Espacios Vectoriales $(V, \mathbb{R}, +, \cdot, \text{Propiedades})$

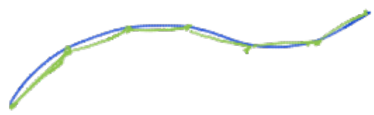
Obs.: por simplicidad denotamos a un e.v. por su conjunto de vectores.

ejm.1: $(\mathbb{R}, \cancel{\mathbb{R}, +, \cdot,} \ldots)$ $\mathbb{R}$

ejm.2: $\mathbb{R}^2$ ✓

ejm.3: $\mathbb{R}^3$ ✓

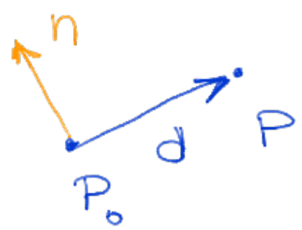
Unit 3



Vector Geometry

Much, if not most, of the graphics pipeline relies on asking geometric questions about objects in a scene. At the modeling stage, it is helpful to know where the middle of a face is or whether four vertices lie in a plane. If we think of light as traveling in rays, then asking where a ray intersects an object is key to understanding the shade of the object and the shadow it casts. The task now is to take the notion of a vector and use it as efficiently as possible to make geometric calculations.

3.1 Lines and planes



3.1.1 Vector description of lines

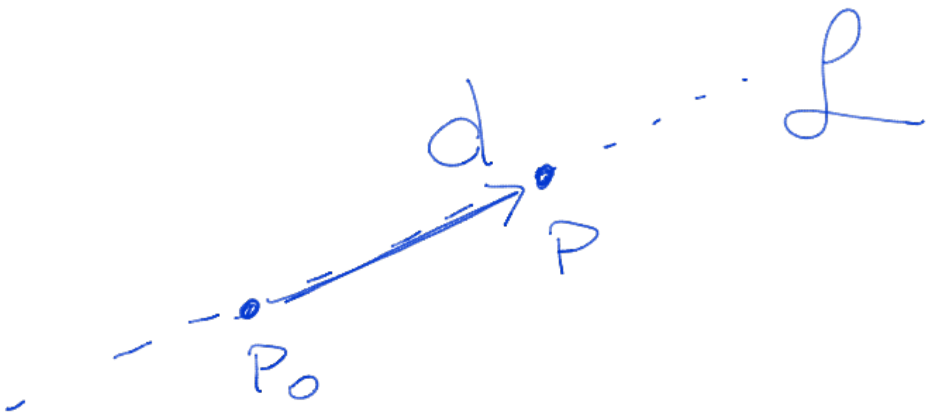
We have two vector descriptions of the two-dimensional line:

$d := P - P_0$

1. With two points, take $d = (P_1 - P_0)$, then any point on the line takes the form $\boxed{P_0 + t \cdot d}$ ec. vectorial

2. With a single point P_0 and a vector n perpendicular to the line (called a *normal vector*), we know $\boxed{n \cdot (P - P_0) = 0}$ describes any point P on the line.

