

Maharashtra State Board 12th Maths Solutions Chapter 6 Differential Equations Ex 6.1

1. Determine the order and degree of each of the following differential equations:

Question (i).

$$dydx_2 + X(dydx) + y = 2\sin x$$

Solution:

The given D.E. is $dydx_2 + X(dydx) + y = 2\sin x$

This D.E. has highest order derivative d_2ydx_2 with power 1.

∴ the given D.E. is of order 2 and degree 1.

Question (ii).

$$1 + (dydx)_2 \sqrt[3]{ } = d_2ydx_2$$

Solution:

The given D.E. is $1 + (dydx)_2 \sqrt[3]{ } = d_2ydx_2$

On cubing both sides, we get

$$1 + (dydx)_2 = (d_2ydx_2)^3$$

This D.E. has highest order derivative d_2ydx_2 with power 3.

∴ the given D.E. is of order 2 and degree 3.

Question (iii).

$$dydx = 2\sin x + 3dydx$$

Solution:

The given D.E. is $dydx = 2\sin x + 3dydx$

$$\therefore (dydx)_2 = 2 \sin x + 3$$

This D.E. has highest order derivative $dydx$ with power 2.

∴ the given D.E. is of order 1 and degree 2.

Question (iv).

$$d_2ydx_2 + dydx + x = 1 + d_3ydx_3 \sqrt{ }$$

Solution:

The given D.E. is $d_2ydx_2 + dydx + x = 1 + d_3ydx_3 \sqrt{ }$

On squaring both sides, we get

$$(d_2ydx_2 + dydx + x)^2 = 1 + d_3ydx_3$$

This D.E. has highest order derivative d_3ydx_3 with power 1.

∴ the given D.E. has order 3 and degree 1.

Question (v).

$$d_2ydt_2 + (dydt)_2 + 7x + 5 = 0$$

Solution:

The given D.E. is $d_2ydt_2 + (dydt)_2 + 7x + 5 = 0$

This D.E. has highest order derivative d_2ydx_2 with power 1.

∴ the given D.E. has order 2 and degree 1.

Question (vi).

$$(y'')^2 + 3y'' + 3xy' + 5y = 0$$

Solution:

The given D.E. is $(y'')^2 + 3y'' + 3xy' + 5y = 0$

This can be written as:

$$(d_3ydx_3)_2 + 3d_2ydx_2 + 3xydx + 5y = 0$$

This D.E. has highest order derivative d_3ydx_3 with power 2.

∴ The given D.E. has order 3 and degree 2.

Question (vii).

$$(d_2ydx_2)_2 + \cos(dydx) = 0$$

Solution:

The given D.E. is $(d_2ydx_2)_2 + \cos(dydx) = 0$

This D.E. has highest order derivative d_2ydx_2

\therefore order = 2

Since this D.E. cannot be expressed as a polynomial in differential coefficients, the degree is not defined.

Question (viii).

$$[1 + (dydx)_2]_{32} = 8 d_2ydx_2$$

Solution:

The given D.E. is $[1 + (dydx)_2]_{32} = 8 d_2ydx_2$

On squaring both sides, we get

$$[1 + (dydx)_2]_3 = 8^2 \cdot (d_2ydx_2)^2$$

This D.E. has highest order derivative d_2ydx_2 with power 2.

\therefore the given D.E. has order 2 and degree 2.

Question (ix).

$$(d_3ydx_3)_{12} \cdot (dydx)_{13} = 20$$

Solution:

The given D.E. is $(d_3ydx_3)_{12} \cdot (dydx)_{13} = 20$

$$\therefore (d_3ydx_3)_3 \cdot (dydx)_2 = 20$$

This D.E. has highest order derivative d_3ydx_3 with power 3.

\therefore the given D.E. has order 3 and degree 3.

Question (x).

$$x + d_2ydx_2 = 1 + (d_2ydx_2)_2 \quad \text{-----} \checkmark$$

Solution:

The given D.E. is $x + d_2ydx_2 = 1 + (d_2ydx_2)_2 \quad \text{-----} \checkmark$

On squaring both sides, we get

$$\left(x + \frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{d^2y}{dx^2}\right)^2$$

$$\therefore x^2 + 2x \frac{d^2y}{dx^2} + \left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{d^2y}{dx^2}\right)^2$$

$$\therefore x^2 + 2x \frac{d^2y}{dx^2} - 1 = 0$$

This D.E. has highest order derivative d_2ydx_2 with power 1.

\therefore the given D.E. has order 2 and degree 1.

Maharashtra State Board 12th Maths Solutions Chapter 6 Differential Equations Ex 6.2

Question 1.

Obtain the differential equation by eliminating the arbitrary constants from the following equations:

$$(i) x^3 + y^3 = 4ax$$

Solution:

$$x^3 + y^3 = 4ax \dots\dots(1)$$

Differentiating both sides w.r.t. x, we get

$$3x^2 + 3y^2 dy/dx = 4a \times 1$$

$$\therefore 3x^2 + 3y^2 dy/dx = 4a$$

Substituting the value of 4a in (1), we get

$$x^3 + y^3 = (3x^2 + 3y^2 dy/dx) x$$

$$\therefore x^3 + y^3 = 3x^3 + 3xy^2 dy/dx$$

$$\therefore 2x^3 + 3xy^2 dy/dx - y^3 = 0$$

This is the required D.E.

$$(ii) Ax^2 + By^2 = 1$$

Solution:

$$Ax^2 + By^2 = 1$$

Differentiating both sides w.r.t. x, we get

$$A \times 2x + B \times 2y dy/dx = 0$$

$$\therefore Ax + By dy/dx = 0 \dots\dots(1)$$

Differentiating again w.r.t. x, we get

$$A \times 1 + B \left[y \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{dy}{dx} \right] = 0$$

$$\therefore A + B \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] = 0$$

$$\therefore A = -B \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right]$$

Substituting the value of A in (1), we get

$$-Bx \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] + By \frac{dy}{dx} = 0$$

$$\therefore -x \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] + y \frac{dy}{dx} = 0$$

$$\therefore -xy \frac{d^2y}{dx^2} - x \left(\frac{dy}{dx} \right)^2 + y \frac{dy}{dx} = 0$$

$$\therefore xy \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y \frac{dy}{dx} = 0$$

This is the required D.E.

Alternative Method:

$$Ax^2 + By^2 = 1 \dots\dots(1)$$

Differentiating both sides w.r.t. x, we get

$$A \times 2x + B \times 2y dy/dx = 0$$

$$\therefore Ax + By dy/dx = 0 \dots\dots(2)$$

Differentiating again w.r.t. x, we get,

$$A \times 1 + B \left[y \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{dy}{dx} \right] = 0$$

$$\therefore A + B \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] = 0 \dots\dots(3)$$

The equations (1), (2) and (3) are consistent in A and B.

\therefore determinant of their consistency is zero.

$$\begin{aligned} & \left| \begin{array}{ccc} x^2 & y^2 & 1 \\ x & y \frac{dy}{dx} & 0 \\ 1 & y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 & 0 \end{array} \right| = 0 \\ \therefore & x^2(0-0) - y^2(0-0) + 1 \left[xy \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 - y \frac{dy}{dx} \right] = 0 \\ \therefore & xy \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 - y \frac{dy}{dx} = 0 \end{aligned}$$

This is the required D.E.

(iii) $y = A \cos(\log x) + B \sin(\log x)$

Solution:

$$y = A \cos(\log x) + B \sin(\log x) \dots\dots (1)$$

Differentiating w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= -A \sin(\log x) \cdot \frac{d}{dx}(\log x) + \\ &\quad B \cos(\log x) \cdot \frac{d}{dx}(\log x) \\ &= \frac{-A \sin(\log x)}{x} + \frac{B \cos(\log x)}{x} \end{aligned}$$

$$\therefore x \frac{dy}{dx} = -A \sin(\log x) + B \cos(\log x)$$

Differentiating again w.r.t. x, we get

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-A \cos(\log x)}{x} - \frac{B \sin(\log x)}{x}$$

$$\begin{aligned} \therefore x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} &= -[A \cos(\log x) + B \sin(\log x)] \\ &= -y \quad \dots [By (1)] \end{aligned}$$

$$\therefore x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0 \text{ is the required D.E.}$$

(iv) $y_2 = (x + c)^3$

Solution:

$$y_2 = (x + c)^3$$

Differentiating w.r.t. x, we get

$$2y \frac{dy}{dx} = 3(x + c)^2 \cdot (1) = 3(x + c)^2$$

$$\therefore (x + c)^2 = \frac{2y}{3} \cdot \frac{dy}{dx}$$

$$\therefore (x + c)^6 = \left(\frac{2y}{3} \cdot \frac{dy}{dx} \right)^3$$

$$\therefore (y^2)^2 = \frac{8y^3}{27} \cdot \left(\frac{dy}{dx} \right)^3 \quad \dots [By (1)]$$

$$\therefore 27y^4 = 8y^3 \left(\frac{dy}{dx} \right)^3$$

$$\therefore 27y = 8 \left(\frac{dy}{dx} \right)^3$$

$$\therefore 8 \left(\frac{dy}{dx} \right)^3 - 27y = 0$$

This is the required D.E.

