Matrix Vector Product

#matrix_multiplication

Symbol	Meaning
$\in$	belongs to
$\mathbb{R}^n$	the set of vectors with n real-valued elements
$\mathbb{R}^{m\times n}$	the set of real-valued matrices with m rows and n columns

Definition of Matrix Vector Multiplication

If $A \in \mathbb{R}^{m\times n}$ has vectors $\vec{a_1}, \ldots, \vec{a_n}$ and $\vec{x} \in \mathbb{R}^n$, then the matrix Vector Product $A\vec{x}$ is a linear combination of the columns of A. It can be shown as shown:

$$A\vec{x} = \begin{pmatrix} | & | & \cdots & | \\ \vec{a_1} & \vec{a_2} & \cdots & \vec{a_n} \\ | & | & \cdots & | \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n} = \begin{pmatrix} \vdots \\ x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n} \\ \vdots \end{pmatrix}$$

NOTE: $A\vec{x}$ is always in the span of A

Existence of Solutions

The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of A.

Example

For what vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the equation have a solution?

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 2 & 8 & 4 & b_2 \\ 0 & 1 & -2 & b_3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 2 & -4 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - \frac{1}{2}b_2 + b_1 \end{array}\right)$$

Solving for b_1,

$$b_1 = \frac{1}{2} - b_3$$

$$\boxed{\vec{b} = \begin{pmatrix} \frac{1}{2} - b_3 \\ b_2 \\ b_3 \end{pmatrix}}$$