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Series 11

Throughout this exercise sheet, let $T \in (0,+\infty)$, let $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$ be a stochastic basis, let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a one-dimensional standard $(\Omega, \mathcal{F}, P, \mathbb{F}_{t\in[0,T]})$-Brownian motion, and let $L\colon [0,T]\times\Omega\to\mathbb{R}$ be a Lévy process.

1. Let $\xi \in \mathbb{R}$, $\mu \in \mathcal{M}\big(\mathcal{B}(\mathbb{R}),\mathcal{B}(\mathbb{R})\big)$, $\sigma \in \mathcal{M}\big(\mathcal{B}(\mathbb{R}),\mathcal{B}(\mathbb{R})\big)$, $\eta \in \mathcal{M}\big(\mathcal{B}(\mathbb{R}),\mathcal{B}(\mathbb{R})\big)$, and consider the Lévy-driven SDE

$$\mathrm{d}X_t = \mu(X_{t-})\,\mathrm{d}t + \sigma(X_{t-})\,\mathrm{d}W_t + \eta(X_{t-})\,\mathrm{d}L_t, \qquad t\in[0,T], \qquad X_0 = \xi. \tag{1}$$

(i) Let $K, N \in \mathbb{N}$. Write a MATLAB function `EulerMaruyamaLevySDE(`T`, `ξ`, `μ`, `σ`, `η`, `W`, `L`)` with inputs $T\in(0,+\infty)$, $\xi\in\mathbb{R}$, $\mu\colon \mathbb{R}^{1\times K}\to\mathbb{R}^{1\times K}$, $\sigma\colon\mathbb{R}^{1\times K}\to\mathbb{R}^{1\times K}$, $\eta\colon\mathbb{R}^{1\times K}\to\mathbb{R}^{1\times K}$, $W\in\mathbb{R}^{(N+1)\times K}$ and $L\in\mathbb{R}^{(N+1)\times K}$ which returns K realizations $Y_N^N(\omega_i)$, $i=1,2,\dots,K$, of the Euler–Maruyama approximation Y_N^N of X_T. The input parameter $W\in\mathbb{R}^{(N+1)\times K}$ is a realization of K independent one-dimensional Brownian motions at the equally spaced time points $\big\{n\Delta t : n = 0,\dots,N\big\}$, i.e.

$$W^{:,i} = \big(W_0, W_{\Delta t}, W_{2\Delta t},\dots, W_{(N-1)\Delta t}, W_{N\Delta t}\big)(\omega_i)$$

for $i=1,2,\dots,K$. Analogously, the input parameter $L\in\mathbb{R}^{(N+1)\times K}$ is a realization of K independent Lévy processes at the equally spaced time points $\big\{n\Delta t : n=0,\dots,N\big\}$, i.e.

$$L^{:,i} = \big(L_0, L_{\Delta t}, L_{2\Delta t},\dots, L_{(N-1)\Delta t}, L_{N\Delta t}\big)(\omega_i)$$

for $i=1,2,\dots,K$.
Hint: You may modify `EMMultiDim.m` from Sheet 6.

(ii) Investigate the convergence rate of the Euler–Maruyama scheme by fixing the parameters $T=1$, $\xi=1$, $\mu(x)=0.5x$, $\sigma(x)=x$, $\eta(x)=0.5x$ and using $K=10^4$, $N=N_\ell=10\cdot 2^\ell$, $\ell\in\{0,1,\dots,4\}$, and a Gamma subordinator with shape parameter $\alpha=2$ and rate $\beta=1$ as driving Levy process. To do so, generate K sample paths of the Brownian motion and of the Gamma subordinator on the finest grid. Then, for every $\ell\in\{0,1,\dots,4\}$ generate K realizations $Y_{N_\ell}^{N_\ell}(\omega_i)$, $i=1,2,\dots,K$, of the Euler–Maruyama approximation $Y_T^{N_\ell}$ of X_T. Hence, for every $\ell\in\{0,1,\dots,4\}$ compute a Monte Carlo approximation E_K^ℓ of

$$\mathbb{E}[|Y_{N_\ell}^{N_\ell} - X_T|^2]^{\frac{1}{2}}$$

based on K samples, and estimate the experimental strong $L^2(P; |\cdot|_{\mathbb{R}})$-convergence rate of the Euler–Maruyama scheme by a linear regression of $\log(E_M^\ell)$ on the log-stepsizes $\log(N_\ell^{-1})$ for $\ell \in \{0, 1, \ldots, 4\}$. **Comment** on the convergence rate observed in your experiment. You can use the template `EulerMaruyamaGammaSDE.m`.
Hint: Use as an approximation of the exact solution X_T an Euler–Maruyama approximation of the SDE on level $\ell = 8$.

2. Let $\mu \in \mathbb{R}$ and $\lambda > 0$, and let $X \sim C_{\mu,\lambda}$ be a Cauchy distributed random variable with parameters μ and λ, i.e. its probability density function is given by

$$f_{\mu,\lambda} \colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{1}{\pi\lambda} \left[\frac{\lambda^2}{(x-\mu)^2 + \lambda^2} \right].$$

(i) Prove that X is infinitely divisible.
Hint: You may use without proof that the characteristic function of X is given by

$$\phi_X(x) = e^{i\mu x - \lambda|x|}, \quad x \in \mathbb{R}.$$

(ii) Show that $\mathbb{E}[X]$ does not exist.