(v) $y = Ae^{5x} + Be^{-5x}$

Solution:

$$y = Ae^{5x} + Be^{-5x} \dots\dots\dots(1)$$

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = Ae^{5x} \times 5 + Be^{-5x} \times (-5)$$

$$\therefore \frac{dy}{dx} = 5Ae^{5x} - 5Be^{-5x}$$

$$\text{and } \frac{d^2y}{dx^2} = 5Ae^{5x} \times 5 - 5Be^{-5x} \times (-5)$$

$$= 25Ae^{5x} + 25Be^{-5x}$$

$$= 25(Ae^{5x} + Be^{-5x}) = 25y \quad \dots\dots \text{[By (1)]}$$

$$\therefore \frac{d^2y}{dx^2} - 25y = 0$$

This is the required D.E.

$$(vi) (y - a)^2 = 4(x - b)$$

Solution:

$$(y - a)^2 = 4(x - b)$$

Differentiating both sides w.r.t. x, we get

$$2(y - a) \cdot ddx(y - a) = 4 ddx(x - b)$$

$$\therefore 2(y - a) \cdot (dydx - 0) = 4(1 - 0)$$

$$\therefore 2(y - a) dydx = 4$$

$$\therefore (y - a) dydx = 2 \dots\dots(1)$$

Differentiating w.r.t. x, we get

$$(y - a) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx} (y - a) = 0$$

$$\therefore (y - a) \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \left(\frac{dy}{dx} - 0 \right) = 0$$

$$\therefore (y - a) \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0$$

$$\therefore \frac{2}{\left(\frac{dy}{dx} \right)} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0 \quad \dots\dots \text{[By (1)]}$$

$$\therefore 2 \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^3 = 0$$

This is the required D.E.

$$(vii) y = a + ax$$

Solution:

$$y = a + ax$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(a + \frac{a}{x} \right) = 0 + a \left(-\frac{1}{x^2} \right)$$

$$\therefore \frac{dy}{dx} = -\frac{a}{x^2}$$

$$\therefore a = -x^2 \frac{dy}{dx}$$

Substituting the value of a in (1), we get

$$y = -x^2 \frac{dy}{dx} + \frac{1}{x} \left(-x^2 \frac{dy}{dx} \right)$$

$$\therefore y = -x^2 \frac{dy}{dx} - x \frac{dy}{dx}$$

$$\therefore (x^2 + x) \frac{dy}{dx} + y = 0$$

$$\therefore x(x + 1) \frac{dy}{dx} + y = 0$$

This is the required D.E.

(viii) $y = c_1 e^{2x} + c_2 e^{5x}$

Solution:

$$y = c_1 e^{2x} + c_2 e^{5x} \dots\dots\dots(1)$$

Differentiating twice w.r.t. x , we get

$$\frac{dy}{dx} = c_1 e^{2x} \times 2 + c_2 e^{5x} \times 5$$

$$\therefore \frac{dy}{dx} = 2c_1 e^{2x} + 5c_2 e^{5x} \dots\dots\dots(2)$$

$$\text{and } \frac{d^2y}{dx^2} = 2c_1 e^{2x} \times 2 + 5c_2 e^{5x} \times 5$$

$$\therefore \frac{d^2y}{dx^2} = 4c_1 e^{2x} + 25c_2 e^{5x} \dots\dots\dots(3)$$

The equations (1), (2) and (3) are consistent in $c_1 e^{2x}$ and $c_2 e^{5x}$

\therefore determinant of their consistency is zero.

$$\begin{array}{|ccc|} \hline & y & 1 & 1 \\ \therefore & \left| \begin{array}{ccc} \frac{dy}{dx} & 2 & 5 \\ \frac{d^2y}{dx^2} & 4 & 25 \end{array} \right| = 0 \\ \therefore & y(50 - 20) - 1\left(25 \frac{dy}{dx} - 5 \frac{d^2y}{dx^2}\right) + 1\left(4 \frac{dy}{dx} - 2 \frac{d^2y}{dx^2}\right) = 0 \\ \therefore & 30y - 25 \frac{dy}{dx} + 5 \frac{d^2y}{dx^2} + 4 \frac{dy}{dx} - 2 \frac{d^2y}{dx^2} = 0 \\ \therefore & 3 \frac{d^2y}{dx^2} - 21 \frac{dy}{dx} + 30y = 0 \end{array}$$

$$\begin{array}{l} \therefore \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 10y = 0 \\ \therefore \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 10y = 0 \end{array}$$

This is the required D.E.

Alternative Method:

$$y = c_1 e^{2x} + c_2 e^{5x}$$

Dividing both sides by e^{5x} , we get

$$e^{-5x} \cdot y = c_1 e^{-3x} + c_2$$

Differentiating w.r.t. x , we get

$$e^{-5x} \cdot \frac{dy}{dx} + y \times e^{-5x} \times (-5) = c_1 e^{-3x} \times (-3) + 0$$

$$\therefore e^{-5x} \left(\frac{dy}{dx} - 5y \right) = -3c_1 e^{-3x}$$

Dividing both sides by e^{-3x} , we get

$$e^{-2x} \left(\frac{dy}{dx} - 5y \right) = -3c_1$$

Differentiating w.r.t. x , we get

$$e^{-2x} \left(\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} \right) + \left(\frac{dy}{dx} - 5y \right) \cdot e^{-2x} (-2) = 0$$

$$\therefore e^{-2x} \left(\frac{d^2y}{dx^2} - 5 \frac{dy}{dx} - 2 \frac{dy}{dx} + 10y \right) = 0$$

$$\therefore \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 10y = 0$$

This is the required D.E.

(ix) $c_1 x_3 + c_2 y_2 = 5$.

Solution:

$$c_1 x_3 + c_2 y_2 = 5 \dots\dots\dots(1)$$

Differentiating w.r.t. x , we get

$$c_1 \times 3x^2 + c_2 \times 2y \frac{dy}{dx} = 0$$

$$\therefore 3c_1 x^2 + 2c_2 y \frac{dy}{dx} = 0 \quad \dots\dots(2)$$

Differentiating again w.r.t. x, we get

$$3c_1 \times 2x + 2c_2 \left[y \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx} \right] = 0$$

$$\therefore 6c_1 x + 2c_2 \left[y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] = 0 \quad \dots\dots(3)$$

The equations (1), (2) and (3) in c_1, c_2 are consistent.

\therefore determinant of their consistency is zero.

$$\therefore \begin{vmatrix} x^3 & y^2 & 5 \\ 3x^2 & 2y \frac{dy}{dx} & 0 \\ 6x & 2y \frac{d^2 y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 & 0 \end{vmatrix} = 0$$

$$\therefore x^3(0 - 0) - y^2(0 - 0) +$$

$$5 \left[6x^2 y \frac{d^2 y}{dx^2} + 6x^2 \left(\frac{dy}{dx} \right)^2 - 12xy \frac{dy}{dx} \right] = 0$$

$$\therefore 6x^2 y \frac{d^2 y}{dx^2} + 6x^2 \left(\frac{dy}{dx} \right)^2 - 12xy \frac{dy}{dx} = 0$$

$$\therefore xy \frac{d^2 y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 - 2y \frac{dy}{dx} = 0$$

This is the required D.E.

$$(x) y = e^{-2x}(A \cos x + B \sin x)$$

Solution:

$$y = e^{-2x}(A \cos x + B \sin x)$$

$$\therefore e^{2x} \cdot y = A \cos x + B \sin x \quad \dots\dots(1)$$

Differentiating w.r.t. x, we get

$$e^{2x} \cdot \frac{dy}{dx} + y \cdot e^{2x} \times 2 = A(-\sin x) + B \cos x$$

$$\therefore e^{2x} \left(\frac{dy}{dx} + 2y \right) = -A \sin x + B \cos x$$

Differentiating again w.r.t. x, we get

$$e^{2x} \left(\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} \right) + \left(\frac{dy}{dx} + 2y \right) \cdot e^{2x} \times 2$$

$$= -A \cos x + B(-\sin x)$$

$$\therefore e^{2x} \left(\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 2 \frac{dy}{dx} + 4y \right) = - (A \cos x + B \sin x)$$

$$\therefore e^{2x} \left(\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4y \right) = -e^{2x} \cdot y \quad \dots [By (1)]$$

$$\therefore \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4y = -y$$

$$\therefore \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 5y = 0$$

This is the required D.E.

Question 2.

Form the differential equation of family of lines having intercepts a and b on the coordinate axes respectively.

Solution:

The equation of the line having intercepts a and b on the coordinate axes respectively, is

$$xa + yb = 1 \quad \dots\dots(1)$$

where a and b are arbitrary constants.

[For different values of a and b, we get, different lines. Hence (1) is the equation of family of lines.]

Differentiating (1) w.r.t. x, we get

$$\frac{1}{a}(1) + \left(\frac{1}{b}\right) \cdot \frac{dy}{dx} = 0$$

$$\therefore \left(\frac{1}{b}\right) \frac{dy}{dx} = -\frac{1}{a} \quad \therefore \frac{dy}{dx} = -\frac{b}{a}$$

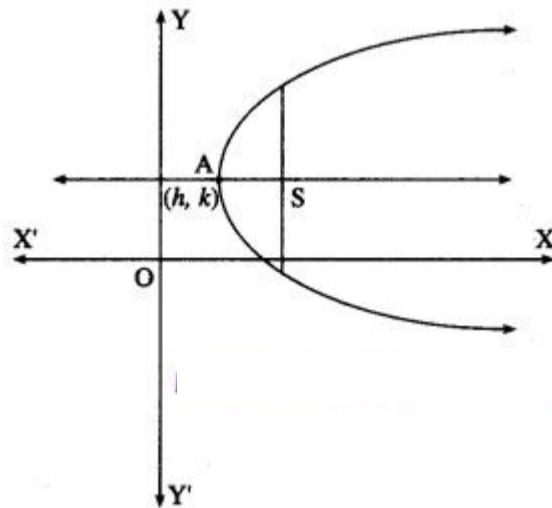
Differentiating again w.r.t. x, we get $\frac{d^2y}{dx^2} = 0$

This is the required D.E.

Question 3.

Find the differential equation all parabolas having length of latus rectum 4a and axis is parallel to the X-axis.

