Linear Transformations

$$T : R^n \to R^m \text{ is linear if } \begin{cases} T(u+v) & = T(u) + T(v) \\ T(cv) & = cT(v) \end{cases}$$

Create a 2×2 matrix A that applies a linear transformation that rotates by an angle θ counterclockwise

$$\begin{aligned} A &= [\vec{a_1}, \vec{a_2}] \\ T(\vec{e_1}) &= \vec{a_1} \\ T(\vec{e_1}) &= \begin{bmatrix} a \\ b \end{bmatrix} & = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \\ \vec{a_1} &= \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \\ T(\vec{e_2}) &= \vec{a_2} \\ T(\vec{e_2}) &= \begin{bmatrix} a \\ b \end{bmatrix} & = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \\ \vec{a_2} &= \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \end{aligned}$$

$$\boxed{A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$$

Geometric Transformations

Reflections

Through $x_1 - \text{axis}$

$$T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Through $x_2 - \text{axis}$

$$T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Through $x_2 = x_1$

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Through $x_2 = -x_1$

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Contractions and Expansions

Horizontal

Contractions ($|k| < 1$)

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} k \\ 0 \end{bmatrix}$$
$$T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

Expansions($|k| > 1$)

$$T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}k\\0\end{bmatrix}$$
$$T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}0\\1\end{bmatrix}$$

So,

$$A = \begin{bmatrix}k & 0\\0 & 1\end{bmatrix}$$

Vertical

Contractions ($|k| < 1$)

$$T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}1\\0\end{bmatrix}$$
$$T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}0\\k\end{bmatrix}$$

So,

$$A = \begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$$

Expansions($|k| > 1$)

$$T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}1\\0\end{bmatrix}$$
$$T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}0\\k\end{bmatrix}$$

So,

$$A = \begin{bmatrix}1 & 0\\0 & k\end{bmatrix}$$

Shears

Horizontal Shear

Left($k < 0$)

$$T(\begin{bmatrix}1\\0\end{bmatrix}) = \begin{bmatrix}1\\0\end{bmatrix}$$
$$T(\begin{bmatrix}0\\1\end{bmatrix}) = \begin{bmatrix}k\\1\end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Right($k > 0$)

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

Vertical Shear

Down($k < 0$)

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ k \end{bmatrix}$$
$$T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Up($k > 0$)

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ k \end{bmatrix}$$
$$T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

Projections

On to the $x_1 - \text{Axis}$

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

On to the $x_2 - \text{Axis}$

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$