LU Factorization

Triangular Matrices

Upper Triangular

If $a_{i,j} = 0$ for $i > j$
In other words the elements below the main diagonal are 0.
Examples:

$$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Lower Triangular

If $a_{i,j} = 0$ for $i < j$
In other words the elements above the main diagonal are 0.
Examples:

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 4 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$$

LU Factorization

If A is an $m \times n$ matrix that can be row reduced to echelon form without row exchanges, then $A = LU$. L is a lower triangular $m \times m$ matrix with 1's on the diagonal, U is an echelon form of A.

For example the LU factorization of $A \in \mathbb{R}^{3\times 2}$ would look like:

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \begin{bmatrix} * & * \\ 0 & * \\ 0 & 0 \end{bmatrix}$$

Algorithm

To solve $A\vec{x} = \vec{b}$ for $\vec{x}$:

1. Construct the LU decomposition of A to obtain L and U.
2. Set $U\vec{x} = \vec{y}$. Forward solve for $\vec{x}$ in $L\vec{y} = \vec{b}$.
3. Backwards solve for $\vec{x}$ in $U\vec{x} = \vec{y}$.

In other words:
Get LU
Then solve $L\vec{y} = \vec{b}$ for $\vec{y}$
Finally use that $\vec{y}$ to solve for $\vec{x}$ in $U\vec{x} = \vec{y}$

Computing LU

You are not allowed to swap rows.
Also Scalar Multiplication is not needed.

$$E_p \cdots E_1 A = U$$

E_j are matrices that perform elementary row operations. Because we did not swap rows, each E_j happens to be lower triangular and invertible

$$\begin{aligned} E_p \cdots E_1 A &= U \\ LE_p \cdots E_1 A &= LU \\ LL^{-1} A &= LU \\ A &= LU \end{aligned}$$

$$\boxed{E_p \cdots E_1 = L^{-1}}$$

To compute the LU decomposition:

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Example

Compute the LU factorization of:

$$A = \begin{bmatrix} 4 & -3 & -1 & 5 \\ -16 & 12 & 2 & -17 \\ 8 & -6 & -12 & 22 \end{bmatrix}$$

Reducing it to echelon form we get:

$$\boxed{U = \begin{bmatrix} 4 & -3 & -1 & 5 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix}}$$

To reduce it we use the fallowing row operations:

$$\begin{gathered} R_2 + 4R_1 \\ R_3 - 2R_1 \\ R_3 - 5R_2 \end{gathered}$$

So L will be such that the same sequence of row operations reduces L to I.

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix}}$$