Solution:



Let A(h, k) be the vertex of the parabola whose length of latus rectum is 4a.

Then the equation of the parabola is $(y - k)^2 = 4a(x - h)$, where h and k are arbitrary constants.

Differentiating w.r.t. x, we get

$$2(y - k) \cdot \frac{d}{dx}(y - k) = 4a \frac{d}{dx}(x - h)$$

$$\therefore 2(y - k) \left(\frac{dy}{dx} - 0 \right) = 4a(1 - 0)$$

$$\therefore 2(y - k) \frac{dy}{dx} = 4a$$

$$\therefore (y - k) \frac{dy}{dx} = 2a \quad \dots\dots(1)$$

Differentiating again w.r.t. x, we get

$$(y - k) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx} (y - k) = 0$$

$$\therefore (y - k) \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \left(\frac{dy}{dx} - 0 \right) = 0$$

$$\therefore (y - k) \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0$$

$$\therefore \frac{2a}{\left(\frac{dy}{dx} \right)} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0 \quad \dots\dots [\text{By (1)}]$$

$$\therefore 2a \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^3 = 0$$

This is the required D.E.

Question 4.

Find the differential equation of the ellipse whose major axis is twice its minor axis.

Solution:

Let 2a and 2b be lengths of major axis and minor axis of the ellipse.

Then $2a = 2(2b)$

$$\therefore a = 2b$$

\therefore equation of the ellipse is

$$x^2/a^2 + y^2/b^2 = 1$$

$$\text{i.e., } x^2/(2b)^2 + y^2/b^2 = 1$$

$$\therefore x^2/4b^2 + y^2/b^2 = 1$$

$\therefore x_2 + 4y_2 = 4b_2$
 Differentiating w.r.t. x, we get
 $2x + 4 \times 2y \frac{dy}{dx} = 0$
 $\therefore x + 4y \frac{dy}{dx} = 0$
 This is the required D.E.

Question 5.

Form the differential equation of family of lines parallel to the line $2x + 3y + 4 = 0$.

Solution:

The equation of the line parallel to the line $2x + 3y + 4 = 0$ is $2x + 3y + c = 0$, where c is an arbitrary constant.

Differentiating w.r.t. x, we get
 $2 \times 1 + 3 \frac{dy}{dx} + 0 = 0$
 $\therefore 3 \frac{dy}{dx} + 2 = 0$

This is the required D.E.

Question 6.

Find the differential equation of all circles having radius 9 and centre at point (h, k).

Solution:

Equation of the circle having radius 9 and centre at point (h, k) is

$$(x - h)^2 + (y - k)^2 = 81 \dots\dots (1)$$

where h and k are arbitrary constant.

Differentiating (1) w.r.t. x, we get

$$2(x - h) \cdot \frac{d}{dx}(x - h) + 2(y - k) \cdot \frac{d}{dx}(y - k) = 0$$

$$\therefore (x - h)(1 - 0) + (y - k) \left(\frac{dy}{dx} - 0 \right) = 0$$

$$\therefore (x - h) + (y - k) \frac{dy}{dx} = 0 \quad \dots\dots (2)$$

Differentiating again w.r.t. x, we get

$$\frac{d}{dx}(x - h) + (y - k) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx}(y - k) = 0$$

$$\therefore (1 - 0) + (y - k) \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \left(\frac{dy}{dx} - 0 \right) = 0$$

$$\therefore (y - k) \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 + 1 = 0$$

$$\therefore (y - k) \frac{d^2y}{dx^2} = - \left[\left(\frac{dy}{dx} \right)^2 + 1 \right]$$

$$\therefore y - k = \frac{- \left[\left(\frac{dy}{dx} \right)^2 + 1 \right]}{\left(\frac{d^2y}{dx^2} \right)} \quad \dots\dots (3)$$

From (2), $x - h = -(y - k) \frac{dy}{dx}$

Substituting the value of $(x - h)$ in (1), we get

$$(y - k)^2 \left(\frac{dy}{dx} \right)^2 + (y - k)^2 = 81$$

$$\therefore \left(\frac{dy}{dx} \right)^2 + 1 = \frac{81}{(y - k)^2}$$

$$\therefore \left(\frac{dy}{dx} \right)^2 + 1 = \frac{81 \cdot \left(\frac{d^2y}{dx^2} \right)^2}{\left[\left(\frac{dy}{dx} \right)^2 + 1 \right]^2} \quad \text{[By (3)]}$$

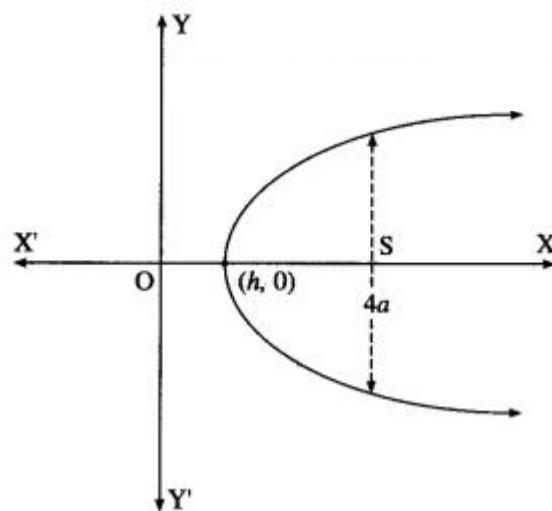
$$\therefore 81 \left(\frac{d^2y}{dx^2} \right)^2 = \left[\left(\frac{dy}{dx} \right)^2 + 1 \right]^3$$

This is the required D.E.

Question 7.

Form the differential equation of all parabolas whose axis is the X-axis.

Solution:



The equation of the parabola whose axis is the X-axis is

$$y^2 = 4a(x - h) \dots\dots (1)$$

where a and h are arbitrary constants.

Differentiating (1) w.r.t. x , we get

$$2y \frac{dy}{dx} = 4a(1 - 0)$$

$$\therefore y \frac{dy}{dx} = 2a$$

Differentiating again w.r.t. x , we get

$$y \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{dy}{dx} = 0$$

$$\therefore y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0.$$

This is the required D.E.

Maharashtra State Board 12th Maths Solutions Chapter 6

Differential Equations Ex 6.3

Question 1.

In each of the following examples verify that the given expression is a solution of the corresponding differential equation.

$$(i) xy = \log y + c; \frac{dy}{dx} = y^2 1 - xy$$

Solution:

$$xy = \log y + c$$

Differentiating w.r.t. x , we get

$$x \cdot \frac{dy}{dx} + y \times 1 = \frac{1}{y} \cdot \frac{dy}{dx} + 0$$

$$\therefore x \frac{dy}{dx} + y = \frac{1}{y} \frac{dy}{dx}$$

$$\left(x - \frac{1}{y} \right) \frac{dy}{dx} = -y$$

$$\therefore \left(\frac{xy - 1}{y} \right) \frac{dy}{dx} = -y$$

$$\therefore \frac{dy}{dx} = \frac{-y^2}{xy - 1} = \frac{y^2}{1 - xy}, \text{ if } xy \neq 1$$

Hence, $xy = \log y + c$ is a solution of the D.E.

$$\frac{dy}{dx} = y^2 1 - xy, xy \neq 1$$

(ii) $y = (\sin^{-1}x)^2 + c; (1-x^2)d_2ydx_2 - xdydx = 2$

Solution:

$$y = (\sin^{-1}x)^2 + c \dots\dots(1)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^2 + 0$$

$$\therefore \frac{dy}{dx} = 2(\sin^{-1}x) \cdot \frac{d}{dx}(\sin^{-1}x)$$

$$= 2\sin^{-1}x \times \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \sqrt{1-x^2}\frac{dy}{dx} = 2\sin^{-1}x$$

$$\therefore (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(\sin^{-1}x)^2$$

$$\therefore (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4(y-c) \quad \dots\dots [\text{By (1)}]$$

Differentiating again w.r.t. x, we get

$$(1-x^2) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2 \cdot \frac{d}{dx}(1-x^2) = 4 \frac{d}{dx}(y-c)$$

$$\therefore (1-x^2) \cdot 2\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 4\left(\frac{dy}{dx} - 0\right)$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2.$$

Hence, $y = (\sin^{-1}x)^2 + c$ is a solution of the D.E.

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2.$$

(iii) $y = e^{-x} + Ax + B; exd_2ydx_2 = 1$

Solution:

$$y = e^{-x} + Ax + B$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = e^{-x} \times (-1) + A \times 1 + 0$$

$$\therefore \frac{dy}{dx} = -e^{-x} + A$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -e^{-x} \times (-1) + 0$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{e^x}$$

$$\therefore exd_2ydx_2 = 1$$

Hence, $y = e^{-x} + Ax + B$ is a solution of the D.E.

$$exd_2ydx_2 = 1$$

(iv) $y = xm; x^2d_2ydx_2 - mx dydx + my = 0$

Solution:

$$y = xm$$

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^m) = mx^{m-1}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}(mx^{m-1}) = m \frac{d}{dx}(x^{m-1}) = m(m-1)x^{m-2}$$

$$\begin{aligned}\therefore x^2 \frac{d^2y}{dx^2} - mx \frac{dy}{dx} + my \\ &= x^2 \cdot m(m-1)x^{m-2} - mx \cdot mx^{m-1} + m \cdot x^m \\ &= m(m-1)x^m - m^2x^m + mx^m \\ &= (m^2 - m - m^2 + m)x^m = 0\end{aligned}$$

This shows that $y = x^m$ is a solution of the D.E.

$$xd_2ydx_2 - mx dydx + my = 0$$

$$(v) y = a + bx; xd_2ydx_2 + 2dydx = 0$$

Solution:

$$y = a + bx$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 0 + b \left(-\frac{1}{x^2} \right) = -\frac{b}{x^2}$$

$$\therefore x^2 \frac{dy}{dx} = -b$$

Differentiating again w.r.t. x, we get

$$x^2 \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x^2) = 0$$

$$\therefore x^2 \frac{d^2y}{dx^2} + \frac{dy}{dx} \times 2x = 0$$

$$\therefore x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 0$$

Hence, $y = a + bx$ is a solution of the D.E.

$$xd_2ydx_2 + 2dydx = 0$$

$$(vi) y = e^{ax}; x dydx = y \log y$$

Solution:

$$y = e^{ax}$$

$$\log y = \log e^{ax} = ax \log e$$

$$\log y = ax \dots\dots(1) \dots\dots [\because \log e = 1]$$

Differentiating w.r.t. x, we get

$$1y \cdot dydx = a \times 1$$

$$\therefore dydx = ay$$

$$\therefore x dydx = (ax)y$$

$$\therefore x dydx = y \log y \dots\dots[\text{By (1)}]$$

Hence, $y = e^{ax}$ is a solution of the D.E.

$$x dydx = y \log y.$$

Question 2.

