Signal, Image and Video Processing
https://doi.org/10.1007/s11760-018-1248-2

ORIGINAL PAPER

Minmax-concave total variation denoising

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Received: 15 September 2017 / Revised: 9 January 2018 / Accepted: 20 January 2018

Abstract

Total variation (TV) denoising is a commonly used method for recovering 1-D signal or 2-D image from additive white Gaussian noise observation. In this paper, we define the Moreau enhanced function of L_1 norm as $\Phi_\alpha(x)$ and introduce the minmax-concave TV (MCTV) in the form of $\Phi_\alpha(Dx)$, where D is the finite difference operator. We present that MCTV approaches $\|Dx\|_0$ if the non-convexity parameter α is chosen properly and apply it to denoising problem. MCTV can strongly induce the signal sparsity in gradient domain, and moreover, its form allows us to develop corresponding fast optimization algorithms. We also prove that although this regularization term is non-convex, the cost function can maintain convexity by specifying α in a proper range. Experimental results demonstrate the effectiveness of MCTV for both 1-D signal and 2-D image denoising.

Keywords Total variation · Signal denoising · Non-convex regularization · L_1 norm

1 Introduction

It is well known that a number of signals and images are not inherently sparse, but demonstrate sparse property in gradient domain. Therefore, total variation (TV) regularization is commonly utilized to promote the sparsity of such signals or images in gradient domain [18], and TV-based methods have received much attention in a variety of signal processing applications, e.g., signal denoising, deblurring, decomposition, reconstruction [2,4,11,12,23–25, 28].

Standard TV term is defined as the L_1 norm of the gradient. It exploits sparsity solely on local, the first-order derivative features. Standard TV has desirable properties such as convexity and the ability to preserve edges. In recent years several other TV forms have been proposed to improve the performance further. High-order TV takes the second-order derivatives into account [24]; non-local TV (NLTV) imposes non-uniform weight on a more global area centered on each pixel [11]; total generalized variation (TGV) of the second order balances between the first- and the second-order derivatives of a function [12,25]. All of the above TV adopt L_1 norm since it induces sparsity most effectively among convex penalties. Nevertheless, L_1 norm penalty tends to underestimate signal values and is not a very good proxy of the L_0 norm. Non-convex penalties can lead to more accurate estimation of the underlying signal [5,6,13,14,16,17,20–22].

Considering it is very difficult to find global minimizers for non-convex optimization problems, Selesnick defined Moreau enhanced TV (METV) in [20] and applied it to 1-D signal denoising problem. This TV form can maintain the convexity of the cost function by specifying the relevant parameter in a proper range. However, it is not practical to use METV in 2-D image denoising since its form is not suitable to develop a fast algorithm for solving corresponding optimization problem.

In this paper, we focus on the issue of TV denoising for 1-D signal and 2-D image. We give the definition of Moreau enhanced function of $f(x)$ and discuss the specific function $\Phi_\alpha(x)$: the Moreau enhanced function of $\|x\|_1$ in detail. By using minmax-concave (MC) penalty, $\Phi_\alpha(x)$ can also be expressed in a pointwise way. We demonstrate that the function $\Phi_\alpha(x)$ is a good proxy of L_0 norm, and introduce MCTV in the form of $\Phi_\alpha(Dx)$ where D is the finite difference operator. MCTV strongly induces signal sparsity in gradient domain. In addition, MCTV has a form allowing us to exploit fast algorithms such as alternating direction method of multipliers (ADMM) for solving 2-D denoising problem. We also prove that although MCTV penalty itself is non-convex, the

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Published online: 31 January 2018

cost function can maintain convexity by specifying the non-convexity parameter α properly. Experimental results on 1-D piecewise constant signals and 2-D synthetic block images demonstrate that our proposed method outperforms standard TV denoising method as well as METV and NLTV.

This paper proceeds as follows: Sect. 2 introduces Moreau envelope, Moreau enhanced function and MCTV. Section 3 gives details of the proposed method of 1-D and 2-D TV denoising problems and algorithms to solve them. In Sect. 4, a series of experimental results is given to demonstrate the effectiveness of MCTV. Finally, the conclusions are drawn in Sect. 5.

2 Moreau enhanced function and MCTV

In this section, we will first give the definition of Moreau envelope and define Moreau enhanced function. Second, we will introduce MC penalty and MCTV. Next, we will explain why MCTV has potential to enhance gradient sparsity further compared to standard and some other non-convex TV penalties. We will compare MCTV with METV and give the reason why our defined TV form fits fast algorithms like ADMM finally.

