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Series 3

1. (i) By direct computation, we have that

$$\begin{aligned}\mathbb{E}_P\big[e^{\beta X}\big] &= \int_{\mathbb{R}} e^{\beta x}\, \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\, dx = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\left(\beta x - \frac{1}{2}x^2\right)}\, dx \\ &= \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}\left(x^2 - 2\beta x + \beta^2 - \beta^2\right)}\, dx = \tfrac{1}{\sqrt{2\pi}}\, e^{\frac{1}{2}\beta^2} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-\beta)^2}\, dx \\ &= e^{\frac{1}{2}\beta^2} \int_{\mathbb{R}} \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\, dy = e^{\frac{1}{2}\beta^2},\end{aligned}$$

where we have used the change of variable $y = x - \beta$.

(ii) Define the test function

$$f : [0,1) \times \mathbb{R} \to \mathbb{R}, \quad (t,y) \mapsto (t-1)y,$$

and let $\widetilde{X} \colon [0,1) \times \Omega \to \mathbb{R}$ be the Itô process given for every $t \in [0,1)$ by

$$\widetilde{X}_t := \int_0^t \frac{1}{s-1} \mathrm{d}W_s.$$

Then, observe that $X_t = f(t, \widetilde{X}(t)) = (t-1)\widetilde{X}_t$ for all $t \in [0,1)$, and apply the time-dependent Itô formula

$$\begin{aligned} X_t &= f(t, \widetilde{X}(t)) - f(0, \widetilde{X}(0)) \\ &= \int_0^t (\widetilde{X}_s + (s-1)\cdot 0)\mathrm{d}s + \int_0^t (s-1)\frac{1}{s-1}\mathrm{d}W_s + \frac{1}{2}\int_0^t 0\mathrm{d}s \\ &= \int_0^t \widetilde{X}_s \mathrm{d}s + \int_0^t \mathrm{d}W_s = \int_0^t \frac{1}{s-1} X_s \mathrm{d}s + W_t, \end{aligned}$$

which proves the desired formula.

2. (i) Define the test function

$$f \colon \mathbb{R}^2 \to (0, +\infty),\ (x,y) \mapsto \exp\left(x - y\right).$$

Then, we have that

$$\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2} = f, \quad \frac{\partial f}{\partial y} = -f \quad \text{and} \quad \frac{\partial f}{\partial y^2} = f.$$

Furthermore, by definition, $\mathcal{E}(X)_t = f(X_t - X_0, \frac{1}{2}\int_0^t Z_s^2 \mathrm{d}s)$ for all $t \in [0,T]$. Thus, the time-dependent Itô formula yields

$$\begin{aligned}\mathcal{E}(X)_t = f(X_t - X_0, -\frac{1}{2}\int_0^t Z_s^2 \mathrm{d}s) &= f(0,0) + \int_0^t \mathcal{E}(X)_s \left(Y_s - \frac{1}{2}Z_s^2\right)\mathrm{d}s \\ &\quad + \int_0^t \mathcal{E}(X)_s Z_s \mathrm{d}W_s + \frac{1}{2}\int_0^t \mathcal{E}(X)_s Z_s^2 \mathrm{d}s \\ &= 1 + \int_0^t \mathcal{E}(X)_s Y_s \mathrm{d}s + \int_0^t \mathcal{E}(X)_s Z_s \mathrm{d}W_s,\end{aligned}$$

which proves item (i).

(ii) Let $t \in [0,T]$. We apply Itô's formula with $f\colon (0,+\infty) \to \mathbb{R}$, $x \mapsto \log(x)$, and we obtain

$$\begin{aligned}\log(X_t) &= \log(X_0) + \int_0^t \frac{1}{X_s}\mu(X_s)\mathrm{d}s + \int_0^t \frac{1}{X_s}\sigma(X_s)\mathrm{d}W_s + \frac{1}{2}\int_0^t \frac{-1}{X_s^2}\sigma(X_s)^2\mathrm{d}s \\ &= 0 + \mathcal{L}(X)_t - \frac{1}{2}\int_0^t \frac{\sigma^2(X_s)}{X_s^2}\mathrm{d}s, \qquad (1)\end{aligned}$$

where the last equality follows by the definition of stochastic logarithm of X. By the definition of stochastic exponential of X and Equation (1), we have that for all $t \in [0,T]$

