STATS 210 **Extra Examples 2**

1. Flashinda Ardent and Bill Clingish are running for President of the Matchstick Corp. As usual, their election campaign turns into a bitter fight. Their biggest argument is about the statistical properties of the opinion polls. Clingish's campaign team contracts Herald Dodgypoll to ask a random sample of 1000 voters whether they plan to vote for Clingish. 410 respondents (41%) say yes. Ardent's team employs another statistical company, Callmy Bluffton, to find another 1000 voters and ask whether they plan to vote for Ardent. 370 respondents (37%) say yes.

   We will perform two hypothesis tests to analyse the results of these polls. Clingish's test will be used as an example. Ardent's test is for you to perform in the tutorial.

   In each case, we wish to **test the hypothesis that the true level of support for the candidate amongst all voters is 40%.** Assume for all the following working that the Matchstick Corp is very very large.

   We use the following formulation:

   (i) For the first poll, X is the number of respondents who say they will vote for Clingish.
   - $X \sim \text{Binomial}(1000, p_C)$ where p_C is the unknown true proportion of Clingish-voters in the Matchstick Corp.
   - *sample answers for the first poll are written in this font at the end of the tutorial.*

   (ii) For the second poll, X is the number of respondents who say they will vote for Ardent.
   - $X \sim \text{Binomial}(1000, p_A)$ where p_A is the unknown true proportion of Ardent-voters in the Matchstick Corp.
   - answers are for you to complete in the tutorial.

(a) Formulate the null hypothesis and alternative hypothesis, in terms of the distribution of X and its parameters. Remember to specify the full distribution of X under the null hypothesis. Use a two-sided test.

(b) Sketch as a curve the shape of the probability function of X under the **null hypothesis.** Your sketch should have axes labelled x and $\mathbb{P}(X = x)$. Mark on the sketch the upper and lower limits of x, (not to scale), and the value of x where the curve peaks.

(c) Mark the observed value of X on your sketch, so that you can see the tail probabilities needed for the p-value. Shade the area under the curve corresponding to the p-value.

(d) Write down the R command required to find the p-value for the hypothesis test.

(e) Running the command in part (d) in R gives `0.05608583`. Interpret this result in terms of strength of evidence against the null hypothesis. You should say, (i) whether we have any evidence against H_0, (ii) whether we have *proof* against H_0, and (iii) whether or not it is true that the observed polling for Ardent is *compatible* with H_0 — in other words, *could it happen* that the support for Ardent really is 40% and we nonetheless see an observed polling of 370/1000?

Now compare your conclusion from part (e) for Ardent with the sample answer for Clingish. Although the example above is fictitious, it is based on real political polls: in NZ we often see the two main candidates or parties polling close to 40% and within about 4 points of each other, and the polls are usually taken over 1000 people.

(f) If an opinion poll has National polling at 41% and Labour polling at 37%, does this mean that National is definitely ahead of Labour? Or could the same result emerge if National and Labour had equal support in the population, or even if Labour were ahead of National? ***Hint: when we say 'polling at 41%', we mean 41% is the sample proportion.***

Note: doing it properly

Our analysis does give us insight into the results of opinion polls. However, in practice we would tend to use a slightly different test, because it is usually a *single* sample of people who are asked to choose between candidates, not two separate polls as in the Ardent and Clingish example. Further, if we did need to analyse two separate polls, we would usually use a two-sample test to test whether the two population proportions could be equal, rather than using two separate hypothesis tests to test separately whether each polling is compatible with a population proportion of 40% support.

---

2. Winsome Peters is also running for President of the Matchstick Corp. His current polling is 80 out of 1000 respondents (8%). If he doesn't get at least 5% of the vote, he will lose his deposit. He wishes to test the evidence against his worst-case-scenario of 5% support, given that 80 out of 1000 respondents said that they would vote for him.

(a) Define an appropriate random variable X, as in Question 1. Formulate the null hypothesis and alternative hypothesis, in terms of the distribution of X and its parameters. Remember to specify the full distribution of X under the null hypothesis. Use a two-sided test.

(b) Sketch as a curve the shape of the probability function of X under the **null hypothesis.** Your sketch should have axes labelled x and $\mathbb{P}(X = x)$. Mark on the sketch the upper and lower limits of x, (not to scale), and the value of x where the curve peaks.

(c) Mark the observed value of X on your sketch, so that you can see the tail probabilities needed for the p-value. Shade the area under the curve corresponding to the p-value.

(d) Write down the R command required to find the p-value for the hypothesis test.

