. Use the Black-Scholes parameters $T = 1, S_0 = 100, r = 0.01, \sigma = 0.1$ and $K_{\text{strike}} = 90$. Estimate the RMSE in equation (3) by generating 10 realizations of the weak error

$$e^{-rT}\mathbb{E}_P\big[f(S_T)\big] - e^{-rT}\frac{1}{K}\sum_{k=1}^{K}\big(2\cdot f\big(Y_{2N}^{2N,k}\big) - f\big(Y_N^{N,k}\big)\big)$$

for each N and K, and average the squared realizations (see the template). Estimate the convergence rates of the weak error with respect to Δt and K.

Hints:

- You may use your MATLAB function `MonteCarloEuler()` from Exercise 1.
- Depending on your workstation this simulation might take a minute or two. Set the parameter `K_RMSE` in the template to `K_RMSE = 1` first to make sure everything works, and then rerun the experiment with `K_RMSE = 10`.

Due: 16:00 o'clock, Monday, 21st November 2022
Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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