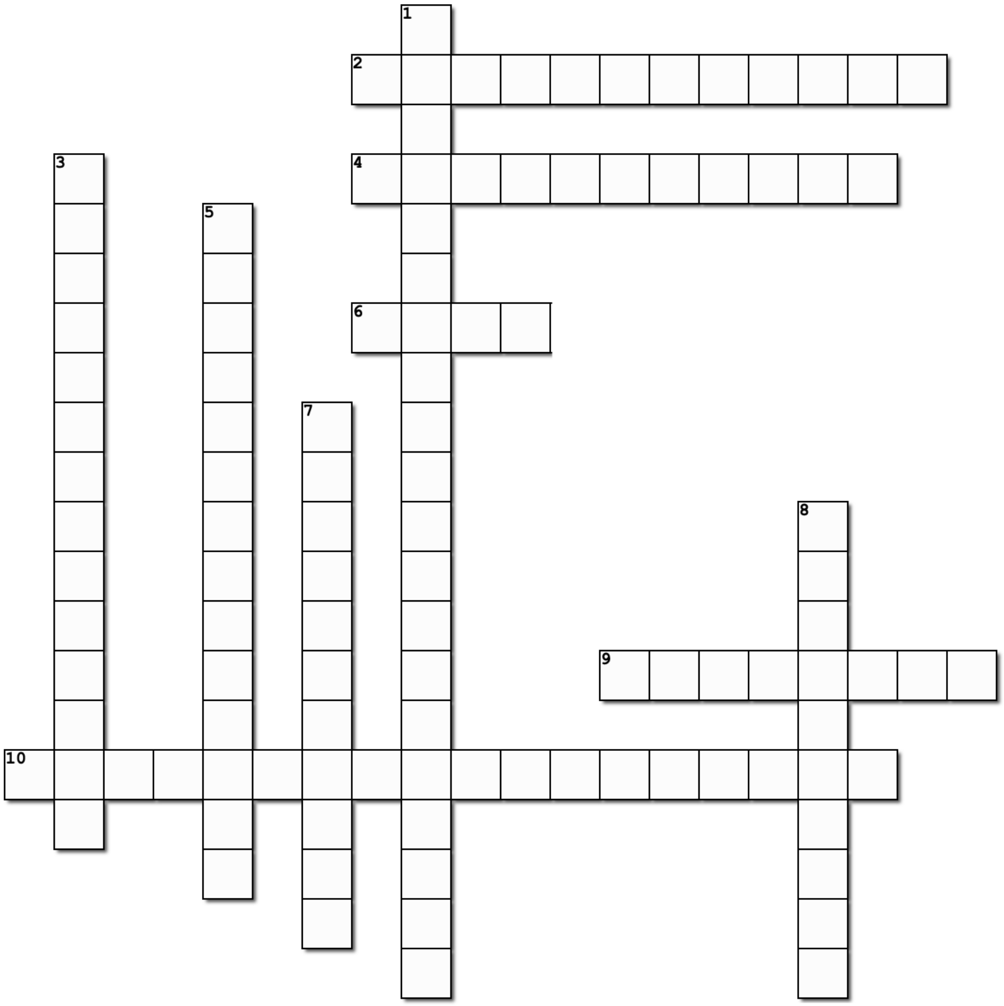
The Cauchy process with shape parameter $\mu \in \mathbb{R}$ and $\lambda > 0$ is defined as the Lévy process $I^{C_{\mu,\lambda}} \colon [0,T] \times \Omega \to [0,\infty)$ such that $I_1^{C_{\mu,\lambda}} \sim C_{\mu,\lambda}$.

(ii) Repeat item (ii) from Exercise 1 by replacing the Gamma subordinator with a Cauchy subordinator with parameters $\mu = 0$ and $\lambda = 1$. **Comment** on the convergence rate observed in your experiment.
Hint: Use the Matlab function `random('stable',1,0,lambda,mu,size(...))` to sample from the Cauchy distribution with parameters μ and λ.

3. Complete the crossword puzzle below:

1. Numerical methods to reduce the computational cost of standard Monte Carlo methods.
2. Almost surely non-decreasing stochastic process.
3. Financial derivative.
4. Extends methods of calculus to stochastic processes.
5. Stochastic process with stationary and independent increments.
6. Stochastic process with jumps.
7. Can be either strong or weak.
8. Increasing sequence of σ-algebras on a measurable space.
9. Numerical scheme with weak convergence of order one.
10. Range of estimates for an unknown parameter.

Siehe nächstes Blatt!



The NASODE team wishes you a Merry Christmas, relaxing holidays, all the best for the exam preparations, and much success for the exam!

Andreas, Francesca, Mateo and Wei

Due: 16:00 o'clock, Monday, 12th December 2022
Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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