

# Linear Transformations

$$T : R^n \rightarrow R^m \text{ is linear if } \begin{cases} T(u + v) &= T(u) + T(v) \\ T(cv) &= cT(v) \end{cases}$$

Create a  $2 \times 2$  matrix  $A$  that applies a linear transformation that rotates by an angle  $\theta$  counterclockwise

$$\begin{aligned} A &= [\vec{a}_1, \vec{a}_2] \\ T(\vec{e}_1) &= \vec{a}_1 \\ T(\vec{e}_1) &= \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \\ \vec{a}_1 &= \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \\ T(\vec{e}_2) &= \vec{a}_2 \\ T(\vec{e}_2) &= \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \\ \vec{a}_2 &= \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \end{aligned}$$

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

## Geometric Transformations

### Reflections

**Through  $x_1$  — axis**

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

**Through  $x_2$  — axis**

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

**Through  $x_2 = x_1$**

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

**Through  $x_2 = -x_1$**

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

## Contractions and Expansions

### Horizontal

**Contractions ( $|k| < 1$ )**

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

**Expansions ( $|k| > 1$ )**

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

## Vertical

### Contractions ( $|k| < 1$ )

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

### Expansions ( $|k| > 1$ )

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

## Shears

### Horizontal Shear

#### Left ( $k < 0$ )

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

**Right**( $k > 0$ )

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

**Vertical Shear**

**Down**( $k < 0$ )

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ k \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

**Up**( $k > 0$ )

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ k \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

**Projections**

**On to the  $x_1$  - Axis**

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

**On to the  $x_2 -$  Axis**

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$