Definition 1 Let $\alpha \geq 0$, $S_\alpha : R^N \rightarrow R$ be defined as Moreau envelope of function f [15,20].

$$S_\alpha(x) = \min_v \left\{ \frac{\alpha}{2}\|x - v\|_2^2 + f(v) \right\}. \tag{1}$$

Definition 2 Let $\alpha \geq 0$, $M_\alpha : R^N \rightarrow R$ be defined as Moreau enhanced function of function f.

$$M_\alpha(x) = f(x) - S_\alpha(x). \tag{2}$$

For function $f(x) = \|x\|_1$, we define the Moreau enhanced function of it as $\Phi_\alpha(x) : R^N \rightarrow R$ with the following form:

$$\Phi_\alpha(x) = \|x\|_1 - \min_v \left\{ \frac{\alpha}{2}\|x - v\|_2^2 + \|v\|_1 \right\}. \tag{3}$$

$\Phi_\alpha(x)$ can also be expressed as

$$\Phi_\alpha(x) = \sum_{i=1}^{N} \varphi_{\mathrm{MC}\_\alpha}(x_i) \tag{4}$$

and

$$\varphi_{\mathrm{MC}\_\alpha}(x) = |x| - \min_v \left\{ \frac{\alpha}{2}(x - v)^2 + |v| \right\}, \tag{5}$$

where $x \in R$. As shown in [19], the solution of $\min_v\{\frac{\alpha}{2}(x - v)^2 + |v|\}$ is Huber function $H_\alpha(x)$:

$$H_\alpha(x) = \begin{cases} \frac{1}{2\alpha}x^2 & |x| \leq \alpha \\ |x| - \frac{\alpha}{2} & |x| > \alpha. \end{cases} \tag{6}$$

Then $\varphi_{\mathrm{MC}\_\alpha}(x)$ can be written as

$$\varphi_{\mathrm{MC}\_\alpha}(x) = \begin{cases} |x| - \frac{1}{2\alpha}x^2 & |x| \leq \alpha \\ \frac{\alpha}{2} & |x| > \alpha, \end{cases} \tag{7}$$

which is the MC penalty [27].

Next, we will compare $\Phi_\alpha(x)$ with L_0 norm, which is $\|x\|_0 = \sum_{i=1}^{N} g_0(x_i)$, where

$$g_0(x) = \begin{cases} 0 & x = 0 \\ 1 & \text{otherwise.} \end{cases} \tag{8}$$

As a convex proxy of $\|x\|_0$, L_1 norm is $\|x\|_1 = \sum_{i=1}^{N} g_1(x_i)$, where

$$g_1(x) = |x|. \tag{9}$$

In Fig. 1 we plot the curves of function $g_0(x)$, $g_1(x)$, $\varphi_{\exp\_\alpha}(x)$ ($\alpha = 1$) and $\varphi_{\mathrm{MC}\_\alpha}(x)$ ($\alpha = 2$). $\varphi_{\exp\_\alpha}(x)$ is defined as

$$\varphi_{\exp\_\alpha}(x) = \frac{1 - \mathrm{e}^{-\alpha|x|}}{\alpha}. \tag{10}$$

It was proposed by Lanza in [14] very recently and applied to 2-D image denoising. Compared with $g_1(x)$ and $\varphi_{\exp\_\alpha}(x)$, $\varphi_{\mathrm{MC}\_\alpha}(x)$ is closer to $g_0(x)$ by selecting proper value of non-convexity parameter α. Therefore, $\Phi_\alpha(x)$ is a good proxy of $\|x\|_0$.

In order to illustrate that $\Phi_\alpha(x)$ can promote sparsity more efficiently than L_1 norm, we added white Gaussian noise to a sparse signal in time domain and recovered it using L_1 norm and $\Phi_\alpha(x)$ as penalty, respectively. Figure 2 shows the original sparse signal, the noisy signal and the recovered signals, from which we can see that $\Phi_\alpha(x)$ works better than L_1 norm.

To promote sparsity in gradient domain, we replace x with its gradient Dx in Eq. (3) (D is the finite difference operator), which leads to our definition of MCTV below.