$\langle n, P - P_0 \rangle = 0$



$$\mathcal{L} := \{ P \in \mathbb{R}^2 \, ; \, P = P_0 + t \cdot d \}$$

$$\mathcal{L} := \{ v \in V \, ; \, v = v_0 + t \cdot d \} \quad d \neq 0$$

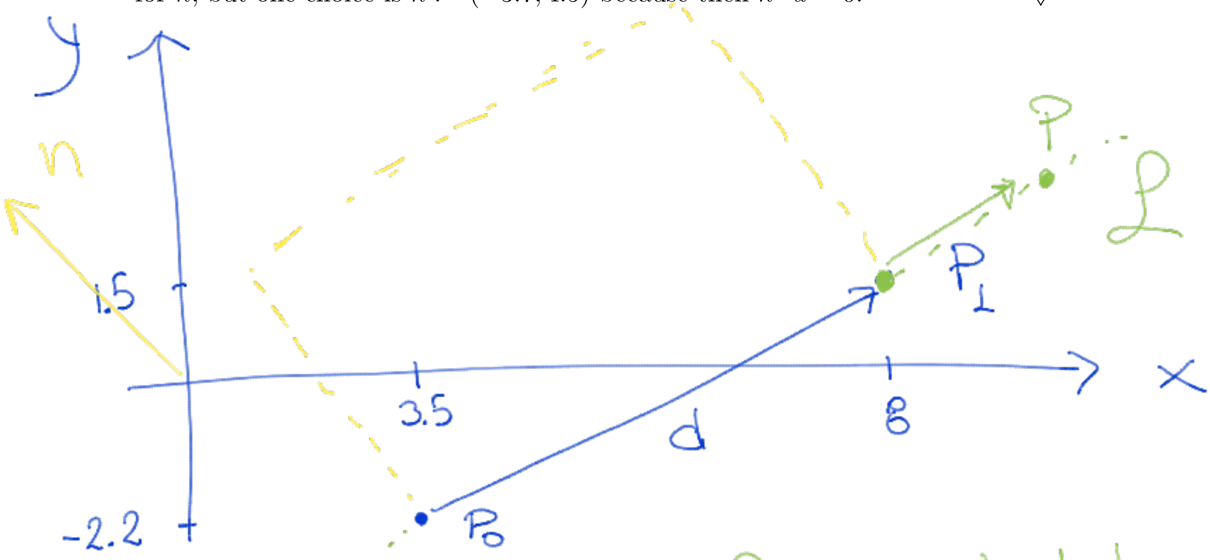
$\Leftrightarrow$

Sólo para rectas en el plano

$$\mathcal{L} := \{ P \in \mathbb{R}^2 \, ; \, \langle n, P - P_0 \rangle = 0 \}$$

$\left\{ P \in \mathbb{R} ;\ \underset{\neq 0}{\alpha} \cdot \underbrace{(P - P_0)} = 0 \right\} = \{P_0\}$

Example 3.1 (Line Perpendicular to a Line Segment)**.** Imagine we are drawing a tilted rectangle where the base edge is the line segment from $P_0 = (3.5, -2.2)$ to $P_1 = (8, 1.5)$. The vector $d := P_1 - P_0 = (4.5, 3.7)$ is parallel to this line segment. The two edges perpendicular to the base are both in the same direction, which is represented by a vector n perpendicular to d. There are many choices for n, but one choice is $n := (-3.7, 4.5)$ because then $n \cdot d = 0$. ♢



$\mathcal{L} := \{ P \in \mathbb{R}^2 ;\ P_0 + t \cdot d \}$

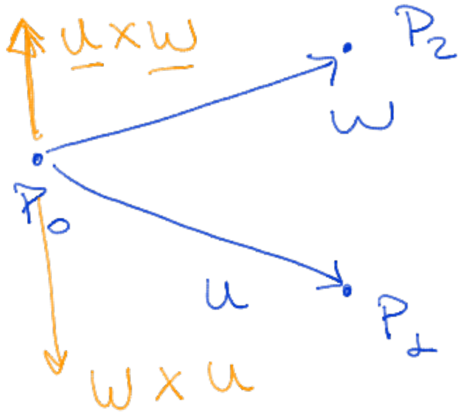
$\mathcal{L} := \{ \quad '' \quad ;\ \langle n, P - P_1 \rangle = 0 \}$

The vector description generalizes easily to three-dimensional lines. In fact, the equation $P = P_0 + t \cdot d$ makes no mention of how many coordinates we have. If there are three coordinates, then d is a three-dimensional vector instead of a two-dimensional vector and we can split the vector equation into three parametric equations, one for each coordinate. If $P_0 = (5, -2, 1)$ and $P_1 = (3, 3, 4)$, then $d = P_1 - P_0 = (-2, 5, 3)$. The vector equation is split into the following coordinate equations: $P := (x, y, z) = P_0 + t \cdot d$

$$\left.\begin{aligned} x &= 5 - 2 \cdot t \\ y &= -2 + 5 \cdot t \\ z &= 1 + 3 \cdot t \end{aligned}\right\} \text{ecc. (ecuación) paramétrica}$$

We were able to give a perpendicular vector for two-dimensional lines and calculate the direction vector by knowing that the dot product of the two had to be zero. In three dimensions, there are an infinite number of vectors perpendicular to any vector. This means we need to specify at least two perpendicular vectors that are not multiples of each other before we have described a direction vector.

