Simple Monte Carlo Evaluation of Integrals

We now explore a totally different method of estimating integrals. Let us introduce this method by asking, ' Suppose the pond is in the middle of a field of known area A. One way to estimate the area of the pond is to throw the stones so that they land at random within the boundary of the field and count the number of splashes that occur when a stone lands in a pond. The area of the pond is approximately the area of the field times the fraction of stones that make a splash. This simple procedure is an example of a *Monte Carlo* method.

More explicitly, imagine a rectangle of height h, width $(b-a)$, and area $A = h(b-a)$ such that the function $f(x)$ is within the boundaries of the rectangle (see Figure 11.3). Compute n pairs of random numbers x_i and y_i with $a \le x_i \le b$ and $0 \le y_i \le h$. The fraction of points x_i, y_i that satisfy the condition $y_i \le f(x_i)$ is an estimate of the ratio of the integral of $f(x)$ to the area of the rectangle. Hence, the estimate F_n in the *hit or miss* method is given by

$$F_n = A\frac{n_s}{n}, \qquad \text{(hit or miss method)} \tag{11.9}$$

where n_s is the number of "splashes" or points below the curve, and n is the total number of points. The number of *trials* n in (11.9) should not be confused with the number of intervals used in the numerical methods discussed in Section 11.1.

Another Monte Carlo integration method is based on the mean-value theorem of calculus, which states that the definite integral (11.1) is determined by the average value of the integrand $f(x)$ in the range $a \le x \le b$. To determine this average, we choose the x_i at random instead of at regular intervals and *sample* the value of $f(x)$. For the one-dimensional integral (11.1), the estimate F_n of the integral in the *sample mean* method is given by

$$F_n = (b-a)\langle f\rangle = (b-a)\frac{1}{n}\sum_{i=1}^{n} f(x_i). \qquad \text{(sample mean method)} \tag{11.10}$$

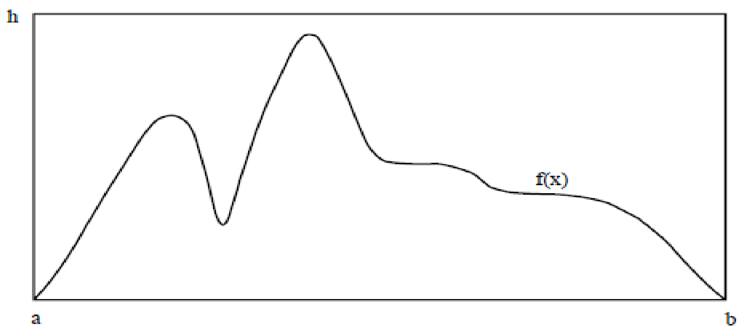

Figure 11.3: The function $f(x)$ is in the domain determined by the rectangle of height H and width $(b-a)$.

The x_i are random numbers distributed uniformly in the interval $a \le x_i \le b$, and n is the number of trials. Note that the forms of (11.3) and (11.10) are formally identical except that the n points are chosen with equal spacing in (11.3) and with random spacing in (11.10). We will find that for low dimensional integrals (11.3) is more accurate, but for higher dimensional integrals (11.10) does better.

A simple program that implements the hit or miss method is given below. Note the use of the `Random` class and the methods `setSeed` and `nextDouble()`. The primary reason that it is desirable to specify the seed rather than to choose it more or less at random from the time (as done by `Math.random()`) is that it is convenient to use the same random number sequence when testing a Monte Carlo program. In Section ??.

Problem 11.3. Monte Carlo integration in one dimension

a. Use the hit or miss Monte Carlo method to estimate F_n, the integral of $f(x) = 4\sqrt{1-x^2}$ in the interval $0 \le x \le 1$ as a function of n. Choose $a = 0$, $b = 1$, $h = 1$, and compute the mean value of the function $\sqrt{1-x^2}$. Multiply the estimate by 4 to determine F_n. Calculate the difference between F_n and the exact result of π. This difference is a measure of the error associated with the Monte Carlo estimate. Make a log-log plot of the error as a function of n. What is the approximate functional dependence of the error on n for large n, for example, $n \ge 10^4$?

b. Estimate the same integral using the sample mean Monte Carlo method (11.10) and compute the error as a function of the number of trials n for $n \ge 10^4$. How many trials are needed to determine F_n to two decimal places? What is the approximate functional dependence of the error on n for large n?

c. Determine the computational time per trial using the two Monte Carlo methods. Which Monte Carlo method is preferable?