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Series 5

Throughout this exercise sheet, let $T \in (0, +\infty)$, let $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$ be a stochastic basis, and let $W\colon [0,T] \times \Omega \to \mathbb{R}$ be a one-dimensional standard $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$-Brownian motion.

1. Let $\xi \in \mathbb{R}$, let $\mu : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz continuous and let $\sigma \in C^1(\mathbb{R};\mathbb{R})$. Consider the one-dimensional SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad t \in [0,T], \quad X_0 = \xi.$$

(i) Let $M, N \in \mathbb{N}$. Write a MATLAB function `Milstein1D`$(T, \xi, \mu, \sigma, \sigma', W)$ with inputs $T \in (0, +\infty)$, $\xi \in \mathbb{R}$, $\mu\colon \mathbb{R}^M \to \mathbb{R}^M$, $\sigma\colon \mathbb{R}^M \to \mathbb{R}^M$, $\sigma'\colon \mathbb{R}^M \to \mathbb{R}^M$, and $W \in \mathbb{R}^{(N+1)\times M}$, which returns M realizations $Y_N^N(\omega_i)$, $i = 1, 2, \dots, M$, of the Milstein approximation Y_N^N of X_T. Note that μ, σ and σ' are function handles, and $W \in \mathbb{R}^{(N+1)\times M}$ is a realization of M independent one-dimensional Brownian motions at the equally spaced time points $\{n\Delta t : n = 0, \dots, N\}$, i.e.

$$W^{:,i} = \big(W_0, W_{\Delta t}, W_{2\Delta t}, \dots, W_{(N-1)\Delta t}, W_{N\Delta t}\big)(\omega_i)$$

for $i = 1, 2, \dots, M$.

Hint: You may modify the solution `EulerMaruyama.m` from Series 4.

(ii) Investigate the strong error of the Milstein scheme for the one-dimensional SDE

$$dX_t = X_t dt + \log\left(1 + X_t^2\right) dW_t, \quad t \in [0,1], \quad X_0 = 1. \tag{1}$$

by using $M = 10^5$ and $N = N_\ell = 10 \cdot 2^\ell$ for $\ell \in \{0, 1, \dots, 4\}$. To this end, for every $\ell \in \{0, 1, \dots, 4\}$ generate M realizations $Y_{N_\ell}^{N_\ell}(\omega_i)$, $i = 1, \dots, M$, of the Milstein approximation $Y_{N_\ell}^{N_\ell}$ of X_T. Then, for every $\ell \in \{0, 1, \dots, 4\}$ compute Monte Carlo approximations

$$\mathbb{E}[|Y_{N_\ell}^{N_\ell} - X_T|] \approx \frac{1}{M} \sum_{i=1}^{M} |Y_{N_\ell}^{N_\ell}(\omega_i) - X_T|$$

and

$$\mathbb{E}[|Y_{N_\ell}^{N_\ell} - X_T|^2]^{\frac{1}{2}} \approx \left(\frac{1}{M} \sum_{i=1}^{M} |Y_{N_\ell}^{N_\ell}(\omega_i) - X_T| \right)^{\frac{1}{2}}.$$

Report on the experimental rates of strong convergence in L^1 and L^2. Use as an approximation of the exact solution a numerical solution of the SDE on level $\underline{\ell = 7}$.

Hint: You may use the template `Milstein_SDE.m`.

(iii) Repeat item (ii) for the SDE

$$dX_t = X_t dt + \sin\left(1 + X_t^2\right) dW_t, \quad t \in [0,T], \quad X_0 = 1. \tag{2}$$

Comment on the results.

2. Let $a, b, \sigma_v > 0$ and $v_0 \geq 0$. Consider the Cox-Ingersoll-Ross process, given as solution to the SDE

$$\mathrm{d}V_t = a(b - V_t)\mathrm{d}t + \sigma_{\mathrm{v}} \sqrt{V_t} \mathrm{d}W_t, \quad V_0 = v_0, \quad t \in [0,T]. \tag{3}$$

(i) Apply the Yamada-Watanabe theorem to prove that the Cox-Ingersoll-Ross process is the unique (up to indistinguishability) solution to the SDE (3).

