

# The Invertible Matrix Theorem

1.  $A$  is invertible.
2.  $A$  is row equivalent to  $I_n$ .
3.  $A$  has  $n$  pivotal columns (all columns are pivotal).
4.  $A\mathbf{x} = \mathbf{0}$  has only the trivial solution.
5. The columns of  $A$  are linearly independent.
6. The equation  $A\mathbf{x} = \mathbf{b}$  has a solution for all  $\mathbf{b} \in \mathbb{R}^n$ .
7. The columns of  $A$  span  $\mathbb{R}^n$ .
8. There is a  $n \times n$  matrix  $C$  so that  $CA = I_n$  ( $A$  has a left inverse.)
9. There is a  $n \times n$  matrix  $D$  so that  $AD = I_n$  ( $A$  has a right inverse.)
10.  $A^T$  is invertible.

Example:

$$\text{Is } \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ 0 & -1 & -1 \end{bmatrix} \text{ invertible?}$$

$$\text{RREF} \left( \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ 0 & -1 & -1 \end{bmatrix} \right) = I_3$$

Every column is pivotal.

$$\text{So, } \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ 0 & -1 & -1 \end{bmatrix} \text{ is invertible!}$$