Home Assignment №1

Due on February 20, 2026

Instructions. Show all essential steps. For programming components, you may use MATLAB or Python. Submit a single PDF report and include your code (as an appendix). Unless stated otherwise, assume image intensities are in $[0, 255]$.

Exercise 1

An image f is characterized by the probability density function (pdf) shown in Fig. 1. (The numbers inside the shaded boxes specify their respective areas. Assume the pdf is piecewise-constant within each box.)

a) Design a 1-bit *uniform* scalar quantizer for this image on the support shown in Fig. 1. Specify the decision threshold(s) and representation levels.

b) Design a 1-bit Max–Lloyd quantizer for the same pdf, using your uniform quantizer from part (a) to initialize the algorithm. Perform *one* full iteration (update representations, then update decision threshold). Report the updated decision threshold and representation levels.

c) Compute the expected mean-squared quantization error (MSE) for (a) and for the one-iteration Max–Lloyd quantizer in (b). Which one is smaller, and why does that make conceptual sense?

d) (Short answer) Under what condition(s) on a pdf would the 1-bit uniform quantizer in (a) coincide with an optimal 1-bit Max–Lloyd quantizer?

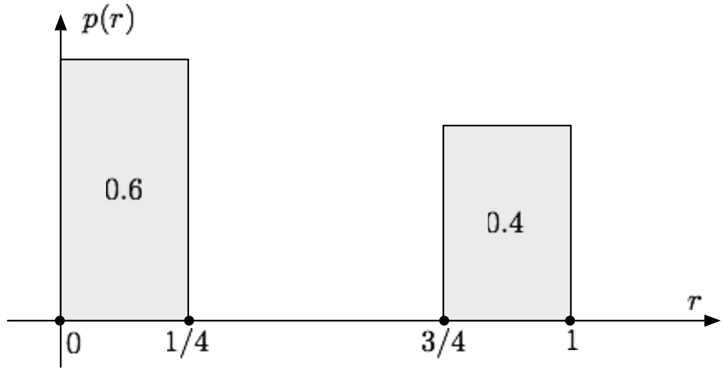

Figure 1: Pertaining to Exercise 1.

Exercise 2

Let f be an image with intensities in $\{0, 1, \ldots, 255\}$. Let $p_f(k)$ denote its normalized histogram (pmf), and

$$F_f(k) = \sum_{i=0}^{k} p_f(i)$$

its cumulative distribution function (cdf). Assume intensity transformations are implemented via a monotone lookup table (LUT)

$$g(x) = T(f(x)),$$

where $T : \{0, \ldots, 255\} \to \{0, \ldots, 255\}$ is non-decreasing.

(a) Equalization as a random-variable transformation

In the continuous case, suppose intensities are modelled as a random variable U with pdf $p_U(u)$ and cdf $F_U(u)$.

i) Using the change-of-variables formula, prove that if

$$V = F_U(U),$$

then V is uniformly distributed on $[0, 1]$.

ii) Explain why in the discrete (8-bit) case histogram equalization cannot, in general, produce a perfectly uniform histogram.

iii) Let $T_{\text{eq}}(k) = 255\,F_f(k)$. Is T_{eq} strictly increasing? Under what condition(s) would it fail to be strictly increasing?

(b) Invertibility and information loss

i) State a necessary and sufficient condition on the LUT T for the mapping $f \mapsto g$ to be invertible.

ii) Is histogram equalization invertible in practice? Explain carefully.

iii) Consider applying histogram equalization to an image f to obtain f_{eq}, and then applying histogram equalization again to f_{eq}. Will the second equalization change the image? Explain.

iv) (Small experiment) Apply histogram equalization to a test image (e.g., `cameraman`). Then construct an approximate inverse mapping using the empirical cdf of the equalized image and apply it. Compare the recovered image with the original. Report PSNR/SSIM and briefly comment on what you observe.

(c) Designing a brightness-constrained transform

Suppose you want to brighten an image while satisfying:

- T is monotone increasing,
- $T(0) = 0$, $T(255) = 255$,
- the average intensity increases by at least 15%.

