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Series 7

1. (i) See file `EM_Heston_solution.m`.

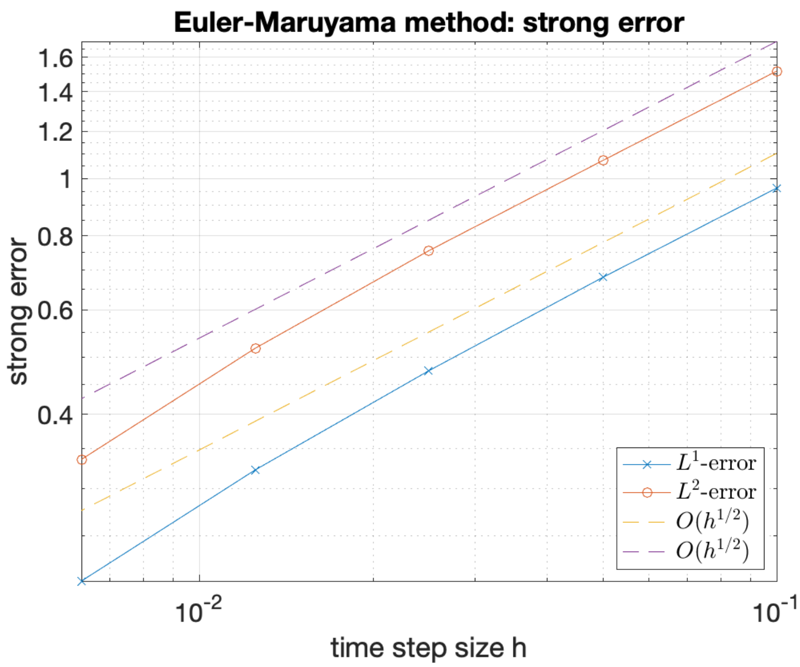

Abbildung 1: Strong convergence of the approximated price process in the Heston model with parameters $T = 1$, $S_0 = 10$, $V_0 = 0.5$, $r = 0.05$, $a = 2$, $b = 0.5$, and $\sigma_v = 0.25$.

```
>> EM_Heston_solution
Strong rate of convergence in L^1: 0.5478
Strong rate of convergence in L^2: 0.53991
```

(ii) See file `EM_Heston_solution.m`.

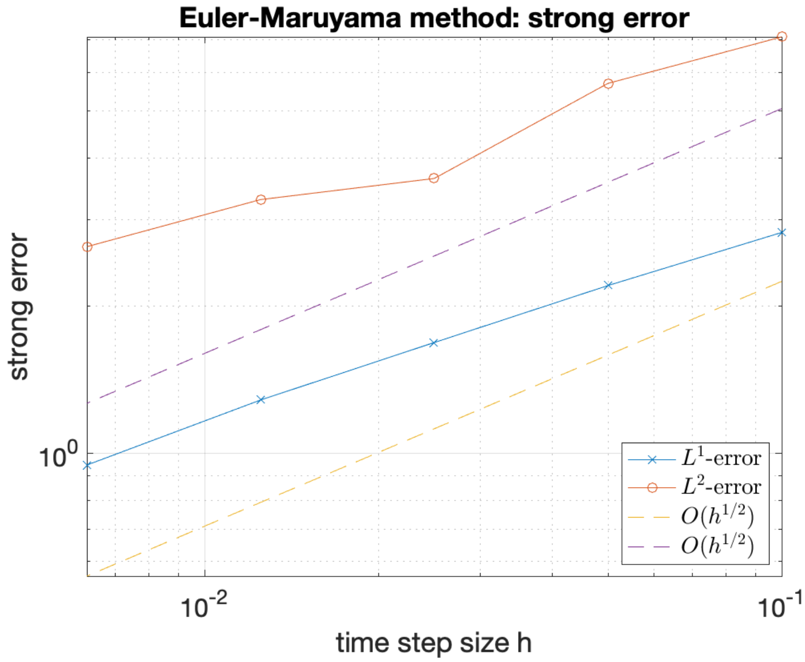

Abbildung 2: Strong convergence of the approximated price process in the Heston model with parameter $T = 1$, $S_0 = 10$, $V_0 = 0.5$, $r = 0.05$, $a = 1$, $b = 0.5$, and $\sigma_{\text{v}} = 2$.

```
>> EM_Heston_solution
Strong rate of convergence in L^1: 0.39335
Strong rate of convergence in L^2: 0.36359
```

2. (i) See file `ErrorEMRichardsonWeak.m`.

```
>> ErrorEMRichardsonWeak
Weak rate of convergence: 2.0283
```

Abbildung 3: Weak convergence of the Euler–Maruyama scheme combined with the Richardson extrapolation method.

