Dr. Andreas Stein
Dr. Francesca Bartolucci

Series 10

Throughout this sheet let $T \in (0, +\infty)$, $m \in \mathbb{N}$, let $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]})$ be a stochastic basis, and let $W\colon [0,T] \times \Omega \to \mathbb{R}^m$ be an m-dimensional $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]})$-Brownian motion.

1. Let $m = 1$. Prove that the stochastic processes $(W_t)_{t\in[0,T]}$ and $(W_t^2 - t)_{t\in[0,T]}$ are $(\mathbb{F}_t)_{t\in[0,T]}$-martingales.
Hint: Use Itô's formula to derive an expression for $W_t^2 - t$.

2. Let $L\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a subordinated Brownian motion with subordinating process $I\colon [0,T] \times \Omega \to [0,\infty)$ of the form

$$L_t = \gamma_1 t + \gamma_2 I_t + \Sigma W_{I_t}, \quad t \in [0,T],$$

where $\gamma_1, \gamma_2 \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times m}$ is a symmetric positive semi-definite matrix with Cholesky factorization $A = \Sigma\Sigma^T$, and let $\varphi_I\colon \mathbb{R} \to \mathbb{C}$ denote the characteristic exponent of the subordinator $I\colon [0,T] \times \Omega \to [0,\infty)$.

(i) [Theorem 6.2.25] Show that L is a Lévy process with characteristic exponent given by

$$\varphi_L(x) = ix^\top\gamma_1 + \varphi_I\left(x^\top\gamma_2 + \frac{i}{2}x^\top Ax\right), \quad x \in \mathbb{R}^m.$$

(ii) Prove that for every $s, t \in [0,T]$, $s < t$,

$$L_t - L_s \stackrel{d}{=} \gamma_1(t-s) + \gamma_2 I_{t-s} + \sqrt{I_{t-s}}\Sigma Z_m,$$

where $Z_m \sim \mathcal{N}_{0,I_m}$, and $\stackrel{d}{=}$ signifies equality in distribution.

3. For every $z > 0$, the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\mathrm{d}x.$$

Let $\alpha, \beta > 0$, and let $X \sim \Gamma_{\alpha,\beta}$ be a Gamma distributed random variable with parameters α and β, i.e. its probability density function $f_{\alpha,\beta}\colon \mathbb{R} \to \mathbb{R}$ is given by

$$f_{\alpha,\beta}(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Bitte wenden!

(i) Prove that X is infinitely divisible.
Hint: You may use without proof that the characteristic function of X is given by

$$\phi_X(x) = \left(1 - i\frac{x}{\beta}\right)^{-\alpha}, \quad x \in \mathbb{R}.$$

The Gamma process with shape parameter $\alpha > 0$ and rate $\beta > 0$ is defined as the Lévy process $I^{\Gamma_{\alpha,\beta}} \colon [0,T] \times \Omega \to [0,\infty)$ such that $I_t^{\Gamma_{\alpha,\beta}} \sim \Gamma_{t\alpha,\beta}$ for every $t \in [0,T] \setminus \{0\}$.

(ii) Determine the characteristic triplet (γ, A, ν) of $I^{\Gamma_{\alpha,\beta}}$.

Hint: You may use without proof that ν is given by

$$\nu(\mathrm{d}z) = \alpha e^{-\beta z} z^{-1} \mathbb{1}_{\{z \geq 0\}} \mathrm{d}z.$$

(iii) Let $m = 1$. Write a Matlab function `VarianceGamma(`$T, N, \gamma_1, \gamma_2, \sigma, \alpha, \beta$`)` with input $T, N \in \mathbb{N}$, γ_1, γ_2, $\sigma \in \mathbb{R}$, $\alpha, \beta > 0$ and output a realization of a sample path $\{L_0, L_{\frac{T}{N}}, \ldots, L_T\}$ of a subordinated Brownian motion L with Gamma subordinator $I^{\Gamma_{\alpha,\beta}} \colon [0,T] \times \Omega \to [0,\infty)$ of the form

$$L_t = \gamma_1 t + \gamma_2 I_t^{\Gamma_{\alpha,\beta}} + \sigma W_{I_t^{\Gamma_{\alpha,\beta}}}, \quad t \in [0,T].$$

Test your Matlab function for the choice of parameters $T = 1, N = 10^3, \gamma_1 = 0, \gamma_2 = 0.5, \sigma = 1$, $\alpha = 10$ and $\beta = 5$.
Hint: You may use the template `VarianceGamma.m` and item (ii) from Exercise 2. You may use the built-in Matlab function `gamrnd(...,...,...)` to simulate a Gamma process.

Due: 16:00 o'clock, Monday, 5th December 2022
Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
Organisation: Francesca Bartolucci, HG G 53.2