$T : \mathbb{R}^2 \to \mathbb{R}^2$

$u \mapsto u + a$

$E : \mathbb{R}^2 \to \mathbb{R}^2$

$u \mapsto E(u)$

Unit 4

$T(0) = a \neq 0$

No es lineal, pero el afín lineal.

E ?

Transformations

Suppose you are in the modeling stage of the graphics pipeline and the task is to build a car for display on the computer screen. Building an object means that we need to find coordinates for points that delineate various features such as the windshield, the hood, the tires, and so on. Many of these features are curves, so we end up approximating them with very small line segments.

Surfaces like the car body are three dimensional, and a mesh of small triangles does the job of approximating their shapes. To specify the line segments and the vertices of triangles, we can use the techniques of vector geometry to make the job considerably easier. However, if the right side of the car looks the same (or almost the same) as the left side of the car, it make sense to simply reflect one side of the car in a plane passing lengthwise down the car's middle.

This process of reflection is a type of transformation. When we reach the tires for the car, we can model one tire finding appropriate vertices, but then just apply another *transformation*, a translation, to copy it from the front to the back of the car. Finally, another reflection transformation will then copy the tire from one side of the car to the other.

Technically, a transformation T is just a function that sends each point (or vertex), A, to another point called $T(A)$. The result is to transform an object into a new object. The new object may have the same shape as the old and just a new position, or it may have an altered shape. Before we can talk about the mechanics of actually performing a transformation, we need to once again consider the differences between vectors and points in order to be careful about h*ow we deal with each. Recall that we decided to represent both vectors and points as a column of numbers, so

$\mathbb{R}^2$ y $M(2,1)$ son isomorfos

$$\text{Vector} v = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad \text{Point} P = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$\mathbb{R}^2$ (a,b) $\dim(\mathbb{R}^2) = 2$ ✓

$M(2,1)$ $\begin{bmatrix} a \\ b \end{bmatrix}$ $\dim(M(2,1)) = 2$ ✓

$T : \mathbb{R}^2 \to M(2,1)$ * biyección

* lineal $\begin{cases} T(u+v) = T(u) + T(v) & \forall u,v \in \mathbb{R}^2 \\ T(\alpha \cdot u) = \alpha T(u) & \forall \alpha \in \mathbb{R} \end{cases}$

* T^{-1} lineal.

$\mathbb{R}^2 \cong M(2,1)$

The vector, however, is a displacement and the point is a position in the plane. We know there is a connection between the two because we determine a vector by subtracting two points. We are most interested in thinking of transformations as moving points to points, but we can also think of them as acting on vectors.

4.1 Linear transformations on $\mathbb{R}^2$

4.1.1 Rotation in two dimensions

4.1.2 Reflection in two dimensions

4.1.3 Scaling in two dimensions

4.1.4 Matrix properties

$\{T : \mathbb{R}^2 \to \mathbb{R}^2 ; T \text{ es lineal}\} =: \mathcal{L}$

$M(2,2) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; a,b,c,d \in \mathbb{R} \right\}$

$\dim(\mathcal{L}) = 4$

Una base de un e.v. es un conjunto linealmente independiente que genera dicho e.v.

ooo

¡ $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ es una <u>base</u> !

$\dim(M(2,2)) = 4$

$T : \mathbb{R}^2 \to \mathbb{R}^2$

$u := (1,2) \mapsto T(u)$

$\underbrace{M \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{} \cong T(u)$

$\mathcal{L} \cong M(2,2)$ ¡ $\mathcal{L}$ y $M(2,2)$ son isomorfos !