(ii) Let $N \in \mathbb{N}$. Assume $[0,T]$ is discretized by a uniform temporal mesh with $N+1$ nodes, i.e. with time step size $\Delta t = T/N$. The *drift-implicit Milstein scheme* for the stochastic process V with step size Δt and initial value $V_0^N = V_0 > 0$ is given for $n = 0, \ldots, N-1$ by

$$V_{n+1}^N = V_n^N + a(b - V_{n+1}^N)\Delta t + \sigma_{\mathrm{v}} \sqrt{V_n^N} (W_{t_{n+1}} - W_{t_n}) + \frac{\sigma_{\mathrm{v}}^2}{4} \left((W_{t_{n+1}} - W_{t_n})^2 - \Delta t\right).$$

Show that if $4ab \geq \sigma_{\mathrm{v}}^2$, then $P(V_n^N > 0) = 1$ for all $n \in \{0, \ldots, N\}$.

(iii) Write a Matlab function `DriftImplicitMilstein(`$T, N, v_0, a, b, \sigma_{\mathrm{v}}$`)` with input $T \in (0, +\infty)$, $N \in \mathbb{N}$, $v_0, a, b, \sigma_{\mathrm{v}} > 0$ and output a realization of the drift-implicit Milstein scheme $\{V_0^N, V_1^N, \ldots, V_N^N\}$ for the Cox-Ingersoll-Ross process V. Then, plot a sample path of the stochastic process V with the choices $T = 1, N = 10^3, v_0 = 0.5, a = 2, b = 0.5$ and $\sigma_{\mathrm{v}} = 0.25$.

3. [Proposition 3.6.6 in the lecture notes] Let $d, N \in \mathbb{N}$, $\mu \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, $\sigma = (\sigma_j)_{j \in \{1,\ldots,m\}} \in C^1(\mathbb{R}^d, \mathbb{R}^{d \times m})$, $\xi \in \mathcal{M}(\mathbb{F}_0, \mathcal{B}(\mathbb{R}^d))$, let $Y^N \colon \{0, 1, \ldots, N\} \times \Omega \to \mathbb{R}^d$ be the Milstein approximation for the SDE

$$\mathrm{d}X_t = \mu(X_t)\, \mathrm{d}t + \sigma(X_t)\, \mathrm{d}W_t, \qquad t \in [0,T], \qquad X_0 = \xi,$$

with time step size $\Delta t = T/N$, and assume for all $x \in \mathbb{R}^d$, $i, j \in \{1, 2, \ldots, m\}$ that

$$(\sigma_i)'(x)\, \sigma_j(x) = (\sigma_j)'(x)\, \sigma_i(x).$$

Prove that for all $n \in \{0, 1, \ldots, N-1\}$ it holds P-a.s. that

$$\begin{aligned} Y_{n+1}^N &= Y_n^N + \mu(Y_n^N)\, \Delta t + \sigma(Y_n^N)\big(W_{t_{n+1}} - W_{t_n}\big) - \frac{\Delta t}{2} \sum_{i=1}^m (\sigma_i)'(Y_n^N)\, \sigma_i(Y_n^N) \\ &+ \frac{1}{2} \sum_{i,j=1}^m (\sigma_i)'(Y_n^N)\, \sigma_j(Y_n^N) \left(W_{t_{n+1}}^{(i)} - W_{t_n}^{(i)}\right) \left(W_{t_{n+1}}^{(j)} - W_{t_n}^{(j)}\right), \qquad Y_0^N = \xi. \end{aligned}$$

Hint: Use Itô's formula to achieve explicit expressions for the iterated integrals

$$\int_{t_0}^{t} W_s^{(1)}\,\mathrm{d}W_s^{(1)} \qquad \text{and} \qquad \int_{t_0}^{t} W_s^{(1)}\,\mathrm{d}W_s^{(2)} + \int_{t_0}^{t} W_s^{(2)}\,\mathrm{d}W_s^{(1)}, \tag{4}$$

where $(W^{(1)}, W^{(2)})\colon [0,T] \times \Omega \to \mathbb{R}^2$ is a 2-dimensional standard $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$-Brownian motion, and where $t, t_0 \in [0,T]$.

Due: 16:00 o'clock, Monday, 31st October 2022
Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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