1.(a) Yes: $X \sim \text{Binomial}(n = 500, p = 0.14)$.

(b) Yes: $X \sim \text{Binomial}(n = 8, p = 0.7)$.

(c) No: not a fixed number of trials (attempts until the first success).

(d) Yes: $X \sim \text{Binomial}(n = 12, p = 0.02)$.

(e) No: not a fixed number of trials (coffees) before she runs out of sugar.

2.(a) $X \sim \text{Binomial}(n = 10,\ p = 0.2)$.

(b)

$$p_2 = \mathbb{P}(X = 2) = \binom{10}{2}(0.2)^2(0.8)^8 = 0.30 \text{ to 2 d.p.}$$

$$p_3 = \mathbb{P}(X = 3) = \binom{10}{3}(0.2)^3(0.8)^7 = 0.20 \text{ to 2 d.p.}$$

(c) $c_0 = \mathbb{P}(X \leq 0) = \mathbb{P}(X = 0) = 0.11$.
$c_4 = \mathbb{P}(X \leq 4) = \mathbb{P}(X \leq 3) + \mathbb{P}(X = 4) = 0.88 + 0.09 = 0.97$.
$c_5 = \mathbb{P}(X \leq 5) = \mathbb{P}(X \leq 4) + \mathbb{P}(X = 5) = 0.97 + 0.02 = 0.99$.

(d) Looking for $\mathbb{P}(X \geq 6)$.
This is $\mathbb{P}(X \geq 6) = 1 - \mathbb{P}(X \leq 5) = 1 - F_X(5)$.
So:

$$\mathbb{P}(\text{get at least 6 out of 10 by guessing}) = 1 - F_X(5) = 1 - c_5 = 1 - 0.99 = 0.01.$$

(e) A sum of independent Binomial random variables, sharing the same value of p, is also Binomial. So $T \sim \text{Binomial}(40, 0.2)$.

(f)

$$\mathbb{P}(T = 10) = \binom{40}{10}(0.2)^{10}(0.8)^{30} = 0.107 \text{ to 3 d.p.}$$

(g) We need:

$$\mathbb{P}(T \geq 20) = 1 - \mathbb{P}(T \leq 19) = 1 - F_T(19).$$

Now

$$\begin{aligned} F_T(19) &= F_T(18) + \mathbb{P}(T = 19) \\ &= 0.9999148 + \binom{40}{19}(0.2)^{19}(0.8)^{40-19} \\ &= 0.9999148 + 0.0000635 \\ &= 0.9999783. \end{aligned}$$

So

$$\mathbb{P}(T \geq 20) = 1 - \mathbb{P}(T \leq 19) = 1 - 0.9999783 = 0.0000217.$$

The probability that Tom passes the multi-choice component just by guessing is about 2 in 100 thousand. No, I wouldn't recommend this strategy. Tom would be better off doing a bit of study.

3.(a) As X takes only values that are greater or equal to one, it is clear that $F_X(x) = \mathbb{P}(X \leq x) = 0$ when $x \in (-\infty, 1)$. For the case when $x \in [1, 9]$, we have:

$$\begin{aligned}
F_X(x) &= \mathbb{P}(X \leq x) \\
&= \mathbb{P}(X \leq \lfloor x \rfloor) \quad [X \text{ takes only integer values}] \\
&= \mathbb{P}(X = 1) + \mathbb{P}(X = 2) + \cdots + \mathbb{P}(X = \lfloor x \rfloor) \\
&= \log_{10}\left(\frac{2}{1}\right) + \log_{10}\left(\frac{3}{2}\right) + \cdots + \log_{10}\left(\frac{\lfloor x \rfloor + 1}{\lfloor x \rfloor}\right) \\
&= \log_{10}\left(\frac{2}{1}\frac{3}{2}\cdots\frac{\lfloor x \rfloor + 1}{\lfloor x \rfloor}\right) \quad [\text{properties } \log_{10}(\cdot)] \\
&= \log_{10}\left(\frac{\lfloor x \rfloor + 1}{1}\right) \\
&= \log_{10}\left(\lfloor x \rfloor + 1\right).
\end{aligned}$$

For $x > 9$, we get: $F_X(x) = \mathbb{P}(X \leq x) = \mathbb{P}(X \leq 9) = F_X(9) = 1$ (see the equations above).

