Exercise sheet 6

The Exercises are meant to be done in groups of three, **please enroll your group in OLAT**. Even if you are doing the exercies on your own, please enroll yourself in an enmpty group in OLAT. Without enrollment, you wont be able to submit.
Submission of your solutions is done via OLAT. Please add the names of all group members to your solution.

Excercise 6.1: Hopfield (5 Punkte)

Consider all patterns y^i, $i = 1, 2, \ldots, 5$, given by the following matrix:

$$Y = \begin{bmatrix} 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(a) Construct the weight matrix of the Hopfield network with Hebb's rule.

(b) Verify if each y^i, $i = 1, \ldots, 5$, is a fixed point of the Hopfield network.

(c) Calculate the weight matrix with the help of the projection rule.

Hint: The weight matrix of a Hopfield network can be constructed via the so called projection rule $W = YY^+$, where Y^+ denotes the pseudo-inverse of Y. The following property of the pseudo-inverse could be helpful:

$$Y \in \mathbb{R}^{m \times n} \text{ has full column rank } \Rightarrow Y^+ = (Y^\top Y)^{-1} Y^\top \in \mathbb{R}^{n \times m}.$$

Excercise 6.2: Lyapunov (5 Punkte)

Let X be a finite set and $f : X \to X$ be given. Assume there exists a function $E : X \to \mathbb{R}$ for which

$$E(f(x)) < E(x)$$

if $x \neq f(x)$. Prove that there exists a fixed point $x^* \in X$, for which $f(x^*) = x^*$, and that any sequence defined by $x_0 \in X$ and $x_{n+1} = f(x_n)$ reaches a fixed point.

Excercise 6.3: Gated Recurrent Units (GRU) (8 Points)

Consider a recurrent neural network called GRU whose dynamics is given as

$$\begin{aligned} z_t &= \sigma(W_z h_{t-1} + U_z x_t + b_z) \\ r_t &= \sigma(W_r h_{t-1} + U_r x_t + b_r) \\ \tilde{h}_t &= \tanh(W_h(r_t \odot h_{t-1}) + U_h x_t + b_h) \\ h_t &= (1 - z_t) \odot h_{t-1} + z_t \odot \tilde{h}_t, \end{aligned}$$

where $x_t \in \mathbb{R}^m$, $h_t \in \mathbb{R}^n$ and $\odot$ denotes the Hadamard product. Further, let W_h and z_t be such that

(i) $\|W_h\| < \frac{1}{L_{\tanh}}$ where $L_{\tanh}$ is the Lipschitz constant of $\tanh$,

(ii) $\exists\ \epsilon > 0$ such that $\epsilon \leq z_t \leq 1 - \epsilon$ for all t

Prove that there exists $h^* \in \mathbb{R}^n$ such that $h_t \to h^*$ exponentially when $x_t \to x^*$ exponentially as $t \to \infty$.
Hint: For the exponential convergence of the input x_t you may consider that $\|x_t - x^*\|_2 \leq C\rho^t$ with $\rho < 1$.

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Excercise 6.4: Project 5 (5 Points)

The objective of this project is to become familiar with sequential prediction problems by use of recurrent structure of networks. The project files are presented as python notebook files in Material/Projects/Proj5 folder. It contains the script file: RNN_GRU.ipynb and the task is to implement the RNN (Task1) and GRU (Task2) networks and compare their results. Note, a simple RNN (recurrent neural network) is given as

$$h_t = \tanh(Wh_{t-1} + Ux_t + b),$$

while the GRU model is as Exercies 6.3.

Submission guidelines:

1. Please submit the solved file (with outputs for each code-block) along with the theory exercise solutions. Please name your submission file as RNN_GRU_solved_group_id.ipynb, where id is your two digit group number. For example, if you belong to Group01 please name the file as: RNN_GRU_solved_group_01.ipynb

2. Please ensure that the submitted ipynb files are runnable and does not produce any errors. The project shall be considered for marking if and only if there are zero errors in the script file.