(e) Running the command in part (d) in R gives `6.97729e-05`. Interpret this result in terms of strength of evidence against the null hypothesis. Is the observed polling for Winsome compatible with the hypothesis that the true level of support for Winsome is 5%?

(f) Compare your result for question 2(e) with your result for question 1(e). In each case, we were testing a hypothesis that the true p was a shift of 3 percentage points away from the observed polling: testing 5% support given 8% polling in Q2; and testing 40% support given 37% polling in Q1. Are the conclusions the same? Can you suggest what difference between the two cases could have caused this difference in the results?

3. President James Abram Garfield (1831-1881) was an American president with lots of children. He had 5 sons and 2 daughters. He also gave his name to the cartoon character Garfield, although the resemblence is not entirely convincing.

   Suppose that each of President Garfield's seven children were boys with probability p and girls with probability $1 - p$, all independently of each other. Define the random variable X to be the number of boys out of Garfield's seven children.

   (a) Write down the distribution of X, in terms of the unknown parameter p.

   (b) Write down x, the observed value of X, from the information above.

   (c) Write down the likelihood function, $L(p\,; x)$, substituting the correct value of x. Remember to state the range of values of p for which your expression applies. In your answer, replace any expression of the form $\binom{a}{b}$ (i.e. ${}^aC_b$) with its actual value.

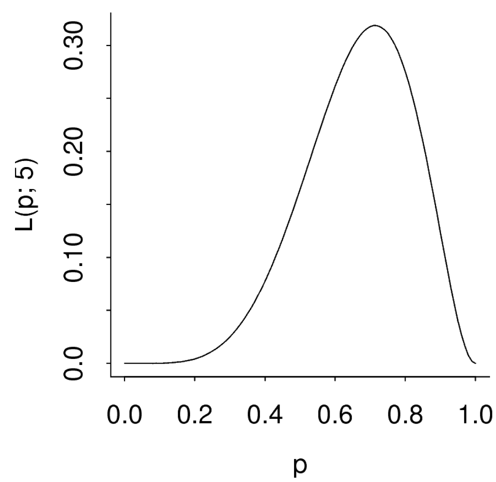
   (d) Find $L\left(\frac{1}{2}\,; 5\right)$.

   (e) Find $L\left(\frac{5}{7}\,; 5\right)$.

   (f) Find $L\,(0.8\,; 5)$.

   (g) The graph of $L(p\,; 5)$ for $0 < p < 1$ is here:

   Redraw the graph as a sketch, and mark on it the answers from parts (d), (e), and (f).



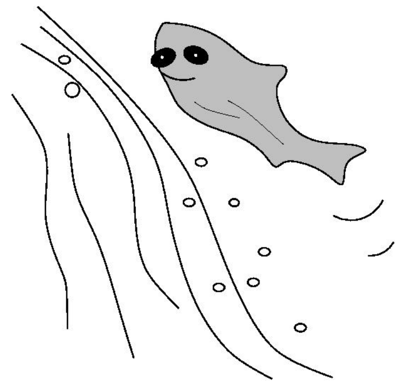
   (h) Show that

$$\frac{dL}{dp} = 21p^4(1-p)(5-7p).$$

   (i) The sketch in part (g) shows that the value of p that maximizes $L(p\,; 5)$ is *not* $p = 0$ or $p = 1$. Use the expression in (h) to find the value of p that *does* maximize the likelihood.

   (j) Write a sentence in English to explain what the maximum likelihood estimate from (i) represents. [Hint: see page 63 of the course book - the comment in blue below "Maximizing the likelihood".]

4. Have you ever seen a salmon leap? Salmon leaps are waterfalls where you can see salmon repeatedly leaping out of the water in their attempts to jump to the top of the waterfall, so they can continue upstream to breed.



Sammie the Salmon is trying to jump up a waterfall. Each time he jumps, he has probability p of making it to the top. All Sammie's jumps are independent. Let X be the total number of jumps that Sammie makes until he has finally made it to the top.

The probability function of X is:

$$\mathbb{P}(X = x) = (1 - p)^{x-1} p,$$

because a total of x jumps requires $x - 1$ failures to begin with (probability $(1 - p)^{x-1}$), followed by a single success (probability p).

