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Series 9

Throughout this sheet let $T \in (0, +\infty)$, $d, N, K \in \mathbb{N}$, let $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]})$ be a stochastic basis, and let $W\colon [0,T] \times \Omega \to \mathbb{R}$ be a one-dimensional standard $(\Omega, \mathcal{F}, P, (\mathbb{F}_t)_{t\in[0,T]})$-Brownian motion.

1. Let $\xi \in \mathbb{R}$, $\mu \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, $\sigma \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, let $X\colon [0,T] \times \Omega \to \mathbb{R}^d$ be a solution process of the SDE

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad t \in [0,T], \qquad X_0 = \xi,$$

and let $f \in \mathcal{M}(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}))$ satisfy $\mathbb{E}_P\big[|f(X_T)|\big] < +\infty$. Write a MATLAB function `MultiLevelMonteCarlo(`T`,`ξ`,`μ`,`σ`,`ε`,`α`,`β`,`γ`,`f`)` with inputs T, ξ, μ, σ, f as above, and simulation input parameters $\varepsilon, \alpha, \beta, \gamma > 0$, that outputs a realization of a multilevel Monte Carlo- Euler approximation of $\mathbb{E}_P\big[f(X_T)\big]$ with tolerance ε.

Recall that the MLMC-Euler estimator is given by

$$E^{\mathrm{ML}}(f(Y_{N_L}^{N_L})) = \sum_{\ell=1}^{L} \frac{1}{K_\ell} \sum_{k=1}^{K_\ell} (f(Y_{N_\ell}^{N_\ell,k}) - f(Y_{N_{\ell-1}}^{N_{\ell-1},k})),$$

where $Y_N^{N,k}$ denotes the k-th sample of the Euler-Maruyama approximation of X_T with stepsize $\Delta t = T/N$, $L = \lceil -\log_2(\varepsilon) \rceil$, $N_\ell = N_0 2^\ell$, $\ell = 1, \dots, L$, with $N_0 = 2T$, and

$$K_\ell = \left\lceil 2^{2\alpha L} \left(\sum_{k=1}^{L} 2^{(\gamma-\beta)k/2} \right) 2^{-(\beta+\gamma)\ell/2} \right\rceil.$$

Hint: You may use the template `MultiLevelMonteCarloEuler.m`.

2. Consider the Black-Scholes model, where the price process of an underlying S is modeled by the SDE

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t, \quad t \in [0,T], \quad S_0 = s_0,$$

for a fixed interest rate $r \in \mathbb{R}$, a volatility parameter $\sigma > 0$ and an initial price $s_0 > 0$.

(i) Test your MATLAB function `MultiLevelMonteCarlo()` from Exercise 1 to evaluate a European call option with strike price $K_{\text{strike}} > 0$ and payoff at T given by $f(S_T) = \max(S_T - K_{\text{strike}}, 0)$. To this end, run the multilevel Monte Carlo scheme with tolerance $\varepsilon \in \{0.05, 0.02, 0.01, 0.005, 0.002\}$ to estimate the root mean squared error (RMSE)

$$\left\| e^{-rT}\mathbb{E}_P\big[f(S_T)\big] - e^{-rT}\sum_{\ell=1}^{L}\frac{1}{K_\ell}\sum_{k=1}^{K_\ell}\big(f(Y_{N_\ell}^{N_\ell,k}) - f(Y_{N_{\ell-1}}^{N_{\ell-1},k})\big)\right\|_{L^2(P;|\cdot|_{\mathbb{R}})}. \tag{1}$$

In equation (1), $Y_N^{N,k}$ denotes the k-th sample of the Euler-Maruyama approximation of S_T with stepsize $\Delta t = T/N$. Use the Black-Scholes parameters $T = 1, S_0 = 100, r = 0.05, \sigma = 0.1$ and $K_{\text{strike}} = 100$. Estimate the RMSE in equation (1) by generating 10 realizations of the weak error

$$e^{-rT}\mathbb{E}_P\big[f(S_T)\big] - e^{-rT}\sum_{\ell=1}^{L}\frac{1}{K_\ell}\sum_{k=1}^{K_\ell}\big(f(Y_{N_\ell}^{N_\ell,k}) - f(Y_{N_{\ell-1}}^{N_{\ell-1},k})\big)$$

for each ε and averaging the squared realizations. Estimate the convergence rates of the weak error and of the overall complexity with respect to ε. **Report** on the results.

