Project 1 Report

Name: CHEN Yufan

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1 Introduction of the Problem (Sod Shock Tube)

1.1 Problem Description and Initial Conditions

The shock tube problem consists initially with the flow at rest with diaphragm separating two uniform regions at different densities and pressures. Place the diaphragm at $x = 1$. Let $\rho = \rho_0$ and $p = p_0$ for $x > 1$ and $\rho = 2\rho_0$ and $p = 2p_0$ for $x \leq 1$, where $\rho_0 = 1$ kg/m^3 and $p_0 = 1\times10^5$ Pa. At $t = 0$, the diaphragm is suddenly broken, sending a normal shock wave forward and an isentropic expansion wave backward. In addition, a contact surface, a jump in density, moves slowly forward.

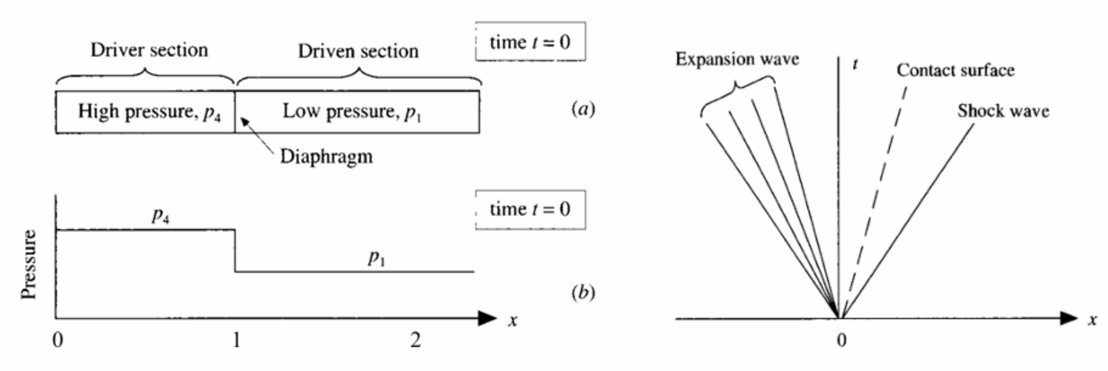

Figure 1 Description of Sod Shock Tube

1.2 Requirements

Use numerical solution to solve the sod shock tube problem based on the unsteady Euler flow model as the equation (1) shown in blown:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho u)}{\partial x} = 0 \\ \dfrac{\partial(\rho u)}{\partial t} + \dfrac{\partial(\rho u^2 + p)}{\partial x} = 0 \\ \dfrac{\partial(\rho E)}{\partial t} + \dfrac{\partial(\rho E u + p u)}{\partial x} = 0 \end{cases} \tag{1}$$

This model can be solved by the numerical scheme (Conservative scheme) as equation (2):

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) \tag{2}$$

The detailed requirements of numerical solutions are listed below:

(1) Compare the numerical solution with the exact solution at different time instants (do not let the wave arrive at the boundaries).

(2) Use different schemes for space discretization (around 100 grid points).
(3) Use first-order difference formula for time discretization.
(4) Use S-W or L-F flux vector splitting for original flux and characteristic flux, and compare their difference.

(5) Choice for time step: $\Delta t \le \frac{0.8\,\Delta x}{(|u|+c)}$.

(6) Submit your report (describe the detailed procedure and attach the source code).

2 Numerical Solution of the Problem

For the sod shock tube problem described in Chapter 1, the finite differential numerical solution **MATLAB codes** of Equation (1) was obtained using **Steger-Warming** (S-W) flux vector splitting method, which decomposes both the **original flux** and the **characteristic flux**. The flux was then calculated using monotonic schemes based on the **Van-Leer and Min-Mod limiters** for differential computation. What's more, the differences between the numerical solution results and the theoretical solutions were compared.

