Inverse of a Matrix

$A \in \mathbb{R}^{n\times n}$ is invertible if there is a $C \in \mathbb{R}^{n\times n}$ so that:

$$AC = CA = I_n$$

If so we write, $C = A^{-1}$

A has an inverse if and only if there is a pivot on every row and column.

For a 2×2

Compute

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Use it to solve a Linear systems

$$\begin{aligned} A\vec{x} &= b \\ A^{-1}A\vec{x} &= A^{-1}b \\ I\vec{x} &= A^{-1}b \\ \vec{x} &= A^{-1}b \end{aligned}$$

Computing A^{-1}

1. Row reduce the augmented matrix $(A \mid I_n)$ to RREF.
2. If the reduction is in the form, $(I_n \mid B)$ then A is invertible and $B = A^{-1}$. Else, A is not invertible.

For example to find $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$:

$$
\begin{aligned}
&= \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\
&= (I_3 \mid A^{-1}) \\
A^{-1} &= \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Why does this work?

Using elementary matrices we can see that when we row reduce the augmented matrix $(A \mid I_n)$ to RREF, we are simple applying row operation, in other words we are simple applying transformations using elementary matrices.
So if,

$$
\begin{gathered}
(E_k \cdots E_3 E_2 E_1) A = I_n \\
E_k \cdots E_3 E_2 E_1 = A^{-1} \\
\text{as,} \\
A^{-1} A = I_n
\end{gathered}
$$

Some properties:

$$
\begin{aligned}
(A^{-1})^{-1} &= A \\
(AB)^{-1} &= B^{-1} A^{-1} \\
(A^T)^{-1} &= (A^{-1})^T
\end{aligned}
$$