Exercise sheet 3

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Contents


```
theory LundfallErfurtEx03
imports Main

begin
```

0.1 Exercise 1

```
theorem a):
  assumes a: (∀ x. P x) ∧ (∀ x. Q x)
  shows ∀ x. P x ∧ Q x
proof –
  {
    fix x
    from a have ∀ x. P x by (rule conjunct1)
    then  have px: P x by (rule allE)
    from a have ∀ x. Q x by (rule conjunct2)
    then have Q x by (rule allE)
    with px have P x ∧ Q x by (rule conjI)
  }
  thus ?thesis by (rule allI)
qed

lemma b):
  assumes ∃ x. P x ∧ Q x
  shows ∃ x. P x
proof –
  from assms obtain x where 1: P x ∧ Q x by (rule exE)
  then have xx: P x by (rule conjunct1)
  then show ?thesis by (rule exI)
qed
```

lemma *c*):
  assumes $\forall\ x.\ P\ x$
  shows $\exists x.\ P\ x$
proof −
  fix x
  from *assms* **have** $P\ x$ **by** (*rule allE*)
  then have $\exists\ x.\ P\ x$ **by** (*rule exI*)
  thus *?thesis* **.**
qed

lemma *d*):
  shows $((\forall\ x.\ P\ x) \wedge (\forall\ x.\ Q\ x)) \longleftrightarrow (\forall\ x.\ P\ x \wedge Q\ x)$
proof −
  {
    assume *a*: $(\forall\ x.\ P\ x) \wedge (\forall\ x.\ Q\ x)$
    {
      fix x
      from *a* **have** $\forall\ x.\ P\ x$ **by** (*rule conjunct1*)
      then have *b*: $P\ x$ **by** (*rule allE*)
      from *a* **have** $\forall\ x.\ Q\ x$ **by** (*rule conjunct2*)
      then have *c*: $Q\ x$ **by** (*rule allE*)
      from *b c* **have** $P\ x \wedge Q\ x$ **by** (*rule conjI*)
    }
    then have $(\forall\ x.\ P\ x \wedge Q\ x)$ **by** (*rule allI*)
  } **note** *lhs* = *this*
  {
    assume *a*: $(\forall\ x.\ P\ x \wedge Q\ x)$
    {
      fix x
      from *a* **have** $P\ x \wedge Q\ x$ **by** (*rule allE*)
      then have $P\ x$ **by** (*rule conjunct1*)
    }
    then have *b*: $\forall\ x.\ P\ x$ **by** (*rule allI*)
    {
      fix x
      from *a* **have** $P\ x \wedge Q\ x$ **by** (*rule allE*)
      then have $Q\ x$ **by** (*rule conjunct2*)
    }
    then have *c*: $\forall\ x.\ Q\ x$ **by** (*rule allI*)
    from *b c* **have** $(\forall\ x.\ P\ x) \wedge (\forall\ x.\ Q\ x)$ **by** (*rule conjI*)
  } **note** *rhs* = *this*
  from *lhs rhs* **show** *?thesis* **by** (*rule iffI*)
qed

0.2 Exercise 2

theorem *a*:
  shows $(\exists\ x.\ \forall\ y.\ P\ x\ y) \longrightarrow (\forall\ y.\ \exists\ x.\ P\ x\ y)$

```
proof −
  {
    assume a: ∃ x. ∀ y. P x y
    {
      fix y
      from a obtain x where 1: ∀ y. P x y by (rule exE)
      from this have P x y by (rule allE)
      from this have ∃ x. P x y by (rule exI)
    }
    from this have ∀ y. ∃ x. P x y by (rule allI)
  }
  from this show ?thesis by (rule impI)
qed

theorem b:
  shows ((∀ x. P x) ⟶ Q) ⟷ ((∃ x. P x) ⟶ Q)
  proof

oops

lemma c:
  shows ((∀ x. P x) ∨ (∀ x. Q x)) ⟷ (∀ x. (P x ∨ Q x))
proof −

oops

lemma d:
  shows ((∃ x. P x) ∨ (∃ x. Q x)) ⟷ (∃ x. (P x ∨ P x))

oops

lemma e:
  shows (∀ x. ∃ y. P x y) ⟶ (∃ y. ∀ x. P x y)

oops

lemma f:
  shows (¬(∀ x. P x)) ⟷ (∃ x. ¬ P x)
proof −
  {
    assume a: ¬(∀ x. P x)
    {
      assume b: ¬(∃ x. ¬ P x)
      {
        fix y
        {
          assume c: ¬P y
          from c have d: ∃ x. ¬ P x by (rule exI)
          from b d have False by (rule notE)
```

```
      }
      from this have e: P y by (rule ccontr)
    }
    from this have d: ∀ x. P x by (rule allI)
    from a d have False by (rule notE)
  }
  from this have ∃ x. ¬ P x by (rule ccontr)
}
note ltor = this
{
  assume 1: ∃ x. ¬ P x
  from 1 obtain a where 2: ¬P a by (rule exE)
  {
    assume 3: ∀ x. P x
    from 3 have 4: P a by (rule allE)
    from 2 4 have False by (rule notE)
  }
  from this have ¬(∀ x. P x) by (rule notI)
}
note rtol = this
from ltor rtol have (¬(∀ x. P x)) ⟷ (∃ x. ¬ P x) by (rule iffI)
thus ?thesis .
qed
```