2.1 Settings and Procedure of S-W Splitting for Original Flux

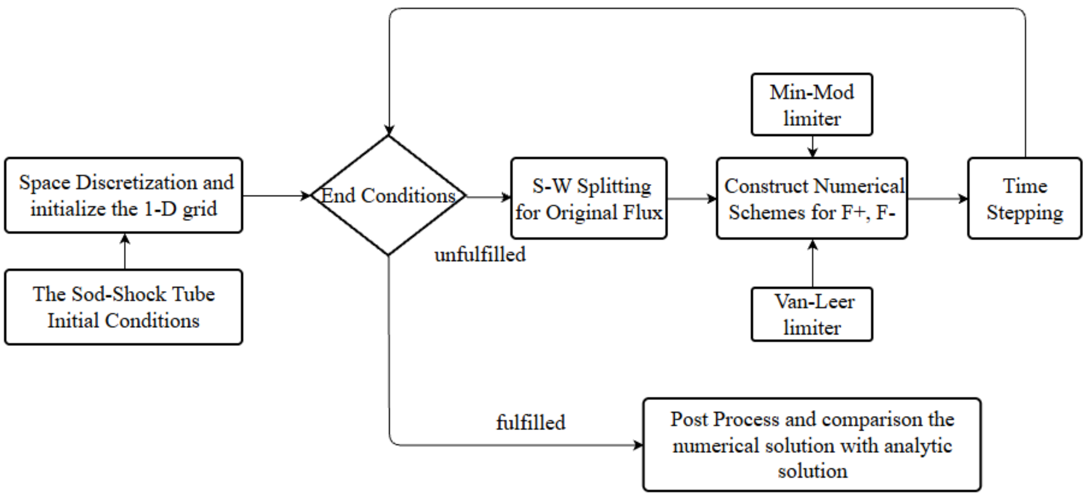

Figure 2 Procedure of S-W Splitting for Original Flux

Settings: The grid points are set as 100 points, and the CFL conditions are set as $\Delta t \le \frac{0.5\,\Delta x}{(|u|+c)}$. The schemes for space discretization are TVD schemes which use Mid-Mod and Van-Leer limiters.

Procedure:

(1) Initialize: Rewrite the equation (1) as the matrix form as shown in equation (3), then initialize the physical variable ρ, u, p at 100 points from x=0 to x=2 based on the section 1.1.

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{3}$$

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho E u + p u \end{pmatrix}, E = \frac{1}{2} u^2 + \frac{p/\rho}{\gamma - 1}$$

(2) Compute Solution Vector and Flux: Use ρ, u, p to compute solution vector U and original flux F on each grid points.

(3) S-W Splitting for Original Flux: Use the S-W splitting method to split the original flux to F+ and F-, the detailed method is shown in equation (4) (in the MATLAB codes this procedure can be realized ***BuildOriginalFlux*** function in the Appendix).

$$\begin{cases} \lambda^{+} = \frac{1}{2}(\lambda + |\lambda|) \\ \lambda^{-} = \frac{1}{2}(\lambda - |\lambda|) \end{cases} \tag{4}$$

Regarding the equation (4) the positive and negative flux can be written as:

$$F^{\pm} = \frac{\rho}{2\gamma} \begin{pmatrix} \lambda_1^{\pm} + 2(\gamma - 1)\lambda_2^{\pm} + \lambda_3^{\pm} \\ (u - a)\lambda_1^{\pm} + 2(\gamma - 1)u\lambda_2^{\pm} + (u + a)\lambda_3^{\pm} \\ (H - ua)\lambda_1^{\pm} + (\gamma - 1)u^2\lambda_2^{\pm} + (H + ua)\lambda_3^{\pm} \end{pmatrix}$$

For $\lambda_1 = u - a, \lambda_2 = u, \lambda_3 = u + a$

(4) Construct Numerical Schemes for F^{+}, F^{-}: This numerical solution codes are using TVD Schemes for F^{+}, F^{-}, the Mid-Mod and Van-Leer limiters were used (in the MATLAB codes this procedure can be realized ***MinMod*** function and ***VanLeer*** function in the Appendix).

$$F_{i+\frac{1}{2}}^{n\ \pm} = F_i^{n\pm} + \frac{1}{2}\varphi(r_i^{\pm})\left(F_i^{n\pm} - F_{i-1}^{n\ \pm}\right) \tag{5}$$

Where r_i can be defined as:

$$r_i = \frac{F_{i+1}^{n\ \pm} - F_i^{n\pm}}{F_i^{n\pm} - F_{i-1}^{n\ \pm}} \tag{6}$$

In the Min-Mod limiter, the φ can be described as the

$$\varphi = \begin{cases} \min(r, 1) & if\ r \geq 0 \\ 0 & if\ r < 0 \end{cases} \tag{7}$$

