STATS 330

Handout 11
A closer look at confidence intervals and prediction intervals

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What is a confidence interval?

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  - NO!

What is a confidence interval?

The “confidence” we have in a confidence interval is from the way confidence intervals perform under repetition of the experiment.

If the experiment is repeated a large number of times, and a 95% confidence interval for β_0 *is calculated each time, then about 95% of those intervals will contain* β_0*. That is, we expect the coverage of those intervals to be about 95%.*

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  - Does this sound familiar?
  - It's precisely what we were doing in the previous handout!

What is a confidence interval?

So, we are "confident" in the confidence interval $(5.18, 46.74)$ in the sense that this type of interval contains β_0 19 times out of 20 under repetition of the experiment, on average.

That is, the confidence we have in a confidence interval is due to the properties of the method that was used to produce the interval.

The interval $(5.18, 46.74)$ may be a "good" interval (i.e., contain β_0), or it could be a "bad" interval (i.e., not contain β_0). We might never know which.

Intervals for μ and new values of Y

In the remainder of this handout we will look at confidence intervals for μ_i and prediction intervals for Y_i. We will continue working with the simple linear model from the previous handout:

$$\mu_i = \beta_0 + \beta_1 x_i,$$
$$Y_i \sim \text{Normal}(\mu_i, \sigma^2),$$

where we have ten observations for which

- $x_1 = 2, x_2 = 4, \ldots, x_{10} = 20,$
- $\beta_0 = 20,$
- $\beta_1 = 3$, and
- $\sigma = 10.$

Intervals for μ and new values of Y

This means the expected value for each observation is

$$\begin{aligned}
\mu_1 &= 20 + 3 \times 2 = 26, \\
\mu_2 &= 20 + 3 \times 4 = 32, \\
&\vdots \\
\mu_{10} &= 20 + 3 \times 20 = 80.
\end{aligned}$$

So the first observation comes from a normal distribution with a mean of 26 and a standard deviation of 10, the second observation comes from a normal distribution with a mean of 32 and a standard deviation of 10, and so on.

We'll further our simulation study to investigate the performance of confidence intervals for μ and prediction intervals for new values of Y when $x = 7$. For $x = 7$, $\mu = 41$.

Simulation study

Confidence intervals for μ

The simulation code requires only minor modification from the previous handout. We now need to use `predict(...,interval="conf")` to get the confidence intervals.

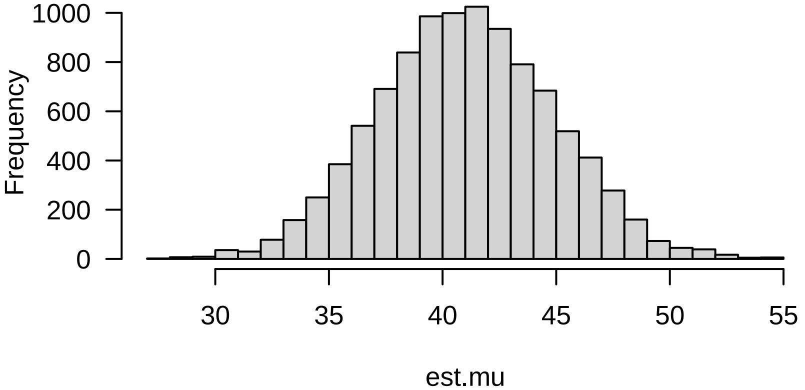
```
...
mu=b0+b1*7
## Creating vector in which to store estimates.
est.mu <- numeric(n.sims)
## Creating matrix in which to store CIs.
ci.mu <- matrix(0, nrow = n.sims, ncol = 2)
## The for loop.
for (i in 1:n.sims) {
  ## Simulating the responses.
  y <- rnorm(10, means, sigma)
  ## Fitting the model.
  fit <- lm(y ~ x)
  ## Extracting muhat and CI.
  pred <- predict(fit, newdata=data.frame(x=7), interval="conf")
  est.mu[i] <- pred[1]
  ci.mu[i, ] <- pred[2:3]
}
```

Simulation study

Estimator performance: Unbiasedness

Here is a histogram of the estimates of $\mu = 41$. Does our estimator appear to be more or less unbiased?

```r
hist(est.mu)
```

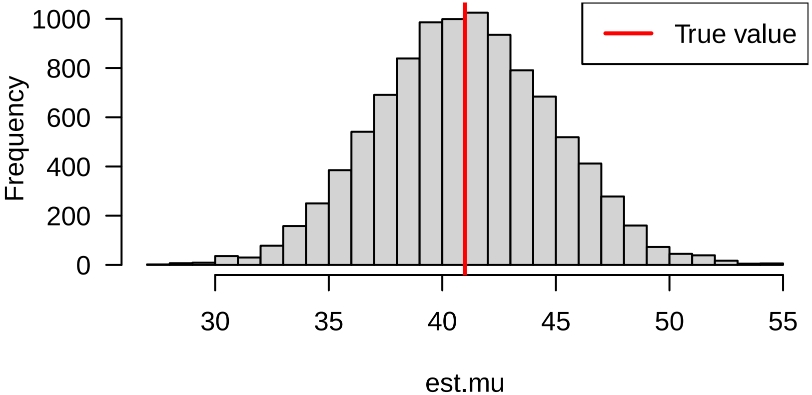


Simulation study

Estimator performance: Unbiasedness

Here is a histogram of the estimates of $\mu = 41$. Does our estimator appear to be more or less unbiased? Yes!

```
hist(est.mu); abline(v = mu, col = "blue")
```



Simulation study

Confidence interval coverage

We can compute if we captured the true value:

```
mu.ci.captured <- (mu >= ci.mu[, 1]) & (mu <= ci.mu[, 2])
table(mu.ci.captured)

## mu.ci.captured
## FALSE  TRUE
##   491  9509

mean(mu.ci.captured)

## [1] 0.9509
```

Out of the 10 000 confidence intervals computed for μ, 9509 of them contained the true value $\mu = 41$.

So our method for calculating 95% confidence intervals works for μ, because approximately 95% of intervals calculated using this method contain the true value.

Prediction intervals for new values of Y

Now, let's examine the performance of prediction intervals.

For 95% prediction intervals, we need the intervals to contain the new value of Y 95% of the time (on average) when both the experiment and the new value of Y are replicated a large number of times.

Simulation study

Prediction intervals for new values of Y

Our simulation code now needs to be modified to use `predict(...,interval="pred")`, and to generate a new value of Y each iteration.

```
...
mu=b0+b1*7
## Creating vector in which to store new Y.
new.y <- numeric(n.sims)
## Creating matrix in which to store PIs.
pi.y <- matrix(0, nrow = n.sims, ncol = 2)
## The for loop.
for (i in 1:n.sims) {
  ## Simulating the responses.
  y <- rnorm(10, means, sigma)
  new.y[i] <- rnorm(1, mu, sigma)
  ## Fitting the model.
  fit <- lm(y ~ x)
  ## Extracting muhat and CI.
  pred <- predict(fit, newdata=data.frame(x=7), interval="pred")
  pi.y[i,] <- pred[2:3]
}
```

Simulation study

Prediction interval coverage

We can compute if we captured the value of Y:

```
y.pi.captured <- (new.y >= pi.y[, 1]) & (new.y <= pi.y[, 2])
table(y.pi.captured)

## y.pi.captured
## FALSE  TRUE
##   495  9505

mean(y.pi.captured)

## [1] 0.9505
```

Out of the 10 000 prediction intervals, 9505 of them contained the new value of Y.

So our method for calculating 95% prediction intervals works because approximately 95% of the intervals calculated using this method contain the new value.