1 Two cards are drawn from a standard deck without replacement. Let X be the number of Kings drawn.

Note: A standard deck contains a total of **52 cards** including **4 kings**.

(a) Determine the probability mass function of X, $f_X(x)$.

Solution: To determine the probability mass function of X, $f_X(x)$:

There are three possible outcomes for X; $X = 0$ (no kings are drawn), $X = 1$ (one of the cards is a king) $X = 2$ (both cards are kings).

$P(X = 0)$: Probability that the first card is not a king $= \frac{48}{52}$ Probability that the second card is also not a king given the first was not $= \frac{47}{51}$ Thus,

$$P(X = 0) = \frac{48}{52} \times \frac{47}{51}$$

$P(X = 1)$: There are two scenarios here - the king is either the first or the second card drawn.

Probability that first card is a king and second is not:

$$\frac{4}{52} \times \frac{48}{51}$$

Probability that first card is not a king but second is:

$$\frac{48}{52} \times \frac{4}{51}$$

Thus,

$$P(X = 1) = \frac{4}{52} \times \frac{48}{51} + \frac{48}{52} \times \frac{4}{51}$$

$P(X = 2)$: Probability that both cards are kings:

$$\frac{4}{52} \times \frac{3}{51}$$

Thus,

$$P(X = 2) = \frac{4}{52} \times \frac{3}{51}$$

Hence, $f_X(x)$ is:

$$f_X(x) = \begin{cases} \frac{48}{52} \times \frac{47}{51} \approx 0.85 & \text{if } x = 0 \\ \frac{4}{52} \times \frac{48}{51} + \frac{48}{52} \times \frac{4}{51} \approx 0.14 & \text{if } x = 1 \\ \frac{4}{52} \times \frac{3}{51} \approx 0.005 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$

(Note that while the decimal approximations come up deficient, the fractions do in fact sum to exactly 1.) [4 marks]

(b) Compute the expected number of Kings drawn, $\mathbb{E}[X]$.

Solution: To compute the expected number of Kings drawn, $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \sum x \times f_X(x)$$

$$\mathbb{E}[X] = 0 \times P(X = 0) + 1 \times P(X = 1) + 2 \times P(X = 2)$$

Substituting in the values we found in part (a):

$$\mathbb{E}[X] = 0\times\left(\frac{48}{52} \times \frac{47}{51}\right)+1\times\left(\frac{4}{52} \times \frac{48}{51} + \frac{48}{52} \times \frac{4}{51}\right)+2\times\left(\frac{4}{52} \times \frac{3}{51}\right)$$

Simplifying, we get:

$$\mathbb{E}[X] \approx 0 \times 0.85 + 1 \times 0.14 + 2 \times 0.005$$

$$\mathbb{E}[X] \approx 0.15$$

So, on average, when drawing two cards from a standard deck without replacement, we expect to draw approximately 0.15 kings.
[2 marks]

(c) Let K_i be the event that the ith card drawn is a king. Find the probability that $X = 2$ given that the first card drawn is a king. That is; find $\mathbb{P}(X = 2|K_1)$.

Solution: Given that the first card drawn is a king, there are now 3 kings left in a deck of 51 cards. The probability that the second card drawn is also a king is:

$$\mathbb{P}(X = 2|K_1) = \frac{3}{51}$$

So, if we know that the first card drawn is a king, the probability that the second card is also a king (and hence $X = 2$) is $\frac{3}{51}$ or approximately 0.0588. [2 marks]

2 Consider two discrete random variables X and Y. X can take on values in the set $\{1, 2, 3\}$ with probabilities

$$P(X = i) = p_i, \text{ for } i \in \{1, 2, 3\}$$

such that

$$p_1 + p_2 + p_3 = 1.$$

Consider a situation when we make repeated observations of X. Let Y be random variable defined as the number of repeated observations of X we need to do *before* an observation of $X = 3$ occurs.

(a) Consider one observation of X. Define the event $A = \{X > 1\}$, find the probability $P(A)$.

Solution: To find $P(A)$:

$$P(A) = P(X = 2) + P(X = 3)$$

$$P(A) = p_2 + p_3$$

[2 marks]

(b) Based on the given probabilities for X, determine the distribution of Y. *Hint:* This is one of the named distributions in the course.

Solution: Given the definition of Y, the number of repeated observations before observing $X = 3$, Y follows a geometric distribution. The distribution describes the number of Bernoulli trials needed for a success to occur. In this case, observing $X = 3$ is our "success". Hence, $Y \sim \text{Geometric}(p_3)$. [2 marks]

(c) Calculate the expected value of Y, $\mathbb{E}[Y]$.

