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Series 2

1. Let $t \in [0,T]$. There exists $n \in \{1,\dots,N\}$ such that $t \in [t_{n-1}, t_n]$. Then, by the triangular inequality

$$\begin{aligned}
&\|\widetilde{W}_t - W_t\|_{L^2(P;|\cdot|_{\mathbb{R}})} = \|\frac{t-t_{n-1}}{\Delta t}W_{t_n} + \frac{t_n - t}{\Delta t}W_{t_{n-1}} - W_t\|_{L^2(P;|\cdot|_{\mathbb{R}})}\\
&= \|\frac{t-t_{n-1}}{\Delta t}W_{t_n} + \frac{t_{n-1}+\Delta t - t}{\Delta t}W_{t_{n-1}} - W_t\|_{L^2(P;|\cdot|_{\mathbb{R}})}\\
&= \|\frac{t-t_{n-1}}{\Delta t}(W_{t_n} - W_{t_{n-1}}) + W_{t_{n-1}} - W_t\|_{L^2(P;|\cdot|_{\mathbb{R}})}\\
&\leq \|\frac{t-t_{n-1}}{\Delta t}(W_{t_n} - W_{t_{n-1}})\|_{L^2(P;|\cdot|_{\mathbb{R}})} + \|W_{t_{n-1}} - W_t\|_{L^2(P;|\cdot|_{\mathbb{R}})}\\
&\leq \|W_{t_n} - W_{t_{n-1}}\|_{L^2(P;|\cdot|_{\mathbb{R}})} + \|W_{t_{n-1}} - W_t\|_{L^2(P;|\cdot|_{\mathbb{R}})},
\end{aligned}$$

where in the last inequality we have used that $|t - t_{n-1}|/\Delta t \leq 1$. So that, we can continue the above chain of inequalities as follows

$$\begin{aligned}
\|\widetilde{W}_t - W_t\|_{L^2(P;|\cdot|_{\mathbb{R}})} &\leq \|W_{t_n} - W_{t_{n-1}}\|_{L^2(P;|\cdot|_{\mathbb{R}})} + \|W_{t_{n-1}} - W_t\|_{L^2(P;|\cdot|_{\mathbb{R}})}\\
&\leq \sqrt{t_n - t_{n-1}} + \sqrt{t - t_{n-1}} \leq 2\sqrt{T/N}.
\end{aligned}$$

Then, we may conclude that

$$\sup_{t\in[0,T]} \|\widetilde{W}_t - W_t\|_{L^2(P;|\cdot|_{\mathbb{R}})} \leq 2\sqrt{T/N},$$

which proves the statement.

2. By the given hint, we know that for random variables $(Y^{(1)}, Y^{(2)})$ such that

$$\begin{pmatrix} Y^{(1)} \\ Y^{(2)} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right),$$

with Σ_{22} with full rank, we have that

$$\left(Y^{(1)} \middle| Y^{(2)} = y\right) \sim \mathcal{N}\left(\mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(y - \mu^{(2)}), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right). \tag{1}$$

Let $0 < r < s < t \leq T$. We start by showing that the random variable $X = \left(W_s | W_t = y\right)$ satisfies $X \sim \mathcal{N}\left(\frac{sy}{t}, \frac{(t-s)s}{t}\right)$. By the independence of increments, $(W_s, W_t - W_s)$ is jointly Gaussian distributed with

$$\begin{pmatrix} W_s \\ W_t - W_s \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s & 0 \\ 0 & t-s \end{pmatrix}\right).$$

Furthermore, since

$$\begin{pmatrix} W_s \\ W_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} W_s \\ W_t - W_s \end{pmatrix},$$

Proposition 0.4.15 in the lecture notes implies that $(W_s, W_t) \sim \mathcal{N}(0, \Sigma)$, where

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & t-s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^\top = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & t-s \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s & s \\ s & t \end{pmatrix}.$$

Therefore, by Equation (1) we conclude that $X = \big(W_s | W_t = y\big) \sim \mathcal{N}\left(\frac{sy}{t}, \frac{(t-s)s}{t}\right)$.