0.3 Exercise 3

```
theorem 3a:
  shows (∃ x. ∀ y. P x y) ⟶ (∀ y. ∃ x. P x y)
proof −
  {
    assume a: ∃ x. ∀ y. P x y
    {
      fix y
      from a obtain x where 1: ∀ y. P x y by (rule exE)
      hence P x y by (rule allE)
      hence ∃ x. P x y by (rule exI)
    }
    hence ∀ y. ∃ x. P x y by (rule allI)
  }
  thus ?thesis by (rule impI)
qed

lemma 3f:
  shows (¬(∀ x. P x)) ⟷ (∃ x. ¬ P x)
proof
  assume a: (¬(∀ x. P x))
  show ∃ x. ¬ P x
  proof (rule ccontr)
    assume b: ¬(∃ x. ¬ P x)
```

```
    {
      fix y
      {
        assume ¬P y
        hence ∃ x. ¬ P x by (rule exI)
        with b have False by (rule notE)
      }
      hence P y by (rule ccontr)
    }
    hence ∀ x. P x by (rule allI)
    with a show False by (rule notE)
  qed
next
  assume 1: ∃ x. ¬ P x
  show ¬(∀ x. P x)
  proof (rule notI)
    from 1 obtain a where 2: ¬P a by (rule exE)
    assume ∀ x. P x
    hence P a by (rule allE)
    with 2 show False by (rule notE)
  qed
qed
```

0.4 Exercise 4

```
lemma allDeMorgan: ¬ (∀ x. P(x)) ⟹ (∃ x. ¬ (P(x))) by simp

lemma disjToImp:
  assumes ¬A ∨ B
  shows A ⟶ B
  proof −
  {
    assume 1: A
      {
        assume 3: ¬A
        {
          assume 2: ¬B
          from 3 1 have False by (rule notE)
        }
        from this have B by (rule ccontr)
      } note case1 = this
      {
        assume B
      } note case2 = this
      from assms case1 case2 have B by (rule disjE)
      }
  from this have A ⟶ B by (rule impI)
  thus ?thesis .
qed
```

lemma *ex4*:
  shows $\exists\ x.\ (D(x) \longrightarrow (\forall\ \ y.\ D(y)))$
proof *cases*
    assume *1*: $\forall\ \ y.\ D(y)$
    {
      assume $D(x)$
      from *1* **have** $(\forall\ \ y.\ D(y))$ **by** $-$
    }
    from *this* **have** *2*: $D(x) \longrightarrow (\forall\ \ y.\ D(y))$ **by** (*rule impI*)
    from *2* **show** *b*: $\exists\ x.\ (D(x) \longrightarrow (\forall\ \ y.\ D(y)))$ **by** (*rule exI*)
next
    assume *4*: $\neg\ (\forall\ \ y.\ D(y))$
    then have *5*: $\exists\ z.\ \neg\ (D(z))$ **by** (*rule allDeMorgan*)
    from *5* **obtain** z **where** *5*: $\neg D(z)$ **by** (*rule exE*)
    then have $\neg D(z) \lor (\forall\ \ y.\ D(y))$ **by** (*rule disjI1*)
    then have *6*: $D(z) \longrightarrow (\forall\ \ y.\ D(y))$ **by** (*rule disjToImp*)
    then show *c*: $\exists\ x.\ (D(x) \longrightarrow (\forall\ \ y.\ D(y)))$ **by** (*rule exI*)
qed

end