Solve the following differential equations.

$$(i) dydx = 1 + y^2 1 + x^2$$

Solution:

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

$$\therefore \frac{1}{1+y^2} dy = \frac{1}{1+x^2} dx$$

Integrating both sides, we get

$$\int \frac{1}{1+y^2} dy = \int \frac{1}{1+x^2} dx$$

$$\therefore \tan^{-1}y = \tan^{-1}x + c$$

This is the general solution.

(ii) $\log(dy/dx) = 2x + 3y$

Solution:

$$\log\left(\frac{dy}{dx}\right) = 2x + 3y$$

$$\therefore \frac{dy}{dx} = e^{2x+3y} = e^{2x} \cdot e^{3y}$$

$$\therefore \frac{1}{e^{3y}} dy = e^{2x} dx$$

Integrating both sides, we get

$$\int e^{-3y} dy = \int e^{2x} dx$$

$$\therefore \frac{e^{-3y}}{-3} = \frac{e^{2x}}{2} + c_1$$

$$\therefore 2e^{-3y} = -3e^{2x} + 6c_1$$

$$\therefore 2e^{-3y} + 3e^{2x} = c, \text{ where } c = 6c_1$$

This is the general solution.

(iii) $y - x dy/dx = 0$

Solution:

$$y - x dy/dx = 0$$

$$\therefore x dy/dx = y$$

$$\therefore x dx = y dy$$

Integrating both sides, we get

$$\int x dx = \int y dy$$

$$\therefore \log|x| = \log|y| + \log c$$

$$\therefore \log|x| = \log|cy|$$

$$\therefore x = cy$$

This is the general solution.

(iv) $\sec 2x \cdot \tan y dx + \sec 2y \cdot \tan x dy = 0$

Solution:

$$\sec 2x \cdot \tan y dx + \sec 2y \cdot \tan x dy = 0$$

$$\therefore \sec 2x \tan y dx + \sec 2y \tan x dy = 0$$

Integrating both sides, we get

$$\int \sec 2x \tan y dx + \int \sec 2y \tan x dy = c_1$$

Each of these integrals is of the type

$$\int f(x) f'(x) dx = \log|f(x)| + C$$

\therefore the general solution is

$$\therefore \log|\tan x| + \log|\tan y| = \log c, \text{ where } c_1 = \log c$$

$$\therefore \log|\tan x \cdot \tan y| = \log c$$

$$\therefore \tan x \cdot \tan y = c$$

This is the general solution.

(v) $\cos x \cdot \cos y dy - \sin x \cdot \sin y dx = 0$

Solution:

$$\cos x \cdot \cos y dy - \sin x \cdot \sin y dx = 0$$

$$\cos y \sin y dy - \sin x \cos x dx = 0$$

Integrating both sides, we get

$$\int \cot y \, dy - \int \tan x \, dx = c_1$$

$$\therefore \log|\sin y| - [-\log|\cos x|] = \log c, \text{ where } c_1 = \log c$$

$$\therefore \log|\sin y| + \log|\cos x| = \log c$$

$$\therefore \log|\sin y \cdot \cos x| = \log c$$

$$\therefore \sin y \cdot \cos x = c$$

This is the general solution.

(vi) $\frac{dy}{dx} = -k$, where k is a constant.

Solution:

$$\frac{dy}{dx} = -k$$

$$\therefore dy = -k dx$$

Integrating both sides, we get

$$\int dy = -k \int dx$$

$$\therefore y = -kx + c$$

This is the general solution.

(vii) $\cos 2y \frac{dy}{dx} + \cos 2x \frac{dx}{dx} = 0$

Solution:

$$\cos 2y \frac{dy}{dx} + \cos 2x \frac{dx}{dx} = 0$$

$$\therefore y \cos 2y \, dy + x \cos 2x \, dx = 0$$

$$\therefore x(1 + \cos 2x) \, dx + y(1 + \cos 2y) \, dy = 0$$

$$\therefore x \, dx + x \cos 2x \, dx + y \, dy + y \cos 2y \, dy = 0$$

Integrating both sides, we get

$$\int x \, dx + \int y \, dy + \int x \cos 2x \, dx + \int y \cos 2y \, dy = c_1 \dots\dots\dots(1)$$

Using integration by parts

$$\begin{aligned} \int x \cos 2x \, dx &= x \int \cos 2x \, dx - \int \left[\frac{d}{dx}(x) \int \cos 2x \, dx \right] dx \\ &= x \left(\frac{\sin 2x}{2} \right) - \int 1 \cdot \frac{\sin 2x}{2} \, dx \\ &= \frac{x \sin 2x}{2} + \frac{1}{2} \cdot \frac{-\cos 2x}{2} = \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} \end{aligned}$$

Similarly,

$$\int y \cos 2y \, dy = \frac{y \sin 2y}{2} + \frac{\cos 2y}{4}$$

\therefore from (1), we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} + \frac{y \sin 2y}{2} + \frac{\cos 2y}{4} = c_1$$

Multiplying throughout by 4, this becomes

$$2x^2 + 2y^2 + 2x \sin 2x + \cos 2x + 2y \sin 2y + \cos 2y = 4c_1$$

$$\therefore 2(x^2 + y^2) + 2(x \sin 2x + y \sin 2y) + \cos 2x + \cos 2y + c = 0, \text{ where } c = -4c_1$$

This is the general solution.

(viii) $y^3 - dy/dx = x^2 dy/dx$

Solution:

$$y^3 - \frac{dy}{dx} = x^2 \frac{dy}{dx}$$

$$\therefore y^3 = \frac{dy}{dx} + x^2 \frac{dy}{dx}$$

$$\therefore y^3 = (1 + x^2) \frac{dy}{dx} \quad \therefore \frac{1}{1 + x^2} dx = \frac{1}{y^3} dy$$

Integrating both sides, we get

$$\int \frac{1}{1 + x^2} dx = \int y^{-3} dy$$

$$\therefore \tan^{-1} x = \frac{y^{-2}}{-2} + c_1$$

$$\therefore \tan^{-1} x = -\frac{1}{2y^2} + c_1$$

$$\therefore 2y^2 \tan^{-1} x = -1 + 2c_1 y^2$$

$$\therefore 2y^2 \tan^{-1} x + 1 = cy^2, \text{ where } c = 2c_1$$

This is the general solution.

(ix) $2e^{x+2y} dx - 3 dy = 0$

Solution:

$$2e^{x+2y} dx - 3 dy = 0$$

$$\therefore 2e^x \cdot e^{2y} dx - 3 dy = 0$$

$$\therefore 2e^x dx - \frac{3}{e^{2y}} dy = 0$$

Integrating both sides, we get

$$2 \int e^x dx - 3 \int e^{-2y} dy = c_1$$

$$\therefore 2e^x - 3 \cdot \frac{e^{-2y}}{(-2)} = c_1$$

$$\therefore 4e^x + 3e^{-2y} = 2c_1$$

$$\therefore 4e^x + 3e^{-2y} = c, \text{ where } c = 2c_1$$

This is the general solution.

(x) $dy/dx = e^{x+y} + x^2 e^y$

Solution:

$$\frac{dy}{dx} = e^{x+y} + x^2 e^y$$

$$\therefore \frac{dy}{dx} = e^x \cdot e^y + x^2 e^y = e^y (e^x + x^2)$$

$$\therefore \frac{1}{e^y} dy = (e^x + x^2) dx$$

Integrating both sides, we get

$$\int e^{-y} dy = \int (e^x + x^2) dx$$

$$\therefore \frac{e^{-y}}{-1} = e^x + \frac{x^3}{3} + c_1$$

$$\therefore e^x + e^{-y} + \frac{x^3}{3} = -c_1$$

$$\therefore 3e^x + 3e^{-y} + x^3 = -3c_1$$

$$\therefore 3e^x + 3e^{-y} + x^3 = c, \text{ where } c = -3c_1$$

This is the general solution.

Question 3.

