1.(a)
$$\begin{aligned}\mathbb{P}(X=x) &= \mathbb{P}(X=x\,|\,G)\mathbb{P}(G)+\mathbb{P}(X=x\,|\,S)\mathbb{P}(S) \quad \text{because } G \text{ and } S \text{ form a partition of } \Omega \\ &= \binom{10}{x}(0.5)^x(0.5)^{10-x}(1-s)+\binom{10}{x}(0.75)^x(0.25)^{10-x}s \quad \text{as required.}\end{aligned}$$

(b)
$$\begin{aligned}\mathbb{P}(X=4) &= (1-s)\binom{10}{4}(0.5)^4(0.5)^{10-4}+s\binom{10}{4}(0.75)^4(0.25)^{10-4} \\ &= (1-s)\times 0.205+s\times 0.016 \\ &= 0.21-0.19s. \qquad \text{(2 d.p.)}\end{aligned}$$

(c) Using the information given, the likelihood is:

$$L(s\,;4,8)=(0.21-0.19s)(0.04+0.24s) \qquad \text{for } 0<s<1.$$

(d)
$$\frac{dL}{ds}=-0.19(0.04+0.24s)+0.24(0.21-0.19s).$$

Solving $\frac{dL}{ds}=0$:

$$\begin{aligned}-0.19(0.04+0.24s)+0.24(0.21-0.19s) &= 0 \\ -0.19\times 0.04+0.24\times 0.21-2\times 0.19\times 0.24s &= 0 \\ s &= \frac{-0.19\times 0.04+0.24\times 0.21}{2\times 0.19\times 0.24} \\ s &= 0.47. \qquad \text{(2 d.p.)}\end{aligned}$$

The MLE of s is therefore

$$\hat{s} = 0.47.$$

(e) The log-likelihood is:

$$\log\Big(L(s\,;4,8)\Big)=\log(0.21-0.19s)+\log(0.04+0.24s) \qquad \text{for } 0<s<1.$$

So

$$\frac{d\log(L)}{ds}=\frac{-0.19}{0.21-0.19s}+\frac{0.24}{0.04+0.24s}\,.$$

Solving $\frac{d\log(L)}{ds} = 0$:

$$\begin{aligned}
\frac{-0.19}{0.21 - 0.19s} + \frac{0.24}{0.04 + 0.24s} &= 0 \\
\Rightarrow \qquad \frac{0.24}{0.04 + 0.24s} &= \frac{0.19}{0.21 - 0.19s} \\
\text{cross-multiply:} \qquad 0.24(0.21 - 0.19s) &= 0.19(0.04 + 0.24s)
\end{aligned}$$

rearrange:

$$\begin{aligned}
0.24 \times 0.21 - 0.24 \times 0.19s - 0.19 \times 0.04 - 0.19 \times 0.24s &= 0 \\
(0.24 \times 0.21 - 0.19 \times 0.04) - s(0.24 \times 0.19 \times 2) &= 0 \\
s &= \frac{0.24 \times 0.21 - 0.19 \times 0.04}{0.24 \times 0.19 \times 2} \\
s &= 0.47. \qquad \text{(2 d.p.)}
\end{aligned}$$

This is the same value of s as obtained in part (c): maximizing the log-likelihood is equivalent to maximizing the likelihood.

2.(a)

$$\# \text{ Swots} = N \times s = 100 \times 0.6 = 60.$$
$$\# \text{ Guessers} = N(1 - s) = 100 \times 0.4 = 40.$$

(b) R commands:

Swots: `rbinom(60, 10, 0.75)`

Guessers: `rbinom(40, 10, 0.5)`

3.(a) MLEs: 0.645 0.598 0.619 0.576

These estimates are all close to the true value of $s = 0.6$. It seems therefore that the MLE method is giving sensible answers.

(b) The dashed line shows the peak of the curve. The dashed line shows the position of the maximum of the log-likelihood, which is the same as the maximum likelihood estimate.

(c) MLEs for $N = 10,000$:

0.599 0.597 0.597 0.604

MLEs for $N = 10$:

0.322 0.630 0.814 0.374

For the large sample size ($N = 10,000$), the MLE is very close to 0.6 and does not vary very much.

For the small sample size ($N = 10$), the MLE is highly variable: it is not clear that it focuses on the correct value 0.6 because it varies so much.
