School of Computer Science
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National University of Engineering

Test 4 - Solution

Topics: convex sets; convex functions; optimality conditions
Subject: Computational Mathematics **Period: 2020.1**

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1. Let $a \in \mathbb{R}^{\mathrm{n}}$, let $d \in \mathbb{R}^{\mathrm{n}} \setminus \{0\}$ and let $c \in \mathbb{R}$. Show that the following sets are convex:

   (a) (1 pt.) $\mathcal{L}(a, d) := \{a + td\ ;\ t \in \mathbb{R}\}$;

   (b) (2 pts.) $\mathcal{H}(d, c) := \{v \in \mathbb{R}^n\ ;\ \langle v, d\rangle = c\}$;

   (c) (2 pts.) $B(a, r) := \{v \in \mathbb{R}^n\ ;\ |v - a| < r\}$.

   Solution.

   (a) Let $x := a + t_1 \cdot d$ and $y := a + t_2 \cdot d$ for some $t_1, t_2 \in \mathbb{R}$ and let $t \in (0, 1)$. Then,

   $$\begin{aligned}(1-t)\cdot x + t\cdot y &= (1-t)\cdot a + t\cdot a + (1-t)\cdot t_1\cdot d + t\cdot t_2\cdot d\\ &= a + s\cdot d\end{aligned}$$

   is in $\mathcal{L}(a, d)$, where $s = (1-t)\cdot t_1 + t\cdot t_2$.

   (b) $\mathcal{H}(d, c)$ could be expressed as the intersection of the semi-spaces: $\mathcal{H}^{-}(d, c)$ and $\mathcal{H}^{-}(-d, -c)$, which are convex; then, $\mathcal{H}(d, c)$ is convex since convexity is invariant under intersections.

(c) Let $x, y \in B(a, r)$ and let $t \in (0, 1)$. Then,

$$\begin{aligned}
|(1-t)\cdot x + t\cdot y - a| &= |(1-t)\cdot x + t\cdot y - (1-t)\cdot a - t\cdot a| \\
&\leq |(1-t)\cdot(x-a)| + |t\cdot(y-a)| \\
&< (1-t)\cdot r + t\cdot r,
\end{aligned}$$

which means that $(1-t)\cdot x + t\cdot y$ is in $B(a, r)$. □

2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = \ln(1+e^x)$.

   (a) (2 pts.) Determine whether f is convex. The assertion must be proven.

   (b) (2 pts.) Determine whether f has a global minimum. The assertion must be proven.

   (c) (1 pt.) Sketch the graph of f.

Solution.

(a) Since f is twice differentiable and

$$\begin{aligned}
f''(x) &= \frac{e^x(1+e^x) - e^x \cdot e^x}{(1+e^x)^2} \\
&= \frac{e^x}{(1+e^x)^2} \\
&> 0
\end{aligned}$$

for every $x \in \mathbb{R}$, it follows that f is convex.

(b) Since f is of $\mathcal{C}^2$ class and $f'' > 0$, f does not have a global minimum; otherwise we must have $\cancel{f''(\bar{x})}$ $f'(\bar{x})$ $= 0$ for some $\bar{x}$ in $\mathbb{R}$, which is impossible.

(c) Since f is convex and

$$f'(x) = \frac{e^x}{1+e^x} > 0,$$

which means that f is monotone increasing, we have:

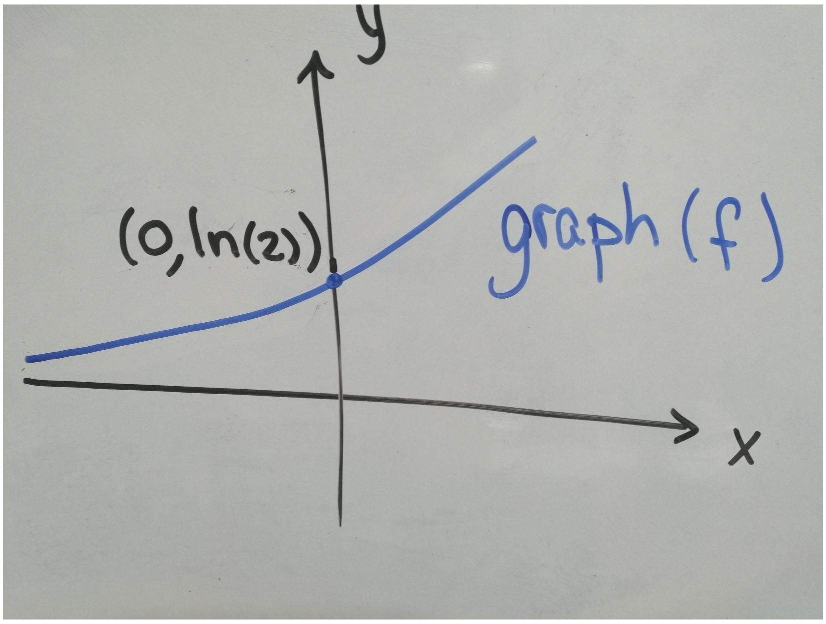

Figure 1: Graph of f.

$\square$

3. Let $C := (-\infty, 1]$ and consider the function $f : C \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 - 1 & \text{for } x < 1, \\ 2 & \text{for } x = 1. \end{cases}$$

   (a) (2 pts.) Determine whether f is convex. The assertion must be proven.

   (b) (2 pts.) Determine whether f has a unique global minimum over C. The assertion must be proven.

   (c) (1 pt.) Sketch the graph of f.

Solution.

   (a) Since f is twice differentiable on $(-\infty, 1)$ and $f'' = 2 > 0$ on this interval, we have that f is convex over this interval. Then, in order to prove that f is convex (over C) it is enough to prove that

$$f\left((1-t) \cdot x + t \cdot 1\right) \leq (1-t) \cdot f(x) + t \cdot f(1)$$

$\forall x, y \in (-\infty, 1]$, $\forall t \in (0, 1)$: $f\left((1-t) \cdot x + t \cdot y\right) \leq (1-t) \cdot f(x) + t \cdot f(y)$

$x, y \in (-\infty, 1)$

$x \neq 1 \quad y = 1 \qquad x < 1, \; y = 1$

for every $x < 1$ and $t \in (0,1)$. Indeed, let $x < 1$ and $t \in (0,1)$. Hence, as $(1-t)\cdot x \leq 1$ and $t^2 < t$, we have that $(1-t) \in (0,1)$

$(1-t)^2 \leq (1-t)$

$$t^2 + 2t(1-t)x \leq 3t,$$

which means that $(1-t)^2 \cdot x^2 + \quad -1 \leq (1-t)^2 \cdot x^2 + 3t - 1$

$\leq (1-t)\cdot x^2 + 3t - 1 =$

$$(1-t)^2 \cdot x^2 + t^2 + 2t\cdot(1-t)\cdot x - 1 \leq (1-t)(x^2-1) + t\cdot 2,$$

and the conclusion follows. $(1-t)\cdot x^2 - (1-t) + t\cdot 2$

(b) Let $g : C \to \mathbb{R}$ given by $g(x) = x^2 - 1$ and let $\bar{x} = 0$. Then, g is convex and of $\mathcal{C}^1$ class. First, because $g'(\bar{x})\cdot(x-\bar{x}) \geq 0$ for every $x \in C$, g has a global minimum over C in $\bar{x}$. Second, as $f = g$ on $(-\infty, 1)$, $f(1) > g(1)$, it follows that f has a global minimum over C in $\bar{x}$. Finally, because $f(x) > f(\bar{x})$ for every $x \in C \setminus \{\bar{x}\}$, f has a unique global minimum.

(c) Since f is convex, $f' < 0$ over $(-\infty, 0)$ and $f' > 0$ over $(0,1)$, we have:

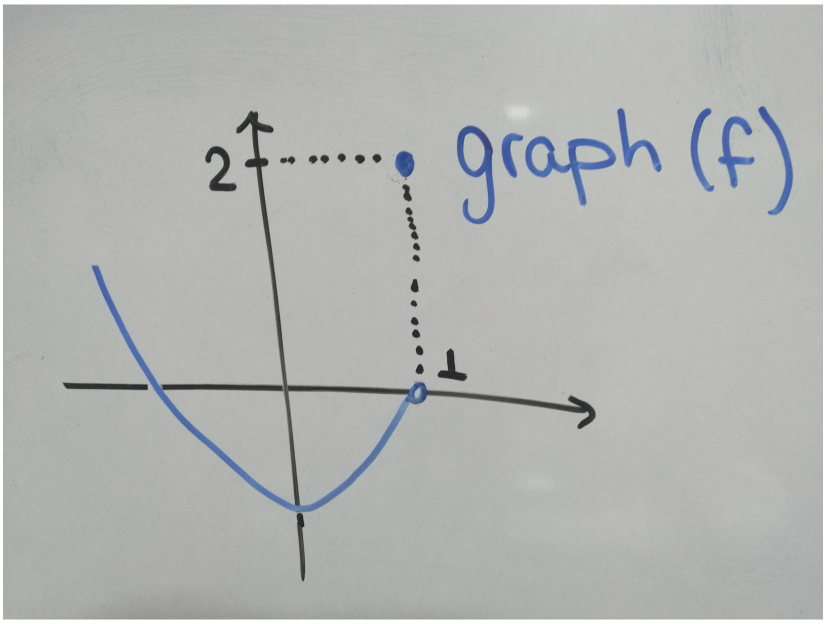

Figure 2: Graph of f.

□

4. Let $C := (0, \infty)$ and let $f : C \to \mathbb{R}$ be such that $f(x) = x + \frac{4}{x}$.

(a) (2 pts.) Determine whether f is convex. The assertion must be proven.

(b) (2 pts.) Determine whether f has a unique global minimum over C. The assertion must be proven.

(c) (1 pt.) Sketch the graph of f.

$f'(x) = 1 - \frac{4}{x^2}$

$f'(2) = 0$

Solution.

(a) Since f is twice differentiable and

$$f''(x) = \frac{8}{x^3} > 0$$

for every $x > 0$, we have that f is convex.

(b) Let $\bar{x} = 2$. Then, as f is convex and $f'(\bar{x}) \cdot (x - \bar{x}) \geq 0$ for every $x \in C$, it follows that f has a global minimum over C in $\bar{x}$. Moreover, if f had another global minimum $\hat{x}$, it would be possible to find a $x \in C$ such that $f'(\hat{x}) \cdot (x - \hat{x}) < 0$. Hence, f has a unique global minimum.

(c) Since f is convex, $f' < 0$ over $(0, 2)$ and $f' > 0$ over $(2, \infty)$, we have:

$$f'(x) = 1 - 4/x^2$$

$$\boxed{f'(\hat{x})\cdot[x-\hat{x}] < 0}$$

$\boxed{\hat{x} < 2}$ $\quad f'(\hat{x}) < 0$

$x > \hat{x}$

$\boxed{\hat{x} > 2}$ $\quad f'(\hat{x}) > 0$

$x < \hat{x}$

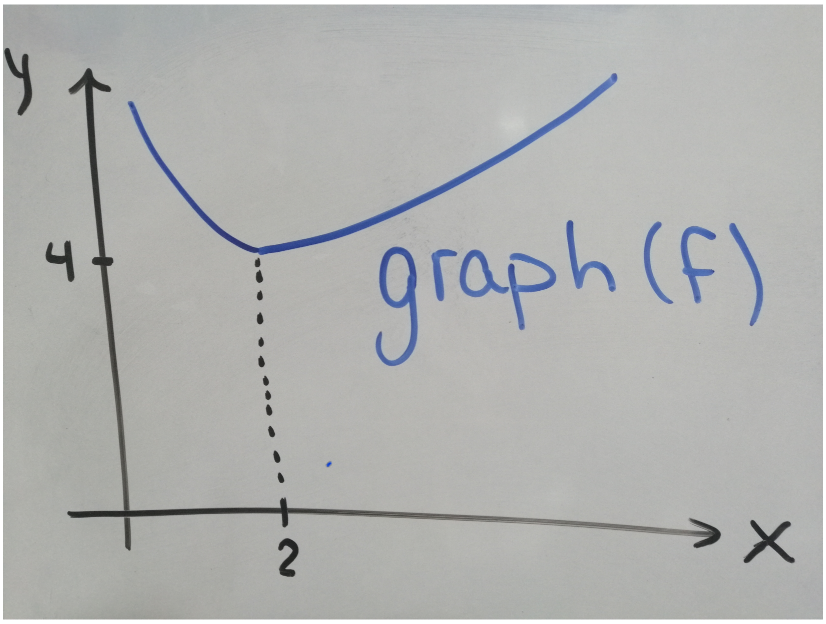

Figure 3: Graph of f.

$\square$

August 13, 2020