Fast simulation of viscous lava flow using Green's functions as a smoothing kernel

Supplementary material

Y. Kedadry & G. Cordonnier

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In this supplementary material we present our integration of the momentum equation.

1 Vertical integration of the momentum equation

Our goal is to compute the vertically averaged velocity $\bar{u}$:

$$\bar{u} = \frac{1}{h}\int_0^h u(z)dz, \tag{1}$$

where the velocity is integrated between the bottom and the top of the lava flow, or, by translation, for an elevation z between 0 and h.

We start with the momentum equation, where we neglect the inertial terms and separate the horizontal components of the Laplacian (for which we use the notation $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$) and the vertical component $\partial^2/\partial z^2$.

$$\vec{0} = \Delta u + \frac{\partial^2 u}{\partial z^2} - \frac{\rho g}{\mu(\theta)}\nabla S. \tag{2}$$

Which can be rewritten:

$$\Delta u + \frac{\partial^2 u}{\partial z^2} = \frac{\rho g}{\mu(\theta)}\nabla S. \tag{3}$$

1.1 Vertical integration of the whole equation

We integrate a first time between an altitude s and h, assuming negligible spatial variations of h:

$$\begin{aligned}\int_s^h \Delta u(z) + \frac{\partial^2 u(z)}{\partial z^2}\, dz &= \int_s^h \frac{\rho g}{\mu(\theta)}\nabla S.\, dz\\ \Delta \int_s^h u(z)\, dz + \left[\frac{\partial u(z)}{\partial z}\right]_s^h &= (h-s)\frac{\rho g}{\mu(\theta)}\nabla S.\end{aligned} \tag{4}$$

The gradient of the velocity is null on the free surface so $\frac{\partial u}{\partial z}(h) = 0$ and equation 4 becomes:

$$\Delta \int_s^h u(z)\, dz - \frac{\partial u(s)}{\partial z} = (h-s)\frac{\rho g}{\mu(\theta)}\nabla S. \tag{5}$$

We integrate again but this time between the bottom of the flow 0 and an arbitrary point s' (note that $u(0) = 0$ a the ground interface):

$$\Delta \int_0^{s'} \int_s^h u(z)\, dz\, ds - u(s') = (hs' - \frac{1}{2}s'^2)\frac{\rho g}{\mu(\theta)}\nabla S. \tag{6}$$

Finally to get the average velocity on the column we integrate between 0 and h before dividing by h.

$$\frac{1}{h}\Delta\left(\int_0^h\int_0^{s'}\int_s^h u(z)\,dz\,ds\,ds'\right)-\bar{u}=\frac{1}{h}(\frac{1}{2}h^3-\frac{1}{6}h^3)\frac{\rho g}{\mu(\theta)}\nabla S. \tag{7}$$

$$=\frac{1}{3}h^2\frac{\rho g}{\mu(\theta)}\nabla S. \tag{8}$$

While we obtain $\bar{u}$ in the second term of Eq. 7, we cannot evaluate the triple integral over u in the first time. Instead, we approximate it.

1.2 Approximation of the integrals of the velocity

To approximate u, we first assume that u approaches u_0, the velocity obtained if we neglect the lateral viscous forces:

$$u_0(z)=-(hz-\frac{1}{2}z^2)\frac{\rho g}{\mu(\theta)}\nabla S. \tag{9}$$

and

$$\bar{u_0}=-\frac{1}{3}h^2\frac{\rho g}{\mu(\theta)}\nabla S. \tag{10}$$

Now, we integrate 9 three times (we omit $dz\,ds\,ds'$ in the integrals for readability):

$$\begin{aligned}
\int_0^h\int_0^{s'}\int_s^h u_0(z) &= \int_0^h\int_0^{s'}\int_s^h\left(-(hz-\frac{1}{2}z^2)\frac{\rho g}{\mu(\theta)}\nabla S.\right)\\
&= -\frac{\rho g}{\mu(\theta)}\nabla S.\left(\int_0^h\int_0^{s'}\left(\frac{h^3}{2}-\frac{h}{2}s^2\right)-\int_0^h\int_0^{s'}\left(\frac{h^3}{6}-\frac{1}{6}s^3\right)\right)\\
&= -\frac{\rho g}{\mu(\theta)}\nabla S.\left(\int_0^h\left(\frac{h^3}{2}s'-\frac{h}{6}s'^3\right)-\int_0^h\left(\frac{h^3}{6}s'-\frac{1}{24}s'^4\right)\right)\\
&= -\frac{\rho g}{\mu(\theta)}\nabla S.\left(\frac{5h^5}{24}-\left[\frac{h^3}{12}s'^2-\frac{1}{120}s'^5\right]_0^h\right)\\
&= -\frac{\rho g}{\mu(\theta)}\nabla S.\left(\frac{5h^5}{24}-\frac{9h^5}{120}\right)\\
&= -\frac{\rho g}{\mu(\theta)}\nabla S.\left(\frac{16h^5}{120}\right)\\
&= -\frac{2}{15}\frac{h^5\rho g}{\mu(\theta)}\nabla S. \qquad (11)
\end{aligned}$$

We reinject Eq. 10 in Eq. 11, yielding

$$\int_0^h\int_0^{z'}\int_z^h u_0=\alpha h^3\bar{u}_0, \tag{12}$$

where $\alpha=\frac{2}{5}$ is a shape factor.

We assume a similar relationship in the general case, yielding the depth-averaged momentum equation with lateral viscosity forces:

$$\frac{1}{h}\Delta\left(\alpha h^3 \bar{u}\right) - \bar{u} = \frac{1}{3}h^2\frac{\rho g}{\mu(\theta)}\nabla S.$$

$$\Delta(h^3\bar{u}) - \frac{h\bar{u}}{\alpha} = \frac{h^3}{3\alpha}\frac{\rho g}{\mu(\theta)}\nabla S. \tag{13}$$

Let's now write $U = h^3\bar{u}$ and $\beta = \frac{1}{\alpha}$. The equation becomes:

$$\Delta U - \beta\frac{U}{h^2} = \beta\frac{h^3\rho g}{3\mu(\theta)}\nabla S. \tag{14}$$