Unit 7

Curves

A straight line segment is a key primitive in computer graphics and, consequently, any flat triangular (or more generally, polygonal) surface is easy to draw on the screen. Yet, curves are indispensable when designing objects like cars, door knobs, archways, and any number of animated characters. Although we end up approximating these forms with many triangles with their straight edges, the problem is still how to find vertices that ultimately give a global look of curvature. If we add the animation stage where we let a car bounce along the dirt roadway or move the camera as though we are flying through the city, then the paths describing these motions rely on various curves.

7.1 Curve descriptions

Definition 7.1 (Curve descriptions in the plane)**.** A two-dimensional curve is a collection of points described by any of the following expressions:

1. Explicit: $y = f(x)$ for some continuous function f.

2. Implicit: $F(x, y) = 0$ for some continuous function F.

3. Parametric: $x = f(t)$ and $y = g(t)$ for continuous functions f and g.

Definition 7.2 (Curve descriptions in the space)**.** A three-dimensional curve is a collection of points described by any of the following expressions:

1. Explicit: $x = f(z)$ and $y = f(z)$ for some continuous function f.

2. Implicit: $F(x, y, z) = 0$ and $G(x, y, z) = 0$ for some continuous functions F and G.

3. Parametric: $x = f(t)$, $y = g(t)$ and $z = h(t)$ for continuous functions f, g and h.

Parametric descriptions are arguably the most useful descriptions for computer graphics. Actually, we have already been using these descriptions because our expression for a line segment (a curve) is $P(t) = t \cdot P_0 + (1 - t)P_1$ where t is in $[0, 1]$ and P_0 and P_1 are points. Here, instead of one equation for each coordinate, we have combined them into one using points.

Example 7.3 (conics)**.** Conics are the familiar two-dimensional curves obtained by intersecting a plane with a cone. They can be implicitly described by

$$A \cdot x^2 + B \cdot x \cdot y + C \cdot y^2 + D \cdot x + E \cdot y + F = 0. \tag{7.1}$$

Various restrictions on the coefficients classify the conics as a pair of straight lines, parabola, ellipse, or hyperbola. For example, the following expression factors nicely:

$$x^2 - 2 \cdot x \cdot y - 3 \cdot y^2 + 3 \cdot x - 5 \cdot y + 2 = (x - 3 \cdot y + 1)(x + y + 2) = 0.$$

This implies that the curve includes points such that $x = 3 \cdot y - 1$ or $x = -y - 2$. Hence the curve is just a pair of straight lines, a degenerate conic. If we wish, we can easily give parametric equations for each of the lines.
If the coefficients are $A = 2$, $B = 0$, $C = 0$, $D = -4$, $E = -1$, $F = 5$, then the description becomes

$$2 \cdot x^2 - 4 \cdot x - y + 5 = 0$$

and this easily rearranges to become

$$y = 2(x - 1)^2 + 3,$$

a parabola. Again, the parametric description is easy, $x = t$ and $y = 3(t - a)^2 + 3$ $\diamondsuit$

Example 7.4. Curves can have more or less convenient descriptions in different coordinate systems. Consider the curve with implicit description

$$x \cdot y - 1 = 0.$$

This is a conic because it fits our general form, but it may not be clear which conic it is. A change of coordinate system will change the description and perhaps put it in a form we recognize. Suppose we wish to describe the curve in a coordinate system where the x- and y-axis have been rotated counterclockwise by θ. The coordinates of a point on the curve will then be rotated clockwise by θ. Letting the new coordinates be (u, v), we convert with the following expressions:

$$x = u \cdot \cos(\theta) - v \cdot \sin(\theta)$$
$$y = u \cdot \sin(\theta) - v \cdot \cos(\theta)$$

Setting $\theta = \pi/4$, we can substitute into the original implicit description.

$$\begin{aligned} x \cdot y - 1 &= \left(\frac{u}{\sqrt{2}} - \frac{v}{\sqrt{2}}\right)\left(\frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}\right) - 1 \\ &= \frac{u^2}{2} - \frac{v^2}{2} - 1 \\ &= 0. \end{aligned}$$

The expression on the right is the implicit description of a hyperbola. If the coefficient for $x \cdot y$ in the general implicit form for a conic is nonzero, it is a good indication that the conic is oriented with axes that are not parallel to the coordinate axes. $\diamondsuit$

Of course, not all curves are conics, and not all curves are described efficiently in Cartesian coordinates. In polar coordinates, a spiral is given by $r = \theta$; converting to Cartesian coordinates gives $x = t \cdot \cos(t)$ and $y = t \cdot \sin(t)$, a little less compact.

