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Series 7

Throughout this exercise sheet, let $T \in (0, +\infty)$, $m \in \mathbb{N}$, let $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$ be a stochastic basis, and let $W = (W^{(1)}, \dots, W^{(m)}): [0,T] \times \Omega \to \mathbb{R}^m$ be a m-dimensional standard $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$-Brownian motion.

1. Consider the Heston model with stochastic volatility for the underlying S, that is for $t \in [0,T]$ the dynamics are given by the system of SDEs

$$\begin{aligned} \mathrm{d}S_t =& rS_t \,\mathrm{d}t + \sqrt{V_t} S_t \,\mathrm{d}W_t^{(1)}, \\ \mathrm{d}V_t =& a(b - V_t)\,\mathrm{d}t + \sigma_\mathrm{v} \sqrt{V_t} \left(\rho \,\mathrm{d}W_t^{(1)} + \sqrt{1-\rho^2}\,\mathrm{d}W_t^{(2)} \right), \end{aligned} \tag{1}$$

with initial values $S_0 > 0$ and $V_0 > 0$. Here, $\rho \in [-1, 1]$ and $r, a, b, \sigma_\mathrm{v} > 0$ are constants. For simplicity, we assume uncorrelatedness, i.e., $\rho = 0$. Then, we set the Heston parameters to $T = 1$, $S_0 = 10$, $V_0 = 0.5$, $r = 0.05$, $a = 2$, $b = 0.5$, and $\sigma_\mathrm{v} = 0.25$.

The *truncated Euler–Maruyama scheme* with step size $\Delta t = T/N$ for the approximation of the SDE (1) is given for $n = 1, \dots, N$ by the iteration

$$\begin{aligned} Y_n^N =& Y_{n-1}^N + rY_{n-1}^N \Delta t + \sqrt{\max(\overline{V}_{n-1}^N, 0)}\, Y_{n-1}^N \left(W_{t_n}^{(1)} - W_{t_{n-1}}^{(1)} \right), \\ \overline{V}_n^N =& \overline{V}_{n-1}^N + a(b - \overline{V}_{n-1}^N)\Delta t + \sigma_\mathrm{v} \sqrt{\max(\overline{V}_{n-1}^N, 0)} \left(W_{t_n}^{(2)} - W_{t_{n-1}}^{(2)} \right), \end{aligned} \tag{2}$$

with initial values $Y_0^N = S_0$ and $\overline{V}_0^N = V_0$. Hence, the discrete volatility process $\overline{V}$ is truncated at zero to avoid negative values in the square roots.

(i) Implement the truncated Euler–Maruyama scheme and investigate the strong error as in Exercise 1 in Series 6. To do this, use the same parameters $M = 10^5$ and $N_\ell = 10 \cdot 2^\ell$ for $\ell = 0, 1, ..., 4$. **Report** on the experimental rates of strong convergence in L^1 and L^2. Use as an approximation of the exact solution a numerical solution of the SDE on the level $\ell = 6$.
Hint: You may use the template `EM_Heston.m` and the solution `EMMultiDim.m` from Series 6.

(ii) Repeat item (i) for the choice of parameters $T = 1$, $S_0 = 10$, $V_0 = 0.5$, $r = 0.05$, $a = 1$, $b = 0.5$, and $\sigma_\mathrm{v} = 2$. **Comment** on the results.

2. Consider the SDE given by

$$\mathrm{d}X_t = \mu_0 X_t \,\mathrm{d}t + \sigma_0 X_t \,\mathrm{d}W_t, \qquad t \in [0,T], \qquad X_0 = \xi, \tag{3}$$

where $\mu_0, \sigma_0 \in \mathbb{R}$ are constant and $\xi \in \mathcal{M}(\mathbb{F}_0, \mathcal{B}(\mathbb{R}^d))$ is such that $\xi \sim \mathcal{N}_{0,1}$.

(i) Investigate the weak convergence of the Euler–Maruyama scheme combined with the Richardson extrapolation method for the SDE (3) with parameters $T = 1$, $\mu_0 = 0.5$, $\sigma_0 = 1$ and for the test function $f(x) = x^2$. To this end, compute the weak error

$$\mathbb{E}[f(X_T)] - \sum_{k=1}^{M} \left(2 \cdot f\left(Y_N^{N,k}\right) - f\left(Y_{N/2}^{N/2,k}\right)\right) \tag{4}$$

for each $N = N_\ell = 10 \cdot 2^\ell$ with $\ell \in \{0, 1, \ldots, 3\}$ and $M = 10^7$. In equation (4), $Y_N^{N,k}$ denotes the k-th sample of the Euler-Maruyama approximation of X_T with stepsize $\Delta t = T/N$.