For each of the following differential equations, find the particular solution satisfying the given condition:

(i) $3e^x \tan y \, dx + (1 + e^x) \sec^2 y \, dy = 0$, when $x = 0, y = \pi$

Solution:

$$3e^x \tan y \, dx + (1 + e^x) \sec^2 y \, dy = 0$$

$$\therefore \frac{3e^x}{1+e^x} \, dx + \frac{\sec^2 y}{\tan y} \, dy = 0$$

Integrating both sides, we get

$$3 \int \frac{e^x}{1+e^x} \, dx + \int \frac{\sec^2 y}{\tan y} \, dy = c_1$$

Each of these integrals is of the type

$$\int \frac{f'(x)}{f(x)} \, dx = \log |f(x)| + c$$

\therefore the general solution is

$$3 \log |1 + e^x| + \log |\tan y| = \log c, \text{ where } c_1 = \log c$$

$$\therefore \log |(1 + e^x)^3 \cdot \tan y| = \log c$$

$$\therefore (1 + e^x)^3 \tan y = c$$

When $x = 0, y = \pi$, we have

$$(1 + e^0)^3 \tan \pi = c$$

$$\therefore c = 0$$

$$\therefore \text{the particular solution is } (1 + e^x)^3 \tan y = 0.$$

(ii) $(x - y^2x) \, dx - (y + x^2y) \, dy = 0$, when $x = 2, y = 0$

Solution:

$$(x - y^2x) \, dx - (y + x^2y) \, dy = 0$$

$$\therefore x(1 - y^2) \, dx - y(1 + x^2) \, dy = 0$$

$$\therefore \frac{x}{1+x^2} \, dx - \frac{y}{1-y^2} \, dy = 0$$

$$\therefore \frac{2x}{1+x^2} - \frac{2y}{1-y^2} \, dy = 0$$

Integrating both sides, we get

$$\int \frac{2x}{1+x^2} \, dx + \int \frac{-2y}{1-y^2} \, dy = c_1$$

Each of these integrals is of the type

$$\int \frac{f'(x)}{f(x)} \, dx = \log |f(x)| + c$$

\therefore the general solution is

$$\log |1 + x^2| + \log |1 - y^2| = \log c, \text{ where } c_1 = \log c$$

$$\therefore \log |(1 + x^2)(1 - y^2)| = \log c$$

$$\therefore (1 + x^2)(1 - y^2) = c$$

When $x = 2, y = 0$, we have

$$(1 + 4)(1 - 0) = c$$

$$\therefore c = 5$$

$$\therefore \text{the particular solution is } (1 + x^2)(1 - y^2) = 5.$$

(iii) $y(1 + \log x) \, dx dy - x \log x = 0$, $y = e^2$, when $x = e$

Solution:

$$y(1 + \log x) \, dx dy - x \log x = 0$$

$$\therefore \frac{1 + \log x}{x \log x} dx - \frac{dy}{y} = 0$$

Integrating both sides, we get

$$\therefore \int \frac{1 + \log x}{x \log x} dx - \int \frac{dy}{y} = c_1 \quad \dots (1)$$

Put $x \log x = t$.

Then $\left[x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x) \right] dx = dt$
 $\therefore \left[\frac{x}{x} + (\log x)(1) \right] dx = dt \quad \therefore (1 + \log x) dx = dt$

$$\therefore \int \frac{1 + \log x}{x \log x} dx = \int \frac{dt}{t} = \log |t| = \log |x \log x|$$

\therefore from (1), the general solution is

$$\log |x \log x| - \log |y| = \log c, \text{ where } c_1 = \log c$$

$$\therefore \log \left| \frac{x \log x}{y} \right| = \log c \quad \therefore \frac{x \log x}{y} = c$$

$$\therefore x \log x = cy$$

This is the general solution.

Now, $y = e^2$, when $x = e$

$$\therefore e \log e = c \cdot e^2 \quad 1 = c \cdot e \quad \dots [\because \log e = 1]$$

$$\therefore c = \frac{1}{e}$$

$$\therefore \text{the particular solution is } x \log x = \left(\frac{1}{e} \right) y$$

$$\therefore y = ex \log x.$$

(iv) $(ey + 1) \cos x + ey \sin x dy/dx = 0$, when $x = \pi/4$, $y = 0$

Solution:

$$(ey + 1) \cos x + ey \sin x dy/dx = 0$$

$$\therefore \frac{\cos x}{\sin x} dx + \frac{ey}{e^y + 1} dy = 0$$

Integrating both sides, we get

$$\int \frac{\cos x}{\sin x} dx + \int \frac{ey}{e^y + 1} dy = c_1 \quad \dots (1)$$

$$\text{Now, } \frac{d}{dx}(\sin x) = \cos x, \frac{d}{dx}(e^y + 1) = e^y \text{ and}$$

$$\int f(x) f'(x) dx = \log|f(x)| + C$$

\therefore from (1), the general solution is

$$\log|\sin x| + \log|ey + 1| = \log c, \text{ where } c_1 = \log c$$

$$\therefore \log|\sin x \cdot (ey + 1)| = \log c$$

$$\therefore \sin x \cdot (ey + 1) = c$$

When $x = \pi/4$, $y = 0$, we get

$$(\sin \pi/4)(e^0 + 1) = c$$

$$\therefore c = 1 \cdot \sqrt{1 + 1} = \sqrt{2}$$

$$\therefore \text{the particular solution is } \sin x \cdot (ey + 1) = \sqrt{2}$$

(v) $(x + 1) dy/dx - 1 = 2e^{-y}$, $y = 0$, when $x = 1$

Solution:

$$(x+1) \frac{dy}{dx} - 1 = 2e^{-y}$$

$$\therefore (x+1) \frac{dy}{dx} = \frac{2}{e^y} + 1 = \frac{2+e^y}{e^y}$$

$$\therefore \frac{e^y}{2+e^y} dy = \frac{1}{x+1} dx$$

Integrating both sides, we get

$$\int \frac{e^y}{2+e^y} dy = \int \frac{1}{x+1} dx$$

$$\therefore \log|2+e^y| = \log|x+1| + \log c$$

$$\dots \left[\because \frac{d}{dy}(2+e^y) = e^y \text{ and } \int \frac{f'(y)}{f(y)} dy = \log|f(y)| + c \right]$$

$$\therefore \log|2+e^y| = \log|c(x+1)|$$

$$\therefore 2+e^y = c(x+1)$$

This is the general solution.

Now, $y = 0$, when $x = 1$

$$\therefore 2 + e^0 = c(1 + 1)$$

$$\therefore 3 = 2c$$

$$\therefore c = \frac{3}{2}$$

\therefore the particular solution is $2 + e^y = \frac{3}{2}(x + 1)$

$$\therefore 2(2 + e^y) = 3(x + 1).$$

$$(vi) \cos(dy/dx) = a, a \in \mathbb{R}, y(0) = 2$$

Solution:

$$\cos(dy/dx) = a$$

$$\therefore dy/dx = \cos^{-1} a$$

$$\therefore dy = (\cos^{-1} a) dx$$

Integrating both sides, we get

$$\int dy = (\cos^{-1} a) \int dx$$

$$\therefore y = (\cos^{-1} a)x + c$$

$$\therefore y = x \cos^{-1} a + c$$

This is the general solution.

Now, $y(0) = 2$, i.e. $y = 2$,

when $x = 0, 2 = 0 + c$

$$\therefore c = 2$$

\therefore the particular solution is

$$\therefore y = x \cos^{-1} a + 2$$

$$\therefore y - 2 = x \cos^{-1} a$$

$$\therefore y - 2x = \cos^{-1} a$$

$$\therefore \cos(y - 2x) = a$$

Question 4.

Reduce each of the following differential equations to the variable separable form and hence solve:

$$(i) dy/dx = \cos(x + y)$$

Solution:

$$\frac{dy}{dx} = \cos(x+y) \quad \dots (1)$$

Put $x+y=u$. Then $1+\frac{dy}{dx}=\frac{du}{dx}$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\therefore (1) \text{ becomes, } \frac{du}{dx} - 1 = \cos u$$

$$\therefore \frac{du}{dx} = 1 + \cos u$$

$$\therefore \frac{1}{1+\cos u} du = dx$$

Integrating both sides, we get

$$\int \frac{1}{1+\cos u} du = \int dx$$

$$\therefore \int \frac{1}{2 \cos^2\left(\frac{u}{2}\right)} du = \int dx$$

$$\therefore \frac{1}{2} \int \sec^2\left(\frac{u}{2}\right) du = \int dx$$

$$\therefore \frac{1}{2} \cdot \frac{\tan\left(\frac{u}{2}\right)}{\left(\frac{1}{2}\right)} = x + c$$

$$\therefore \tan\left(\frac{x+y}{2}\right) = x + c$$

This is the general solution.

(ii) $(x-y)^2 dy/dx = a^2$

Solution:

$$(x-y)^2 \frac{dy}{dx} = a^2 \quad \dots (1)$$

$$\text{Put } x-y=u \quad \therefore x-u=y \quad \therefore 1-\frac{du}{dx}=\frac{dy}{dx}$$

$$\therefore (1) \text{ becomes, } u^2 \left(1 - \frac{du}{dx} \right) = a^2$$

$$\therefore u^2 - u^2 \frac{du}{dx} = a^2$$

$$\therefore u^2 - a^2 = u^2 \frac{du}{dx} \quad \therefore dx = \frac{u^2}{u^2 - a^2} du$$

Integrating both sides, we get

$$\int dx = \int \frac{(u^2 - a^2) + a^2}{u^2 - a^2} du$$

$$\therefore x = \int 1 du + a^2 \int \frac{du}{u^2 - a^2} + c_1$$

$$= u + a^2 \cdot \frac{1}{2a} \log \left| \frac{u-a}{u+a} \right| + c_1$$

$$\therefore x = x - y + \frac{a}{2} \log \left| \frac{x-y-a}{x-y+a} \right| + c_1$$

$$\therefore -c_1 + y = \frac{a}{2} \log \left| \frac{x-y-a}{x-y+a} \right|$$

$$\therefore -2c_1 + 2y = a \log \left| \frac{x-y-a}{x-y+a} \right|$$

$$\therefore c + 2y = a \log \left| \frac{x-y-a}{x-y+a} \right|, \text{ where } c = -2c_1$$

This is the general solution.

