Linear Transformations

$$T : R^n \to R^m \text{ is linear if } \begin{cases} T\left(u+v\right) & = T\left(u\right) + T\left(v\right) \\ T\left(cv\right) & = cT\left(v\right) \end{cases}$$

Create a 2×2 matrix A that applies a linear transformation that rotates by an angle θ counterclockwise

$$\begin{aligned} A &= [\vec{a_1}, \vec{a_2}] \\ T\left(\vec{e_1}\right) &= \vec{a_1} \\ T\left(\vec{e_1}\right) &= \begin{bmatrix} a \\ b \end{bmatrix} & &= \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \\ \vec{a_1} &= \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \\ T\left(\vec{e_2}\right) &= \vec{a_2} \\ T\left(\vec{e_2}\right) &= \begin{bmatrix} a \\ b \end{bmatrix} & &= \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \\ \vec{a_2} &= \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \end{aligned}$$

$$\boxed{A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

Geometric Transformations

Reflections

Through $x_1 - \text{axis}$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Through $x_2 - \text{axis}$

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Through $x_2 = x_1$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Through $x_2 = -x_1$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Contractions and Expansions

Horizontal

Contractions \left($|k| < 1$\right)

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} k \\ 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Expansions\left($|k| > 1$\right)

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}k\\0\end{bmatrix}$$
$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}$$

So,

$$A=\begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$$

Vertical

Contractions \left($|k| < 1$\right)

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$$
$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\k\end{bmatrix}$$

So,

$$A=\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$$

Expansions\left($|k| > 1$\right)

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$$
$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}0\\k\end{bmatrix}$$

So,

$$A=\begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$$

Shears

Horizontal Shear

Left\left($k < 0$\right)

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$$
$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}k\\1\end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Right\left($k > 0$\right)

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Vertical Shear

Down\left($k < 0$\right)

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ k \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Up\left($k > 0$\right)

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ k \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Projections

On to the $x_1 - \text{Axis}$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$
$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$$

So,

$$A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

On to the $x_2 - \mathrm{Axis}$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$$
$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

So,

$$A = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$