Matrix Vector Product

#matrix_multiplication

Symbol	Meaning
$\in$	belongs to
$\mathbb{R}^n$	the set of vectors with n real-valued elements
$\mathbb{R}^{m\times n}$	the set of real-valued matrices with m rows and n columns

Definition of Matrix Vector Multiplication

If $A \in \mathbb{R}^{m\times n}$ has vectors $\mathbf{a_1}, \dots, \mathbf{a_n}$ and $\mathbf{x} \in \mathbb{R}^n$, then the matrix Vector Product $A\mathbf{x}$ is a linear combination of the columns of A. It can be shown as shown:

$$A\mathbf{x} = \begin{bmatrix} | & | & \dots & | \\ \mathbf{a_1} & \mathbf{a_2} & \dots & \mathbf{a_n} \\ | & | & \dots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n} = \begin{bmatrix} \vdots \\ x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n} \\ \vdots \end{bmatrix}$$

NOTE: $A\mathbf{x}$ is always in the span of A

Existence of Solutions

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Example

For what vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ does the equation have a solution?

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 2 & 8 & 4 & b_2 \\ 0 & 1 & -2 & b_3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 2 & -4 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - \frac{1}{2}b_2 + b_1 \end{array}\right]$$

Solving for b_1 from the last row,

$$b_1 = \frac{1}{2} - b_3$$

$$\boxed{\mathbf{b} = \begin{bmatrix} \frac{1}{2} - b_3 \\ b_2 \\ b_3 \end{bmatrix}}$$