(iii) $x + y \frac{dy}{dx} = \sec(x^2 + y^2)$

Solution:

$$x + y \frac{dy}{dx} = \sec(x^2 + y^2) \quad \dots (1)$$

$$\text{Put } x^2 + y^2 = u \quad \therefore 2x + 2y \frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore x + y \frac{dy}{dx} = \frac{1}{2} \cdot \frac{du}{dx}$$

$$\therefore (1) \text{ becomes, } \frac{1}{2} \cdot \frac{du}{dx} = \sec u \quad \therefore \frac{1}{\sec u} du = 2 \cdot dx$$

Integrating both sides, we get

$$\int \cos u du = 2 \int dx$$

$$\therefore \sin u = 2x + c$$

$$\therefore \sin(x^2 + y^2) = 2x + c$$

This is the general solution.

(iv) $\cos^2(x - 2y) = 1 - 2 \frac{dy}{dx}$

Solution:

$$\cos^2(x - 2y) = 1 - 2 \frac{dy}{dx} \quad \dots (1)$$

$$\text{Put } x - 2y = u. \text{ Then } 1 - 2 \frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore (1) \text{ becomes, } \cos^2 u = \frac{du}{dx}$$

$$\therefore dx = \frac{1}{\cos^2 u} du$$

Integrating both sides, we get

$$\int dx = \int \sec^2 u du$$

$$\therefore x = \tan u + c$$

$$\therefore x = \tan(x - 2y) + c$$

This is the general solution.

$$(v) (2x - 2y + 3) dx - (x - y + 1) dy = 0, \text{ when } x = 0, y = 1$$

Solution:

$$(2x - 2y + 3) dx - (x - y + 1) dy = 0$$

$$\therefore (x - y + 1) dy = (2x - 2y + 3) dx$$

$$\therefore \frac{dy}{dx} = 2(x-y) + 3(x-y) + 1 \dots\dots\dots(1)$$

Put $x - y = u$, Then $1 - \frac{dy}{dx} = \frac{du}{dx}$

$$\therefore \frac{dy}{dx} = 1 - \frac{du}{dx}$$

$$\therefore (1) \text{ becomes, } 1 - \frac{du}{dx} = \frac{2u + 3}{u + 1}$$

$$\therefore \frac{du}{dx} = 1 - \frac{2u + 3}{u + 1} = \frac{u + 1 - 2u - 3}{u + 1}$$

$$\therefore \frac{du}{dx} = \frac{-u - 2}{u + 1} = -\left(\frac{u + 2}{u + 1}\right)$$

$$\therefore \frac{u + 1}{u + 2} du = -dx$$

Integrating both sides, we get

$$\int \frac{u + 1}{u + 2} du = - \int 1 dx$$

$$\therefore \int \frac{(u + 2) - 1}{u + 2} du = - \int 1 dx$$

$$\therefore \int \left(1 - \frac{1}{u + 2}\right) du = - \int 1 dx$$

$$\therefore u - \log|u + 2| = -x + c$$

$$\therefore x - y - \log|x - y + 2| = -x + c$$

$$\therefore (2x - y) - \log|x - y + 2| = c$$

This is the general solution.

Now, $y = 1$, when $x = 0$.

$$\therefore (0 - 1) - \log|0 - 1 + 2| = c$$

$$\therefore -1 - 0 = c$$

$$\therefore c = -1$$

∴ the particular solution is

$$(2x - y) - \log|x - y + 2| = -1$$

$$\therefore (2x - y) - \log|x - y + 2| + 1 = 0$$

Maharashtra State Board 12th Maths Solutions Chapter 6 Differential Equations Ex 6.4

I. Solve the following differential equations:

Question 1.

$$x \sin(yx) dy = [y \sin(yx) - x] dx$$

Solution:

$$x \sin\left(\frac{y}{x}\right) dy = \left[y \sin\left(\frac{y}{x}\right) - x \right] dx$$

$$\therefore \frac{dy}{dx} = \frac{y \sin\left(\frac{y}{x}\right) - x}{x \sin\left(\frac{y}{x}\right)} \quad \dots (1)$$

$$\text{Put } y = vx \quad \therefore \frac{dy}{dx} = v + x \frac{dv}{dx} \text{ and } \frac{y}{x} = v$$

$$\therefore (1) \text{ becomes, } v + x \frac{dv}{dx} = \frac{vx \sin v - x}{x \sin v}$$

$$\therefore x \frac{dv}{dx} = \frac{v \sin v - 1}{\sin v} - v$$

$$\therefore x \frac{dv}{dx} = \frac{v \sin v - 1 - v \sin v}{\sin v} = \frac{-1}{\sin v}$$

$$\therefore \sin v dv = -\frac{1}{x} dx$$

Integrating both sides, we get

$$\int \sin v dv = - \int \frac{1}{x} dx$$

$$\therefore -\cos v = -\log x - c$$

$$\therefore \cos\left(\frac{y}{x}\right) = \log x + c$$

This is the general solution.

Question 2.

$$(x_2 + y_2) dx - 2xy . dy = 0$$

Solution:

$$(x_2 + y_2) dx - 2xy dy = 0$$

$$\therefore 2xy dy = (x_2 + y_2) dx$$

$$\therefore dy/dx = x_2 + y_2 / 2xy \dots\dots\dots(1)$$

$$\text{Put } y = vx \quad \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\therefore (1) \text{ becomes, } v + x \frac{dv}{dx} = \frac{x^2 + v^2 x^2}{2x(vx)}$$

$$\therefore v + x \frac{dv}{dx} = \frac{1 + v^2}{2v}$$

$$\therefore x \frac{dv}{dx} = \frac{1 + v^2}{2v} - v = \frac{1 + v^2 - 2v^2}{2v}$$

$$\therefore x \frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\therefore \frac{2v}{1-v^2} dv = \frac{1}{x} dx$$

Integrating both sides, we get

$$\int \frac{2v}{1-v^2} dv = \int \frac{1}{x} dx$$

$$-\int \frac{-2v}{1-v^2} dv = \int \frac{1}{x} dx$$

$$\therefore -\log|1-v^2| = \log x + \log c_1$$

$$\dots \left[\because \frac{d}{dv}(1-v^2) = -2v \text{ and } \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c \right]$$

$$\therefore \log \left| \frac{1}{1-v^2} \right| = \log c_1 x$$

$$\therefore \log \left| \frac{1}{1-\left(\frac{y^2}{x^2}\right)} \right| = \log c_1 x$$

$$\therefore \log \left| \frac{x^2}{x^2-y^2} \right| = \log c_1 x$$

$$\therefore \frac{x^2}{x^2-y^2} = c_1 x$$

$$\therefore x^2 - y^2 = \frac{1}{c_1} x$$

$$\therefore x^2 - y^2 = cx, \text{ where } c = \frac{1}{c_1}$$

This is the general solution.

Question 3.

$$(1+2e^{xy}) + 2e^{xy}(1-xy)dy/dx = 0$$

Solution:

$$(1 + 2e^y) + 2e^y \left(1 - \frac{x}{y}\right) \frac{dy}{dx} = 0$$

$$\therefore (1 + 2e^y) + 2e^y \left(1 - \frac{x}{y}\right) \cdot \frac{1}{\left(\frac{dx}{dy}\right)} = 0$$

$$\therefore (1 + 2e^y) \frac{dx}{dy} + 2e^y \left(1 - \frac{x}{y}\right) = 0 \quad \dots (1)$$

$$\text{Put } \frac{x}{y} = u \quad \therefore x = uy$$

$$\therefore \frac{dx}{dy} = u + y \frac{du}{dy}$$

$$\therefore (1) \text{ becomes, } (1 + 2e^u) \left(u + y \frac{du}{dy}\right) + 2e^u (1 - u) = 0$$

$$\therefore u + 2ue^u + y \left(1 + 2e^u\right) \frac{du}{dy} + 2e^u - 2ue^u = 0$$

$$\therefore (u + 2e^u) + y(1 + 2e^u) \frac{du}{dy} = 0$$

$$\therefore \frac{dy}{y} + \frac{1 + 2e^u}{u + 2e^u} du = 0$$

Integrating both sides, we get

$$\int \frac{1}{y} dy + \int \frac{1 + 2e^u}{u + 2e^u} du = c_1$$

$$\therefore \log|y| + \log|u + 2e^u| = \log c, \text{ where } c_1 = \log c$$

$$\dots \left[\because \frac{d}{du}(u + 2e^u) = 1 + 2e^u \text{ and} \right]$$

$$\int \frac{f'(u)}{f(u)} du = \log |f(u)| + c$$

$$\therefore \log|y(u + 2e^u)| = \log c$$

$$\therefore y(u + 2e^u) = c$$

$$\therefore y \left(\frac{x}{y} + 2e^y \right) = c$$

$$\therefore x + 2ye^y = c$$

This is the general solution.