Example 3.2 (Three-dimensional Line)**.** Suppose $P_0 = (-7, 5, 8)$, $P_1 = (-3, 9, 0)$, and $P_2 = (1, 6, 8)$ are three points in space. To find the equation of the line through P_0 perpendicular to both segments P_0P_1 and P_0P_2, we find vectors $u := P_1 - P_0 = (4, 4, -8)$ and $w := P_2 - P_0 = (8, 1, 0)$. Then, since the direction vector d is perpendicular to u and w it is parallel to $u \times w = \boxed{(8, -64, -28)}$. We can take $\boxed{d := (2, -16, -7)}$ because it is a multiple of the cross product. The equation of the line is then... ♢



$$\mathcal{L} := \{P \in \mathbb{R}^3 ; P = P_0 + t \cdot d, t \in \mathbb{R}\}.$$

3.1.2 Vector description of planes

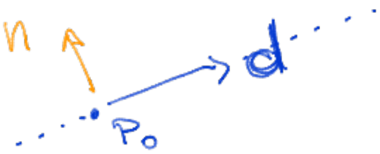
A similar vector approach works to describe planes in three dimensions. Two points determine a line, and a single vector (parallel to the line) determines its direction. It takes three points to determine a plane and two vectors (parallel to the plane) to determine its orientation. Start with three points on the plane, P_0, P_1, P_2. The two vectors $v_1 := P_1 - P_0$ and $v_2 := P_2 - P_0$ determine the position of the plane (as long as the three points are not collinear). The cross product $n := v_1 \times v_2$ is a vector that is perpendicular to the plane; n is a normal vector. This perpendicular vector alone determines the plane's orientation, whereas it takes two vectors parallel to the plane to do the same job.

v_1, v_2 l.i. $\equiv [\forall \alpha, \beta \in \mathbb{R} : \alpha \cdot v_1 + \beta \cdot v_2 = 0 \Rightarrow \alpha = \beta = 0]$

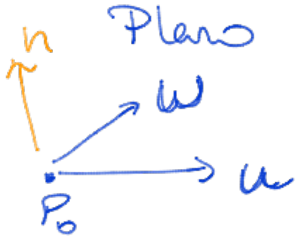
P_0

$$\mathcal{P} := \{P \in \mathbb{R}^3 ; P = P_0 + t_1 \cdot v_1 + t_2 \cdot v_2, t_1, t_2 \in \mathbb{R}\}$$

Recta



Plano



$\{u, w \text{ l.i}\}$

$P_0 + t\cdot d$ $\quad$ $P_0 + s\cdot u + t\cdot w$

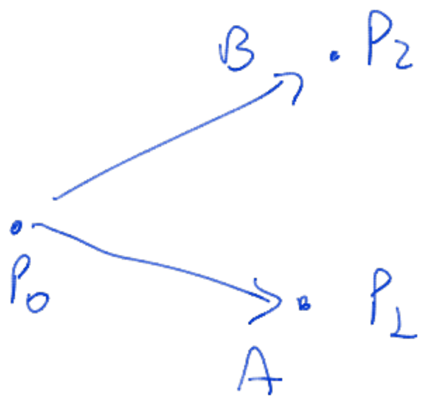
$$n \cdot (P - P_0) = 0$$

$\{P \in \mathbb{R}^2 ; \langle n, P - P_0 \rangle = 0\}$ $\quad$ $\{P \in \mathbb{R}^3 ; \langle n, P - P_0 \rangle\}$

Las rectas son los hiperplanos de $\mathbb{R}^2$ $\quad$ Los planos son los hiperplanos de $\mathbb{R}^3$

Los hiperplanos de un espacio euclidiano n-dimensional tienen dimensión "n-1"

Example 3.3 (Plane Containing Three Points)**.** With the three points, $P_0 :=$ $(1, 1, 1)$, $P_1 = (4, -2, 5)$, and $P_2 = (3, 8, -1)$, form the vectors $A := P_1 - P_0 =$ $(3, -3, 4)$ and $B := P_2 - P_0 = (2, 7, -2)$. ♢



