Unit-5

Graphs and trees

Graphs

Let $G = \{V, E\}$ be a graph Here $V = \{V_1, V_2 \cdots V_n\}$

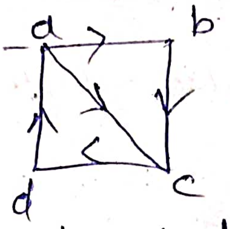
$E = \{(V_1 V_2)(V_2, V_3) \ldots (V_2 V_n)\}$.

* $|V|$ denotes the no of vertices of graph.

$|E|$ denotes the size of the graph.

* If the graph has direction, then the graph is said to be directed graph

Ex:-



In a directed graph we have indegree and outdegree for the vertices

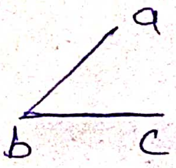
* indegree of a vertex means the no of edges which are incident to a vertex, it is denoted by $deg^+(V)$

* outdegree of a vertex means the number of edges which are incident from a vertex, it is ~~dento~~ denoted by $deg^-(V)$

* degree spectrum is the collection of indegree and outdegree for the same vertex.
it is denoted by (i, j) where i represent indegree of a vertex j represents outdegree of a vertex.

* A graph is said to be undirected if it has no direction.

Ex:- consider a graph $G = \{V, E\}$ where $V = \{a, b, c\}$

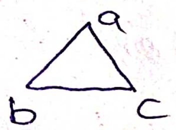


* we have different types of undirected graphs they are :- 1) cyclic graph, 2) wheel 3) Path graph 4) Bipartite graph 5) null graph 6) complete graph 7) planar graph 8) Euler's graph 9) Hamiltonian graph 10) regular graph.

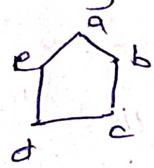
1) cyclic graph (C_n) :-

A graph G is said to be cyclic if it has n vertexs and n edges which forms a circuit.

Ex:- C_3



C_5



2) Path graph (P_n) :-

If we are removing one edge from the cyclic graph C_n we get the path graph P_n

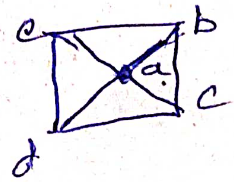
Ex:- P_5

3) Wheel graph (W_n):—

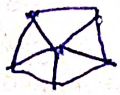
A single new vertex is adjacent remaining $n-1$ vertices which forms a circuit

minimum 4 vertices

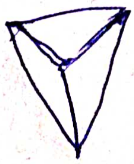
Ex:- W_5



W_6



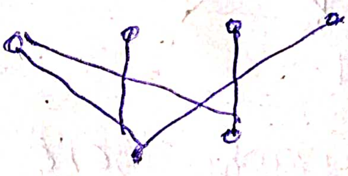
W_4



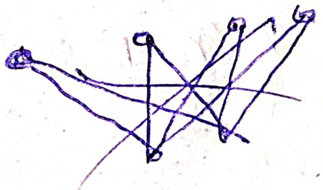
4) Bipartite graph

a graph is said to be bipartite the vertex set be partition to two set M, n such that the vertices of M are adjacent to vertices of n only. it is denoted by $K_{m,n}$

$m=4$ $n=2$

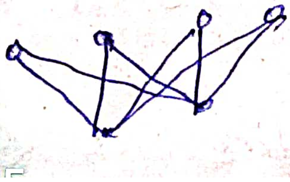


Ex:-



a bipartite graph is said to be complete if all vertices of M are adjacent to all vertices of n

Ex:—



Null graph

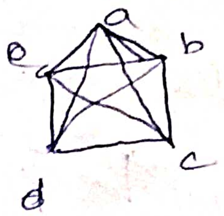
A graph is said to null graph if it has vertex set only. denoted by N_n

Ex:- N_3

complete graph

A graph is said to be complete if very vertex is adjacent to remaining $n-1$ vertices it is denoted by K_n

Ex:- K_3 K_5

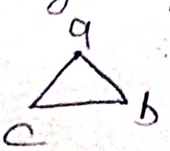


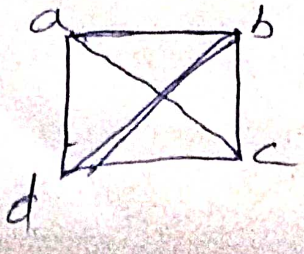
In complete graphs there are $\frac{n(n-1)}{2}$ edges