Hints: You may use template `MultiLevelMonteCarloBSCall.m`. The exact value of the call price $e^{-rT}\mathbb{E}_P\big[f(S_T)\big]$ is given by the Black Scholes formula in Series 4. You may also use the MATLAB function `blsprice()`. Set the parameter `M_RMSE` in the template to `M_RMSE = 1` first to make sure everything works, and then rerun the experiment with `M_RMSE = 10`.

(ii) Compare your MATLAB function `MultiLevelMonteCarlo()` from Exercise 1 with your MATLAB function `MonteCarloEuler()` from Sheet 8 to evaluate a European call option with strike price $K_{\text{strike}} > 0$ and payoff at T given by $f(S_T) = \max(S_T - K_{\text{strike}}, 0)$. To this end, run the Monte Carlo-Euler scheme to compute an approximation of $\mathbb{E}_P\big[f(S_T)\big]$ with tolerance $\varepsilon \in \{0.05, 0.02, 0.01, 0.005, 0.002\}$. Adjust the number of samples K for each step size $\Delta t = \varepsilon$ so that the statistical error and the discretization bias in the root mean squared error (RMSE)

$$\left\| e^{-rT}\mathbb{E}_P\big[f(S_T)\big] - e^{-rT}\frac{1}{K}\sum_{k=1}^{K} f\big(Y_N^{N,k}\big)\right\|_{L^2(P;|\cdot|_{\mathbb{R}})} \tag{2}$$

are balanced. In Equation (2), $Y_N^{N,k}$ denotes the k-th sample of the Euler-Maruyama approximation of S_T with stepsize $\Delta t = T/N$. Use the Black-Scholes parameters $T = 1, S_0 = 100, r = 0.05, \sigma = 0.1$ and $K_{\text{strike}} = 100$. Estimate the RMSE in equation (2) by generating 10 realizations of the weak error

$$e^{-rT}\mathbb{E}_P\big[f(S_T)\big] - e^{-rT}\frac{1}{K}\sum_{k=1}^{K} f\big(Y_N^{N,k}\big)$$

for each ε and averaging the squared realizations. Estimate the convergence rates of the weak error and of the overall complexity with respect to ε, and plot the

Siehe nächstes Blatt!

estimated computational times against ε in a logarithmic diagram. **Report** on the results.

Hints: You may use template `MLMCvsMCEBSCall.m`. Set the parameter `M_RMSE` in the template to `M_RMSE = 1` first to make sure everything works, and then rerun the experiment with `M_RMSE = 10`.

(iii) Repeat item (ii) with $\varepsilon \in \{0.02, 0.01, 0.005, 0.002, 0.001\}$. **Report** on the results.

Hint: Depending on your workstation, this simulation might take up to half an hour!

3. Let $C\colon [0,T] \times \Omega \to \mathbb{R}^m$ be a compound Poisson process with rate $\lambda > 0$ and jump measure μ on $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$. Recall that C is given by

$$C_t := \sum_{k=1}^{N(t)} X_k, \quad t \in [0,T], \tag{3}$$

where $N\colon [0,T] \times \Omega \to \mathbb{N}_0$ is a Poisson process with rate λ, and $(X_k, k \in \mathbb{N})$ is a sequence of i.i.d. $\mathbb{R}^m$-valued random variables with distribution μ, such that $(X_k, k \in \mathbb{N})$ and N are independent.

(i) Show that the characteristic function of C is given by

$$\mathbb{E}_P\big[e^{\mathbf{i}x^\top C_t}\big] = \exp\left(\lambda t \int_{\mathbb{R}^m} (e^{\mathbf{i}x^\top z} - 1)\mu(dz)\right), \quad x \in \mathbb{R}^m. \tag{4}$$

(ii) Now assume $m = 1$ and that $(X_k, k \in \mathbb{N})$ is such that $\mathbb{E}_P\big[|X_1|^2\big] < \infty$. Use item (i) to show that

$$\mathbb{E}_P\big[C_t\big] = t\lambda\mathbb{E}_P\big[X_1\big], \quad \text{and} \quad \mathrm{Var}_P(C_t) = \lambda t(\mathbb{E}_P\big[X_1\big]^2 + \mathrm{Var}_P(X_1)). \tag{5}$$

Due: 16:00 o'clock, Monday, 28th November 2022
Webpage: `https://moodle-app2.let.ethz.ch/course/view.php?id=17423`
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