D-MATH Numerical Analysis of Stochastic ODEs Autumn 2021
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Series 9

1. See file `MultiLevelMonteCarlo.m`.

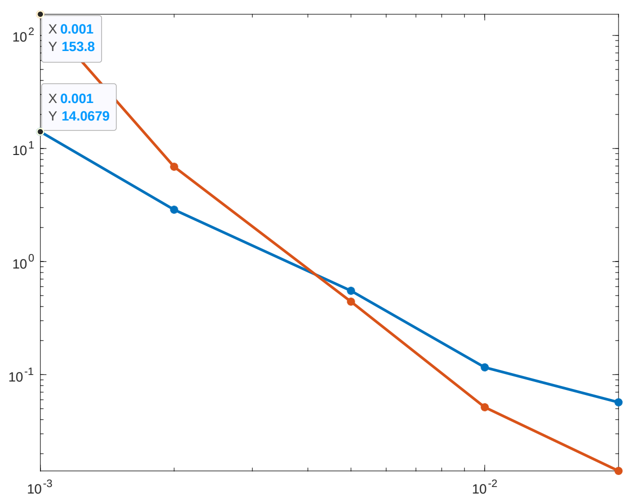
2. (i) See file `MultiLevelMonteCarloBSCall.m`.

```
>> MultiLevelMonteCarloBSCall
Convergence rate of the Multi Level Monte Carlo scheme w . r . t . to epsilon: 0.96121
Overall complexity of the Multi Level Monte Carlo method w . r . t . to epsilon: -1.8331
```

(ii) See file `MLMCvsMCEBSCall.m`.

```
>> MLMCvsMCEBSCall
Convergence rate of the Multi Level Monte Carlo scheme w . r . t . to epsilon:  1.0312
Overall complexity of the Multi Level Monte Carlo method w . r . t . to epsilon:-1.8525
Convergence rate of the Monte Carlo Euler scheme w . r . t . to epsilon:0.99745
Overall complexity of the Monte Carlo Euler method w . r . t . to epsilon: -2.8123
```

(iii) See file `MLMCvsMCEBSCall.m`.



3. We use the tower property of conditional expectations and the independence of N and the X_k to obtain for any $x \in \mathbb{R}^m$ and $n \in \mathbb{N}$

$$\begin{aligned}
\mathbb{E}_P\big[e^{\mathbf{i}x^\top C_t}\big] &= \sum_{n\in\mathbb{N}_0} \mathbb{E}_P\big[e^{\mathbf{i}x^\top C_t} \,|\, N_t = n\big] P(N_t = n) \\
&= \sum_{n\in\mathbb{N}_0} \mathbb{E}_P\left[\exp\left(\mathbf{i}x^\top \sum_{k=1}^{N_t} X_k\right) \,\Big|\, N_t = n\right] P(N_t = n) \\
&= \sum_{n\in\mathbb{N}_0} \mathbb{E}_P\left[e^{\mathbf{i}x^\top X_1}\right]^n P(N_t = n) \\
&= \sum_{n\in\mathbb{N}_0} \left(\int_{\mathbb{R}} e^{\mathbf{i}x^\top z}\mu(dz)\right)^n e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \exp\left(-\lambda t \int_{\mathbb{R}} \mu(dz)\right) \exp\left(\lambda t \int_{\mathbb{R}^m} e^{\mathbf{i}x^\top z}\mu(dz)\right),
\end{aligned}$$

which proves the first part of the claim.

Given that μ has a second moment, the expressions for mean and variance in the one-dimensional case follow from the moment generating property of $\mathbb{E}_P\big[e^{\mathbf{i}x^\top C_t}\big]$ via

$$\mathbb{E}_P\big[C_t\big] = (-\mathbf{i}) \cdot \frac{d}{dx}\mathbb{E}_P\big[e^{\mathbf{i}x^\top C_t}\big]\big|_{x=0}, \quad \text{and} \quad \mathbb{E}_P\big[C_t^2\big]) = (-1) \cdot \frac{d^2}{dx^2}\mathbb{E}_P\big[e^{\mathbf{i}x^\top C_t}\big]\big|_{x=0}.$$

Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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