K-regular graph

A graph is said to K regular or regular of degree K if $\Delta(G) = \delta(G) = K$ where $\Delta(G)$ denotes the maximum of degree of all vertices of graph G. ~~cr~~ $\delta(G)$ dendtes the minimum of degree of all vertices of G

Ex:- 2-regular graph 3-regular graph

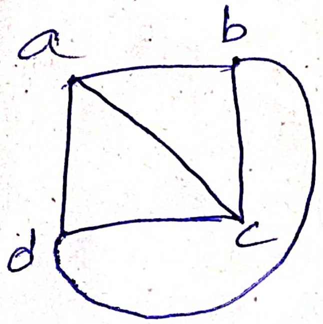


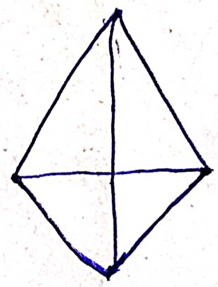


planar graph

A graph G is said be planar it can be drawn on plane with out any edge cross overs.

Ex: K_3, C_3 K_4 are planar

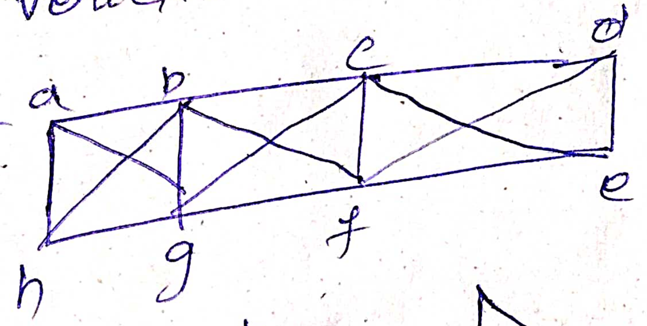


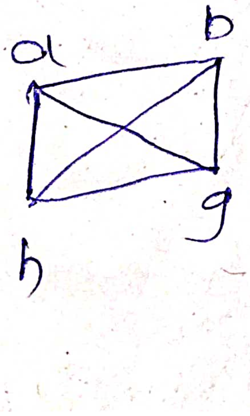


critical planar graph

if we are removing consider the sub graph of G if we are removing one vertex's. then we get a planar graph.

Ex:





Euler formula for planar graphs:—

Statement :- In a connected planar graph G,

$|V| - |E| + |R| = 2$ where $|V|$ denotes the no of vertices

$|E|$: The no of edges, $|E|$ denotes the no of region.

Proof :- we prove this theorem using mathematical Induction on no of regions

Bases of induction

we have to show that the stmt is true for

$|R| = 1$

W.K.T a tree with n vertices have exactly n-1 edges and it has only one region

i.e $|V| = n$ and $|E| = n-1$, $|R| = 1$

consider

$|V| - |E| + |R|$

$= n - n + 1 + 1$

$= 2.$

∴ The given stmt is true for $|R| = 1$.

Inductive hypothesis

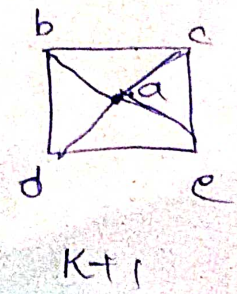
Assume that $|R| = K$ i.e $|V| - |E| + |R| = 2$ is True

Inductive step

We have to show that the given stmt is True for $|R| = K+1$

consider a connected planar graph G' having K+1 regions it has the vertices $|V'|$ and edges $|E'|$ and regions $|R'|$

G1



K+1

If we are removing any one edge common to the boundary of two regions we get a graph G having $|V|$ vertices and $|E|$ edges and $|R|$ regions

Here $|V| = |V'|$

$|E| = |E'| - 1$

$|R| = |R'| - 1$

consider $|V'| - |E'| + |R'|$

$= |V| - |E| - \cancel{1} + |R| + \cancel{1}$

$= 2$

$\therefore$ the given stmt is True for $|R| = K+1$.