In the Van-Leer limiter, the φ can be described as the

$$\varphi = \frac{r + |r|}{1 + |r|} \tag{7}$$

(5) Time Stepping: Using first-order time stepping, which can be written as:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}}^{n\ \ +} + F_{i+\frac{1}{2}}^{n\ \ -} - F_{i-\frac{1}{2}}^{n\ \ +} - F_{i-\frac{1}{2}}^{n\ \ -}\right) \tag{8}$$

(6) End condition and post-processing: set the end condition as timestep< 500 or t< 2ms to ensure that the wave did not propagate to the boundary of the shock tube. And use the ***plot*** function in MATLAB to visualize the results

2.2 Settings and Procedure of S-W Splitting for Characteristic Flux

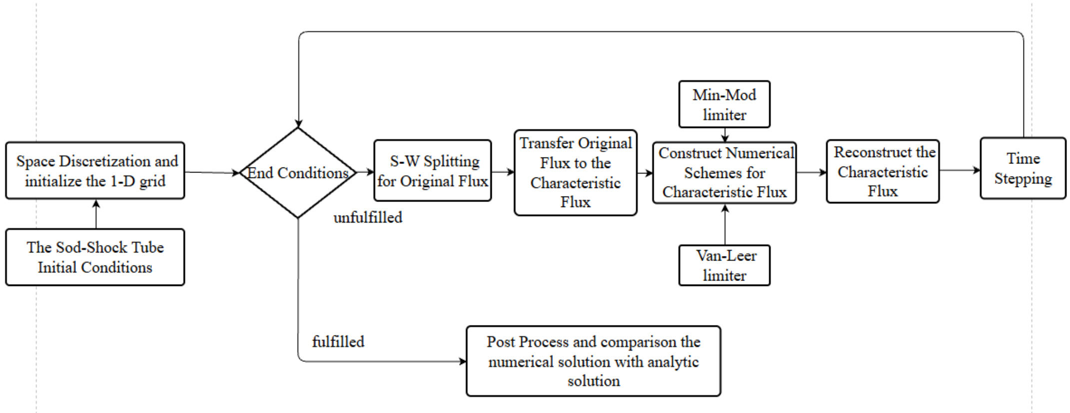

Figure 3 Procedure of S-W Splitting for Characteristic Flux

Settings: The grid points are set as 100 points, and the CFL conditions are set as $\Delta t \le \frac{0.1\,\Delta x}{(|u|+c)}$. The schemes for space discretization are TVD schemes which use Mid-Mod and Van-Leer limiters.

Procedure:

(1) Initialize: Initialize the physical variable ρ, u, p at 100 points from x=0 to x=2 based on the section 1.1.

(2) Compute Solution Vector and Flux: Use ρ, u, p to compute solution vector U and original flux F on each grid points based on equation (3).

(3) S-W Splitting for Original Flux: Use the S-W splitting method to split the original flux to F+ and F-, the detailed method is shown in equation (4) (in the MATLAB codes this procedure can be realized ***BuildOriginalFlux*** function in the Appendix).

(4) Transfer Original Flux to the Characteristic Flux: Determine diagonalization matrix $\boldsymbol{R}^{1}$ and $\boldsymbol{R}$ at $(i+1/2)$ based on the U at $(i+1/2)$:

$$U_{i+\frac{1}{2}} = \frac{1}{2}(U_i + U_{i+1}) \tag{9}$$

$$A_{i+\frac{1}{2}} = R_{i+\frac{1}{2}}\Lambda_{i+\frac{1}{2}}R^{-1}_{i+\frac{1}{2}} \tag{10}$$

Then transfer the original flux F+ and F- to characteristic flux $\tilde{F}^{\pm}$

$$\tilde{F}^{\pm}{}_k = R^{-1}_{i+\frac{1}{2}}F^{\pm}{}_k \tag{11}$$

(in the MATLAB codes this procedure can be realized ***BuildCharacteristicFlux*** function in the Appendix).

(5) Construct Numerical Schemes Characteristic Flux: This numerical solution codes are using TVD Schemes for $\tilde{F}^{\pm}$, the Mid-Mod and Van-Leer limiters were used (in the MATLAB codes this procedure can be realized ***MinMod*** function and ***VanLeer*** function in the Appendix) as equations (5)-(7).

