Dr. Andreas Stein
Dr. Francesca Bartolucci

Series 2

1. Let $T \in (0,\infty)$, let $(\Omega,\mathcal{F},P)$ be a probability space, and let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion. Let $N \in \mathbb{N}$, and set $\Delta t = T/N$ and $t_n = n\Delta t$ for $n = 0,\dots,N$. Given $W_{t_0}, W_{t_1},\dots,W_{t_N}$, define the linear interpolation $\widetilde{W}\colon [0,T]\times\Omega\to\mathbb{R}$ by

$$\widetilde{W}_t = \frac{t-t_{n-1}}{\Delta t} W_{t_n} + \frac{t_n - t}{\Delta t} W_{t_{n-1}}, \qquad t \in [t_{n-1}, t_n], \quad n = 1,\dots,N. \tag{1}$$

Show that $\sup_{t\in[0,T]} \|\widetilde{W}_t - W_t\|_{L^2(P;|\cdot|_\mathbb{R})} = \mathcal{O}(\sqrt{T/N})$.

2. Let $T \in (0,+\infty)$, let $(\Omega,\mathcal{F},P)$ be a probability space, let $W\colon [0,T]\times\Omega\to\mathbb{R}$ be a one-dimensional standard Brownian motion, and let $0 < r < s < t \le T$. Show that the conditional random variables $X := \big(W_s | W_t = y\big)$ and $\widetilde{X} := \big(W_s | W_r = x, W_t = y\big)$, which means that X is conditioned on the values of W_t and $\widetilde{X}$ is conditioned on the values of W_r and W_t, satisfy

$$X \sim \mathcal{N}\left(\frac{sy}{t}, \frac{(t-s)s}{t}\right) \quad \text{and} \quad \widetilde{X} \sim \mathcal{N}\left(\frac{(t-s)x + (s-r)y}{(t-r)}, \frac{(s-r)(t-s)}{(t-r)}\right).$$

The random variable $\widetilde{X}$ is also called *Brownian bridge* of W at s, conditional on the values $W_r = x$ and $W_t = y$.

Hint: You may use Corollary 1.3.5 together with the following result (without proof) applied to the random vectors $\big(W_s, W_t\big)$ and $\big(W_r, W_s, W_t\big)$, respectively:

Let $m,n \in \mathbb{N}$, let $\mu^{(1)} \in \mathbb{R}^m, \mu^{(2)} \in \mathbb{R}^n$, and let $\Sigma_{11} \in \mathbb{R}^{m\times m}, \Sigma_{12} \in \mathbb{R}^{m\times n}, \Sigma_{21} \in \mathbb{R}^{n\times m}, \Sigma_{22} \in \mathbb{R}^{n\times n}$ be matrices such that

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}$$

is symmetric positive semi-definite and Σ_{22} has full rank. Let $(Y^{(1)}, Y^{(2)})$ be a random vector with the $(m+n)$-dimensional normal distribution

$$\begin{pmatrix} Y^{(1)} \\ Y^{(2)} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right).$$

Then, for every $y \in \mathbb{R}^n$ the conditional distribution of $Y^{(1)}$ on the value $Y^{(2)} = y$ is given by

$$\big(Y^{(1)} \big| Y^{(2)} = y\big) \sim \mathcal{N}\left(\mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}\big(y - \mu^{(2)}\big), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right).$$

Bitte wenden!

3. Let $T \in (0, +\infty)$, let $(\Omega, \mathcal{F}, P)$ be a filtered probability space, and let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a one-dimensional standard Brownian motion. Let $N \in \mathbb{N}$, set $\Delta t := T/N$ and $t_i := i\Delta t$ for $i = 0, \ldots, N$. Given $W_{t_0}, W_{t_1}, \ldots, W_{t_N}$, define the linear interpolation $\widetilde{W}\colon [0,T] \times \Omega \to \mathbb{R}$ for $t \in [t_i, t_{i+1}]$, $i = 0, \ldots, N-1$, by

$$\widetilde{W}_t := \frac{t_{i+1} - t}{\Delta t} W_{t_i} + \frac{t - t_i}{\Delta t} W_{t_{i+1}}.$$

Recall from the lecture that $\widetilde{W}$ is a time-continuous approximation of W based on the $N+1$ discrete realizations $W_{t_0}, W_{t_1}, \ldots, W_{t_N}$.

(i) Write a MATLAB function `BMIncrInterp`(t, T, N) that returns a realization of $\widetilde{W}_t$ for arbitrary $t \in [0, T]$. The realizations of $W_{t_0}, W_{t_1}, \ldots, W_{t_N}$ should be simulated by summation of the increments $W_{t_1} - W_{t_0}, \ldots, W_{t_N} - W_{t_{N-1}}$. Set $T = 1$ and plot a path of $\widetilde{W}$ for each $N \in \{2^7, 2^8, 2^9\}$ on the y-axis against the vector $t = 0 : 2^{-10} : 1$ on the x-axis.

Hint: You may use the built in MATLAB function `interp1` for the linear interpolation.

(ii) Write a MATLAB function `BMBridgeInterp`(t, T, N) that returns a realization of W_t (<u>not</u> $\widetilde{W}_t$) for arbitrary $t \in [0, T]$. To achieve this, generate $W_{t_0}, W_{t_1}, \ldots, W_{t_N}$ as in item (i) and then, draw samples of the Brownian bridge $(W_t | W_{t_i} = x, W_{t_{i+1}} = y)$ conditional on the given values $W_{t_i} = x, W_{t_{i+1}} = y$ for $t \in [t_i, t_{i+1}]$ and $x, y \in \mathbb{R}$. Set $T = 1$ and plot a path of W for each $N \in \{2^7, 2^8, 2^9\}$ on the y-axis against $t = 0 : 2^{-10} : 1$ on the x-axis.

Hint: Use Exercise 2 and the function `BMIncrInterp`(t, T, N) from item (i).

Due: 16:00 o'clock, Monday, 10th October 2022
Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
Organisation: Francesca Bartolucci, HG G 53.2