Hints:

- You may modify the solution `ErrorEMWeak.m` from Exercise 2 in Series 6.
- Depending on your workstation this simulation might take two or three minutes.

(ii) **Remark 4.4.2.** Let Y^N be the Euler-Maruyama approximation of X with $N \in \mathbb{N}$ time steps and let f be a smooth function with polynomially bounded derivatives. Let $M \geq 1$. We define the higher-order extrapolation A_M recursively via

$$A_0 := f(\overline{Y}_T^N), \quad A_m(N) := \frac{2^m A_{m-1}(2N) - A_{m-1}(N)}{2^m - 1}, \; m = 1, \ldots, M.$$

Prove that for every $M \geq 1$

$$|\mathbb{E}[f(X_T)] - \mathbb{E}[A_M(N)]| \leq C N^{-(M+1)}$$

for a $C > 0$ and large enough $N \in \mathbb{N}$.

Hints: You may use without proof the higher order expansions

$$\mathbb{E}[f(X_T)] - \mathbb{E}[f(\overline{Y}_T^N)] = \sum_{m=1}^{M} C_m N^{-m} + \mathcal{O}(N^{-(M+1)}),$$

for every $M \in \mathbb{N}$, where $C_1, \ldots, C_M \in \mathbb{R}$ are constants independent of N.

3. Let $g\colon (1, +\infty) \to \mathbb{R}$, $x \mapsto x^{-3}$, and define the integral

$$\mu := \int_1^{+\infty} g(x) \mathrm{d}x.$$

In this exercise we estimate the value of μ by a Monte Carlo algorithm and we derive the corresponding confidence intervals.

Siehe nächstes Blatt!

(i) Let U be a $\mathcal{U}_{(0,1)}$-distributed random variable. Determine a function $h\colon (0,1) \to \mathbb{R}$ such that

$$\mu = \mathbb{E}_P[h(U)]$$

and compute the value of μ.

(ii) Write a MATLAB function `MCEstimatorCI(h,N, alpha)` with inputs h from item (i), $N \in \mathbb{N}$ and $\alpha \in [0,1]$ that generates N independent realizations $(U_i)_{i=1}^N$ of U and returns as outputs

1. the sample mean E_N for the set $(h(U_i))_{i=1}^N$;
2. the (unbiased) sample variance V_N for the set $(h(U_i))_{i=1}^N$;
3. the (asymptotically valid) α-confidence interval $[a_N^{\mathrm{Ch}}, b_N^{\mathrm{Ch}}]$ centered at E_N based on the Chebyshev inequality;
4. the (asymptotically valid) α-confidence interval $[a_N^{\mathrm{CLT}}, b_N^{\mathrm{CLT}}]$ centered at E_N based on the CLT.

For $N \in \{2^8, 2^9, \ldots, 2^{16}\}$ plot in a logarithmic diagram the true value of μ, the estimate E_N, and the 95%-confidence intervals $[a_N^{\mathrm{CLT}}, b_N^{\mathrm{CLT}}]$ and $[a_N^{\mathrm{Ch}}, b_N^{\mathrm{Ch}}]$ on the y-axis, against the values of N on the x-axis.

(iii) For $N \in \{2^8, 2^9, \ldots, 2^{16}\}$ and $\alpha = 0.95$, run $M = 10^3$ times the MATLAB function `MCEstimatorCI(h,N, alpha)` from item (ii), and generate two plots containing:

1. the values of $N^{-1/2}$, the average of $|E_N - \mu|$ over all M, and the average of $\sqrt{V_N/N}$ over all M on the y-axis, against the values of N on the x-axis;
2. the percentages of $\mu \in [a_N^{\mathrm{CLT}}, b_N^{\mathrm{CLT}}]$ and $\mu \in [a_N^{\mathrm{Ch}}, b_N^{\mathrm{Ch}}]$ over all M on the y-axis, against the values of N on the x-axis.

Comment on the results in the plots.

Hint: the MATLAB *function* `norminv()` *returns the inverse of the standard normal cumulative distribution function.*

Due: 16:00 o'clock, Monday, 14th November 2022
Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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