Definition 3 Let $\alpha \geq 0$, $\|x\|_{\mathrm{MCTV}}$ be defined as MCTV.

$$\|x\|_{\mathrm{MCTV}} = \|Dx\|_1 - \min_v \left\{ \frac{\alpha}{2}\|Dx - v\|_2^2 + \|v\|_1 \right\}. \tag{11}$$

If x is a vector, then $Dx_i = (x_{i+1} - x_i)$.
If x is an image, then $Dx_{i,j} = (x_{i+1,j} - x_{i,j}, x_{i,j+1} - x_{i,j})$.

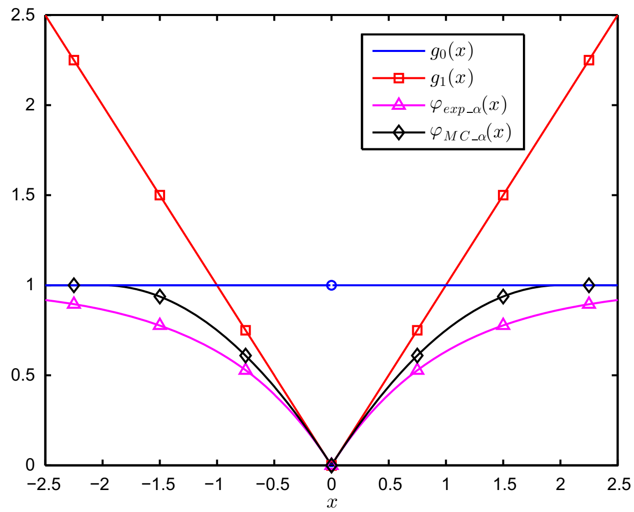

Fig. 1 Curves of function $g_0(x)$, $g_1(x)$, $\varphi_{\exp\_\alpha}(x)$ ($\alpha = 1$) and $\varphi_{MC\_\alpha}(x)$ ($\alpha = 2$)

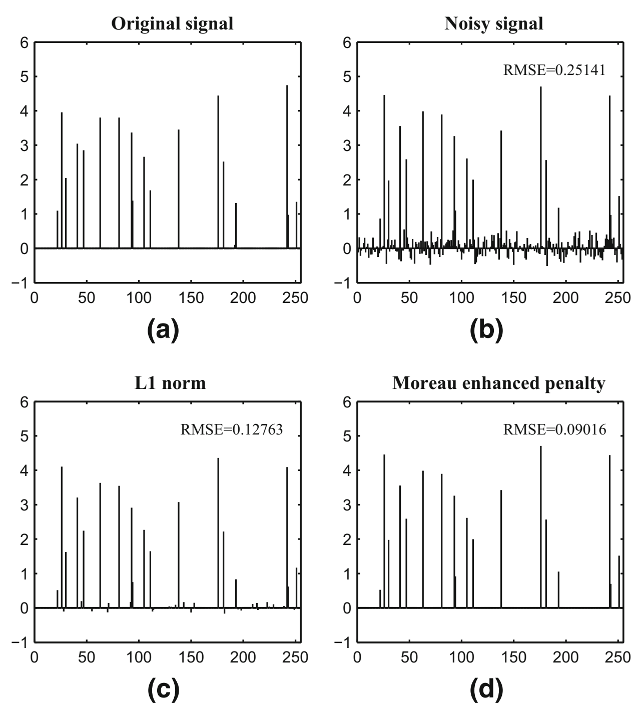

Fig. 2 **a** Original signal; **b** noisy signal ($\sigma = 0.25$); **c** denoised signal using L_1 norm; **d** denoised signal using Moreau enhanced penalty ($\Phi_\alpha(x)$)

The definition of METV in [20] is given below:

$$\|x\|_{\text{METV}} = \|Dx\|_1 - \min_v \left\{ \frac{\alpha}{2} \|x - v\|_2^2 + \|Dv\|_1 \right\}. \quad (12)$$

We can see that v is the approximation to Dx in Eq. (11), while the approximation to x in Eq. (12). Though these two definitions look similar, 2-D MCTV minimization problems can be efficiently solved by ADMM via replacing z with Dx as shown in Sect. 3.3, while it is impossible to do so in solving METV denoising problem. So far, METV is only used in 1-D signal denoising. Moreover, experimental results in 1-D TV denoising show that MCTV outperforms METV with regard to root-mean-square error (RMSE) in Sect. 4.

3 Signal denoising with MCTV regularization

In this section, a MCTV denoising model will be proposed in Sect. 3.1, and the fast algorithms for 1-D and 2-D MCTV denoising will be developed in Sects. 3.2 and 3.3, respectively.

