1 SIR Model with natural deaths and births

Let's rewrite our system of equations (**??**) – (**??**) with an addition of the death and birth processes. Consider these rates are equal to each other, we introduce new variable D:

$$\begin{cases} \dfrac{ds(t)}{dt} = -\dfrac{\beta}{N}s(t)i(t) + D(N - s(t)) & (1) \\ \dfrac{di(t)}{dt} = (\dfrac{\beta}{N}s(t) - \gamma - D)i(t) & (2) \\ \dfrac{dr(t)}{dt} = \gamma i(t) - Dr(t) & (3) \end{cases}$$

As we did before, we rewrite it into one equation:

$$i = \frac{s' - D(N - s)}{-\tilde{\beta}s} \tag{4}$$

Again we differentiate both sides (with $\tilde{\beta} = \frac{\beta}{N}$ substitution):

$$i' = -\frac{1}{\tilde{\beta}}(\frac{(s'' + Ds')s - (s' + Ds - DN)s'}{s^2}), \tag{5}$$

$$i' = -\frac{1}{\tilde{\beta}}(\frac{s''}{s} - \frac{s'^2 - DNs'}{s^2}) \tag{6}$$

Now we can insert (4) into (2):

$$i' = (\tilde{\beta}s - \gamma - D)(\frac{s' - D(N - s)}{-\tilde{\beta}s}) \tag{7}$$

$$i' = -\frac{1}{\tilde{\beta}s}(\tilde{\beta}s - \gamma - D)(s' + Ds - DN) \tag{8}$$

And comparing (8) and (6) we get next equation:

$$ss'' - s'^2 + (-\tilde{\beta}s^2 + (\gamma + D)s + DN)s' - \tilde{\beta}Ds^3 - (\tilde{\beta}DN + D(\gamma + D))s^2 - (\gamma + D)DNs = 0 \tag{9}$$

We can see that coefficients at s^3, s^2 and s are *consts* so we substitute them with
$A = -\tilde{\beta}D, B = -(\tilde{\beta}DN + D(\gamma + D)), C = -(\gamma + D)DN$:

$$ss'' - s'^2 + (-\tilde{\beta}s^2 + (\gamma + D)s + DN)s' + As^3 + Bs^2 + Cs = 0 \tag{10}$$

Now we introduce a function $\phi = \frac{dt}{ds}$, so the equation will be:

$$-\phi' - \frac{\phi}{s} + (-\tilde{\beta}s^2 + (\gamma + D)s + DN)\frac{\phi^2}{s} + (As^2 + Bs + C)\phi^3 = 0 \tag{11}$$

And by substitution $v = \phi s, v' = \phi' s + \phi$ the following equation occur:

$$v' + (-\tilde{\beta} + \frac{(\gamma + D)}{s} + \frac{DN}{s^2})v^2 + (A + \frac{B}{s} + \frac{C}{s^2})v^3 = 0 \tag{12}$$

Then we assume these three substitutions:

$$\begin{cases} r(s) = \dfrac{1}{v(s)}, & (13) \\ \xi(s) = (\tilde{\beta} - \dfrac{(\gamma + D)}{s} - \dfrac{DN}{s^2}), & (14) \\ \eta(s) = -(A + \dfrac{B}{s} + \dfrac{C}{s^2}) & (15) \end{cases}$$

and we get Abel-type first-order second-kind equation:

$$rr' = \xi(s)r + \eta(s) \tag{16}$$