SIR models for the spread of COVID-19

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1 Introduction

Write smth about covid-19, motivation, SIR model, Kermack and McKendrick etc.

2 Mathematical model of epidemics

Let's consider a group of N people and classify them into these types:

- **S** – Susceptible
- **I** – Infected
- **R** – Recovered

This is called the SIR model for the spread of epidemic diseases that describes the changes in numbers of the three types of individuals. We denote the number of susceptible persons by $s(t)$, the number of infected individuals by $i(t)$ and the number of recovered people by $r(t)$. The time t is measured in days. Also we suppose that each person is in contact with m persons per day average. Hence the number of contacts between the susceptible and infected people becomes $\frac{m}{N}i(t)s(t)$. If we set the probability of infection for each contact as p then the number of newly infected individuals within Δt days becomes

$$\frac{m}{N}i(t)s(t)p\Delta t \tag{1}$$

in total. Let $\beta = mp$, the number of non-infected people **(S)** from the t-th day to $(t+\Delta t)$-th day changes as

$$s(t+\Delta t) - s(t) = -\frac{\beta}{N}s(t)i(t)\Delta t \tag{2}$$

When $\Delta t \to 0$, we can rewrite it as differential equation

$$\frac{ds(t)}{dt} = -\frac{\beta}{N}s(t)i(t) \tag{3}$$

Meanwhile, the infected individuals recover at a removal rate γ per day. Subsequently, increase in the number of the recovered persons becomes

$$\frac{dr(t)}{dt} = \gamma i(t) \tag{4}$$

Respectively, $\dfrac{1}{\gamma}$ is the expected duration of infection. Also the number of recovered people includes the amount of deceased persons because they cannot possibly infect others.

As well as the total number of individuals is N, we can express $i(t)$

$$i(t) = N - s(t) - r(t) \tag{5}$$

The change in the number of infected people can be written as

$$\frac{di(t)}{dt} = -\frac{ds(t)}{dt} - \frac{dr(t)}{dt} = \frac{\beta}{N}s(t)i(t) - \gamma i(t) = \frac{\beta}{N}(s(t) - \gamma)i(t) \tag{6}$$

Gathering the equations above, we have

$$\begin{cases} \dfrac{ds(t)}{dt} = -\dfrac{\beta}{N}s(t)i(t) & (7) \\ \dfrac{di(t)}{dt} = \dfrac{\beta}{N}(s(t) - \gamma)i(t) & (8) \\ \dfrac{dr(t)}{dt} = \gamma i(t) & (9) \end{cases}$$

As initial conditions, we use

$$\begin{cases} s(0) = N_1 \\ i(0) = N_2 \\ r(0) = 0 \end{cases} \tag{10}$$

If we introduce the following transformations of variables

$$\tilde{s}(t) = \frac{s(t)}{N}, \tilde{i}(t) = \frac{i(t)}{N}, \tilde{r}(t) = \frac{r(t)}{N}, \tilde{t} = \beta t$$

Then system of the equations becomes

$$\begin{cases} \dfrac{d\tilde{s}(t)}{d\tilde{t}} = -\tilde{s}(t)\tilde{i}(t) \\ \dfrac{d\tilde{i}(t)}{d\tilde{t}} = (\tilde{s}(t) - \dfrac{1}{R_0})\tilde{i}(t) \\ \dfrac{d\tilde{r}(t)}{d\tilde{t}} = \dfrac{1}{R_0}\tilde{i}(t) \end{cases} \tag{11}$$

The number $R_0 = \dfrac{\beta}{\gamma}$ is known as the basic reproduction number. The number of infected people increases when $R_0 > 1$ and decreases when $R_0 < 1$.

3 Analytical solution of SIR model

When model is defined, we can solve the system of equations (7) – (9). For simplicity forget about function arguments. Firstly, rewrite the equation (7) as

$$i = -\frac{1}{\tilde{\beta}}(\frac{s'}{s}) \tag{12}$$

where $\tilde{\beta} = \dfrac{\beta}{N}$. Then differentiate both sides

$$i' = -\frac{1}{\tilde{\beta}}(-\frac{s'^2}{s^2} + \frac{s''}{s}) \tag{13}$$

Next insert the equation (12) into (8)

$$i' = -(\tilde{\beta}s - \gamma)\frac{1}{\tilde{\beta}}(\frac{s'}{s}) \tag{14}$$

Comparing equations (13) and (14) we have

$$s\frac{d^2s}{dt^2} - (\frac{ds}{dt})^2 + (\gamma - \tilde{\beta}s)s\frac{ds}{dt} = 0 \tag{15}$$

Now we introduce the following function

$$\phi = \frac{dt}{ds} \tag{16}$$

Afterward (15) becomes

$$\frac{d\phi}{ds} + \frac{\phi}{s} = (\gamma - \tilde{\beta}s)\phi^2 \tag{17}$$

This is a Bernoulli differential equation. Divide both parts by ϕ^2

$$\frac{\phi'}{\phi^2} + \frac{1}{\phi s} = \gamma - \tilde{\beta}s \tag{18}$$

Then make a substitution

$$z = \frac{1}{\phi}, z' = -\frac{\phi'}{\phi^2} \tag{19}$$

Now solve the first-order linear ordinary equation

$$-z' + \frac{z}{s} = \gamma - \tilde{\beta}s \tag{20}$$

with general solution where C is constant

$$z = -\gamma s \ln s + \tilde{\beta}s^2 + Cs \tag{21}$$

Returning back to ϕ

$$\phi = \frac{1}{s(C - \gamma \ln s + \tilde{\beta}s)} \tag{22}$$

From the relation of inverse function in equation (16), we have

$$\frac{1}{\phi} = \frac{ds}{dt} \tag{23}$$

Using equation (7), we obtain

$$i(t) = -\frac{1}{\tilde{\beta}}(C - \gamma \ln s(t) + \tilde{\beta}s(t)) \tag{24}$$