3.1 Proposed model

In many cases, the data acquisition of a signal can be simply modeled as

$$y = x + w, \quad (13)$$

where x is the desired signal and usually demonstrates sparsity in some domain; w is the disturbance or noise; and y is the measured signal.

It is very common to recover the signal x by minimizing the following function with standard TV penalty.

$$\widehat{x} = \arg\min_x \left\{ \frac{1}{2} \|y - x\|_2^2 + \lambda \|x\|_{\text{TV}} \right\}, \quad (14)$$

where $\|x\|_{TV} = \|Dx\|_1$.

Since MCTV promotes the gradient sparsity more effectively than standard TV, here we propose to recover x by using MCTV as a regularization term.

$$\widehat{x} = \arg\min_x \left\{ \frac{1}{2} \|y - x\|_2^2 + \lambda \|x\|_{\text{MCTV}} \right\}. \quad (15)$$

In accordance with the definition of $\|x\|_{\text{MCTV}}$, the cost function of Eq. (15) has the following form:

$$\begin{aligned} G_{\text{MCTV}\_\alpha}(x) &= \frac{1}{2} \|y - x\|_2^2 \\ &\quad + \lambda \left\{ \|Dx\|_1 - \min_v \left\{ \frac{\alpha}{2} \|Dx - v\|_2^2 + \|v\|_1 \right\} \right\}. \end{aligned} \quad (16)$$

Although the MCTV penalty is non-convex, as we present in Theorem 1, the cost function $G_{\text{MCTV}\_\alpha}(x)$ maintains convexity by specifying $0 \le \alpha < \frac{1}{4\lambda}$.

Theorem 1 *Let $\alpha \ge 0, \lambda > 0$. If $0 \le \alpha < \frac{1}{4\lambda}$, then $G_{\text{MCTV}\_\alpha}(x)$ defined by Eq. (16) is convex.*

Proof

$$\begin{aligned}
&G_{\mathrm{MCTV_}\alpha}(x)\\
&=\frac{1}{2}\|y-x\|_2^2\\
&\quad+\lambda\left\{\|Dx\|_1-\min_v\left\{\frac{\alpha}{2}\|Dx-v\|_2^2+\|v\|_1\right\}\right\}\\
&=\max_v\left\{\frac{1}{2}\|y-x\|_2^2+\lambda\|Dx\|_1-\frac{\lambda\alpha}{2}\|Dx-v\|_2^2-\lambda\|v\|_1\right\}\\
&=\max_v\left\{\frac{1}{2}x^T(I-\lambda\alpha D^TD)x+\lambda\|Dx\|_1+g(x,v)\right\}\\
&=\frac{1}{2}x^T(I-\lambda\alpha D^TD)x+\lambda\|Dx\|_1+\max_v g(x,v),
\end{aligned} \tag{17}$$

where the last term $g(x, v)$ is affine in x. $g(x, v)$ is convex as it is the pointwise maximum of a set of convex functions. The cost function $G_{\mathrm{MCTV\_}\alpha}(x)$ is a convex function if matrix $(I-\lambda\alpha D^TD)$ is positive definite.

For $0 \le \alpha < \frac{1}{4\lambda}$, the matrix $(I-\lambda\alpha D^TD)$ is positive definite. This is proved in Theorem 2. □

Theorem 2 *Let $\alpha \ge 0, \lambda > 0$. If $0 \le \alpha < \frac{1}{4\lambda}$, then matrix $A = (I-\lambda\alpha D^TD)$ is positive definite, where D is the $(N-1)\times N$ matrix*

$$\begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}.$$

Proof We rewrite matrix A as

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \lambda\alpha \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & & \\ & & & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda\alpha & \lambda\alpha & & & \\ \lambda\alpha & 1-2\lambda\alpha & \lambda\alpha & & \\ & \lambda\alpha & 1-2\lambda\alpha & \lambda\alpha & \\ & & \ddots & & \\ & & & \lambda\alpha & 1-\lambda\alpha \end{bmatrix}.$$

Let a_{ij} denote the entry of the matrix A; if $a_{ii} > \sum_{j\ne i}|a_{ij}|$, $i = 1, 2, ..., N$, then A is positive definite according to Theorem 3 and Theorem 4.