$V := \{f : \mathbb{R} \to \mathbb{R} ; f \text{ cont}\}$

Example 3.4 (Plane Perpendicular to a Line)**.** To find a plane containing $P_0 := (-10, 3, 5)$ and perpendicular to the line through P_0 and $P_1 := (2, 7, 2)$, note that the vector $n := P_1 - P_0 = (12, 4, -3)$ is normal to the plane. So the vector equation is $(12, 4, -3) \cdot (P - (-10, 3, 5)) = 0$ and the implicit coordinate equation is

$$12 \cdot x + 4 \cdot y - 3 \cdot z = -123.$$

♢

$\mathcal{P} = \{P \in \mathbb{R}^3 ; \langle n, P - P_0 \rangle = 0\}$

(x, y, z)

x, y, z

$\mathcal{P}$

3.2 Distances

Def (distancia entre dos vectores) Sean $u, v \in \mathbb{R}^3 \ (\mathbb{R}^2)$ definimos la distancia entre estos vectores como la expresión:

$$\|u-v\|.$$

Def (Norma) Definimos la norma de un vector $v \in \mathbb{R}^3 \ (\mathbb{R})$ como

$$\|v\| := \sqrt{\langle v, v\rangle}$$

Def (Producto Interno) Definimos el producto interno de dos vectores $u, v \in \mathbb{R}^3 \ (\mathbb{R}^2)$ como

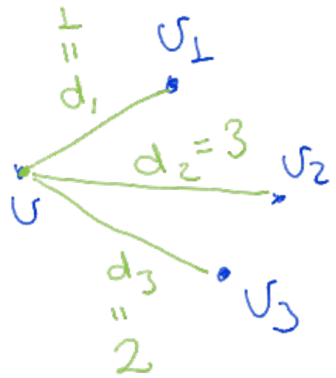
$u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$

$$\boxed{\langle u, v\rangle = \sum_{j=1}^{n} u_j \cdot v_j .}$$

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$$
$$(u, v) \longmapsto \cdot$$

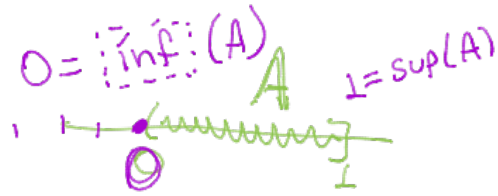
$d(v, S) = 1$

$S := \{v_1, v_2, v_3\}$



Def (distancia de un pto. a un conjunto) Sean $v \in \mathbb{R}^3$, $S \subseteq \mathbb{R}^3$, definimos la distancia de v a S como

$$d(v, S) := \inf \{\|v-u\| ; u \in S\}$$



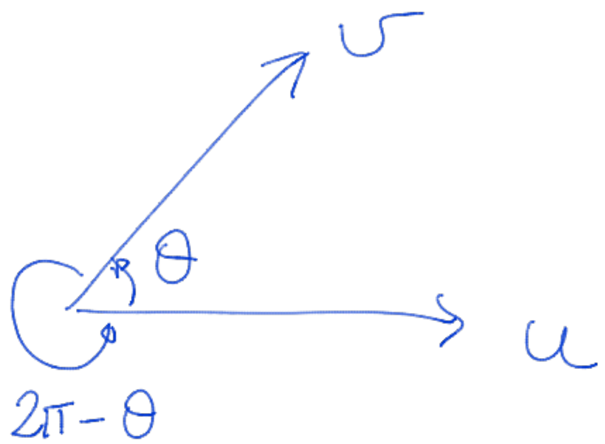
Para hablar de <u>ángulo</u> formado por dos vectores, necesitamos tener <u>producto interno</u>.

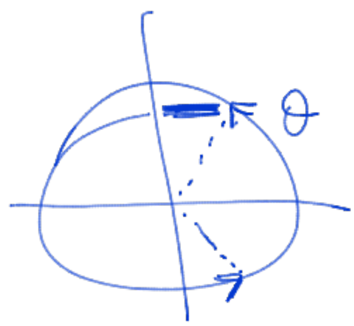
<u>Def</u> (ángulo entre dos vectores)

Sean $u, v \in \mathbb{R}^3$ $(\mathbb{R}^2)$ <u>NO</u> nulos. Definimos el ángulo entre dos los vectores u y v como aquel ángulo θ t.q. $= \pi/2$

$(0 \leq \theta \leq \pi)$

$$\cos(\theta) := \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|} > 0$$





<u>Def</u> (Vectores ortogonales)

Decimos que dos vectores $u, v \in \mathbb{R}^3$ son <u>ortogonales</u> si $\langle u, v \rangle = 0$.