Question 4.

$$y_2 dx + (xy + x_2) dy = 0$$

Solution:

$$y_2 dx + (xy + x_2) dy = 0$$

$$\therefore (xy + x_2) dy = -y_2 dx$$

$$\therefore dy/dx = -y_2/x_2 \dots\dots(1)$$

Put $y = vx$

$$\therefore dy/dx = v + x dv/dx$$

Substituting these values in (1), we get

$$v + x \frac{dv}{dx} = \frac{-v^2x^2}{x \cdot vx + x^2} = \frac{-v^2}{v+1}$$

$$\therefore x \frac{dv}{dx} = \frac{-v^2}{v+1} - v = \frac{-v^2 - v^2 - v}{v+1}$$

$$\therefore x \frac{dv}{dx} = \frac{-2v^2 - v}{v+1} = -\left(\frac{2v^2 + v}{v+1}\right)$$

$$\therefore \frac{v+1}{2v^2+v} dv = -\frac{1}{x} dx$$

Integrating both sides, we get

$$\int \frac{v+1}{2v^2+v} dv = - \int \frac{1}{x} dx$$

$$\therefore \int \frac{v+1}{v(2v+1)} dv = - \int \frac{1}{x} dx$$

$$\therefore \int \frac{(2v+1)-v}{v(2v+1)} dv = - \int \frac{1}{x} dx$$

$$\therefore \int \left(\frac{1}{v} - \frac{1}{2v+1} \right) dv = - \int \frac{1}{x} dx$$

$$\therefore \int \frac{1}{v} dv - \int \frac{1}{2v+1} dv = - \int \frac{1}{x} dx$$

$$\therefore \log|v| - \frac{1}{2} \log|2v+1| = -\log|x| + \log c$$

$$\therefore 2\log|v| - \log|2v+1| = -2\log|x| + 2\log c$$

$$\therefore \log|v^2| - \log|2v+1| = -\log|x^2| + \log c^2$$

$$\therefore \log \left| \frac{v^2}{2v+1} \right| = \log \left| \frac{c^2}{x^2} \right|$$

$$\therefore \frac{v^2}{2v+1} = \frac{c^2}{x^2}$$

$$\therefore \frac{\left(\frac{y^2}{x^2}\right)}{2\left(\frac{y}{x}\right)+1} = \frac{c^2}{x^2} \quad \therefore \frac{y^2}{x(2y+x)} = \frac{c^2}{x^2}$$

$$\therefore xy^2 = c^2(x+2y)$$

This is the general solution.

Question 5.

$$(x_2 - y_2) dx + 2xy dy = 0$$

Solution:

$$(x^2 - y^2)dx + 2xy dy = 0$$

$$\therefore -2xy dy = (x^2 - y^2)dx$$

$$\therefore \frac{dy}{dx} = \frac{x^2 - y^2}{-2xy} \quad \dots(1)$$

put $y = vx$

$$\therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\therefore (1) \text{ becomes, } v + x \frac{dv}{dx} = \frac{x^2 - v^2 x^2}{-2x(vx)}$$

$$\therefore v + x \frac{dv}{dx} = \frac{1 - v^2}{-2v}$$

$$\therefore x \frac{dv}{dx} = \frac{1 - v^2}{-2v} - v = \frac{1 - v^2 + 2v^2}{-2v}$$

$$\therefore x \frac{dv}{dx} = \frac{1 + v^2}{-2v}$$

$$\therefore \frac{-2v}{1 + v^2} dv = \frac{1}{x} dx$$

Integrating both sides, we get

$$\therefore \int \frac{-2v}{1 + v^2} dv = \int \frac{1}{x} dx$$

$$\therefore \log|1 + v^2| = \log x + \log c_1$$

$$\dots \left[\because \frac{d}{dx}(1 + v^2) = 2v \text{ and } \int \left[\frac{f'(x)}{f(x)} dx = \log|f(x)| + c \right] \right]$$

$$\therefore \log \left| \frac{1}{1 + v^2} \right| = \log c_1 x$$

$$\therefore \log \left| \frac{1}{1 + \left(\frac{y^2}{x^2} \right)} \right| = \log c_1 x$$

$$\therefore \log \left| \frac{x^2}{x^2 + y^2} \right| = \log c_1 x$$

$$\therefore \frac{x^2}{x^2 + y^2} = c_1 x$$

$$\therefore x^2 + y^2 = \frac{1}{c_1} x$$

$$\therefore x^2 + y^2 = cx \text{ where } c = \frac{1}{c_1}$$

This is the general solution.

Question 6.

$$dy/dx + x - 2y/2x - y = 0$$

Solution:

$$\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$$

$$\therefore \frac{dy}{dx} = -\left(\frac{x-2y}{2x-y}\right) \quad \dots\dots\dots(1)$$

$$\text{Put } y = vx. \quad \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\therefore (1) \text{ becomes, } v + x \frac{dv}{dx} = -\left(\frac{x-2vx}{2x-vx}\right)$$

$$\therefore v + x \frac{dv}{dx} = -\left(\frac{1-2v}{2-v}\right)$$

$$\therefore x \frac{dv}{dx} = -\left(\frac{1-2v}{2-v}\right) - v$$

$$\therefore x \frac{dv}{dx} = \frac{-1+2v-2v+v^2}{2-v}$$

$$\therefore x \frac{dv}{dx} = \frac{v^2-1}{2-v}$$

$$\therefore \frac{2-v}{v^2-1} dv = \frac{1}{x} dx$$

Integrating both sides, we get

$$\int \frac{2-v}{v^2-1} dv = \int \frac{1}{x} dx$$

$$\therefore 2 \int \frac{1}{v^2-1} dv - \frac{1}{2} \int \frac{2v}{v^2-1} dv = \int \frac{1}{x} dx$$

$$\therefore 2 \times \frac{1}{2} \log \left| \frac{v-1}{v+1} \right| - \frac{1}{2} \log |v^2-1| = \log |x| + \log c_1$$

$$\dots \left[\because \frac{d}{dv} (v^2-1) = 2v \text{ and } \int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c \right]$$

$$\therefore \log \left| \frac{v-1}{v+1} \right| - \log |(v^2-1)^{\frac{1}{2}}| = \log |c_1 x|$$

$$\therefore \log \left| \frac{v-1}{v+1} \cdot \frac{1}{\sqrt{v^2-1}} \right| = \log |c_1 x|$$

$$\therefore \frac{v-1}{v+1} \cdot \frac{1}{\sqrt{v^2-1}} = c_1 x$$

$$\therefore \frac{\frac{y}{x}-1}{\frac{y}{x}+1} \cdot \frac{1}{\sqrt{\frac{y^2}{x^2}-1}} = c_1 x$$

$$\therefore \frac{y-x}{y+x} \cdot \frac{x}{\sqrt{y^2-x^2}} = c_1 x$$

$$\therefore \frac{y-x}{y+x} = c_1 \sqrt{y^2-x^2}$$

$$\therefore \frac{y-x}{y+x} = c_1 \sqrt{y-x} \cdot \sqrt{y+x}$$

$$\therefore \sqrt{y-x} = c_1 (y+x)^{\frac{3}{2}}$$

$$\therefore y-x = c_1^2 (x+y)^3$$

$$\therefore y-x = c(x+y)^3, \text{ where } c=c_1^2$$

$$\therefore y = c(x+y)^3 + x$$

This is the general solution.

Question 7.

$$x \frac{dy}{dx} - y + x \sin(y/x) = 0$$

Solution:

$$x \frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0 \quad (1)$$

$$\text{Put } y = vx \quad \therefore \frac{dy}{dx} = v + x \frac{dv}{dx} \text{ and } \frac{y}{x} = v$$

$$\therefore (1) \text{ becomes, } x\left(v + x \frac{dv}{dx}\right) - vx + x \sin v = 0$$

$$\therefore vx + x^2 \frac{dv}{dx} - vx + x \sin v = 0$$

$$\therefore x^2 \frac{dv}{dx} + x \sin v = 0$$

$$\therefore \frac{1}{\sin v} dv + \frac{1}{x} dx = 0$$

Integrating, we get

$$\int \csc v dv + \int \frac{1}{x} dx = c_1$$

$$\therefore \log |\csc v - \cot v| + \log |x| = \log c, \text{ where } c_1 = \log c$$

$$\therefore \log |x(\csc v - \cot v)| = \log c$$

$$\therefore x\left(\frac{1}{\sin v} - \frac{\cos v}{\sin v}\right) = c$$

$$\therefore x(1 - \cos v) = c \sin v$$

$$\therefore x\left[1 - \cos\left(\frac{y}{x}\right)\right] = c \sin\left(\frac{y}{x}\right)$$

This is the general solution.

Question 8.

$$(1+e_{xy})dx + e_{xy}(1-x_y)dy = 0$$

Solution:

$$(1 + e^{x/y}) dx + e^{x/y} \left(1 - \frac{x}{y}\right) dy = 0$$

$$\therefore (1 + e^{x/y}) \frac{dx}{dy} + e^{x/y} \left(1 - \frac{x}{y}\right) = 0 \quad \dots (1)$$

$$\text{Put } \frac{x}{y} = u \quad \therefore x = uy \quad \therefore \frac{dx}{dy} = u + y \frac{du}{dy}$$

$$\therefore (1) \text{ becomes, } (1 + e^u) \left(u + y \frac{du}{dy}\right) + e^u (1 - u) = 0$$

$$\therefore u + ue^u + y(1 + e^u) \frac{du}{dy} + e^u - ue^u = 0$$

$$\therefore (u + e^u) + y(1 + e^u) \frac{du}{dy} = 0$$

$$\therefore \frac{dy}{y} + \frac{1 + e^u}{u + e^u} du = 0$$

$$\therefore \int \frac{dy}{y} + \int \frac{1 + e^u}{u + e^u} du = c_1 \quad \dots (2)$$

$$\frac{d}{du} (u + e^u) = 1 + e^u \text{ and } \int \frac{f'(u)}{f(u)} du = \log|f(u)| + c$$

\therefore from (2), the general solution is

$$\log|y| + \log|u + e^u| = \log c, \text{ where } c_1 = \log c$$

$$\therefore \log|y(u + e^u)| = \log c \quad \therefore y(u + e^u) = c$$

$$\therefore y \left(\frac{x}{y} + e^{x/y} \right) = c \quad \therefore x + ye^{x/y} = c$$

This is the general solution.