7.1.1 Lagrange interpolation

Example 7.5 (Curve to interpolate three points)**.** Take $P_0 = (-1, 3)$, $P_1 = (2, 5)$, and $P_2 = (4, 1)$. Then our interpolation procedure gives the following description:

$$\begin{aligned} P(t) &= \frac{1}{2}(t-1)(t-2)\begin{bmatrix} -1 \\ 3 \end{bmatrix} - t(t-2)\begin{bmatrix} 2 \\ 5 \end{bmatrix} + \frac{1}{2}t(t-1)\begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2}(t^2 - 7 \cdot t + 2) \\ -3 \cdot t^2 + 5 \cdot t + 3 \end{bmatrix} \end{aligned}$$

The coordinate parametric equations are both quadratic as our method predicts from the way we chose the blending functions. A quick check when $t = 0, 1, 2$ shows that the curve does pass through the three specified points.
From Equation (7.1), we know this curve is a conic, and if we use the procedure we saw earlier for converting a parametric description to an implicit description, we can verify that this curve is a parabola (since $B^2 - 4 \cdot A \cdot C = 0$). $\diamondsuit$

7.1.2 Matrix form for curves

7.2 Bézier curves

7.2.1 Properties for two-dimensional Bézier curves

7.2.2 Joining Bézier curve segments

7.2.3 Three-dimensional Bézier curves

7.3 B-splines

7.3.1 Linear uniform B-splines

7.3.2 Quadratic uniform B-Splines

7.3.3 Cubic uniform B-Splines

7.3.4 B-spline properties

7.4 Exercises

Exercise 7.1. Example 7.3 shows the general form of a conic. Suppose we translate the curve by making the substitution $x^* := x - h$ and $y^* := y - k$. Show that we can pick h and k appropriately so the general form of the new curve has $D^* = E^* = 0$. Also show that $A^* = A$, $B^* = B$, and $C^* = C$.

Exercise 7.2. Assuming that the conic is not a pair of lines, the quantity $\delta = B^2 - 4 \cdot A \cdot C$ determines whether it is a parabola ($\delta = 0$), ellipse ($\delta < 0$), or hyperbola ($\delta > 0$). Show that if the conic is a parabola, then $A \cdot x^2 + B \cdot x \cdot y + C \cdot y^2$ is a perfect square.

Exercise 7.3. Consider the ellipse $4 \cdot x^2 + y^2 = 1$. Using the technique shown in Example 7.4, rotate the axes $\pi/6$ radians counterclockwise and give the new description of the ellipse.

Exercise 7.4. Given a conic description $A \cdot x^2 + B \cdot x \cdot y + C \cdot y^2 + D \cdot x + E \cdot y + F = 0$, show that an axis rotation of θ where

$$\cot\left(\frac{A - C}{B}\right)$$

will make $B = 0$.

Exercise 7.5. The circle of radius a can be parameterized as $x = a \cdot \cos(t)$ and $y = a \cdot \sin(t)$. Find $r(t)$ and $r'(t)$. Determine the unit tangent $T(t)$ and find its value at various points on the circle including $t = 0$, $t = \pi/4$, and $t = \pi/2$.

Exercise 7.6. Verify that the curve in Example 7.5 is a parabola by finding the implicit expression of the curve and then calculating $B^2 - 4 \cdot A \cdot C$ (see Exercise 7.2).

Exercise 7.7. Let $P_0 = (4, 2)$ and $P_1 = (8, -16)$. Consider the curve $P(t) = (1 - t) \cdot P_0 + t^2 \cdot P_1$. This is not an affine combination, but it does interpolate the two points. By translating the points and curve two units to the right, show that this curve is not affine-invariant.

Exercise 7.8. Using the Lagrange interpolation method, find a curve through the points $(0, 1)$, $(3, -1)$, $(4, 3)$, and $(6, 5)$. Verify that the blending functions sum to 1.

Exercise 7.9. Let $P_0 := (-3, 0)$, $P_1 := (-1, 4)$, $P_2 := (2, 3)$ and $P_3 := (4, 1)$. Find the parametric description $P(t)$ of the curve that interpolates these points using Lagrange interpolation.

Exercise 7.10. With the same notation of the previous exercise:

1. Find the parametric description $P(t)$ of the cubic Bézier curve with control points: P_0, P_1, P_2 and P_3.

2. Find the slope of the tangents at the first and last control points by finding the derivative of the blending functions.

3. Verify that the slopes match the slopes of the line segments P_0P_1 and P_2P_3.

Exercise 7.11. Give the complete description of the blending function $N_{(n,1)}(t)$.

Exercise 7.12. Construct a uniform quadratic B-spline using the control points $(-1,0)$, $(1,4)$, $(3,-2)$, and $(4,3)$. Find the point on the curve at $t = 3.5$.

Exercise 7.13. Use the de Boor algorithm to verify the point on the curve at $t = 3.5$ in Exercise 7.12.

Exercise 7.14. Let $P_0 := (-1,0)$, $P_1 := (1,4)$, $P_2 := (3,-2)$, $P_3 := (4,3)$ and $P_4 := (6,1)$:

1. Construct a uniform cubic B-spline using the control points P_0, P_1, P_2, P_3 and P_4. Find the parametric expressions for the coordinates x and y.

2. Verify by finding the derivatives that, at the joining point between the first and second segment, the first and second derivatives match.

Exercise 7.15. Verify that the blending functions for the uniform cubic B-spline sum to 1.

Exercise 7.16. Write a program that shows:

1. the interpolating curve of Exercise 7.9 with its interpolating points;

2. the Bézier curve of Exercise 7.10 with its control points;

3. the uniform cubic B-spline of Exercise 7.14 with its control points.