Theorem 2:-

Stmt: In a connected planar graph G, $|E| > 1$

then (i) $|E| \leq 3|V| - 6$

(ii) There is a vertex V of G $\ni \deg(v) \leq 5$

By Euler's formula we have

$|V| - |E| + |R| = 2$

i.e $|V| + |R| = 2 + |E| \longrightarrow$ (A)

we know that in any connected planar graph G each region is bounded by atleast 3 edges and each edge belongs to exactly 2 regions

$2|E| \geq 3|R|$

$\Rightarrow |R| \leq \frac{2}{3}|E|$

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$$|V| + |R| \leq |V| + \frac{2}{3}|E| \longrightarrow (B)$$

~~from from~~

~~from~~ from (A) and B

$$2 + |E| \leq |V| + \frac{2}{3}|E|$$

$$|E| - \frac{2}{3}|E| \leq |V| - 2$$

$$\frac{1}{3}|E| \leq |V| - 2$$

$$|E| \leq 3|V| - 6.$$

ii). suppose on the contrary assume that

$$\deg(V) \geq 6$$

w.K.T. in any ~~connected~~ planar undirected graph

$$\sum_{i=1}^{n} \deg(V_i) = 2|E|$$

$$6|V| \leq 2|E$$

$$|V| \leq \frac{1}{3}|E| \longrightarrow (A)$$

w.K.T ~~in any conn~~ $|V| - |E| + |R| = 2$

in any connected Planar

alleast 3

each edges belongs to exactly 2 regions

i.e $2|E| \geq 3|R|.$

$$|R| \leq \frac{2}{3}|E| \longrightarrow (B)$$

in a connected planar graph G we have

$$|V| - |E| + |R| = 2$$

$$\Rightarrow |V| + |R| = |E| + 2 \longrightarrow (1)$$

from (A) and (B)

~~Consider~~

$$|V| + |E| = \frac{1}{3}|E| + \frac{2}{3}|E|$$

$$|E| + 2 \leq |E|$$

which is contradiction fact that

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$2 \leq 0$

$\therefore$ our assumption is wrong

There is a vertex v of G $\ni$ $deg(v) \leq 5$

Theorem 3:-

stmt :- A complete graph K_n is planar for $n \leq 4$.

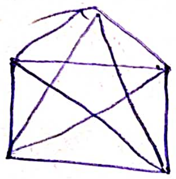
We have S.T

K_n is planar for $n \leq 4$

i.e it is enough to show that

K_n is not planar for $n \geq 5$

assume that K_5 is planar



Here $|V| = 5$, $\therefore$ w.K.T complete graph have $|E| = \frac{n(n-1)}{2} = 10$

By Eulers formula we have

$$|V| - |E| + |R| = 2$$

$$|R| = 2 - |V| + |E|$$

$$= 2 - 5 + 10$$

$$|R| = 7$$

w.K.T in any connected planar graph G

~~as~~ each region is bounded by atleast 3 edges and each edge is belongs to 2 regions

$2|E| \geq 3|R|$

$\Rightarrow 20 \geq 21$

which is a contradiction fact that

$20 \leq 21$

$\therefore K_5$ is not a planar graph

K_n is planar for $n \leq 4$.

Theorem:-4

stmt :- A complete bipartite graph $K_{m,n}$ is planar for $m \leq 2$ or $n \leq 2$

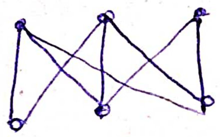
we have to S.T

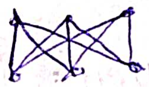
$K_{m,n}$ is planar for $m \leq 2$ or $n \leq 2$

i.e it is enough to show that $K_{m,n}$ is not planar for $m \geq 2$ or $n \geq 2$

suppose on contrary assume that

$K_{3,3}$ is a planar graph





$|V| = 6$

$|E| = 9$

By Euler formula we have

$|R| = 2 - |V| + |E|$

$= 5.$

w.k.T in any complete bipartite graph

there can be ~~cycle~~ length ≥ 4

i.e $2|E| \geq 4|R|$

$18 \geq 20$

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which is a contradiction fact that $18 \leq 20$

∴ $K_{3,3}$ is not a planar graph.

