Linear Transformations

$$T:R^{n}
ightarrow R^{m} ext{ is linear if } egin{cases} T\left(u+v
ight) &=T\left(u
ight)+T\left(v
ight) \ T\left(cv
ight) &=cT\left(v
ight) \end{cases}$$

Create a 2×2 matrix A that applies a linear transformation that rotates by an angle θ counterclockwise

$$A = [ec{a_1}, ec{a_2}]$$
 $T\left(ec{e_1}
ight) = ec{a_1}$
 $T\left(ec{e_1}
ight) = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$
 $ec{a_1} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$
 $T\left(ec{e_2}
ight) = ec{a_2}$
 $T\left(ec{e_2}
ight) = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$
 $ec{a_2} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$
 $ec{A} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

Geometric Transformations

Reflections

Through $x_1 - axis$

$$T\left(egin{bmatrix}1\\1\end{bmatrix}
ight)=egin{bmatrix}1\\-1\end{bmatrix}$$

So,

$$A = egin{bmatrix} 1 & 0 \ 0 & -1 \end{bmatrix}$$

Through $x_2 - axis$

$$T\left(egin{bmatrix}1\\1\end{bmatrix}
ight)=egin{bmatrix}-1\\1\end{bmatrix}$$

So,

$$A = egin{bmatrix} -1 & 0 \ 0 & 1 \end{bmatrix}$$

Through $x_2 = x_1$

$$T \left(egin{bmatrix} 1 \ 0 \end{bmatrix}
ight) = egin{bmatrix} 0 \ 1 \end{bmatrix} \ T \left(egin{bmatrix} 0 \ 1 \end{bmatrix}
ight) = egin{bmatrix} 1 \ 0 \end{bmatrix}$$

So,

$$A = egin{bmatrix} 0 & 1 \ 1 & 0 \end{bmatrix}$$

Through $x_2 = -x_1$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix}$$
 $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix}$

So,

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Contractions and Expansions

Horizontal

Contractions ($\left|k\right|<1$)

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}k\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\1\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} k & 0 \ 0 & 1 \end{bmatrix}$$

Expansions ($\left|k\right|>1$)

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}k\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\1\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} k & 0 \ 0 & 1 \end{bmatrix}$$

Vertical

Contractions (|k| < 1)

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}1\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\k\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} 1 & 0 \ 0 & k \end{bmatrix}$$

Expansions ($\left|k\right|>1$)

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}1\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\k\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} 1 & 0 \ 0 & k \end{bmatrix}$$

Shears

Horizontal Shear

Left(k < 0)

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}1\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}k\\1\end{bmatrix} \end{aligned}$$

$$A = egin{bmatrix} 1 & k \ 0 & 1 \end{bmatrix}$$

Right(k > 0)

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}1\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}k\\1\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} 1 & k \ 0 & 1 \end{bmatrix}$$

Vertical Shear

Down(k < 0)

$$T \left(egin{bmatrix} 1 \ 0 \end{bmatrix}
ight) = egin{bmatrix} 1 \ k \end{bmatrix}$$
 $T \left(egin{bmatrix} 0 \ 1 \end{bmatrix}
ight) = egin{bmatrix} 0 \ 1 \end{bmatrix}$

So,

$$A = egin{bmatrix} 1 & 0 \ k & 1 \end{bmatrix}$$

Up(k > 0)

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}1\\k\end{bmatrix} \ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\1\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} 1 & 0 \ k & 1 \end{bmatrix}$$

Projections

On to the $x_1 - \mathrm{Axis}$

$$egin{aligned} T\left(egin{bmatrix}1\\0\end{bmatrix}
ight) &= egin{bmatrix}1\\0\end{bmatrix}\ T\left(egin{bmatrix}0\\1\end{bmatrix}
ight) &= egin{bmatrix}0\\0\end{bmatrix} \end{aligned}$$

So,

$$A = egin{bmatrix} 1 & 0 \ 0 & 0 \end{bmatrix}$$

On to the $x_2 - \mathrm{Axis}$

$$T\begin{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$T\begin{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = egin{bmatrix} 0 & 0 \ 0 & 1 \end{bmatrix}$$