For the first and the last rows, we solve

$$a_{ii} > \sum_{j\ne i}|a_{ij}|\ (i = 1, N)$$

or

$$1-\lambda\alpha > \lambda\alpha;$$

then,

$$\alpha < \frac{1}{2\lambda}. \tag{18}$$

For the rest of the rows,

$$a_{ii} > \sum_{j\ne i}|a_{ij}|\ (i = 2, ..., N-1)$$

or

$$1-2\lambda\alpha > 2\lambda\alpha;$$

then,

$$\alpha < \frac{1}{4\lambda}. \tag{19}$$

Combining (18) and (19), we have

$$\alpha < \frac{1}{4\lambda}. \tag{20}$$

□

Theorem 3 *Let A be a real $N \times N$ symmetric matrix; A is positive definite if and only if all its eigenvalues are positive.*

Theorem 4 *For non-singular matrix $A = (a_{ij})_{N\times N} \in R^{N\times N}$, if $a_{ii} > \sum_{j\ne i}|a_{ij}|$, $i = 1, 2, ..., N$, then the real eigenvalues of A are positive.*

3.2 1-D TV denoising

In order to find the minimizer of $G_{\mathrm{MCTV\_}\alpha}(x)$, we use iterative forward–backward splitting (FBS) algorithm from [7]. FBS minimizes a function with the form

$$F(x) = f_1(x) + f_2(x) \tag{21}$$

with the condition of both f_1 and f_2 being convex and ∇f_1 being Lipschitz continuous. FBS iteration step consists of the following two equations:

$$z^k = x^k - \mu\nabla f_1\left(x^k\right) \tag{22}$$

$$x^{k+1} = \arg\min_x\left\{\frac{1}{2}\left\|z^k-x\right\|_2^2 + \mu f_2(x)\right\}. \tag{23}$$

The x^{k+1} generated by Eq. (23) converges to a minimizer of $F(x)$. In order to apply FBS, we set

$$f_1(x) = \frac{1}{2}\|y - x\|_2^2 - \lambda \min_{v} \left\{ \frac{\alpha}{2}\|Dx - v\|_2^2 + \|v\|_1 \right\} \tag{24}$$

$$f_2(x) = \lambda \|Dx\|_1. \tag{25}$$

$G_{\mathrm{MCTV}\_\alpha}(x)$ is convex as long as the parameter is set as $0 \le \alpha < \frac{1}{4\lambda}$. It is easy to prove

$$\nabla f_1(x) = x - y - \lambda\alpha \left(D^T Dx - D^T l_1 \left(Dx; \frac{1}{\alpha} \right) \right), \tag{26}$$

where

$$l_1(y; \lambda) = \arg\min_{x} \left\{ \frac{1}{2}\|y - x\|_2^2 + \lambda \|x\|_1 \right\} \tag{27}$$

which can be efficiently solved by iterative shrinkage threshold algorithm (ISTA) [1,9]. Setting μ=1, the 1-D TV denoising algorithm is given in Algorithm 1. Line 5 is standard TV denoising minimization, and we calculated it using the fast algorithm proposed by Condat in [8].

Algorithm 1

Input y
1. **Initialize** $x^0 = y$
2. **Initialize** $\lambda > 0, 0 \le \alpha < \frac{1}{4\lambda}$
3. **Do**
4. $z^k = y + \lambda\alpha(D^T Dx^k - D^T l_1(Dx^k; \frac{1}{\alpha}))$
5. $x^{k+1} = \arg\min_x \{\frac{1}{2}\|z^k - x\|_2^2 + \lambda\|Dx\|_1\}$
6. **Until convergence**

Output x

3.3 2-D TV denoising

For image denoising, x is a 2-D image in $G_{\mathrm{MCTV}\_\alpha}(x)$. We set $z = Dx$; then, the augmented Lagrangian form of (16) is given by

$$\begin{aligned} L_p(x, z, u) = &\frac{1}{2}\|y - x\|_2^2 + \lambda \Phi_\alpha(z) \\ &- \lambda u^T (z - Dx) + \frac{\lambda\rho}{2}\|z - Dx\|_2^2 \end{aligned} \tag{28}$$

where $\Phi_\alpha(z)$ is defined in Eq. (3). According to ADMM [3], the minimizer of function (28) can be obtained by following iteration steps.