Solution: For a geometric distribution with success probability p, the expected value is given by:

$$\mathbb{E}[Y] = \frac{1-p}{p}$$

Given that $Y \sim \text{Geometric}(p_3)$:

$$\mathbb{E}[Y] = \frac{1-p_3}{p_3}.$$

[2 marks]

(d) We keep on drawing from X repeatedly, to produce several observations of Y. Let Y_i represent the number of trials needed to observe $X = 3$ for the i^{th} time. Now, let

$$Z = Y_1 + Y_2 + \cdots + Y_5.$$

Determine $E(Z)$, the expected total number of trials to observe $X = 3$ five times.

Solution: Each Y_i follows a geometric distribution with parameter p_3. Hence, $E[Y_i] = \frac{1}{p_3}$ for each i.
Using the linearity of expectation, we get:

$$E[Z] = E[Y_1] + E[Y_2] + \cdots + E[Y_5]$$

$$E[Z] = 5 \times \frac{1 - p_3}{p_3}$$

This could also be solved by thinking of Z as a Negative Binomial random variable, which leads to the same conclusion. [3 marks]

3 Three friends, passionate about Doctor Who (a famous British sci-fi TV show), have decided to jointly collect Doctor Who figurines sold *individually* in opaque packaging.

They've acquired all the figurines for their collection, except for the super-rare *K-9* of which they have none. In order for each of them to get a full set, they need three K-9 figurines in total.

Each package costs $c = \$5$, each package contains at most one K-9, and the probability of a package containing a K-9 is $p = \frac{1}{100}$.

(a) Determine the expected number of packages they need to purchase jointly to find three "K-9" figurines.

Solution: The problem relates to the negative binomial distribution, which models the number of failures before a set number k of successes. Here, the "success" is finding a K-9 figurine, and we're interested in the number of failures before finding three K-9 figurines.

Thus the expected value of Y is:

$$\mathbb{E}[Y] = \frac{k(1-p)}{p} + k$$

Where k is the number of successes (here, 3 K-9 figurines) and p is the probability of success on a single trial (here $\frac{1}{100}$).

Substituting in the given values:

$$\mathbb{E}[Y] = \frac{3(1-\frac{1}{100})}{\frac{1}{100}} + 3 = 297 + 3 = 300$$

Thus, they are expected to purchase 300 packages jointly to find three "K-9" figurines. [2 marks]

(b) Use some value or values from Table 1 to calculate the probability that they will need to buy more than $n = 240$ packages to find three "K-9" figurines.

Solution: The probability that the friends will need to buy more than 240 packages to find three "K-9" figurines can be calculated as:

$$P(Y > 237) = 1 - P(Y \leq 237)$$

Using the provided table, $P(Y \leq 237)$ is given as $F_Y(237) = 0.4308$. Thus:

$$P(Y > 237) = 1 - 0.4308 = 0.5692$$

Therefore, there's a 56.92% chance that they will need to buy more than 240 packages to find three "K-9" figurines. [3 marks]

n =	235	236	237	238	239	240	241
$f_X(n)$	0.00152	0.00152	0.00151	0.00151	0.00150	0.00150	0.00149
$f_Y(n)$	0.00264	0.00263	0.00263	0.00262	0.00262	0.00261	0.00261
$F_X(n)$	0.5452	0.5468	0.5483	0.5498	0.5513	0.5528	0.5543
$F_Y(n)$	0.4256	0.4282	0.4308	0.4334	0.4361	0.4387	0.4413

Table 1: pdf and cdf for $X \sim$ Geometric($\frac{1}{300}$) and $Y \sim$ NegativeBinomial(3, $\frac{1}{100}$) for selected values of x

(c) Let C be the total amount of money that the 3 friends need to spend in order to find the final 3 figurines. Calculate the value of C.

Solution: The expected number of non-K-9 packages the friends need to purchase before finding 3 K-9 figurines is:

$$E[Y] = 3 \times \frac{99}{1} = 297$$

Including the 3 K-9 packages they aim to purchase, the total expected number of packages is 300.
Given that each package costs \$5:

$$C = 300 \times \$5 = \$1500$$

Thus, the friends are expected to spend \$1500 to find the three "K-9" figurines. [2 marks]

(d) An online store is selling K-9 figurines in perfect condition for \$400 each. Use the results from part (b) and (c) to explain why the friends should or should not buy the figurines this way instead.

Solution: Financially, buying the K-9 figurines directly from the online store seems to be the better choice. From part (b), there's over a 50% chance they would spend at least \$1200 (240 packages at \$5 each) trying to find three K-9 figurines through random purchasing. Moreover, as deduced in part (c), their expected expenditure is \$1500, which is equivalent to buying all three K-9 figurines from the online store at \$400 each and still saving \$300.

However, there's an intangible value in the thrill and experience of finding the rare K-9 figurine themselves. If the thrill of the hunt is worth the potential extra cost, they might choose to continue buying the packages. Otherwise, from a purely financial perspective, buying directly from the online store is the wiser choice.

[1 marks]