(ii) Let Y^N be the Euler-Maruyama approximation of X with $N \in \mathbb{N}$ time steps and let f be a smooth function with polynomially bounded derivatives. Let $M > 1$. We define the higher-order extrapolation A_M recursively via

$$A_0 := f(\overline{Y}_T^N), \quad A_m(N) := \frac{2^m A_{m-1}(2N) - A_{m-1}(N)}{2^m - 1}, \; m = 1, \dots, M.$$

We want to show that

$$|\mathbb{E}[f(X_T)] - \mathbb{E}[A_M(N)]| \le C N^{-(M+1)}$$

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for some constant $C > 0$ and large enough $N \in \mathbb{N}$. By the given hint, we know that the higher order expansions

$$\mathbb{E}[f(X_T)] - \mathbb{E}[f(\overline{Y}_T^N)] = \sum_{k=1}^{K} C_k N^{-k} + \mathcal{O}(N^{-(K+1)}) \tag{1}$$

hold true for every $K \in \mathbb{N}$, where $C_1, \ldots, C_K \in \mathbb{R}$ are constants independent of N. We prove that for every $M \geq 0$ the expansions

$$\mathbb{E}[f(X_T)] - \mathbb{E}[A_M(N)] = \sum_{k=M+1}^{K} C_k N^{-k} + \mathcal{O}(N^{-(K+1)}) \tag{2}$$

hold true for every $K \in \mathbb{N}$, $K > M$, and constants $C_{M+1}, \ldots, C_K \in \mathbb{R}$ independent of N. By (1), equation (2) holds true for $M = 0$. Now, we consider $M \geq 1$ and we suppose that the expansions

$$\mathbb{E}[f(X_T)] - \mathbb{E}[A_{M-1}(N)] = \sum_{k=M}^{K} C_k N^{-k} + \mathcal{O}(N^{-(K+1)})$$

hold true for every $K \in \mathbb{N}$, $K > M - 1$, and constants $C_M, \ldots, C_K \in \mathbb{R}$ independent of N. Then, we show that the expansions

$$\mathbb{E}[f(X_T)] - \mathbb{E}[A_M(N)] = \sum_{k=M+1}^{K} C_k N^{-k} + \mathcal{O}(N^{-(K+1)})$$

hold true for every $K \in \mathbb{N}$, $K > M$, and constants $C_{M+1}, \ldots, C_K \in \mathbb{R}$ independent of N. Let $K > M$, we have that

$$\begin{aligned}
&\mathbb{E}[f(X_T)] - \mathbb{E}[A_M(N)] = \frac{2^M}{2^M - 1}(\mathbb{E}[f(X_T)] - \mathbb{E}[A_{M-1}(2N)]) - \frac{1}{2^M - 1}(\mathbb{E}[f(X_T)] - \mathbb{E}[A_{M-1}(N)]) \\
&= \frac{2^M}{2^M - 1} \sum_{k=M}^{K} C_k (2N)^{-k} + \frac{2^M}{2^M - 1}\mathcal{O}((2N)^{-(K+1)}) - \frac{1}{2^M - 1} \sum_{k=M}^{K} C_k N^{-k} - \frac{1}{2^M - 1}\mathcal{O}(N^{-(K+1)}) \\
&= \frac{2^M}{2^M - 1} \sum_{k=M+1}^{K} C_k (2N)^{-k} - \frac{1}{2^M - 1} \sum_{k=M+1}^{K} C_k N^{-k} + \mathcal{O}(N^{-(K+1)}) \\
&= \sum_{k=M+1}^{K} \frac{2^{M-k} - 1}{2^M - 1} C_k N^{-k} + \mathcal{O}(N^{-(K+1)}) \\
&= \sum_{k=M+1}^{K} C'_k N^{-k} + \mathcal{O}(N^{-(K+1)}),
\end{aligned}$$

where $C'_{M+1}, \ldots, C'_K \in \mathbb{R}$ are independent of N. Hence, the expansions (2) hold true for every $M \geq 0$ and for every $K \in \mathbb{N}$, $K > M$. In particular, if $K = M + 1$, equation (2) becomes

