笔记

Husimi图

针对Husimi函数值的一些简单推导，以及一些猜想和结论。

流算符的本征态

$$\langle \mathbf{r}|\mathbf{r}_0,\mathbf{k}_0,\sigma\rangle = \frac{1}{\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{4\sigma^2}+\mathrm{i}\mathbf{k}_0\cdot\mathbf{r}}$$

平面波$\psi_A = \mathrm{e}^{i\mathbf{k}_A\cdot\mathbf{r}}$的Husimi函数的解析值

$$\begin{aligned}
\langle\psi|\mathbf{r}_0,\mathbf{k}_0,\sigma\rangle &= \frac{1}{\sigma\sqrt{2\pi}}\int \mathrm{e}^{-i\mathbf{k}_A\cdot\mathbf{r}}\mathrm{e}^{-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{4\sigma^2}+\mathrm{i}\mathbf{k}_0\cdot\mathbf{r}}\,\mathrm{d}\mathbf{r}\\
&= \frac{1}{\sigma\sqrt{2\pi}}\int \mathrm{e}^{-\frac{(\mathbf{r}-\mathbf{r}_0)^2}{4\sigma^2}+\mathrm{i}(\mathbf{k}_0-\mathbf{k}_A)\cdot\mathbf{r}}\,\mathrm{d}\mathbf{r}\\
&= \frac{4\sigma}{\sqrt{2\pi}}\int \mathrm{e}^{-(\mathbf{R}-\mathbf{R}_0)^2+\mathrm{i}\mathbf{K}\cdot\mathbf{R}}\,\mathrm{d}\mathbf{R} \quad \text{其中}\mathbf{R}=\frac{\mathbf{r}}{2\sigma},\mathbf{K}=2\sigma(\mathbf{k}_0-\mathbf{k}_A)\\
&= \frac{4\sigma}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{R}_0}\int \mathrm{e}^{-\mathbf{R}'^2+\mathrm{i}\mathbf{K}\cdot\mathbf{R}'}\,\mathrm{d}\mathbf{R}' \qquad \text{其中}\mathbf{R}'=\mathbf{R}-\mathbf{R}_0\\
&= \frac{4\sigma}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}\mathbf{K}\cdot\mathbf{R}_0-\frac{\mathbf{K}^2}{4}}\,\pi\\
&= 2\sigma\sqrt{2\pi}\mathrm{e}^{-\sigma^2(\mathbf{k}_0-\mathbf{k}_A)^2+\mathrm{i}(\mathbf{k}_0-\mathbf{k}_A)\cdot\mathbf{r}_0}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Hu}(\mathbf{r}_0,\mathbf{k}_0,\sigma;\psi) &= |\langle\psi|\mathbf{r},\mathbf{k}_0,\sigma\rangle|^2\\
&= 8\pi\sigma^2\mathrm{e}^{-2\sigma^2(\mathbf{k}_A-\mathbf{k}_0)^2}
\end{aligned}$$

Δk和Δx可能的解析表达式

其中$x=\mathbf{r}_x-\mathbf{r}_{0x}$，$k_x=\mathbf{k}_x-\mathbf{k}_{0x}$，平移高斯包的中心点的到圆点处的位置，这样平均值为零，以方便后面不确定度的计算。

$$
\begin{aligned}
\langle \mathbf{r}_0, \mathbf{k}_0, \sigma | \hat{x} | \mathbf{r}_0, \mathbf{k}_0, \sigma \rangle &= 0 \qquad \text{（奇函数，对称区间内的积分为零）} \\
\langle \mathbf{r}_0, \mathbf{k}_0, \sigma | \hat{x}^2 | \mathbf{r}_0, \mathbf{k}_0, \sigma \rangle &= \frac{1}{2\pi\sigma^2} \iint x^2 \mathrm{e}^{-\frac{x^2+y^2}{2\sigma^2}} \,\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2\pi\sigma^2} \int \mathrm{e}^{-\frac{y^2}{2\sigma^2}} \,\mathrm{d}y \int x^2 \mathrm{e}^{-\frac{x^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \frac{1}{2\pi\sigma^2} \sqrt{\frac{\pi}{\alpha}} \left(\frac{\mathrm{d}}{\mathrm{d}\alpha} \sqrt{\frac{\pi}{\alpha}} \right) \qquad \text{其中}\alpha = \frac{1}{2\sigma^2} \\
&= \frac{1}{4\pi\sigma^2} \sigma\sqrt{2\pi}\, \sigma^3 \sqrt{8\pi} \\
&= \sigma^2 \\
\Delta x &= \sqrt{\langle \mathbf{r}_0, \mathbf{k}_0, \sigma | \hat{x}^2 | \mathbf{r}_0, \mathbf{k}_0, \sigma \rangle} \\
&= \sigma
\end{aligned}
$$

$$
\begin{aligned}
\langle \mathbf{r}_0, \mathbf{k}_0, \sigma | \hat{k}_x | \mathbf{r}_0, \mathbf{k}_0, \sigma \rangle &= \frac{1}{2\pi\sigma^2} \iint [-\mathrm{i}\frac{\partial}{\partial x} \mathrm{e}^{-\frac{x^2+y^2}{4\sigma^2} + \mathrm{i}(k_{0x}x + k_{0y}y)}] \mathrm{e}^{-\frac{x^2+y^2}{4\sigma^2} + \mathrm{i}(k_{0x}x + k_{0y}y)} \,\mathrm{d}x\mathrm{d}y - k_{0x} \\
&= 0 \\
\langle \mathbf{r}_0, \mathbf{k}_0, \sigma | \hat{k}_x^2 | \mathbf{r}_0, \mathbf{k}_0, \sigma \rangle &= \frac{1}{2\pi\sigma^2} \iint |(-\mathrm{i}\frac{\partial}{\partial x} - k_{0x}) \mathrm{e}^{-\frac{x^2+y^2}{4\sigma^2} + \mathrm{i}(k_{0x}x + k_{0y}y)}|^2 \mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2\pi\sigma^2} \iint |\mathrm{i}\frac{x}{2\sigma^2} \mathrm{e}^{-\frac{x^2+y^2}{4\sigma^2} + \mathrm{i}(k_{0x}x + k_{0y}y)}|^2 \mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2\pi\sigma^2} \cdot \frac{1}{4\sigma^4} \int \mathrm{e}^{-\frac{y^2}{2\sigma^2}} \,\mathrm{d}y \int x^2 \mathrm{e}^{-\frac{x^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \frac{1}{8\pi\sigma^6} \sqrt{\frac{\pi}{\alpha}} \left(\frac{\mathrm{d}}{\mathrm{d}\alpha} \sqrt{\frac{\pi}{\alpha}} \right) \qquad \text{其中}\alpha = \frac{1}{2\sigma^2} \\
&= \frac{1}{16\pi\sigma^6} \sigma\sqrt{2\pi}\, \sigma^3 \sqrt{8\pi} \\
&= \frac{1}{4\sigma^2} \\
\Delta k &= \sqrt{(\Delta k)^2} = \frac{1}{2\sigma}
\end{aligned}
$$

- 动量算符：$\hat{\mathbf{p}} = -\mathrm{i}\hbar\nabla \rightarrow \hat{\mathbf{k}} = -\mathrm{i}\nabla$
- 猜测：$\sigma = \frac{1}{2\Delta k} = \frac{k}{2\Delta k}$
- 需要用到的积分公式：

$$
\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha x^2} \,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}}
$$