$$\begin{aligned}\mathcal{E}(\mathcal{L}(X))_t &= \exp\left(\mathcal{L}(X)_t - \frac{1}{2}\int_0^t \frac{\sigma^2(X_s)}{X_s^2}\mathrm{d}s\right) \\ &= \exp\left(\log(X_t) + \frac{1}{2}\int_0^t \frac{\sigma^2(X_s)}{X_s^2}\mathrm{d}s - \frac{1}{2}\int_0^t \frac{\sigma^2(X_s)}{X_s^2}\mathrm{d}s\right) \\ &= X_t,\end{aligned}$$

which concludes the exercise.

3. (i) Consider the process $\widetilde{Y}\colon [0,T]\times\Omega \to \mathbb{R}$ defined by

$$\widetilde{Y}_t := \int_0^t \mu_1(s)\mathrm{d}s + \int_0^t \sigma_1(s)\mathrm{d}W_s, \quad t \in [0,T].$$

Since every continuous function on a compact interval is bounded, it holds that

$$\int_0^T |\mu_1(s)| + |\sigma_1(s)|^2 dt \le \left(\sup_{s\in[0,T]} |\mu_1(s)| + \sup_{s\in[0,T]} |\sigma_1(s)|^2\right) T < +\infty.$$

Then, by definition, $\widetilde{Y}$ is an Itô process and the stochastic exponential $\mathcal{E}(\widetilde{Y})$ of $\widetilde{Y}$ is given by

$$\begin{aligned}\mathcal{E}(\widetilde{Y})_t &= \exp\left(\widetilde{Y}_t - \frac{1}{2}\int_0^t \sigma_1(s)^2\mathrm{d}s\right) \\ &= \exp\left(\int_0^t \mu_1(s) - \frac{\sigma_1(s)^2}{2}\mathrm{d}s + \int_0^t \sigma_1(s)\mathrm{d}W_s\right).\end{aligned}$$

Furthermore, by Exercise 2, $\mathcal{E}(\widetilde{Y})$ satisfies the identity

$$\mathcal{E}(\widetilde{Y})_t = 1 + \int_0^t \mu_1(s)\mathcal{E}(\widetilde{Y})_s \mathrm{d}s + \int_0^t \sigma_1(s)\mathcal{E}(\widetilde{Y})_s \mathrm{d}W_s, \quad t \in [0,T],$$

and therefore $Y := \mathcal{E}(\widetilde{Y})$ is a solution process of the SDE

$$\mathrm{d}Y_t = \mu_1(t)Y_t\mathrm{d}t + \sigma_1(t)Y_t\mathrm{d}W_t, \quad t \in [0,T], \quad Y_0 = 1. \tag{2}$$

By the given hint, the uniqueness (up to indistinguishability) follows by showing the global Lipschitz continuity of the SDE coefficients together with a linear growth condition. For every $t \in [0,T]$ it holds that

$$|x \cdot \mu_1(t) - y \cdot \mu_1(t)| = |\mu_1(t)||x-y| \leq C|x-y|, \quad \forall x, y \in \mathbb{R},$$

where $C = \max\{|\mu_1(t)| : t \in [0,T]\} < +\infty$, which proves that the SDE coefficient $\mu\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ defined by $\mu(t,x) = x \cdot \mu_1(t)$ is global Lipschitz continuous in the second variable, uniformly in t. The proof of the global Lipschitz continuity of $\sigma\colon [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma(t,x) = x \cdot \sigma_1(t)$, is analogous. We finally prove that the coefficients μ and σ are of linear growth in the second variable, uniformly in t. For all $t \in [0,T]$ we have that

$$|\mu(t,x)| + |\sigma(t,x)| = |\mu_1(t)||x| + |\sigma_1(t)||x| \leq C|x|, \quad \forall x \in \mathbb{R}$$

where $C = \max\{|\mu_1(t)| + |\sigma_1(t)| : t \in [0,T]\} < +\infty$, which concludes item (i).

