Solutions to the stone getting away from launch point problem

1 Method 1: $\dfrac{\mathrm{d}r}{\mathrm{d}t} > 0$

Write the parametric equations of motion of the particle with respect to time:

$$x(t) = v_0 t \cos\theta, \qquad y(t) = v_0 t \sin\theta - \frac{1}{2} g t^2$$

Now, let $\vec{r}$ be the position vector of the particle. As it is always going away from the starting point, r must always increase in magnitude, so $\dfrac{\mathrm{d}r}{\mathrm{d}t} > 0$ must always hold. It is easy to obtain the expression of $r(t)$ using the Pythagorean Theorem:

$$r^2(t) = x^2(t) + y^2(t) \implies r(t) = \sqrt{v_0^2 t^2 - v_0 g t^3 \sin\theta + \frac{1}{4} g t^4}$$

To evaluate the condition $\dfrac{\mathrm{d}r}{\mathrm{d}t} > 0$, it is sufficient to evaluate $\dfrac{\mathrm{d}r^2}{\mathrm{d}t} > 0$ thanks to the power rule for functions. This is equivalent to:

$$2v_0^2 t - 3v_0 g t^2 \sin\theta + g^2 t^3 > 0$$

.

Rearranging gives:

$$\sin\theta < \frac{2v_0^2 t + g^2 t^3}{3 v_0 g t^2} \equiv f(t)$$

The easiest thing to do here, in my opinion, is to find the minimum of $f(t)$ w.r.t. time: $\dfrac{\mathrm{d}f}{\mathrm{d}t} = 0 \implies t = \dfrac{v_0\sqrt{2}}{g}$, which gives $\sin\theta < \frac{2\sqrt{2}}{3}$. $\sin(x)$ is strictly increasing on $\left[0, \dfrac{\pi}{2}\right]$ and therefore $\theta < \arcsin\left(\frac{2\sqrt{2}}{3}\right)$ and therefore $\boxed{\theta < 70.528^\circ}$.

2 Non-derivative solution making use of the dot-product

The radial velocity, $v_r = \dfrac{\mathrm{d}r}{\mathrm{d}t}$ can also be written in dot-product form as $\vec{r} \cdot \vec{v}$, where $\vec{v}$ is the instantaneous velocity. Now, one has to write $\vec{r}(t)$ and $\vec{v}(t)$ in terms of $\vec{v_0}, t, \vec{g}$ as follows:

$$\vec{r}(t) = \vec{v_0} t + \frac{\vec{g}}{2} t^2, \qquad \vec{v}(t) = \vec{v_0} + \vec{g} t$$

And therefore, evaluating the dot product and dividing both sides by $\dfrac{v_0 t^2 g\sqrt{2}}{2} \neq 0$ one finds that:

$$\frac{v_0\sqrt{2}}{gt} + \frac{gt}{v_0\sqrt{2}} > \frac{3\sin\theta\sqrt{2}}{2}$$

Rearranging gives:

$$\sin\theta < \frac{\sqrt{2}}{3}\left(\frac{v_0\sqrt{2}}{gt} + \frac{gt}{v_0\sqrt{2}}\right)$$

Notice that $\dfrac{v_0\sqrt{2}}{gt} = \left(\dfrac{gt}{v_0\sqrt{2}}\right)^{-1}$, and therefore one can use the famous inequality $x + x^{-1} \geq 2$, with the equality case only holding for $x = 1 \iff v_0\sqrt{2} = gt$ (identical to the t that minimises $f(t)$ in the previous solution). This gives $\sin\theta < \dfrac{2\sqrt{2}}{3}$ so again, $\boxed{\theta < 70.528^\circ}$.