4.2 Three dimensions linear transformations

4.2.1 Rotations in three dimensions

4.2.2 Reflections in three dimensions

4.2.3 Scaling and shear in three dimensions

Def (transformación lineal)

Sean U, V espacios vectoriales. Una transformación lineal de U a V es una aplicación $T: U \to V$ tal que $\forall u, v \in U$, $\forall \alpha \in \mathbb{R}$:

$$\begin{cases} T(u+v) = T(u) + T(v) \\ T(\alpha \cdot u) = \alpha \cdot T(u) \end{cases}$$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T: \mathbb{R}^3 \to \mathbb{R}^3$

$T: \mathbb{R} \to \mathbb{R}^3$

4.3 Affine transformations (afín-lineal)

4.3.1 Transforming homogeneous coordinates

4.3.2 Perspective transformations

4.3.3 Transforming normals

Def Sean U y V espacios vectoriales. Una transformación afín lineal de U a V es una aplicación

$$A : U \to V \quad \text{t.q.}$$

$A = b + L$, donde $L : U \to V$ es lineal y $b \in V$

Obs: Toda transformación lineal es una transformación afín lineal ($b = 0$).

4.4 Exercises

Exercise 4.1. Find the transformation matrix for the linear transformation that sends $(3, -1)$ to $(2, 4)$ and $(5, 1)$ to $(3, 8)$.

Exercise 4.2. Let a triangle have vertices $A = (-2, -3)$, $B = (4, 1)$, and $C = (2, 5)$. Find the transformed vertices when the triangle is rotated by $\pi/4$ clockwise around vertex A.

Exercise 4.3. Start with vertices $A = (30, 6)$ and $B = (52, 10)$. Find vertex C so that the three points form an equilateral triangle.

Exercise 4.4. Let a triangle have one vertex at the origin. Show that, if the determinant of a 2×2 matrix is 1, then the associated linear transform preserves the area of this triangle. Expand this to verify that the area of any triangle is preserved under the transformation.

Exercise 4.5. Reflect the point $(8, -2)$ in the line through the origin and the point $(4, 5)$.

Exercise 4.6. Reflect the point $(6, -1, 3)$ in the plane through the origin with normal $(-1, 5, 2)$.

Exercise 4.7. Give the transformation matrix for the two-dimensional linear transformation that projects everything on the x-axis. Use this matrix and a rotation to find the transformation matrix that projects everything onto the line $y = x$.

Exercise 4.8. Find the two-dimensional linear transformation that reflects points in the line $y = 3 \cdot x + 7$.

Exercise 4.9. The 2×2 matrix M can be thought of as having two vectors: $u := (a, b)$ and $v := (c, d)$ forming the columns. If the vectors are unit vectors and they are perpendicular, then the matrix is called **orthogonal**.

1. Show that we can set $a = \pm\cos(\theta)$ and $b = \pm\sin(\theta)$.
2. Show further that $d = a$ or $d = -a$, which implies $c = -b$ or $c = b$.
3. Explain why rotation and reflection matrices are examples of orthogonal matrices and that products of these two types are also orthogonal.
4. Intuitively, we know that rotations and reflections should not change the area of triangles and, indeed, (show that) the determinant of an orthogonal matrix is ± 1.

Exercise 4.10. The reflection of the point (x, y, z) in the origin is $(-x, -y, -z)$. Find the 4×4 matrix for homogeneous coordinates that will reflect a point in the point $(2, 5, -1)$.

Exercise 4.11. The unit cube with vertices (a, b, c), where each component is 0 or 1, is rotated by $\pi/6$ counterclockwise around the diagonal through $P_0 :=$ $(0, 0, 0)$ and $(1, 1, 1)$.

1. Find the transformation matrix.

2. Find the coordinates of the transformed cube.

Exercise 4.12. Show that a rotation of $2\pi/3$ clockwise around the line from $P_0 := (0, 0, 0)$ to $(1, 1, 1)$ is the product of two rotations around coordinate axes.

Exercise 4.13. The vertices $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, and $(-1, -1, 1)$ form a tetrahedron with equal sides. In Example 3.24 from Chapter 3, there is another set of vertices for a tetrahedron with equal sides. Find the transformation matrix that takes the first tetrahedron to the second.

Exercise 4.14. A two-dimensional transformation reflects in the x-axis and then reflects in the line through the origin and $(3, 4)$. Show that the resulting transformation is a rotation, and give the angle of rotation.

Exercise 4.15. Show that the three-dimensional shear transformation given by

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

preserves volume by explaining what it does to a unit cube.

Exercise 4.16. Write a program to present

1. the unit cube of question two;

2. the transformed cube of question two.