(6) Reconstruct the Characteristic Flux: make the characteristic flux multiply right eigenvectors to reconstruct the flux:

$$\hat{F}_{i+\frac{1}{2}} = \hat{F}^{+}_{i+\frac{1}{2}} + \hat{F}^{-}_{i+\frac{1}{2}} = R_{i+\frac{1}{2}}\tilde{F}^{+}{}_{i+\frac{1}{2}} + R_{i+\frac{1}{2}}\tilde{F}^{-}{}_{i+\frac{1}{2}} \tag{12}$$

(in the MATLAB codes this procedure can be realized ***ReconstructFlux*** function in the Appendix)

(7) Time Stepping: Using first-order time stepping, which can be written as:

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}\left(\hat{F}^n_{i+\frac{1}{2}} - \hat{F}^n_{i-\frac{1}{2}}\right) \tag{13}$$

(8) End condition and post-processing: set the end condition as timestep< 500 or t< 2ms to ensure that the wave did not propagate to the boundary of the shock tube. And use the plot function in MATLAB to visualize the results.

2.3 Analytic Solution for Sod Shock Tube

The analytic (exact) solution of sod shock tube can be shown in the Figure 4, pressure in the region 2 p^* can be compute by equation (14), due to the equation (14) is nonlinear, the Newton's iterative method was used to solve this equation (in the MATLAB codes this procedure can be realized ***Newton*** function in the Appendix).

$$f(p^*, p_i, \rho_i) = \begin{cases} \dfrac{p^* - p_i}{\rho_i c_i [\frac{\gamma+1}{2\gamma}(\frac{p^*}{p_i}) + \frac{\gamma-1}{2\gamma}]^{1/2}}, & p^* > p_i \\ \dfrac{2c_i}{\gamma-1}[(\frac{p^*}{p_i})^{\frac{\gamma-1}{2\gamma}} - 1], & p^* < p_i \end{cases}$$

$$u_1 - u_2 = f(p^*, p_1, \rho_1) + f(p^*, p_2, \rho_2) \tag{14}$$

After pressure in region 2 are computed, velocity and density in region 2 can be get from Rankine-Hugoniot relation, and the density in region 3 can be get from the isentropic condition. The information between region 3 and 4 can be computed by Riemann invariant of left running characteristic line. Then all physical parameter distributions through the shock tube at the any time before the wave reach the wall of shock tube can be easily depicted. (in the MATLAB codes this procedure can be realized ***Analytic*** function in the Appendix).

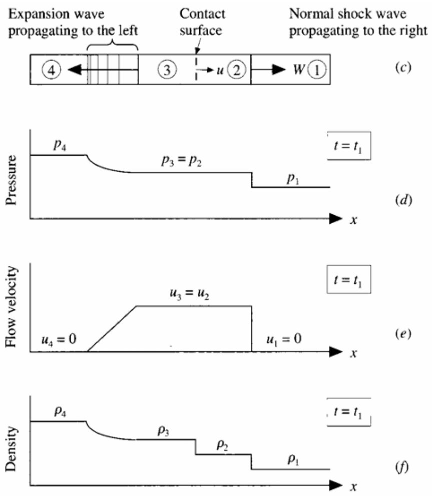

Figure 4 Analytic (Exact) Solution of Sod Shock Tube

3 Results

3.1 Comparison of S-W Splitting for Characteristic Flux and Original Flux

(1) MinMod limiter:

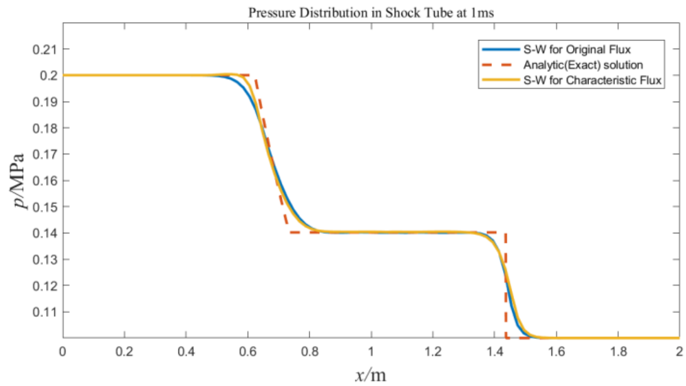

Figure 5 Pressure Distribution in Shock Tube at 1ms (MinMod limiter)