3.3 Exercises

In the following exercises, use vector techniques to find solutions.

Exercise 3.1. Find the fourth vertex P_3 of the rectangle in Example 3.1.

Exercise 3.2. A line goes through $P_0 = (10, 8)$ and $P_1 = (7, -3)$. Find an equation for the line perpendicular to this line through a point two-thirds of the way from P_0 to P_1.

Exercise 3.3. The vertices $A = (30, 6)$ and $B = (52, 10)$ form the base of an isosceles triangle (two sides equal). Find vertex C so that the height of the triangle is 40.

Exercise 3.4. Find the equation of the line through $(-8, 12, 7)$ and the mid-point of the segment from $(0, 2, 5)$ to $(-4, -4, 2)$.

Exercise 3.5. Find the implicit coordinate equation of the plane through $(1, -4, 1)$, $(3, 6, 5)$, and $(-2, 2, 6)$.

Exercise 3.6. How close is the point $(10, 15)$ to the line through $(11, 6)$ and $(24, 30)$?

Exercise 3.7. Two planes have the same normal $(13, -2, 6)$. One contains the point $P_1 = (3, 3, 9)$ and the other contains $P_2 = (-7, 0, 6)$. How far apart are the planes?

Exercise 3.8. Two planes have the same normal $(10, 12, -5)$. One contains the origin. Find the vector equation of the second plane so that it is 32 units from the first one. (Two possible answers.)

Exercise 3.9. One face of a rectangular box in space is a plane with normal $(2, -1, -1)$. The face contains the vertex $A = (8, 3, 6)$ and the adjacent vertex $B = (5, 12, -9)$. Find the equation of all three planes meeting at vertex A.

Exercise 3.10. In each of the following cases, A and B are the end points of one line segment and C and D form a second segment. Determine if the two segments intersect, and if they do, find the point of intersection.

1. $A = (50, 240)$, $B = (500, 115)$, $C = (80, 100)$, $D = (400, 130)$
2. $A = (-10, 8)$, $B = (110, -17)$, $C = (200, 6)$, $D = (16, -50)$
3. $A = (100, 24, 19)$, $B = (-8, -3, -8)$, $C = (-11, -18, 21)$, $D = (17, 2, 1)$
4. $A = (-20, 31, 6)$, $B = (15, -12, 18)$, $C = (-34, -10, 12)$, $D = (10, 17, -1)$.

Exercise 3.11. The vertices $(4, 5)$, $(30, 8)$, and $(25, 18)$ form a triangle. Determine if the point $(15, 16)$ is inside or outside the triangle.

Exercise 3.12. Determine whether the line through the points $(1,3,3)$ and $(-2,4,8)$ intersects the plane with the implicit equation $3 \cdot x + 6 \cdot y - 2 \cdot z = 8$. If it does intersect, find the point of intersection.

Exercise 3.13. Your eye is at position $(2,5,-1)$ looking in direction $(1,1,3)$. A sphere with radius 3 is currently centered at $(9,8,12)$. We move the sphere perpendicular to the ray from your eye until the ray just touches the sphere. Determine the new center of the sphere.

Exercise 3.14. Find the area of the triangle in two dimensions with vertices $(10,24)$, $(22,38)$, and $(15,4)$. If the vertices of a triangle have integer coordinates, what must be true about the area?

Exercise 3.15. A line in space passes through $(3,1,-9)$ with direction $(2,-6,2)$ and a second line passes through $(7,2,1)$ with direction $(1,0,-2)$. Find the points on the two lines that are closest together.

Exercise 3.16. Write a method (function) that takes the four end points of two line segments (two dimensions) as input and outputs whether or not the segments intersect.

Exercise 3.17. Write a method (function) that takes the eight vertices of a box (not necessarily a parallelepiped) plus a point as input and outputs whether the point is inside the box. You must include a code to determine whether the box is well formed.