D-MATH Numerical Analysis of Stochastic ODEs Autumn 2022

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Series 4

Throughout this exercise sheet, let $T \in (0, +\infty)$, let $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$ be a stochastic basis, and let $W\colon [0,T] \times \Omega \to \mathbb{R}$ be a one-dimensional standard $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$-Brownian motion.

1. Consider the Black-Scholes model, where the price process of an underlying S is modeled by the SDE

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t, \quad S_0 = s_0 > 0, \quad t \in [0,T],$$

for fixed interest rate $r \in \mathbb{R}$ and volatility parameter $\sigma > 0$. Let

$$f(S_T) = \max\{S_T - K, 0\}$$

be the payoff function of a European call option with strike price $K > 0$. Derive the Black-Scholes formula

$$e^{-rT}\,\mathbb{E}_P[f(S_T)] = S_0\,\Phi\left(\frac{\left(r+\frac{\sigma^2}{2}\right)T+\ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}}\right) - K\,e^{-rT}\,\Phi\left(\frac{\left(r-\frac{\sigma^2}{2}\right)T+\ln\left(\frac{S_0}{K}\right)}{\sigma\sqrt{T}}\right),$$

where $\Phi\colon \mathbb{R} \to \mathbb{R}$ denotes the $\mathcal{N}_{0,1}$-distribution function, i.e.

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\mathrm{d}y, \qquad x \in \mathbb{R}.$$

Hint: You may use item (i) in Exercise 1 from Sheet 3.

2. Let $\xi \in \mathcal{L}^p(P|_{\mathbb{F}_0}; |\cdot|)$ for some $p \geq 2$. We consider the SDE

$$\mathrm{d}X_t = \log(1 + X_t^2)\mathrm{d}t + \mathbf{1}_{\{X_t>0\}}X_t\mathrm{d}W_t, \quad t \in [0,T], \quad X_0 = \xi. \tag{1}$$

(i) Show that the SDE (1) admits a unique (up to indistinguishability) solution process $X\colon [0,T] \times \Omega \to \mathbb{R}$.

(ii) Let $M, N \in \mathbb{N}$. Write a MATLAB function `EulerMaruyama(`T`,`ξ`,`W`)` with inputs $T \in (0, +\infty)$, $\xi \in \mathbb{R}$ and $W \in \mathbb{R}^{(N+1)\times M}$, which returns M realizations $Y_N^N(\omega_i)$, $i = 1, 2, \ldots, M$, of the Euler–Maruyama approximation Y_N^N of X_T. The input parameter $W \in \mathbb{R}^{(N+1)\times M}$ is a realization of M independent one-dimensional Brownian motions at the equally spaced time points $\{n\Delta t : n = 0, \ldots, N\}$, i.e.

$$W^{:,i} = \left(W_0, W_{\Delta t}, W_{2\Delta t}, \ldots, W_{(N-1)\Delta t}, W_{N\Delta t}\right)(\omega_i)$$

for $i = 1, 2, \ldots, M$. You can use the template `EulerMaruyama.m`.

Bitte wenden!

(iii) Investigate the convergence rate of the Euler–Maruyama scheme by fixing the parameters $T = 1$, $\xi = 1$, and using $M = 10^5$ and $N = N_\ell = 10 \cdot 2^\ell$ for $\ell \in \{0, 1, \ldots, 4\}$. To do so, generate M sample paths of the Brownian motion on the finest grid. Then, for every $\ell \in \{0, 1, \ldots, 4\}$ generate M realizations $Y_{N_\ell}^{N_\ell}(\omega_i)$, $i = 1, 2, \ldots, M$, of the Euler–Maruyama approximation $Y_{N_\ell}^{N_\ell}$ of X_T. Hence, for every $\ell \in \{0, 1, 2, 3\}$ compute a Monte Carlo approximation E_M^ℓ of

$$\mathbb{E}[|Y_{N_{\ell+1}}^{N_{\ell+1}} - Y_{N_\ell}^{N_\ell}|^2]^{\frac{1}{2}} \approx \left(\frac{1}{M} \sum_{i=1}^{M} |Y_{N_{\ell+1}}^{N_{\ell+1}}(\omega_i) - Y_{N_\ell}^{N_\ell}(\omega_i)|^2 \right)^{\frac{1}{2}} =: E_M^\ell$$

based on M samples, and estimate the experimental strong L^2 convergence rate of the Euler–Maruyama scheme by a linear regression of $\log(E_M^\ell)$ on the log-stepsizes $\log(N_\ell^{-1})$ for $\ell \in \{0, 1, 2, 3\}$ (for this you may use the MATLAB function `polyfit`). **Comment** on the convergence rate observed in your experiment. You can use the template `ErrorEM.m`.