(ii) Let $Z^{(1)}, Z^{(2)}\colon [0,T] \times \Omega \to \mathbb{R}$ be predictable processes satisfying

$$P\left(\int_0^T |Z_t^{(1)}| + |Z_t^{(2)}|^2 dt < +\infty\right) = 1. \tag{3}$$

Then, $Z\colon [0,T] \times \Omega \to \mathbb{R}$ given by

$$Z_t := X_0 + \int_0^t Z_s^{(1)} ds + \int_0^t Z_s^{(2)} dW_s, \quad t \in [0,T],$$

is an Itô process. Let Y be the unique (up to indistinguishability) solution process of the homogeneous SDE (2) from item (i) and define $X = YZ$. Then, the two-dimensional Itô formula applied to the process (Y,Z) and the function $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(y,z) \mapsto yz$, yields

$$\begin{aligned} X_t - X_0 &= Y_t Z_t - Y_0 Z_0 \\ &= \int_0^t Z_s \mu_1(s) Y_s + Y_s Z_s^{(1)} ds + \int_0^t Z_s \sigma_1(s) Y_s + Y_s Z_s^{(2)} dW_s + \int_0^t \sigma_1(s) Y_s Z_s^{(2)} ds \\ &= \int_0^t \mu_1(s) X_s + Y_s\big(Z_s^{(1)} + \sigma_1(s) Z_s^{(2)}\big) ds + \int_0^t \sigma_1(s) X_s + Y_s Z_s^{(2)} dW_s. \end{aligned} \tag{4}$$

Hence, it remains to choose $Z^{(1)}, Z^{(2)}$ in a suitable way such that $X = YZ$ is a solution process of the linear non-homogeneous SDE

$$\mathrm{d}X_t = (\mu_1(t)X_t + \mu_2(t))\mathrm{d}t + (\sigma_1(t)X_t + \sigma_2(t))\mathrm{d}W_t, \quad t \in [0,T], \quad X_0 = \xi. \tag{5}$$

By Item (i), we know that

$$Y_t = \exp\left(\int_0^t \mu_1(s) - \frac{\sigma_1(s)^2}{2} ds + \int_0^t \sigma_1(s) dW_s\right), \quad t \in [0,T].$$

Then, Y has P-a.s. positive and continuous sample paths, and therefore Y is predictable. Moreover, the process $Y^{-1} := (Y_t^{-1}, t \in [0,T])$ is well-defined and has P-a.s. positive and continuous sample paths. Hence, Y^{-1} is also predictable. By Equations (4) and (5), we set

$$Z_t^{(1)} := Y_t^{-1}(\mu_2(t) - \sigma_1(t)\sigma_2(t)), \quad Z_t^{(2)} := Y_t^{-1}\sigma_2(t), \quad t \in [0,T]. \tag{6}$$

Thus, by this choice of $Z^{(1)}$ and $Z^{(2)}$, we have that

$$Y_s\big(Z_s^{(1)} + \sigma_1(s) Z_s^{(2)}\big) = \mu_2(s), \quad Y_s Z_s^{(2)} = \sigma_2(s), \quad s \in [0,T],$$

and Equation (4) equals (5), i.e. $X = YZ$ satisfies

$$X_t - X_0 = \int_0^t \mu_1(s) X_s + \mu_2(s) ds + \int_0^t \sigma_1(s) X_s + \sigma_2(s) dW_s. \tag{7}$$

By construction, $Z^{(1)}$ and $Z^{(2)}$ are predictable processes, but we need to show that $Z^{(1)}$ and $Z^{(2)}$ satisfy Equation (3). We recall that Y^{-1} has P-a.s. positive, continuous sample paths on $[0,T]$. Moreover, by hypothesis, the coefficients $\mu_1, \mu_2, \sigma_1, \sigma_2 : [0,T] \to \mathbb{R}$ are globally Lipschitz continuous functions. Therefore, the processes $Z^{(1)}$ and $Z^{(2)}$ given by (6) have P-a.s. continuous sample paths on $[0,T]$. Since continuous functions on compact intervals are bounded, it holds P-a.s that

$$\int_0^T |Z_t^{(1)}| + |Z_t^{(2)}|^2 dt \leq \left(\sup_{t\in[0,T]} |Z_t^{(1)}| + \sup_{t\in[0,T]} |Z_t^{(2)}|^2\right) T < +\infty,$$

and Equation (3) is satisfied. Therefore, by Equation (7), we have that $X = YZ$ is a solution process of the linear non-homogeneous SDE (5).

Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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