Dr. Andreas Stein
Dr. Francesca Bartolucci

Series 6

1. (i) See file `EMMultiDim.m`.

(ii) See file `ErrorEM2dGBM.m`.

2. (i) We first check the conditions of the existence and uniqueness theorem: We have that $\xi : \Omega \to \mathbb{R}$ is $\mathbb{F}_0$-measurable by assumption, and that $\mathbb{E}_P[|\xi|^p] < \infty$ for all $p \in (0, \infty)$. Moreover, the coefficients μ and σ are linear functions, thus globally Lipschitz in $\mathbb{R}$. Hence, there exists a unique strong solution X such that $\mathbb{E}_P[|X_t|^p] < \infty$ for all $p \in [1, \infty)$ by Theorem 2.5.1.

By exercise 3 on Series 3, the unique solution is given by the geometric Brownian motion

$$X_t = \xi \exp\left((\mu_0 - \frac{\sigma_0^2}{2})t + \sigma_0 W_t\right), \quad t \in [0, T].$$

Thus, it follows for all $p \in (0, \infty)$ and all $t \in [0, T]$ that

$$\begin{aligned}
\mathbb{E}_P\left[X_t^p\right] &= \mathbb{E}_P\left[\xi^p \exp\left(p(\left(\mu_0 - \frac{\sigma_0^2}{2}\right)t + p\sigma_0 W_t\right)\right] \\
&= \mathbb{E}_P\left[\xi^p\right] \exp\left(p\left(\mu_0 - \frac{\sigma_0^2}{2}\right)t\right) \mathbb{E}_P\left[\exp\left(p\sigma_0 W_t\right)\right] \\
&= \mathbb{E}_P\left[\xi^p\right] \exp\left(p\left(\mu_0 - \frac{\sigma_0^2}{2}\right)t\right) \exp\left(p^2 \frac{\sigma_0^2}{2} t\right) \\
&= \mathbb{E}_P\left[\xi^p\right] \exp\left(\left(p\mu_0 + (p^2 - p)\frac{\sigma_0^2}{2}\right)t\right).
\end{aligned}$$

The second identity holds since ξ is independent of W_t and the third line follows by Exercise 1(i) on Series 3.

(ii) We have to check that the assumptions of Theorem 4.2.4 are satisfied. It holds that $\xi \in \cap_{p\in(0,\infty)} \mathcal{L}(P|_{\mathbb{F}_0}; |\cdot|_{\mathbb{R}})$ since ξ is $\mathbb{F}_0$-measurable and all moments of normal distribution are finite. Moreover, as the coefficients μ and σ are linear functions, they are in particular Lipschitz continuous and four times continuously differentiable with polynomial bounded derivatives (the first derivative is constant, all higher derivatives vanish). Moreover, for $f(x) := x^n$ with $n \in \mathbb{N}$ there holds

$$\begin{aligned}
&|f^{(i)}(x)| = \left| \left(\prod_{j=0}^{i-1} (n - i) \right) x^{n-i} \right| \le C(1 + |x|^{n-i}), \quad i = 1, \dots, n, \\
&|f^{(i)}(x)| = 0, \quad i \ge n + 1,
\end{aligned}$$

where the constant $C = C(n, i)$ depends on the indicated parameters. Hence, Theorem 4.2.4 applies and the Euler scheme converges weakly with order one.

3. (i) See file `ErrorEMWeak.m`. The estimate rate of convergence is $\alpha \approx 0.97531$, so it is very close to the expected weak rate of $\alpha = 1$.

(ii) See file `ErrorMilsteinWeak.m`. The estimate rate of convergence is $\alpha \approx 1.0052$. Thus, we don't see any increase in the weak convergence rate by the Milstein scheme, although the coefficients μ and σ are smooth with bounded derivatives and $f \in C_p^\infty(\mathbb{R}; \mathbb{R})$.

(iii) See file `ErrorEMWeak2.m`. The exact mean is given for $t = 1$ and $c = 5$ by the formula

$$\begin{aligned}
\mathbb{E}_P[f(X_t)] &= P\left(X_t \geq c\right) \\
&= P\left(\xi \exp\left((\mu_0 - \frac{\sigma_0^2}{2})t + \sigma_0 W_t\right) \geq c\right) \\
&= P\left(W_t \geq \frac{1}{\sigma_0}\left(\log(c) - \left(\mu_0 - \frac{\sigma_0^2}{2}\right)t\right)\right) \\
&= 1 - \Phi\left(\frac{1}{\sqrt{t}\sigma_0}\left(\log(c) - \left(\mu_0 - \frac{\sigma_0^2}{2}\right)t\right)\right),
\end{aligned}$$

where Φ is the cdf of the standard normal distribution, and we have used that $W_t \sim N_{0,t}$.

We now see an deteriorated rate of $\alpha \approx 0.13252$. This could be explained since the test function f is now discontinuous, and therefore the conditions of Theorem 4.2.4 are violated.

Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
Organisation: Francesca Bartolucci, HG G 53.2