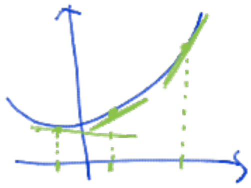
Unit

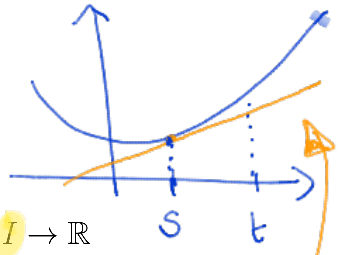
~~Week~~ 11

Nonlinear Optimization

11.3 Convexity



→ ↑ pendiente



11.3.4 Convex functions of a single variable

Remark 11.1. Let convex sets of $\mathbb{R}$ are precisely the intervals.

Property 11.2. Let I be a non-empty open interval of $\mathbb{R}$ and let $\theta : I \to \mathbb{R}$ be a differentiable function. Then the following propositions are equivalent:

1. θ is convex;
2. $(t - s) \cdot (\theta'(t) - \theta'(s)) \geq 0$ for every $s, t \in I$; θ' es monótona creciente
3. $\theta(t) \geq \theta(s) + (t - s) \cdot \theta'(s)$ for every $s, t \in I$.

el gráfico de la función está por encima de los gráficos de las rectas tangentes a la gráfica de la función.

◇

Remark 11.3. The Property 11.2 gives two characterization of convex differentiable functions:

$\emptyset = (a, a)$

$\emptyset =]a, a[$

- item 2 says that the differentiable functions θ that are convex are precisely the ones such that θ' is monotone increasing, i.e., for every $s, t \in I$:

$$\left[s < t \quad \Rightarrow \quad \theta'(s) \leq \theta'(t); \right] \equiv \left[\overset{<0}{(t - s)} \cdot \underbrace{(\theta'(t) - \theta'(s))}_{\leq 0} \geq 0 \right]$$

- item 3 says that the differentiable functions θ that are convex are precisely the ones such that the graph of θ is above the tangent line to the graph of θ at every point of the graph.

$s > t \qquad \theta'(s) - \theta'(t) \geq 0$

Moreover, we also have a characterization of convexity for more regular functions. $\diamondsuit$

Property 11.4. With the same notation of Property 11.2, let us assume that θ is twice differentiable. Then θ is convex if and only if $\theta''(t) \geq 0$ for every $t \in I$. $\diamondsuit$

11.3.5 Convex functions of several variables

The necessary and sufficient conditions of convexity of first and second order we saw for functions of a single variable could be extended to functions of several variables. First, let us recall the following concept.

Definition 11.5. We say that a symmetric matrix A of order n is **positive semi-definite** (**positive definite**) if $x^\top \cdot A \cdot x \geq 0$ ($x^\top \cdot A \cdot x > 0$) for every $x \in \mathbb{R}^n$ ($x \in \mathbb{R}^n \setminus \{0\}$) and we denote it by $A \geq 0$ ($A > 0$). $\diamondsuit$

Property 11.6 (Characterizations of convexity of first order)**.** Let C be a non-empty convex and open set of $\mathbb{R}^n$ and let $f : C \to \mathbb{R}$ be a differentiable function. Then the following propositions are equivalent:

1. f is convex;
2. $\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0$ for every $x, y \in C$; monotonicidad del gradiente
3. $f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$ for every $x, y \in C$.

$\diamondsuit$

Property 11.7 (Characterization of convexity of second order)**.** With the same notation of Property 11.6, let us assume that f is twice differentiable. Then f is convex if and only if $\nabla^2 f(x) \geq 0$ for every $x \in C$. $\diamondsuit$

$\forall u \in \mathbb{R}^n: \quad \underline{u^T \cdot \nabla^2 f(x) \cdot u} \geq 0$

11.4 Optimality conditions

First, let us formally define the notions of global minimum and local minimum.

Definition 11.8 (Global and local minima)**.** The general formulation of a minimization problem is

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in C, \end{array}$$

$C \subseteq \Omega$ abto

where $C \subseteq \mathbb{R}^n$ is the **feasible set**, $f : \Omega \to \mathbb{R}$ is the **objective function** and $\Omega \subseteq \mathbb{R}^n$ is a non-empty open set such that $\Omega \supseteq C$. Then, we say that f has a **global minimum** (**strict global minimum**) over C in $\bar{x} \in C$ if $f(\bar{x}) \leq f(x)$ ($f(\bar{x}) < f(x)$) for every $x \in C$ ($x \in C \setminus \{\bar{x}\}$). Moreover, we say that f has a **local minimum** (**strict local minimum**) over C in $\bar{x} \in C$ if there exists an open set V such that $V \ni \bar{x}$ and $f(\bar{x}) \leq f(x)$ ($f(\bar{x}) < f(x)$) for every $x \in C \cap V$ ($x \in (C \cap V) \setminus \{\bar{x}\}$). Finally, when $C = \mathbb{R}^n$ we shall omit the preposition 'over'. ◇

Hereafter until the end of this unit we shall continue with the notations of Definition 11.8.

Remark 11.9.

