INF554 - Lab 2 - Solutions

1) Low rank approximation

Question 1

Following the notation in the jupyter notebook the necessary calculations are the following,

$$\begin{aligned}
\mathbb{V}\left[C_i U\right] &= \mathbb{V}\left[(U^T C_i^T)^T\right] \\
&= \mathbb{V}\left[(U^T C_i^T)\right]^T \\
&= U^T \mathbb{V}\left[C_i\right] U \\
&= \Lambda,
\end{aligned}$$

where Λ contains the eigenvalues of the covariance matrix on the diagonal and has zero entries in all other locations. Hence, since the off-diagonal elements of $\mathbb{V}\left[C_i U\right]$ equal 0 the principal components are uncorrelated. Furthermore, their variance can be read off to equal the eigenvalues of the covariance matrix.

Question 2

- The quality of the low rank approximation improves as k increases.
- The best performing methods are PCA and MDS.

2) Orthogonal transformation of the input data

Question 3

Following the notation defined in the jupyter notebook, the principal components obtained from the original image are $C_i U_j$. The centered version of the transformed image $\tilde{C}_i$ is related to the centered version of the untransformed image C_i as follows,

$$\tilde{C}_i = \tilde{X}_i - \tilde{M}_i = X_i Q - M_i Q = C_i Q. \quad (1)$$

Hence, the eigendecomposition of $\tilde{C}_i^T \tilde{C}_i$ is equal to $Q^T U \Lambda U^T Q$. Therefore, the principal components of the transformed image are equal to $\tilde{C}_i Q^T U_j = C_i Q Q^T U_j = C_i U_j$, as required. This relation can also be proved more formally by observing that $C_i^T C_i$ and $\tilde{C}_i^T \tilde{C}_i$ are similar matrices via the similarity matrix Q.

3) Image Denoising

This section was tricky! We have added Gaussian noise to the two images. In the second image we transformed the noise to lie in orthogonal complement of the space spanned by the first 75 eigenvectors of the uncorrupted image. Therefore, when reconstructing the second image using the first 75 eigenvectors, you should find that no noise is present in their reconstruction at all. This can be observed in a direct comparison of the low rank approximation of the original image using the first 75 eigenvectors and the denoised version of image 2.

More explicitly, for a given channel i, let X_i denotes the original image, $X_i^{(1)}$ denotes the noisy image 1, $X_i^{(2)}$ denotes the noisy image 2, N_1, N_2 denote Gaussian noise matrices matching X_i in dimension and U denotes the matrix contating the first 75 eigenvectors of the centered image C_i as columns, then,

$$X_i^{(1)} = X_i + N_1,$$
$$X_i^{(2)} = X_i + N_2(I - UU^T).$$

Question 4

- The low rank approximation decreases the amount of noise in the images.
- Using only a few of the basis elements to reconstruct the full image reduces the overall level of noise in the image. This effect is amplified in our methodology since we use the basis elements of the logo without noise. If too many basis elements are used then the images start to appear more noisy due to the noise in the original image being preserved in the reconstruction.

Question 5

The noise in image 2 is only present in the space spanned by the eigenvectors with indices larger than 75 (as described at the beginning of this section).