Moreover, from equations (7) and (9), we get

$$\frac{dr}{dt} = -\frac{\gamma}{\tilde{\beta}}(\frac{s'}{s}) \tag{25}$$

Subsequently, the relation between s(t) and r(t) becomes

$$r(t) = -\frac{\gamma}{\tilde{\beta}} \ln \frac{s(t)}{C_1} \tag{26}$$

where C_1 is a constant. According to our initial conditions (10)

$$C_1 = N_1 \tag{27}$$

From the relation $s(0) + i(0) + r(0) = N$, we have

$$C = -\tilde{\beta}N + \gamma \ln N_1 \tag{28}$$

If we substitute C into equation (22), we obtain

$$\frac{dt}{ds} = \frac{1}{s(-\tilde{\beta}N - \gamma \ln \frac{s}{N_1} + \tilde{\beta}s)} \tag{29}$$

Integrate this and express t as a function of s

$$t = \int_{s(0)}^{s(t)} \frac{d\varepsilon}{\varepsilon(-\tilde{\beta}N - \gamma \ln \frac{\varepsilon}{N_1} + \tilde{\beta}\varepsilon)} \tag{30}$$

For convenience, we change a variable

$$\xi = \frac{\varepsilon}{N_1} \tag{31}$$

Rewrite the equation (30)

$$t(s) = \int_{1}^{\frac{s(t)}{N_1}} \frac{d\xi}{\xi(-\beta - \gamma \ln \xi + \beta\xi \frac{N_1}{N})} \tag{32}$$

Now we can calculate $t(s)$ using numerical integration with a small step size and $s(t)$ as a parameter like that

$$\int_a^b f(\xi)\, d\xi \simeq \sum_{i=0}^{n-1} f(a + ih)h, \text{ where } h = \frac{b-a}{n} \tag{33}$$

If $t(s)$ is obtained, then $i(t)$ and $r(t)$ can be calculated from equations (24) and (26) respectively.

4 SIR Model with natural deaths and births

Let's rewrite our system of equations (7) – (9) with an addition of the death and birth processes. Consider these rates are equal to each other, we introduce new variable D:

$$\begin{cases} \dfrac{ds(t)}{dt} = -\dfrac{\beta}{N}s(t)i(t) + D(N - s(t)) & (34) \\ \dfrac{di(t)}{dt} = (\dfrac{\beta}{N}s(t) - \gamma - D)i(t) & (35) \\ \dfrac{dr(t)}{dt} = \gamma i(t) - Dr(t) & (36) \end{cases}$$

As we did before, we rewrite it into one equation:

$$i = \frac{s' - D(N - s)}{-\tilde{\beta}s} \tag{37}$$

Again we differentiate both sides (with $\tilde{\beta} = \frac{\beta}{N}$ substitution):

$$i' = -\frac{1}{\tilde{\beta}}(\frac{(s'' + Ds')s - (s' + Ds - DN)s'}{s^2}), \tag{38}$$

$$i' = -\frac{1}{\tilde{\beta}}(\frac{s''}{s} - \frac{s'^2 - DNs'}{s^2}) \tag{39}$$

Now we can insert (37) into (35):

$$i' = (\tilde{\beta}s - \gamma - D)(\frac{s' - D(N - s)}{-\tilde{\beta}s}) \tag{40}$$

$$i' = -\frac{1}{\tilde{\beta}s}(\tilde{\beta}s - \gamma - D)(s' + Ds - DN) \tag{41}$$

And comparing (41) and (39) we get next equation:

$$ss'' - s'^2 + (-\tilde{\beta}s^2 + (\gamma + D)s + DN)s' - \tilde{\beta}Ds^3 - (\tilde{\beta}DN + D(\gamma + D))s^2 - (\gamma + D)DNs = 0 \tag{42}$$

We can see that coefficients at s^3, s^2 and s are *consts* so we substitute them with $A = -\tilde{\beta}D, B = -(\tilde{\beta}DN + D(\gamma + D)), C = -(\gamma + D)DN$:

$$ss'' - s'^2 + (-\tilde{\beta}s^2 + (\gamma + D)s + DN)s' + As^3 + Bs^2 + Cs = 0 \tag{43}$$

Now we introduce a function $\phi = \frac{dt}{ds}$, so the equation will be:

$$-\phi' - \frac{\phi}{s} + (-\tilde{\beta}s^2 + (\gamma + D)s + DN)\frac{\phi^2}{s} + (As^2 + Bs + C)\phi^3 = 0 \tag{44}$$

And by substitution $v = \phi s, v' = \phi' s + \phi$ the following equation occur:

$$v' + (-\tilde{\beta} + \frac{(\gamma + D)}{s} + \frac{DN}{s^2})v^2 + (A + \frac{B}{s} + \frac{C}{s^2})v^3 = 0 \tag{45}$$

Then we assume these three substitutions:

$$\begin{cases} r(s) = \dfrac{1}{v(s)}, & (46) \\ \xi(s) = (\tilde{\beta} - \dfrac{(\gamma + D)}{s} - \dfrac{DN}{s^2}), & (47) \\ \eta(s) = -(A + \dfrac{B}{s} + \dfrac{C}{s^2}) & (48) \end{cases}$$

and we get Abel-type first-order second-kind equation:

$$rr' = \xi(s)r + \eta(s) \tag{49}$$