# **Linear Transformations**

$$T:R^{n}
ightarrow R^{m} ext{ is linear if } egin{cases} T\left(u+v
ight) &=T\left(u
ight)+T\left(v
ight) \ T\left(cv
ight) &=cT\left(v
ight) \end{cases}$$

Create a  $2 \times 2$  matrix A that applies a linear transformation that rotates by an angle  $\theta$  counterclockwise

$$A = [ec{a_1}, ec{a_2}]$$
 $T\left(ec{e_1}
ight) = ec{a_1}$ 
 $T\left(ec{e_1}
ight) = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ 
 $ec{a_1} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ 
 $T\left(ec{e_2}
ight) = ec{a_2}$ 
 $T\left(ec{e_2}
ight) = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$ 
 $ec{a_2} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$ 
 $ec{A} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ 

## **Geometric Transformations**

#### Reflections

Through  $x_1 - axis$ 

$$T\left(egin{bmatrix}1\\1\end{bmatrix}
ight)=egin{bmatrix}1\\-1\end{bmatrix}$$

So,

$$A = egin{bmatrix} 1 & 0 \ 0 & -1 \end{bmatrix}$$

Through  $x_2 - axis$ 

$$T\left(egin{bmatrix}1\\1\end{bmatrix}
ight)=egin{bmatrix}-1\\1\end{bmatrix}$$

So,

$$A = egin{bmatrix} -1 & 0 \ 0 & 1 \end{bmatrix}$$

Through  $x_2=x_1$ 

$$T \left( egin{bmatrix} 1 \ 0 \end{bmatrix} 
ight) = egin{bmatrix} 0 \ 1 \end{bmatrix} \ T \left( egin{bmatrix} 0 \ 1 \end{bmatrix} 
ight) = egin{bmatrix} 1 \ 0 \end{bmatrix}$$

So,

$$A = egin{bmatrix} 0 & 1 \ 1 & 0 \end{bmatrix}$$

Through  $x_2 = -x_1$ 

$$T \left( egin{bmatrix} 1 \ 0 \end{bmatrix} 
ight) = egin{bmatrix} 0 \ -1 \end{bmatrix} \ T \left( egin{bmatrix} 0 \ 1 \end{bmatrix} 
ight) = egin{bmatrix} -1 \ 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

# **Contractions and Expansions**

## **Horizontal**

Contractions  $\ensuremath{\mathsf{left}}(|k| < 1\ensuremath{\mathsf{lright}})$ 

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}k\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\1\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} k & 0 \ 0 & 1 \end{bmatrix}$$

Expansions\left(|k|>1\right)

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}k\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\1\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} k & 0 \ 0 & 1 \end{bmatrix}$$

#### **Vertical**

# Contractions $\ensuremath{\mathsf{left}}(|k| < 1\ensuremath{\mathsf{lright}})$

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}1\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\k\end{bmatrix} \end{aligned}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

## Expansions\left(|k|>1\right)

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}1\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\k\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} 1 & 0 \ 0 & k \end{bmatrix}$$

## **Shears**

#### **Horizontal Shear**

## Left\left(k < 0\right)

$$egin{aligned} T\left(egin{bmatrix}1\0\end{bmatrix}
ight) &= egin{bmatrix}1\0\end{bmatrix}\ T\left(egin{bmatrix}0\1\end{bmatrix}
ight) &= egin{bmatrix}k\1\end{bmatrix} \end{aligned}$$

$$A = egin{bmatrix} 1 & k \ 0 & 1 \end{bmatrix}$$

 $\textbf{Right} \\ \textbf{left} \\ (k > 0 \\ \textbf{right})$ 

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}1\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}k\\1\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} 1 & k \ 0 & 1 \end{bmatrix}$$

## **Vertical Shear**

 ${\bf Down \ left \ } (k < 0 \ left)$ 

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}1\\k\end{bmatrix} \ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\1\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} 1 & 0 \ k & 1 \end{bmatrix}$$

 $Up \left( k > 0 \right)$ 

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}1\\k\end{bmatrix} \ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\1\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} 1 & 0 \ k & 1 \end{bmatrix}$$

# **Projections**

On to the  $x_1 - Axis$ 

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}1\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\0\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} 1 & 0 \ 0 & 0 \end{bmatrix}$$

On to the  $x_2 - \mathrm{Axis}$ 

$$T\begin{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$T\begin{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = egin{bmatrix} 0 & 0 \ 0 & 1 \end{bmatrix}$$