Partitioned Matrices

Imagine a matrix A,

$$A = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 6 \\ 7 & 9 & 1 & 6 \end{bmatrix}$$

Partitioned it could look like this,

$$\begin{aligned} A &= \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 5 \\ 6 \\ 6 \end{bmatrix} \\ \begin{bmatrix} 7 & 9 & 1 \end{bmatrix} & \begin{bmatrix} 6 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} I_3 & U \\ V & X \end{bmatrix} \end{aligned}$$

We can even perform matrix multiplication,

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} I_2 & X \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} \\ &= I_2U + XY \\ &= \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Find the Inverse

Compute equations the inverse $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$.

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = I = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}$$

$$\begin{aligned} 0W + CY &= 0 \\ CY &= 0 \\ C^{-1}CY &= C^{-1}0 \\ Y &= 0 \end{aligned}$$

$$\begin{aligned} 0X + CZ &= I_n \\ C^{-1}CZ &= C^{-1}I_n \\ Z &= C^{-1} \end{aligned}$$

We know $Y = 0$ as it was calculated above

$$\begin{aligned} AW + BY &= I_n \\ AW + B0 &= I_n \\ A^{-1}AW &= A^{-1}I_n \\ W &= A^{-1} \end{aligned}$$

$$\begin{aligned} AX + BZ &= 0 \\ A^{-1}AX &= -A^{-1}BZ \\ X &= -A^{-1}BZ \\ X &= -A^{-1}BC^{-1} \end{aligned}$$

So, putting this back into a matrix:

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}^{-1} = \boxed{\begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix}}$$