(b) The following calculations are straightforward:

$$\begin{aligned}
\mathbb{P}(X + Y = 10) &= \sum_{x=1}^{9} \mathbb{P}(X = x, Y = 10 - x) \\
&= \sum_{x=1}^{9} \mathbb{P}(X = x)\mathbb{P}(Y = 10 - x) \; [X, Y \text{ are independent}] \\
&= \frac{1}{9}\sum_{x=1}^{9} \mathbb{P}(X = x) \; [\text{use pmf of } Y] \\
&= \frac{1}{9} \; [\text{use pmf of } X]
\end{aligned}$$

4.(a) Observe that Z can take only two values: 0 and 1. We have:

$$\begin{aligned}
\mathbb{P}(Z = 0) &= \mathbb{P}(X = 0, Y = 0) + \mathbb{P}(X = 1, Y = 1) \; [\text{definition } \oplus] \\
&= \mathbb{P}(X = 0)\mathbb{P}(Y = 0) + \mathbb{P}(X = 1)\mathbb{P}(Y = 1) \; [X, Y \text{ are independent}] \\
&= \left(1 - \frac{1}{10}\right)\frac{1}{2} + \frac{1}{10}\frac{1}{2} \; [\text{use pmf's of } X, Y] \\
&= \frac{1}{2}.
\end{aligned}$$

It is clear that $\mathbb{P}(Z = 1) = 1 - \mathbb{P}(Z = 0) = 1 - \frac{1}{2} = \frac{1}{2}.$

For completeness, we also present the calculations for $\mathbb{P}(Z = 1)$:

$$\begin{aligned}
\mathbb{P}(Z = 1) &= \mathbb{P}(X = 0, Y = 1) + \mathbb{P}(X = 1, Y = 0) \text{ [definition } \oplus\text{]} \\
&= \mathbb{P}(X = 0)\mathbb{P}(Y = 1) + \mathbb{P}(X = 1)\mathbb{P}(Y = 0) \text{ [}X, Y \text{ are independent]} \\
&= \left(1 - \frac{1}{10}\right)\frac{1}{2} + \frac{1}{10}\left(1 - \frac{1}{2}\right) \text{ [use pmf's of } X, Y\text{]} \\
&= \frac{1}{2}.
\end{aligned}$$

Hence, $Z \sim \text{Bernoulli}(1/2)$.

(b) We have:

$$\begin{aligned}
\mathbb{P}(Y = 0, Z = 0) &= \mathbb{P}(X = 0, Y = 0) \text{ [definition } \oplus\text{]} \\
&= \mathbb{P}(X = 0)\mathbb{P}(Y = 0) \text{ [}X, Y \text{ are independent]} \\
&= \left(1 - \frac{1}{10}\right)\left(1 - \frac{1}{2}\right) \\
&= \frac{9}{20} = 0.45.
\end{aligned}$$

(c) As we have already computed $\mathbb{P}(Y = 0, Z = 0)$ in part (b), it is natural to calculate the following product:

$$\begin{aligned}
\mathbb{P}(Y = 0)\mathbb{P}(Z = 0) &= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right) \text{ [see part (a)]} \\
&= \frac{1}{4} = 0.25.
\end{aligned}$$

From the calculations above, we have that $\mathbb{P}(Y = 0, Z = 0) \neq \mathbb{P}(Y = 0)\mathbb{P}(Z = 0)$. As there exists a pair (y, z) for which $\mathbb{P}(Y = y, Z = z) \neq \mathbb{P}(Y = y)p(Z = z)$, we conclude that the random variables Y and Z are not independent.