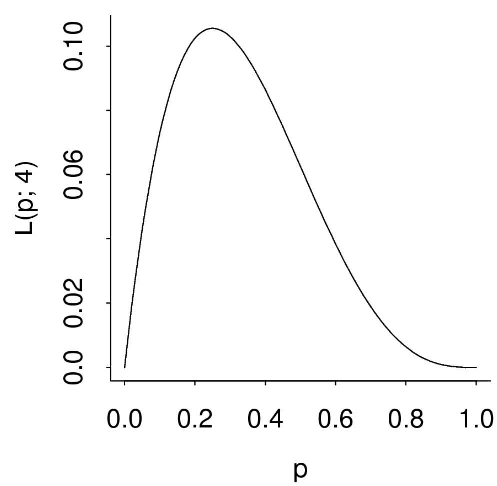
Suppose that Sammie manages to jump to the top of the waterfall after $X = 4$ jumps.

(a) Write down the likelihood function, $L(p\,; 4)$. State the range of values of p for which the function is defined.

(b) Show that

$$\frac{dL}{dp} = (1 - p)^2(1 - 4p).$$

(c) The likelihood function is plotted below. Using the expression in (b), find the maximum likelihood estimate of p.



(d) Explain why the maximum likelihood estimate is also a common-sense estimate of p, remembering that Sammie first succeeded on his fourth jump.

Sample answers for the Clingish poll in Question 1

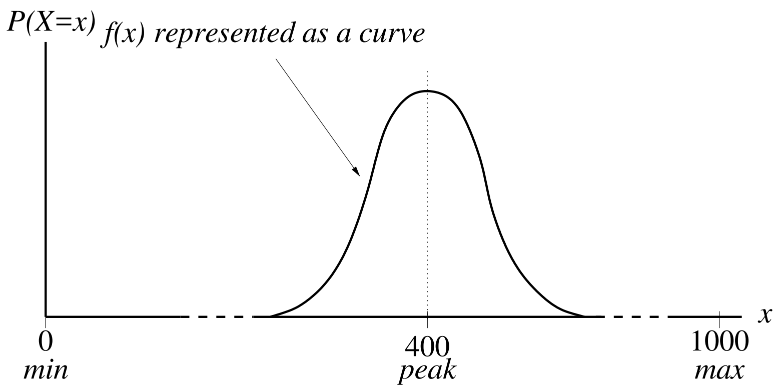
(a) *Let X be the number of Clingish-voters out of 1000 respondents.*

$$
\begin{aligned}
X &\sim \textit{Binomial}(1000,\ p_C) \\
H_0 &: p_C = 0.4. \\
H_1 &: p_C \neq 0.4. \qquad \textit{(two-sided test)}
\end{aligned}
$$

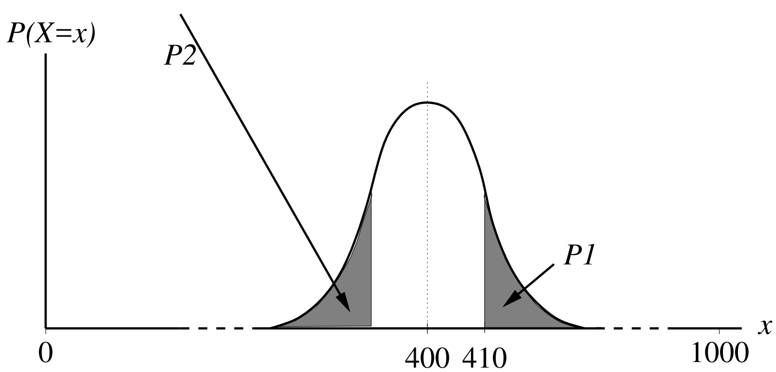
(b) *Under H_0, we have $X \sim$ Binomial$(1000, 0.4)$.*

The probability function of X peaks at about $1000 \times 0.4 = 400$.

Sketch:



(c)



(d) *For the p-value,*

$$
\begin{aligned}
\mathbb{P}(X \geq 410) &= 1 - \mathbb{P}(X < 410) \\
&= 1 - \mathbb{P}(X \leq 409) \\
&= 1 - F_X(409).
\end{aligned}
$$

Thus the R command for the p-value is: `2 * (1 - pbinom(409, 1000, 0.4))`

(e) *For the Clingish test, running the command in (d) in R gives the result* `0.5388626`.

Interpretation: over 53% of the time when the true value of p is 0.4, the polling of 1000 people will give an answer as extreme as $x = 410$. This p-value is large and therefore we have no evidence against the null hypothesis that $p = 0.4$. The observed polling is compatible with the possibility that the true support for Clingish is 40%.

(Note that this does not imply that p really is 0.4; it just tells us that the observation is **compatible** *with this possibility.)*