Recall that the average image intensity can be written as

$$\mathbb{E}[f] = \sum_{k=0}^{255} k\, p_f(k).$$

i) Propose a parametric family of transforms T_θ (e.g., power-law, logarithmic, or piecewise linear) that satisfies the boundary constraints.

ii) Express the new mean intensity as

$$\mathbb{E}[T_\theta(f)] = \sum_{k=0}^{255} T_\theta(k)\, p_f(k).$$

Explain clearly why this formula holds.

iii) Derive a condition on θ that guarantees

$$\mathbb{E}[T_\theta(f)] \geq 1.15\,\mathbb{E}[f].$$

iv) Explain qualitatively which intensity ranges must expand in order for the mean to increase, and which ranges must compress.

Include your MATLAB/Python code for the computational components.

Exercise 3

We consider the linear model

$$\bar{g} = \mathbf{H}\bar{f} + \bar{\xi},$$

where f is an $N \times M$ image, $\bar{f} = \mathrm{vec}(f) \in \mathbb{R}^K$, $K = NM$, and $\mathbf{H}$ represents 2D convolution with a kernel h. The noise $\bar{\xi}$ is zero-mean white Gaussian.

In this exercise you will recover f using optimization methods, without ever forming the matrix $\mathbf{H}$ explicitly.

Data generation

- Load the image `cameraman`.
- Generate a blur kernel using `fspecial('gaussian',[7 9],1.5)`.
- Form the blurred image using zero-padded convolution using function `conv2`:

  $$g = h * f,$$

  and retain the output of size $(N+6)\times(M+8)$ (i.e., "full" convolution).
- Add zero-mean Gaussian noise with standard deviation $\sigma = 5$.

The resulting image g is your observation.

Part A: Convolution as a linear operator and its adjoint

a) Suppose $\mathbf{H}$ represents full convolution with kernel h. Show that the adjoint operator $\mathbf{H}^T$ corresponds to convolution with a flipped version of h. Explicitly state what transformation is applied to h.

b) Explain why, when $\mathbf{H}$ corresponds to "full" convolution, $\mathbf{H}^T$ corresponds to "valid" convolution.

c) Verify numerically that for arbitrary images f and u,

$$\langle \mathbf{H}\bar{f}, \bar{u} \rangle = \langle \bar{f}, \mathbf{H}^T \bar{u} \rangle.$$

(Use inner products of vectorized images.)

Part B: Tikhonov regularization

Consider

$$\min_{\bar{f}} \left\{ \frac{1}{2} \|\mathbf{H}\bar{f} - \bar{g}\|_2^2 + \frac{\lambda}{2} \|\bar{f}\|_2^2 \right\}.$$

where $\|\cdot\|_2$ stands for the standard Euclidean norm.

a) Compute the gradient of the objective. Your final expression should involve $\mathbf{H}^T$.

b) Write the gradient descent update rule.

c) Implement gradient descent using only `conv2` (with appropriate boundary handling). Do not form matrices.

d) Experiment with at least five values of $\lambda \in [10^{-4}, 10^{-1}]$. For each case report PSNR/SSIM and display the reconstruction.

e) Comment on the effect of λ. Relate your observations to the singular values of $\mathbf{H}$.

Part C: Total Variation (TV) regularization

Consider

$$\min_{\bar{f}} \left\{ \frac{1}{2} \|\mathbf{H}\bar{f} - \bar{g}\|_2^2 + \lambda \|f\|_{\mathrm{TV}} \right\}.$$

Use the discrete isotropic TV

$$\|f\|_{\mathrm{TV}} = \sum_{i,j} \sqrt{(D_x f_{i,j})^2 + (D_y f_{i,j})^2 + \epsilon^2},$$

where D_x and D_y are forward differences and $\epsilon = 10^{-3}$ for numerical stability.

a) Derive the gradient of the smoothed TV functional. Express your answer using discrete divergence operators.

b) Implement gradient descent using only finite differences and `conv2`.

c) Compare the results with Tikhonov regularization for comparable PSNR/SSIM levels.

d) Comment on:

- edge preservation,
- ringing artifacts,
- oversmoothing,
- convergence behavior.

Part D: Boundary effects

Repeat the recovery experiment using circular convolution. Compare the reconstructions near image borders. Explain the differences using the interpretation of $\mathbf{H}^T$.

Include all MATLAB/Python code.