Remark: For $N_{\ell+1} > N_\ell$, the triangle inequality yields

$$\mathbb{E}[|Y_{N_\ell+1}^{N_\ell+1} - Y_{N_\ell}^{N_\ell}|^2]^{\frac{1}{2}} \leq \mathbb{E}[|Y_{N_\ell+1}^{N_\ell+1} - X_T|^2]^{\frac{1}{2}} + \mathbb{E}[|Y_{N_\ell}^{N_\ell} - X_T|^2]^{\frac{1}{2}} \leq 2CN_\ell^{-\alpha},$$

for some positive constant C, where $\alpha > 0$ is the convergence rate of the Euler-Maruyama scheme. Hence, for $N_\ell \approx N_{\ell+1} > 0$, we may assume that

$$\log(2C) - \alpha \log(N_\ell) \approx \frac{1}{2} \log \left(\mathbb{E}[|Y_{N_\ell+1}^{N_\ell+1} - Y_{N_\ell}^{N_\ell}|^2] \right).$$

3. Let $\sigma > 0$ and $\xi \in \mathcal{L}^p(P|_{\mathbb{F}_0}; |\cdot|)$ for some $p \geq 2$. We consider the *stochastic Ginzburg-Landau equation*

$$\mathrm{d}X_t = \left(\frac{1}{2}\sigma^2 X_t - X_t^3 \right) \mathrm{d}t + \sigma X_t \mathrm{d}W_t, \quad t \in [0, T], \quad X_0 = \xi, \tag{2}$$

and we denote by X the unique (up to indistinguishability) solution to the SDE (2).

(i) Let $M, N \in \mathbb{N}$. Write a MATLAB function `EulerMaruyamaGL(`T`,`ξ`,`W`)` with inputs $T \in (0, +\infty)$, $\xi \in \mathbb{R}$ and $W \in \mathbb{R}^{(N+1)\times M}$, which returns M realizations $Y_N^N(\omega_i)$, $i = 1, 2, \ldots, M$, of the Euler–Maruyama approximation Y_N^N of X_T. The input parameter $W \in \mathbb{R}^{(N+1)\times M}$ is a realization of M independent one-dimensional Brownian motions at the equally spaced time points $\{n\Delta t : n = 0, \ldots, N\}$, i.e.

$$W^{:,i} = \left(W_0, W_{\Delta t}, W_{2\Delta t}, \ldots, W_{(N-1)\Delta t}, W_{N\Delta t}\right)(\omega_i)$$

for $i = 1, 2, \ldots, M$. You can modify the template `EulerMaruyama.m` in Exercise 2.

(ii) Choose $\sigma = 7$, $\xi = 1$ and $T = 3$. Write a MATLAB script, which calls the function `EulerMaruyamaGL(`T`,`ξ`,`W`)`, to compute a Monte Carlo approximation E_M of

$$\mathbb{E}[|Y_N^N|^2] \approx \frac{1}{M} \sum_{i=1}^{M} |Y_N^N(\omega_i)|^2 =: E_M$$

based on $M = 10^5$ samples, where $Y_N^N(\omega_i)$, $i = 1, 2, \ldots, M$, denote M realizations of the Euler–Maruyama approximation Y_N^N of X_T with $N = 10^3$ time steps. **Comment** on the result.

(iii) Repeat items (i) and (ii) replacing the Euler–Maruyama scheme with the increment-tamed Euler–Maruyama scheme with time step size $\Delta t = T/N$

$$Y_{n+1}^N = Y_n^N + \frac{\mu(Y_n^N)\Delta t + \sigma(Y_n^N)(W_{t_{n+1}} - W_{t_n})}{\max\{1, \Delta t|\mu(Y_n^N)\Delta t + \sigma(Y_n^N)(W_{t_{n+1}} - W_{t_n})|\}}, \quad n = 0, \ldots, N-1, \tag{3}$$

and with initial condition $Y_0^N = \xi$, where $\mu\colon \mathbb{R} \to \mathbb{R}$ and $\sigma\colon \mathbb{R} \to \mathbb{R}$ denote respectively the drift and the diffusion coefficients of the SDE (2). We refer also to Definition 3.5.7 in the lecture notes.

Due: 16:00 o'clock, Monday, 24th October 2022
Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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