

# Homogeneous Coordinates

Translations of points in  $\mathbb{R}^n$  does not correspond directly to a linear transform. Homogeneous coordinates are used to model translations using matrix multiplication.

## Homogeneous Coordinates in $\mathbb{R}^2$

Each point  $(x, y)$  in  $\mathbb{R}^2$  can be identified with the point  $(x, y, H)$ ,  $H \neq 0$ , on the plane in  $\mathbb{R}^3$  that lies  $H$  units above the  $xy$ -plane.

$(x, y) \rightarrow (x + h, y + k)$  can be represented by,

$$\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + h \\ y + k \\ 1 \end{pmatrix}$$

Now rotate a triangle  $((1, 1), (2, 4), (3, 1))$  by  $\frac{\pi}{2}$  radians counterclockwise about the point  $(0, 1)$ .

$$d = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Shift down by 1,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} d = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Now rotate,

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

Shift up by 1,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -3 & 0 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 0 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

This give us the points,  $(0, 2), (-3, 3), (0, 4)$

## In $\mathbb{R}^3$

So,  $(x, y, z) \rightarrow (x + h, y + k, z + l)$  can be represented by,

$$\begin{pmatrix} 1 & 0 & 0 & h \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + h \\ y + k \\ z + l \\ 1 \end{pmatrix}$$

## Rotation in $\mathbb{R}^3$

about  $x_2$ -axis by  $\pi$  rads.

To find  $A = (a_1, a_2, a_3)$ . We can find  $T(e_1)$  as  $T(e_1) = Ae_1 = a_1$ . We can similarly find all the columns of  $A$ .

$$T(e_1) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T(e_3) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

## Projection

Onto the plane  $x_3 = 4$

What should we do?

1. Shift everything down by 4 (Homogeneous Coordinates)
2. Apply the projection
3. Shift everything back up by 4 (Homogeneous Coordinates)

Amusing a vector  $\vec{v}$ ,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \vec{v}$$