$$\mathbb{E}[f(X_T)] - \mathbb{E}[A_M(N)] = C_{M+1} N^{-(M+1)} + \mathcal{O}(N^{-(M+2)}),$$

and consequently

$$\limsup_{N\to+\infty} \frac{|\mathbb{E}[f(X_T)] - \mathbb{E}[A_M(N)]|}{N^{-(M+1)}} \le \limsup_{N\to+\infty} \frac{|C_{M+1}|N^{-(M+1)} + |\mathcal{O}(N^{-(M+2)})|}{N^{-(M+1)}}$$
$$= |C_{M+1}| < \infty,$$

which is equivalent to prove that there exists a constant $C > 0$ such that

$$|\mathbb{E}[f(X_T)] - \mathbb{E}[A_M(N)]| \le CN^{-(M+1)}$$

for large enough $N \in \mathbb{N}$.

3. (i) We make the change of variable $u = x^{-1}$. Hence, $dx = -u^{-2}du$ and we obtain

$$\mu = \int_1^\infty x^{-3}\mathrm{d}x = \int_1^0 (u^{-1})^{-3}(-u^{-2})\mathrm{d}u = \int_0^1 u\mathrm{d}u = \mathbb{E}_P[U].$$

Hence, $\mu = \mathbb{E}_P[h(U)]$ with $h\colon (0,1) \to \mathbb{R}$, $u \mapsto u$. Moreover, $\mu = \mathbb{E}_P[U] = 1/2$.

(ii) See files `MCEstimatorCI.m` and `MCCIPlot.m`. The output is

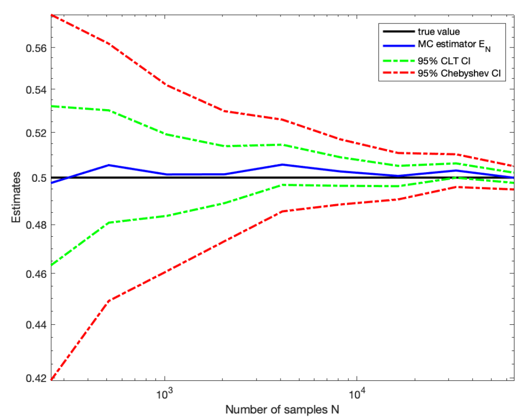
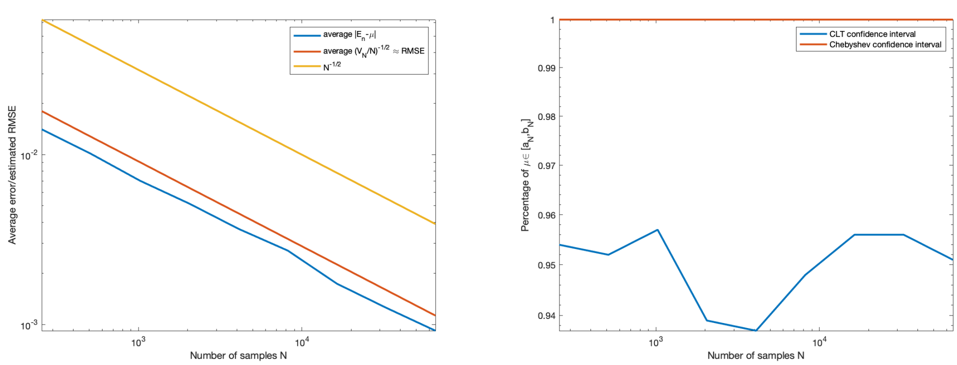


Abbildung 4: The intervals based on Chebyshev's inequality always contain the confidence intervals based on the central limit theorem.

(iii) See file `MCCIPlot2.m`. The output is



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We observe that the (average) Monte Carlo error converges with a rate of $N^{-1/2}$, which we would expect from theory. Moreover, concerning the confidence intervals based on the CLT, we see that $\mu \in [a_N^{\mathrm{CLT}}, b_N^{\mathrm{CLT}}]$ in roughly 95% of the cases. On the other hand, we see that $\mu \in [a_N^{\mathrm{Cb}}, b_N^{\mathrm{Cb}}]$ in any case for every N. These experiments are based on the fact that the intervals $[a_N^{\mathrm{Cb}}, b_N^{\mathrm{Cb}}]$ and $[a_N^{\mathrm{CLT}}, b_N^{\mathrm{CLT}}]$ are asymptotically valid 95%-confidence intervals for μ.

Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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