Step 1 Update x^{k+1} with z^k and u^k fixed:

$$\begin{aligned} x^{k+1} = \arg\min_{x} \Big\{ &\frac{1}{2}\|y - x\|_2^2 + \lambda (u^k)^T Dx \\ &+ \frac{\lambda\rho}{2}\|z^k - Dx\|_2^2 \Big\}. \end{aligned} \tag{29}$$

The optimal solution is given by the equation below:

$$\left(I + \lambda\rho D^T D\right)x^{k+1} = y + \lambda\rho D^T z^k - \lambda D^T u^k. \tag{30}$$

Since $D^T Dx = 4x_{i,j} - x_{i,j-1} - x_{i,j+1} - x_{i-1,j} - x_{i+1,j}$, using the Gauss–Seidel method applied in [10], if $x_{i,j}^{k+1} = G_{i,j}^k$, Eq. (30) can be equivalently written as

$$\begin{aligned} G_{i,j}^k = &\frac{\lambda\rho}{1 + 4\lambda\rho} \Big(x_{i,j-1}^k + x_{i,j+1}^k + x_{i-1,j}^k + x_{i+1,j}^k \\ &+ 2z_{i,j}^k - z_{i,j+1}^k - z_{i+1,j}^k \Big) \\ &- \frac{\lambda}{1 + 4\lambda\rho} \left(2u_{i,j}^k - u_{i,j+1}^k - u_{i+1,j}^k \right) \\ &+ \frac{1}{1 + 4\lambda\rho} y_{i,j}. \end{aligned} \tag{31}$$

Step 2 Update z^{k+1} with x^{k+1} and u^k fixed:

$$\begin{aligned} z^{k+1} &= \arg\min_{z} \left\{ \lambda\Phi_\alpha(z) - \lambda(u^k)^T z + \frac{\lambda\rho}{2}\|z - Dx^{k+1}\|_2^2 \right\} \\ &= \arg\min_{z} \left\{ \Phi_\alpha(z) + \frac{\rho}{2}\left\|z - \left(Dx^{k+1} + \frac{u^k}{\rho} \right)\right\|_2^2 \right\}. \end{aligned} \tag{32}$$

Parameter α controls the convexity of the cost function as shown in Theorem 5.

Theorem 5 *Let $\alpha \ge 0, \lambda > 0$; define $G_{L1\_\alpha}(x)$ as*

$$G_{L1\_\alpha}(x) = \frac{1}{2}\|y - x\|_2^2 + \lambda\Phi_\alpha(x). \tag{33}$$

If $0 \le \alpha < \frac{1}{\lambda}$, then $G_{L1\_\alpha}(x)$ is convex.

Proof

$$\begin{aligned} G_{L1\_\alpha}(x) &= \frac{1}{2}\|y - x\|_2^2 \\ &\quad + \lambda \left\{ \|x\|_1 - \min_{v} \left\{ \frac{\alpha}{2}\|x - v\|_2^2 + \|v\|_1 \right\} \right\} \\ &= \max_{v} \left\{ \frac{1}{2}\|y - x\|_2^2 + \lambda\|x\|_1 - \frac{\alpha\lambda}{2}\|x - v\|_2^2 - \lambda\|v\|_1 \right\} \\ &= \frac{1}{2}(1 - \lambda\alpha)\|x\|_2^2 + \lambda\|x\|_1 + \max_{v} g(x, v). \end{aligned} \tag{34}$$

$g(x, v)$ is affine function of x, and it is convex since it is the pointwise maximum of a set of convex functions. Therefore, $G_{L1\_\alpha}(x)$ is a convex function if $1-\lambda\alpha > 0$, or $0 \le \alpha < \frac{1}{\lambda}$.

Based on Theorem 5, setting the parameter $0 \le \alpha < \rho$, z^{k+1} generated by FBS in Eq. (36) converges to the solution of optimization problem (32). We write the iteration procedure below:

Do

$$t^k = Dx^{k+1} + \frac{1}{\rho}u^k + \frac{\alpha}{\rho}\left(z^k - l_1\left(z^k; \frac{1}{\alpha}\right)\right) \tag{35}$$

$$z^{k+1} = l_1\left(t^k; \frac{1}{\rho}\right) \tag{36}$$

Until convergence.

Step 3 Update u^{k+1} with x^{k+1} and z^{k+1} fixed:

$$u^{k+1} = u^k + (Dx^{k+1} - z^{k+1}). \tag{37}$$

A brief summary of the proposed 2-D TV denoising algorithm is presented in Algorithm 2.

