

Linear Transformations

$$T : R^n \rightarrow R^m \text{ is linear if } \begin{cases} T(u + v) &= T(u) + T(v) \\ T(cv) &= cT(v) \end{cases}$$

Create a 2×2 matrix A that applies a linear transformation that rotates by an angle θ counterclockwise

$$\begin{aligned} A &= [\vec{a}_1, \vec{a}_2] \\ T(\vec{e}_1) &= \vec{a}_1 \\ T(\vec{e}_1) &= \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \\ \vec{a}_1 &= \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \\ T(\vec{e}_2) &= \vec{a}_2 \\ T(\vec{e}_2) &= \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \\ \vec{a}_2 &= \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \end{aligned}$$

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Geometric Transformations

Reflections

Through x_1 — axis

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Through x_2 — axis

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Through $x_2 = x_1$

$$T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Through $x_2 = -x_1$

$$T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Contractions and Expansions

Horizontal

Contractions ($|k| < 1$)

$$T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

$$T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Expansions ($|k| > 1$)

$$T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

$$T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Vertical

Contractions ($|k| < 1$)

$$T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Expansions ($|k| > 1$)

$$T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

Shears

Horizontal Shear

Left ($k < 0$)

$$T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Right($k > 0$)

$$\begin{aligned} T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} k \\ 1 \end{bmatrix} \end{aligned}$$

So,

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Vertical Shear

Down($k < 0$)

$$\begin{aligned} T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} 1 \\ k \end{bmatrix} \\ T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Up($k > 0$)

$$\begin{aligned} T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) &= \begin{bmatrix} 1 \\ k \end{bmatrix} \\ T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Projections

On to the x_1 - Axis

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

On to the $x_2 -$ Axis

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$