∴ $K_{m,n}$ is planar for $m \le 2$ or $n \le 2$

Theorem:-5

~~Stmt~~ : sum of degrees theorem or hand shaking theorem

Stmt

In any directed graph G we have $\sum_{i=1}^{n} deg(V_i) = \sum_{i=1}^{n} deg^+(V_i)$

$+ \sum_{i=1}^{n} deg^-(V_i)$

$= 2|E|.$

In an undirected graph G we have

$$\sum_{i=1}^{n} deg(V_i) = 2|E|$$

Proof consider a directed graph G having the vertices $V = \{V_1 \dots V_n\}$

Assume that the indegree of V_1 is n_1

indegree of V_2 is n_2

⋮

Similarly indegree of V_n is n_n

i.e $\sum_{i=1}^{n} deg^+(V_i) = n_1 + n_2 + n_3 + \dots + n_n$

$= |E|.$

Similarly

Assume that outdegree of V_1 is n_1

outdegree of V_2 is n_2

⋮

Similarly outdegree of V_n is n_n

i.e $\sum_{i=1}^{n} deg^-(v_i) = n_1 + n_2 + \cdots + n_n$

$= |E|$

w.K.T in a directed graph G every vertex has indegree and outdegree i.e

$\sum_{i=1}^{n} deg(v_i) = \sum_{i=1}^{n} deg^+(v_i) + \sum_{i=1}^{n} deg^-(v_i)$

$= 2|E|$.

(ii) in an undirected graph G there is no indegree and outdegree for the vertices here (a, b) means there is an edge from a to b and b to a $\therefore \sum_{i=1}^{n} deg(v_i) = 2|E|$.

Theorem:- 6

Stmt

In any non directed graph G there is an even number of vertices of odd degree.

Proof

consider an directed graph G having n vertices if possible assume that K vertices has degree odd then the remaining n−K vertices has degree even

w.K.T the sum of even degree vertices is equal to even

Here total no of edges is equal to odd + even = odd.

$\therefore$ our Assumption is wrong

In an undirected graph G, K vertices has even degree.

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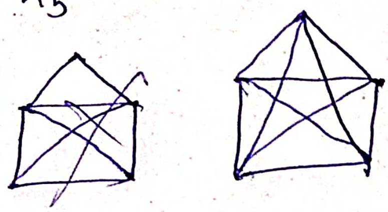
Hamiltonian graph

A graph is said to be hamiltonian if it contains hamiltonian circuit

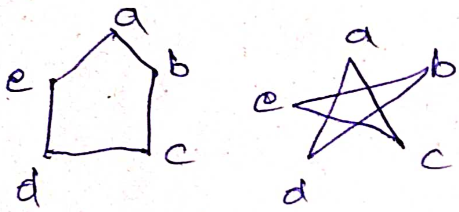
Hamiltonian circuit must contain n vertices and n edges

Hamiltonian path must contain n vertices and n-1 edges.

Ex:- K_5



The ~~H~~ Hamiltonian circuits for graph K_5



While constructing the hamiltonian circuits all the unused edges are delete. And every ~~very~~ vertice has degree 2.

Euler's graph

Ex:- cycli graphs

A Graph is said to Euler's Graph if it has Euler's circuit.

Euler's circuit must cover each and every edge of given graph G.

Every cyclic graph is a Euler's graph

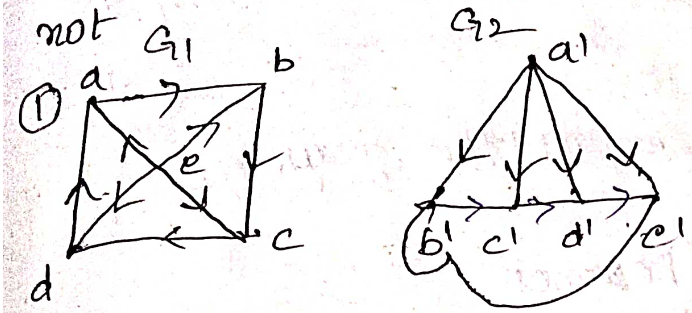
isomorphism
~~Homomorphism~~ of graphs

Two types 1. Directgraphs
2. undirected graph

1. Directed graph ~~homo~~ is omorphism

Let G_1 and G_2 be two graphs we say that G_1 is isomorphism to G_2. if there exist a function $f: V(G_1) \rightarrow V(G_2)$ such that f is bijective