Algorithm 2

Input y
1. **Initialize** $x^0 = y, z^0 = 0, u^0 = 0$
2. **Initialize** $K, \lambda > 0, \rho > 0$ and $0 \le \alpha < \rho$
3. **For** $k = 0 : K$
4. $x^{k+1} = G^k$
5. **Do**
6. $t^k = Dx^{k+1} + \frac{u^k}{\rho} + \frac{\alpha}{\rho}(z^k - l_1(z^k; \frac{1}{\alpha}))$
7. $z^{k+1} = l_1(t^k; \frac{1}{\rho})$
8. **Until convergence**
9. $u^{k+1} = u^k + (Dx^{k+1} - z^{k+1})$
10. **End for**

Output x

The standard TV denoising problem in (14) can also be solved efficiently with ADMM [26]. The whole process is similar to our proposed algorithm, with a difference that instead of calculating line 5 to line 8, standard TV solves the following problem:

$$z^{k+1} = \arg\min_z \left\{ \frac{\rho}{2}\|z - \left(Dx^{k+1} + \frac{u^k}{\rho}\right)\|_2^2 + \|z\|_1 \right\}. \tag{38}$$

□

4 Experimental results

In this section, we will present experimental results of the proposed method and then compare them with standard TV, METV and NLTV results. All of our experiments were implemented on MATLAB R2014a on a PC equipped with a 1.7 GHz CPU and 8 GB RAM. The quality and accuracy of denoised signals were evaluated by RMSE, and those of denoised images were by the peak signal-to-noise ratio (PSNR) and the relative error (Err).

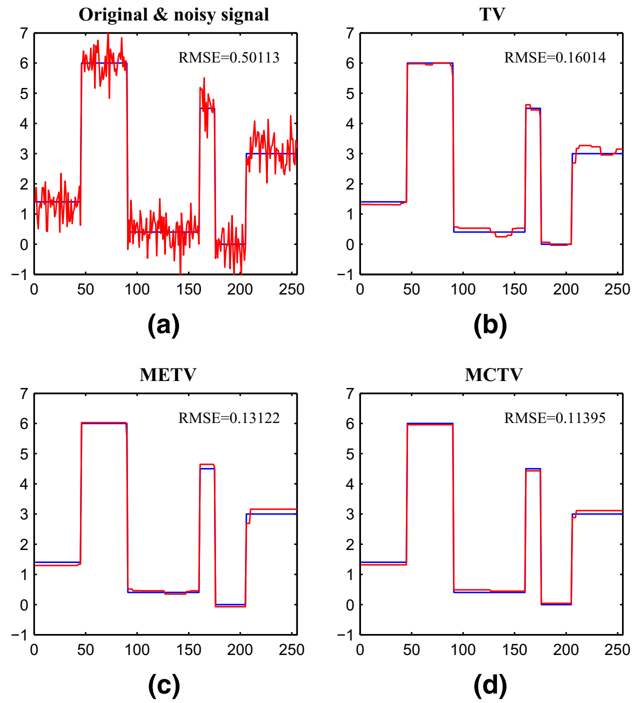

Fig. 3 **a** Original signal (blue) and noisy signal (red) with $\sigma = 0.5$; **b** denoised signal using standard TV penalty (red); **c** denoised signal using METV penalty (red); **d** denoised signal using MCTV penalty (red) (color figure online)

First we applied the proposed denoising method to the 1-D piecewise constant signal (length $N = 256$) with additive white Gaussian noise ($\sigma = 0.5$) shown in Fig. 3a. Standard TV, METV and MCTV were utilized, and the denoised results are presented in Fig. 3b–d, respectively. For each TV penalty, we set the regularization parameter to $\lambda = 0.25\sqrt{N}\sigma = 2$; the value of α for METV was chosen as $\alpha = 0.7/\lambda = 0.35$ according to [20]; and for MCTV, we set $\alpha = 0.25/\lambda = 0.125$. The results show that standard TV underestimates the amplitudes of jump discontinuities, while MCTV estimates jump discontinuities most accurately, especially those occurring near other jump discontinuities of opposite sign.