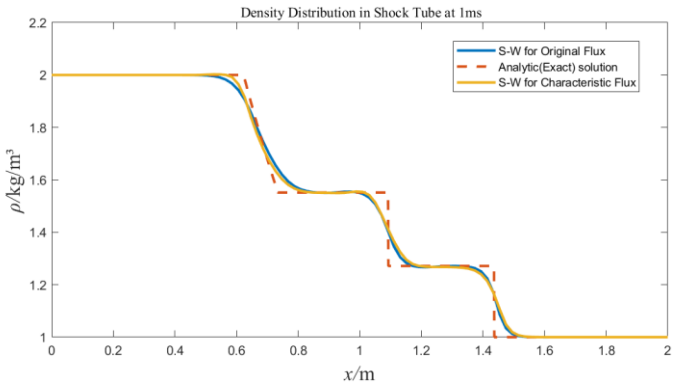

Figure 6 Density Distribution in Shock Tube at 1ms (MinMod limiter)

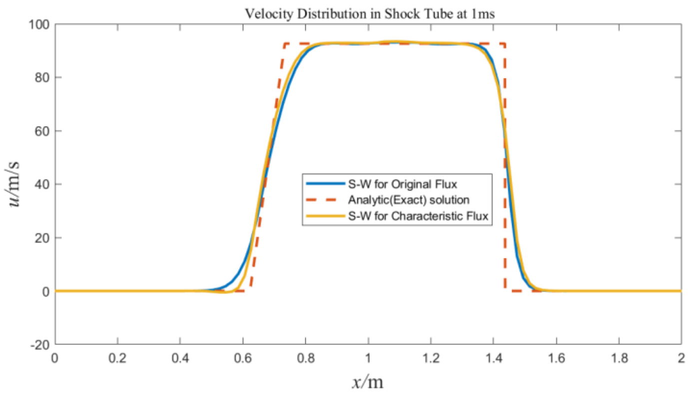

Figure 7 Velocity Distribution in Shock Tube at 1ms (MinMod limiter)

(2) VanLeer limiter:

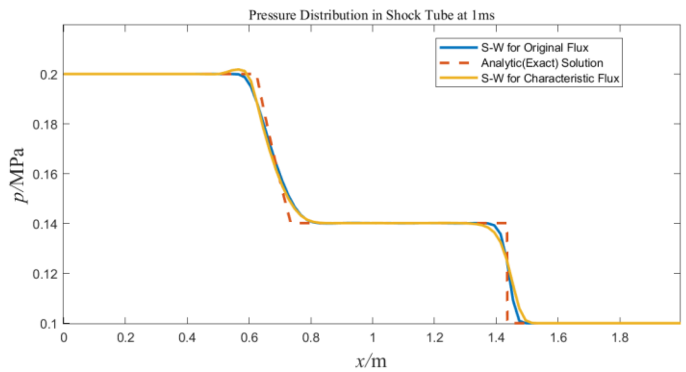

Figure 8 Pressure Distribution in Shock Tube at 1ms (VanLeer limiter)

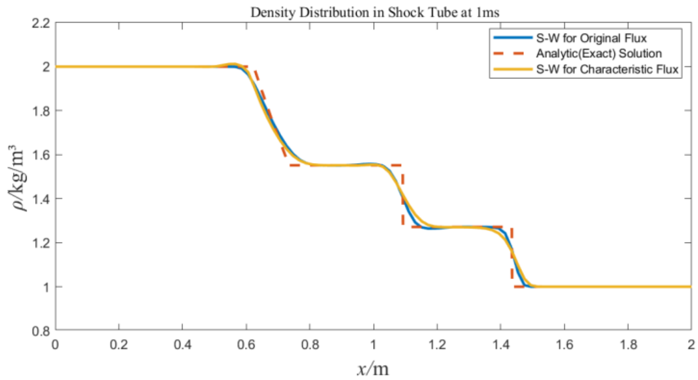

Figure 9 Density Distribution in Shock Tube at 1ms (VanLeer limiter)