*verify that the following graphs are isomorphic or not



sol Given that $G_1 = (V_1, E_1)$

where $V_1 = \{a, b, c, d, e\}$

$|V_1| = 5$

$E_1 = \{(a,b), (b,c), (c,d) (d,a) (e,a) (e,b) (e,c) (e,d)$

$|E_1| = 8$

G.T $G_2 = (V_2, E_2)$

$V_2 = \{a', b', c', d', e'\}$

$|V_2| = 5$

$E_2 = \{(a',b') (a',c') (a',d'), (a',e') (b',c') (c',d') (d',e')$

$(e',b')\}$

$|E_2| = 8$

the given graphs G_1 and G_2 has same no. of vertices and same no of edges

G_1		G_2	
V_1	Degree seqtrum	V_2	Degree seqtrum
a	(2,1)	a'	(0,4)
b	(2,1)	b'	(2,1)
c	(2,1)	c'	(2,1)
d	(2,1)	d'	(2,1)
e	(0,4)	e'	(2,1)

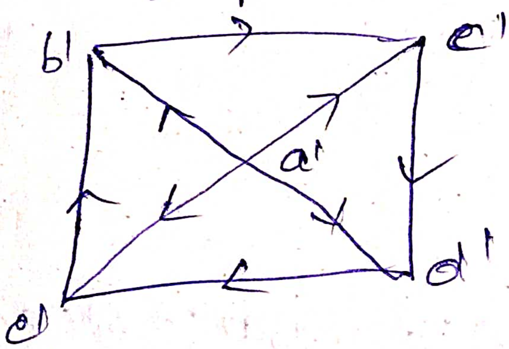
Here both the graphs G_1 and G_2 have same degree sepectram.

In graph G_1 the vertex e is replaced by a'

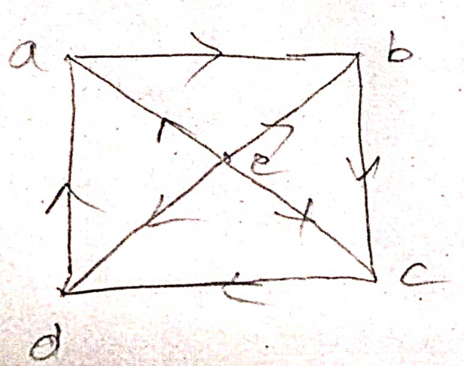
a is replaced by b'
b is by c'
c by d'
d by e'.

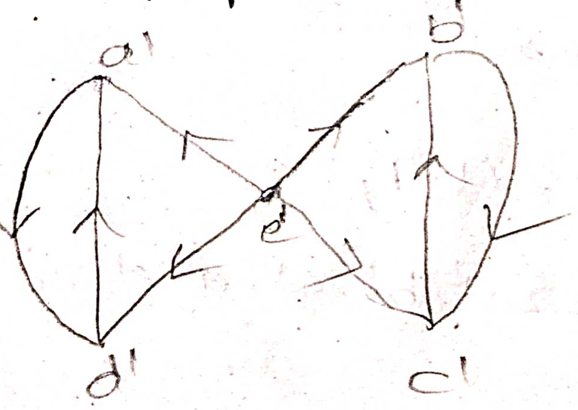
we get the graph g_2

then graph G' becomes



2) verify that the given graph is isomorphic or not





Sol Given that

The $G_1 = (V_1, E_1)$

The vertices are $V_1 = \{a, b, c, d, e\}$

$|V_1| = 5$

$E_1 = \{(a,b), (b,c) (c,d) (d,a) (e,a) (e,b) (e,c) (e,d)\}$

$|E_1| = 8$

The $G_2 = (V_2, E_2)$

The vertices are $V_2 = \{a', b', c', d', e'\}$

$|V_2| = 5$

$E_2 = \{(a', d') (e' d') \cancel{(c' d')} (d' a') (e' a') (e' b') (e' c') (c', b')\}$

(d', c')

$|E_2| = 5$

G_1		G_2	
V_1	DS	V_2	DS
a	(2,1)	a'	(2,1)
b	(2,1)	b'	(2,1)
c	(2,1)	c'	(2,1)
d	(2,1)	d'	(2,1)
e	(0,4)	e'	(0,4)

Here both the graphs G_1 and G_2 have same degree sepectrum

In graph G_1
vertex e is replaced by e'
a " by a'
b b'
c c'
d d'

Here $(a, b) \in E_1$
$(a' b') \notin E_2$

hence graph G_2 contains loops
is not isomorphic $(G_1 \ncong G_2)$

Observation

1. If the both the given graphs are simple then the graphs are isomorphic
2. If one graph is simple and other graph is loop or cyclic then graphs are not isomorphic