Similarly, we have that

$$\begin{pmatrix} W_s - W_r \\ W_r \\ W_t - W_s \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s-r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & t-s \end{pmatrix}\right),$$

and

$$\begin{pmatrix} W_s \\ W_r \\ W_t \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} W_s - W_r \\ W_r \\ W_t - W_s \end{pmatrix}.$$

Then, $(W_s, W_r, W_t) \sim \mathcal{N}(0, \Sigma)$, where

$$\Sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} s-r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & t-s \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s & r & s \\ r & r & r \\ s & r & t \end{pmatrix}.$$

Then, observing that $r(t-r) \neq 0$ and using the fact that

$$\begin{pmatrix} r & r \\ r & t \end{pmatrix}^{-1} = \frac{1}{r(t-r)} \begin{pmatrix} t & -r \\ -r & r \end{pmatrix},$$

we can see from Equation (1) that $\widetilde{X} = \big(W_s | W_r = x, W_t = y\big)$ is normally distributed with mean given by

$$0 + \begin{pmatrix} r & s \end{pmatrix} \frac{1}{r(t-r)} \begin{pmatrix} t & -r \\ -r & r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{(t-s)x + (s-r)y}{(t-r)},$$

and variance given by

$$s - \begin{pmatrix} r & s \end{pmatrix} \frac{1}{r(t-r)} \begin{pmatrix} t & -r \\ -r & r \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \frac{(s-r)(t-s)}{(t-r)}.$$

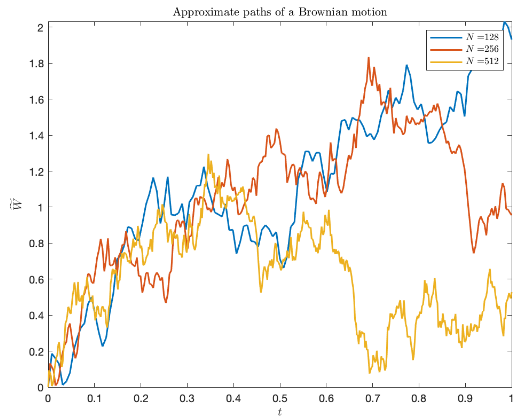
That is,

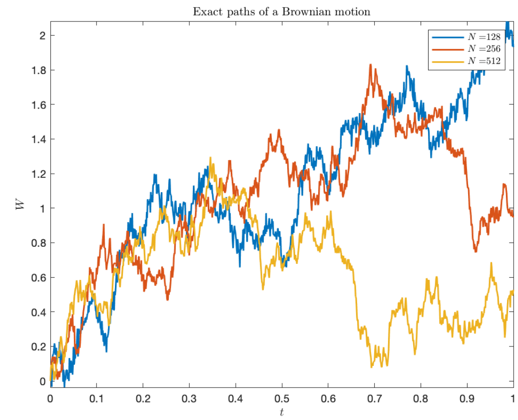
$$\widetilde{X} \sim \mathcal{N}\left(\frac{(t-s)x + (s-r)y}{(t-r)}, \frac{(s-r)(t-s)}{(t-r)}\right),$$

which concludes the exercise.

Siehe nächstes Blatt!

3. See MATLAB files `BMIncInterp`, `BMBridgeInterp` and `BMPlot`. The output is:





Paths of a Brownian motions generated by linear interpolation (left) as in item (i) and by Brownian bridge simulation as in item (ii) (right).

Remark: By Exercise 2, we know that for $t \in [t_i, t_{i+1}]$, $i = 0, \ldots, N-1$, and $x, y \in \mathbb{R}$,

$$(W_t | W_{t_i} = x, W_{t_{i+1}} = y) \sim \mathcal{N}\left(\frac{(t_{i+1}-t)x + (t-t_i)y}{\Delta t}, \frac{(t-t_i)(t_{i+1}-t)}{\Delta t}\right).$$

Then, we observe that the value of the mean is given by the linear interpolation of the values $W_{t_i} = x$ and $W_{t_{i+1}} = y$ at t.

Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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