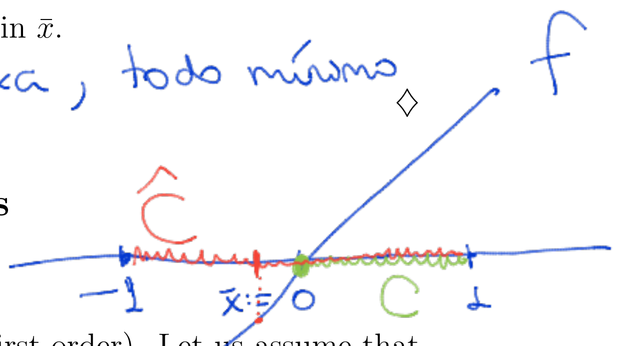
1. If f has a global minimum over C in $\bar{x}$, then f has a local minimum over C in $\bar{x}$.

2. The property of having a global or local minimum is invariant if we shrink the feasible region, e.g., if $\tilde{C} \subseteq C$ and f has a local minimum over C in $\bar{x}$, then f has a local minimum over $\tilde{C}$ in $\bar{x}$.

   $\exists\, V \subseteq \mathbb{R}^n$ abto t.q. $V \ni \bar{x}$, $\forall x \in V \cap C \supseteq V \cap \tilde{C}$ $f(x) \geq f(\bar{x})$

3. The property of having a global or local minimum is relative if we expand the feasible set, e.g., if $\bar{x} = 0$, $C = [0, 1]$, $\hat{C} = [-1, 1]$, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$, then we have that f has local minimum over C in $\bar{x}$, but does not have a local minimum over $\bar{C}$ in $\bar{x}$.

4. Si $f : C \to \mathbb{R}$ es convexa, todo mínimo local es global. ◇



11.4.1 Case without restrictions

It is the case where $\boxed{C = \mathbb{R}^n}$.

Property 11.10 (Optimality conditions of first order)**.** Let us assume that f is of $\mathcal{C}^1$ class over $\mathbb{R}^n$.

$\boxed{\forall\, V \subseteq \mathbb{R} \text{ abto}}$ t.q. $V \ni \bar{x}$

$f : C \subseteq \mathbb{R}^n \to \mathbb{R}$

$f : \mathbb{R}^n \to \mathbb{R}$

$f \in C^1(\mathbb{R}^n, \mathbb{R})$

$\exists\, x \in \hat{C} \cap V$ t.q. $f(x) < f(\bar{x})$

$x < 0$

entonces f tiene una singularidad en $\bar{x}$

Condición necesaria 1. If f has a local minimum in $\bar{x}$, then $\nabla f(\bar{x}) = 0$.

" suficiente 2. If f is convex and $\nabla f(\bar{x}) = 0$, then f has a global minimum in $\bar{x}$.

$\diamondsuit$

Property 11.11 (Optimality conditions of second order)**.** Let us assume that f is of $\mathcal{C}^2$ class over $\mathbb{R}^n$.

Cond. necesaria 1. If f has a local minimum in $\bar{x}$, then $\nabla f(\bar{x}) = 0$ and $\nabla^2 f(\bar{x}) \geq 0$.

" suficiente 2. If $\nabla f(\bar{x}) = 0$ and $\underline{\nabla^2 f(\bar{x}) > 0}$, then f has a strict local minimum in $\bar{x}$.

$\diamondsuit$

11.4.2 Case where the feasible set is convex

It is the case where C is a convex set.

Property 11.12 (Optimality conditions of first order)**.** Let us assume that f is of $\mathcal{C}^1$ class over G and $\bar{x}$ is in C. para algún cjto. G t.q. $G \supseteq C$.

1. If f has a local minimum over C in $\bar{x}$, then $\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0$ for every $x \in C$.
2. If f is convex and $\langle \nabla f(\bar{x}), x - \bar{x} \rangle \geq 0$ for every $x \in C$, then f has a global minimum over C in $\bar{x}$.

$\diamondsuit$

11.5 Exercises

Exercise 11.~~1~~6. Show a function $f : \mathbb{R} \to \mathbb{R}$ that is not convex, but is concave.

Exercise 11.~~2~~7. Show (by an example) that in general the sufficient condition of Property 11.10 is not a necessary condition.

Exercise 11.~~3~~8. Show (by an example) that in general the necessary condition of Property 11.10 is not a sufficient condition.

Exercise 11.4. Show (by an example) that in general the sufficient condition of Property 11.11 is not a necessary condition.

Exercise 11.5. Show (by an example) that in general the necessary condition of Property 11.11 is not a sufficient condition.

Exercise 11.6. Determine whether the functions $f, g : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x + \sqrt{x^2 + 4}$ and $g(x) = \ln(1 + e^x)$, are convex.

Exercise 11.7. Consider the function $f : (-\infty, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 - 1 & \text{for } x < 1, \\ 2 & \text{for } x = 1. \end{cases}$$

Determine whether f is convex.

Exercise 11.8. Determine whether the function $f : [1, \infty) \to \mathbb{R}$, given by $f(x) = -\sqrt{x - 1}$, is convex.

Exercise 11.9. Let $C := 0, \infty)$ and let $f : C \to \mathbb{R}$ be such that $f(x) = x + \frac{5}{x}$.

1. Determine whether f is convex.
2. Study whether f has a unique global minimum over C.