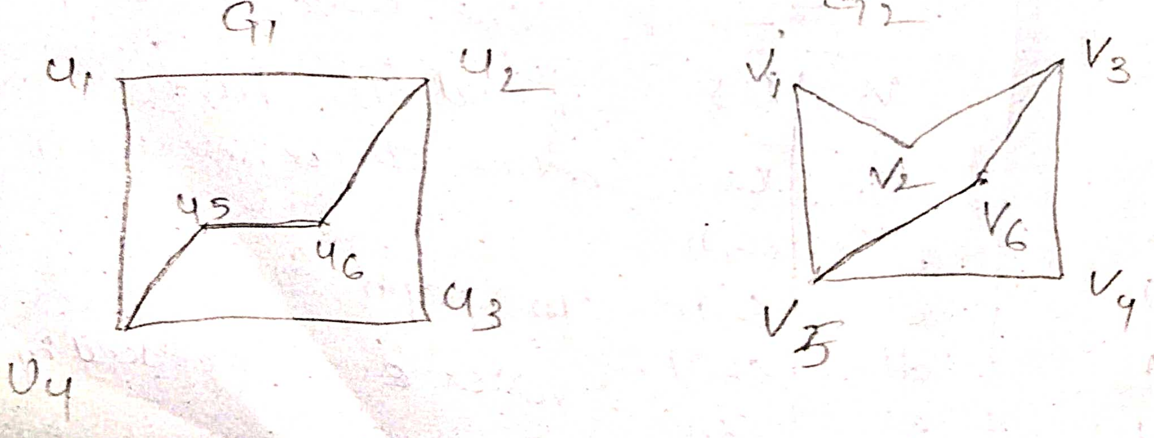
Isomorphism for undirected graph

Procedure for testing isomorphism of undirected graph

1. verify the vertices in both the graphs are equal or not
2. Verify the edges $|E_1| = |E_2|$
3. identify no of loops in the both the graphs
4. verify the degree sequence.
5. ~~E is a mapping from~~

   $f: V(G_1) \rightarrow V(G_2)$ is bijective or not
6. If all the above conditions are satisfied then the given graphs are isomorphic

1. Determine whether the graphs are isomorphic or not



Sol Given that

~~$V_1 = \{v_1$~~

$|V_1| = 6$

$|V_2| = 6$

$|E_1| = 7$

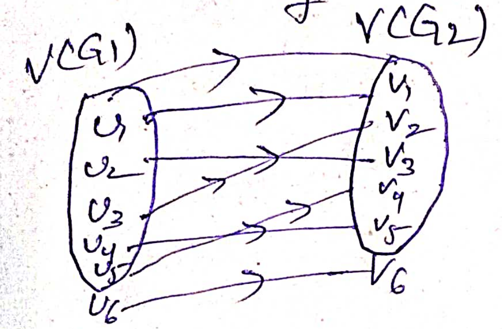
$|E_2| = 7$

loop in $G_1 = 0 =$ loops in G_2

degree sequence in G_1

G_1 2, 3, 2, 3, 2, 2

G_2 2, 2, 3, 2, 3, 2



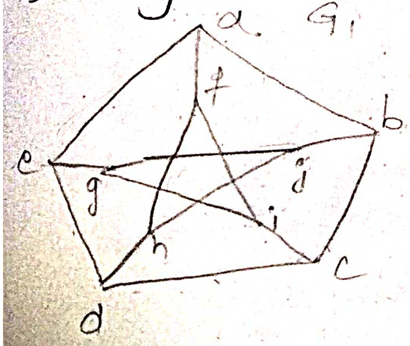
f is 1-1

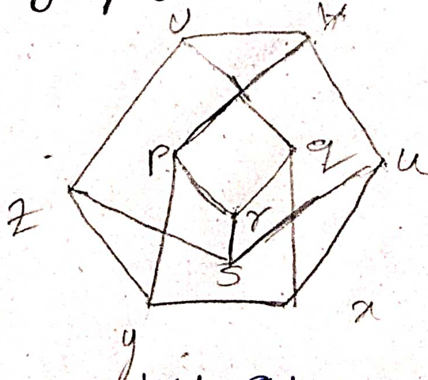
Range ~~of~~ codomain = 6

∴ f is bijective

$G_1 \cong G_2$

2) Verify whether the graphs are isomorphic or not





$|V_1| = 10$

$|E_1| = 15$

$|V_1| = \cancel{8}10$

$|E_1| = 15$