Then we changed the magnitude of noise and plot the curves of average RMSE versus the noise standard variation for three different TV penalties in Fig. 4 to further compare them. For a large range of σ ($0.1 \le \sigma \le 0.5$), the proposed MCTV denoising method yielded the lowest average RMSE. Both Figs. 3 and 4 illus-

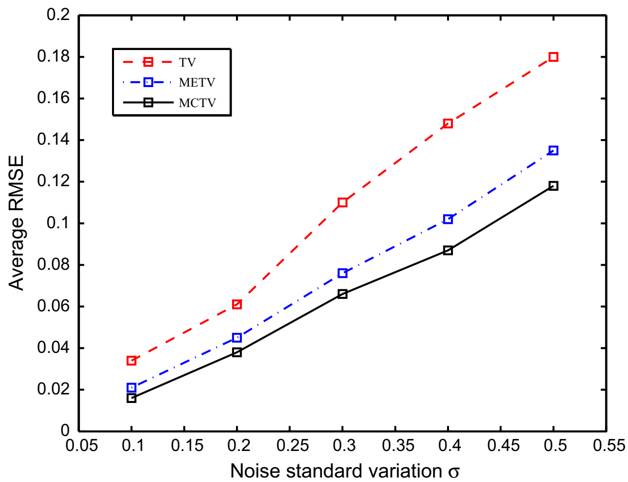

Fig. 4 Average RMSE versus the noise standard variation for different TV penalties

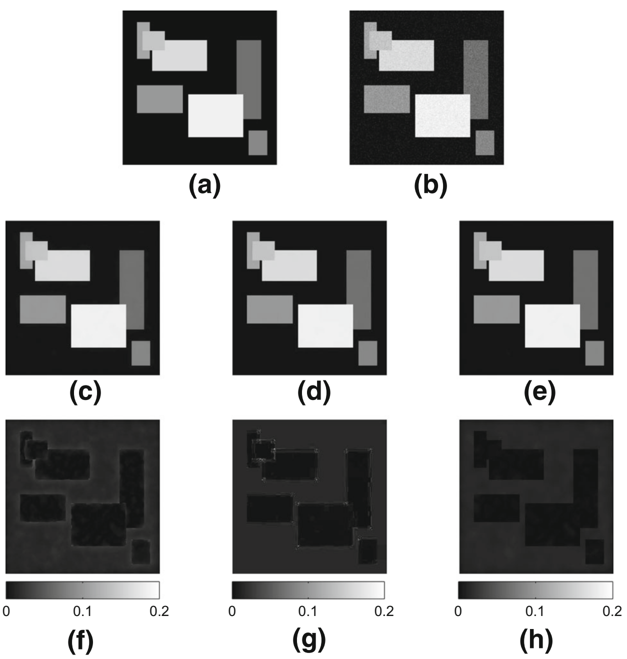

Fig. 5 **a** Original image; **b** noisy image; **c** denoised image using standard TV penalty; **d** denoised image using NLTV penalty; **e** denoised image using MCTV penalty; **f** difference between (**a**) and (**c**); **g** difference between (**a**) and (**d**); **h** difference between (**a**) and (**e**)

trate the effectiveness of MCTV in 1-D signal denoising.

After testing the MCTV denoising in 1-D piecewise constant signals, we extended its application to 2-D images. A 256×256 synthetic block image was generated with sparsity in gradient domain shown in Fig. 5a. A white Gaussian noise is added in Fig. 5b, the PSNR of which was 29.97 dB. Standard TV, NLTV and MCTV were applied to recover the desired image, and the results are shown in Fig. 5c–e. The parameters of NLTV were set as suggested by [4,28]; the relevant parameters of the other two TV were set as $\lambda = 0.05$, $\rho = 100$ (TV), and $\lambda = 0.1$, $\rho = 50$ and $\alpha = 0.5\rho = 2.5$ (MCTV). The Err of them was 3.45%, 3.37% and 2.94%, and the PSNR values were 37.06, 37.91 and 38.47 dB, respectively. In Fig. 5f–h, difference images were shown to further compare the capability of three TV penalties. Comparing Fig. 5h with f and g, we observed that MCTV can further denoise the image and smooth almost all parts of the blocks in image better than standard TV as well as NLTV. All of these results demonstrate that MCTV is a proper regularization term for gradient-sparse image denoising with better performance.

5 Conclusion

In this paper, we give the general definition of Moreau enhanced function, introduce MCTV and propose a new denoising method. MCTV has a form which fits fast algorithm FBS and ADMM and can maintain the convexity of the cost function in each iteration. Through a series of experimental results, we demonstrate that the sparse-encouraging property of MCTV outperforms standard TV in both 1-D and 2-D denoising, and it performs even better than METV in smoothing 1-D piecewise constant signals and NLTV in recovering 2-D synthetic block images.

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