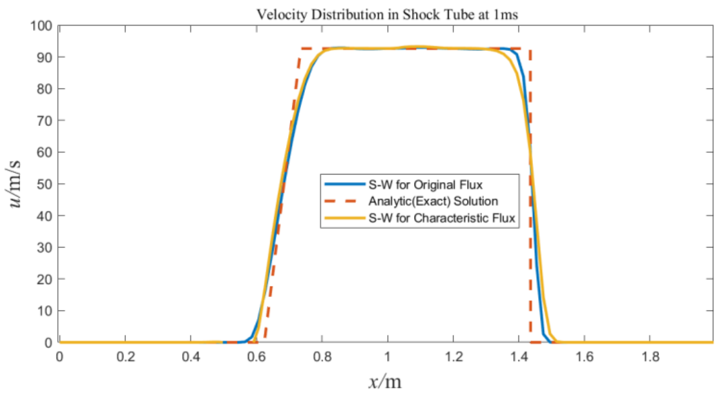

Figure 10 Velocity Distribution in Shock Tube at 1ms (VanLeer limiter)

(3) Gas State in Shock Tube at Deferent Time (SW splitting with VanLeer limiter as example)

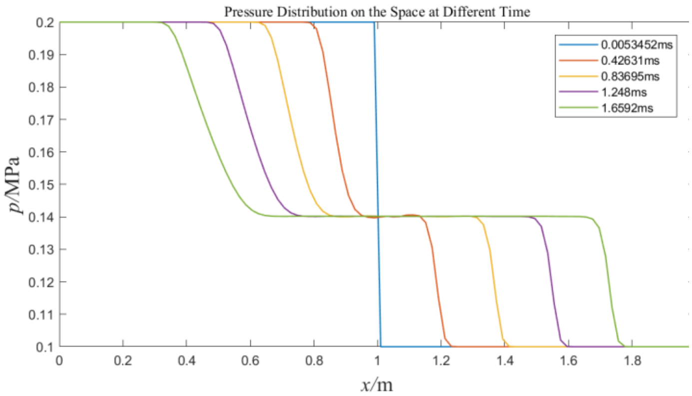

Figure 11 Pressure Distribution in the Space at Different Time

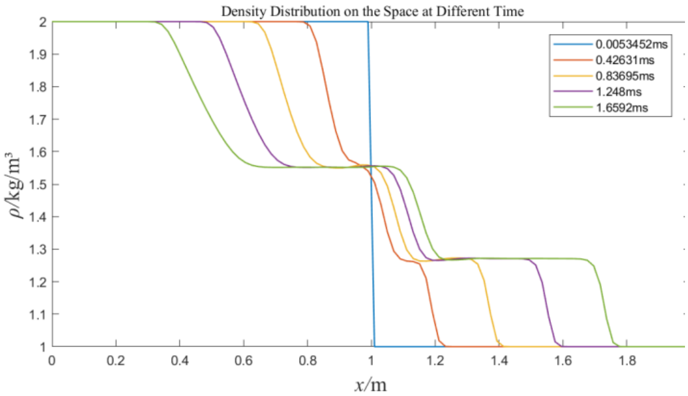

Figure 12 Density Distribution in the Space at Different Time

3.2 summary

Comparing the analytic solution with numerical solutions which utilize S-W splitting for both characteristic flux and original flux based on the sod-shock tube problem, we can find that:

(1) The S-W splitting for characteristic flux method is more accurate when dealing with sod-shock tube problem and discontinuities because it captures the wave properties better (which can be seen in Figure 5 and 6).
(2) The S-W splitting for original flux method may get better result when dealing with smooth section, but perform poorer in dealing discontinuities, even introduce more dissipation and dispersion when using different numerical method.

Comparing the results of using different limiters i.e. MinMod and VanLeer, we can find that:

(1) The Van Leer limiter provides higher accuracy in smooth sections but may introduce some dissipation (which can be seen at Figure 8).

(2) The MinMod limiter is very conservative at discontinuities, which can be seen as using 1st order upwind scheme, but it is effective in preventing oscillations but may lead to a loss of accuracy in smooth regions.

The error analysis:

(1) There is an error in the initial distribution of physical variables in the numerical solution (which can be seen in Figure 11, bule line), and this error leads to a certain amount of dispersion.
(2) Despite the use of limiters, no matter VanLeer or MinMod is used, when dealing with strong discontinuities, it employs scheme downscaling to 1st order upwind to prevent divergence and oscillations, causing dissipation, and the use of higher-order scheme at smooth sections introduces dispersion.

Appendix

All Matlab functions, main scripts, and original figure files are in ***Sod_ShockTube*** folder.