Question 9.

$$y^2 - x^2 dy/dx = xy dy/dx$$

Solution:

$$y^2 - x^2 \frac{dy}{dx} = xy \frac{dy}{dx}$$

$$\therefore x^2 \frac{dy}{dx} + xy \frac{dy}{dx} = y^2$$

$$\therefore (x^2 + xy) \frac{dy}{dx} = y^2$$

$$\therefore \frac{dy}{dx} = \frac{y^2}{x^2 + xy} \quad \dots (1)$$

$$\text{Put } y = vx \quad \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\therefore (1) \text{ becomes, } v + x \frac{dv}{dx} = \frac{v^2 x^2}{x^2 + x \cdot vx} = \frac{v^2}{1+v}$$

$$\therefore x \frac{dv}{dx} = \frac{v^2}{1+v} - v = \frac{v^2 - v - v^2}{1+v}$$

$$\therefore x \frac{dv}{dx} = \frac{-v}{1+v} \quad \therefore \frac{1+v}{v} dv = -\frac{1}{x} dx$$

Integrating, we get

$$\int \left(\frac{1+v}{v} \right) dv = - \int \frac{1}{x} dx$$

$$\int \left(\frac{1}{v} + 1 \right) dv = - \int \frac{1}{x} dx$$

$$\therefore \int \frac{1}{v} dv + \int 1 dv = - \int \frac{1}{x} dx$$

$$\therefore \log|v| + v = -\log|x| + c$$

$$\therefore \log \left| \frac{y}{x} \right| + \frac{y}{x} = -\log|x| + c$$

$$\therefore \log|y| - \log|x| + \frac{y}{x} = -\log|x| + c$$

$$\therefore \frac{y}{x} + \log|y| = c$$

This is the general solution.

Question 10.

$$xy \frac{dy}{dx} = x^2 + 2y^2, y(1) = 0$$

Solution:

$$xy \frac{dy}{dx} = x^2 + 2y^2$$

$$\therefore \frac{dy}{dx} = \frac{x^2 + 2y^2}{xy} \quad \dots (1)$$

Put $y = vx$. Then $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\therefore (1) \text{ becomes, } v + x \frac{dv}{dx} = \frac{x^2 + 2v^2 x^2}{x \cdot vx} = \frac{1 + 2v^2}{v}$$

$$\therefore x \frac{dv}{dx} = \frac{1 + 2v^2}{v} - v = \frac{1 + 2v^2 - v^2}{v}$$

$$\therefore x \frac{dv}{dx} = \frac{1 + v^2}{v}$$

$$\therefore \frac{v}{1 + v^2} dv = \frac{1}{x} dx$$

$$\int \frac{v}{1 + v^2} dv = \int \frac{1}{x} dx$$

$$\therefore \frac{1}{2} \int \frac{2v}{1 + v^2} dv = \int \frac{1}{x} dx + \log c_1$$

$$\therefore \frac{1}{2} \log |1 + v^2| = \log |x| + \log c_1$$

$$\therefore \log |1 + v^2| = 2 \log |x| + 2 \log c_1$$

$$\therefore \log |1 + v^2| = \log |x^2| + \log c_1^2$$

$$\therefore \log |1 + v^2| = \log |cx^2|, \text{ where } c = c_1^2$$

$$\therefore 1 + v^2 = cx^2$$

$$\therefore 1 + \frac{y^2}{x^2} = cx^2$$

$$\therefore \frac{x^2 + y^2}{x^2} = cx^2$$

$$\therefore x^2 + y^2 = cx^4$$

This is the general solution.

Now, $y(1) = 0$, i.e. when $x = 1$, $y = 0$, we get

$$1 + 0 = c(1) \quad \therefore c = 1$$

\therefore the particular solution is $x^2 + y^2 = x^4$.

Question 11.

$x dy + 2y \cdot dx = 0$, when $x = 2$, $y = 1$

Solution:

$$\therefore x dy + 2y \cdot dx = 0$$

$$\therefore x dy = -2y dx$$

$$\therefore xy dy = -2x dx$$

Integrating, we get

$$\int \frac{1}{y} dy = -2 \int \frac{1}{x} dx$$

$$\therefore \log |y| = -2 \log |x| + \log c$$

$$\therefore \log |y| = -\log |x^2| + \log c$$

$$\therefore \log |y| = \log \left| \frac{c}{x^2} \right|$$

$$\therefore y = \frac{c}{x^2} \quad \therefore x^2 y = c$$

This is the general solution.

When $x = 2, y = 1$, we get

$$4(1) = c$$

$$\therefore c = 4$$

\therefore the particular solution is $x^2y = 4$.

Question 12.

$$x^2 \frac{dy}{dx} = x^2 + xy + y^2$$

Solution:

$$x^2 \frac{dy}{dx} = x^2 + xy + y^2$$

$$\therefore dy/dx = x^2 + xy + y^2 \dots\dots\dots(1)$$

Put $y = vx$

$$\therefore dy/dx = v + x \frac{dv}{dx}$$

$$\therefore (1) \text{ becomes, } v + x \frac{dv}{dx} = \frac{x^2 + x \cdot vx + v^2 x^2}{x^2}$$

$$\therefore v + x \frac{dv}{dx} = 1 + v + v^2$$

$$\therefore x \frac{dv}{dx} = 1 + v^2$$

$$\therefore \frac{1}{1+v^2} dv = \frac{1}{x} dx$$

Integrating, we get

$$\int \frac{1}{1+v^2} dv = \int \frac{1}{x} dx + c$$

$$\therefore \tan^{-1} v = \log|x| + c$$

$$\therefore \tan^{-1} \left(\frac{y}{x} \right) = \log|x| + c$$

This is the general solution.

Question 13.

$$(9x + 5y) dy + (15x + 11y) dx = 0$$

Solution:

$$(9x + 5y) dy + (15x + 11y) dx = 0$$

$$\therefore (9x + 5y) dy = -(15x + 11y) dx$$

$$\therefore dy/dx = -(15x+11y)/(9x+5y) \dots\dots\dots(1)$$

Put $y = vx$

$$\therefore dy/dx = v + x \frac{dv}{dx}$$

$$\therefore (1) \text{ becomes, } v + x \frac{dv}{dx} = \frac{-(15x + 11vx)}{9x + 5vx}$$

$$\therefore v + x \frac{dv}{dx} = \frac{-(15 + 11v)}{9 + 5v}$$

$$\therefore x \frac{dv}{dx} = \frac{-(15 + 11v)}{9 + 5v} - v = \frac{-15 - 11v - 9v - 5v^2}{9 + 5v}$$

$$\therefore x \frac{dv}{dx} = \frac{-5v^2 - 20v - 15}{9 + 5v} = -\left(\frac{5v^2 + 20v + 15}{5v + 9}\right)$$

$$\therefore \frac{5v + 9}{5v^2 + 20v + 15} dv = -\frac{1}{x} dx \quad \dots (2)$$

Integrating, we get

$$\frac{1}{5} \int \frac{5v + 9}{v^2 + 4v + 3} dv = - \int \frac{1}{x} dx$$

$$\text{Let } \frac{5v + 9}{v^2 + 4v + 3} = \frac{5v + 9}{(v + 3)(v + 1)} = \frac{A}{v + 3} + \frac{B}{v + 1}$$

$$\therefore 5v + 9 = A(v + 1) + B(v + 3)$$

Put $v + 3 = 0$, i.e. $v = -3$, we get

$$-15 + 9 = A(-2) + B(0)$$

$$\therefore -6 = -2A \quad \therefore A = 3$$

Put $v + 1 = 0$, i.e. $v = -1$, we get

$$-5 + 9 = A(0) + B(2)$$

$$\therefore 4 = 2B \quad \therefore B = 2$$

$$\therefore \frac{5v + 9}{v^2 + 4v + 3} = \frac{3}{v + 3} + \frac{2}{v + 1}$$

∴ (2) becomes,

$$\frac{1}{5} \int \left(\frac{3}{v + 3} + \frac{2}{v + 1} \right) dv = - \int \frac{1}{x} dx$$

$$\therefore \frac{3}{5} \int \frac{1}{v + 3} dv + \frac{2}{5} \int \frac{1}{v + 1} dv = - \int \frac{1}{x} dx$$

$$\therefore \frac{3}{5} \log|v + 3| + \frac{2}{5} \log|v + 1| = -\log|x| + c_1$$

$$\therefore 3 \log|v + 3| + 2 \log|v + 1| = -5 \log x + 5c_1$$

$$\therefore \log|(v + 3)^3| + \log|(v + 1)^2| = -\log|x^5| + \log c,$$

where $5c_1 = \log c$

$$\therefore \log|(v + 3)^3(v + 1)^2| = \log \left| \frac{c}{x^5} \right|$$

$$\therefore (v + 3)^3(v + 1)^2 = \frac{c}{x^5}$$

$$\therefore \left(\frac{y}{x} + 3\right)^3 \left(\frac{y}{x} + 1\right)^2 = \frac{c}{x^5}$$

$$\therefore \frac{(y + 3x)^3}{x^3} \times \frac{(y + x)^2}{x^2} = \frac{c}{x^5}$$

$$\therefore (x + y)^2(3x + y)^3 = c$$

This is the general solution.

Question 14.

$$(x_2 + 3xy + y_2) dx - x_2 dy = 0$$

Solution:

$$(x_2 + 3xy + y_2) dx - x_2 dy = 0$$

$$\therefore x_2 dy = (x_2 + 3xy + y_2) dx$$

$$\therefore dy/dx = x^2 + 3xy + y^2x^2 \dots\dots\dots(1)$$

$$\text{Put } y = vx \quad \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\therefore (1) \text{ becomes, } v + x \frac{dv}{dx} = \frac{x^2 + 3x \cdot vx + v^2 x^2}{x^2}$$

$$\therefore v + x \frac{dv}{dx} = 1 + 3v + v^2$$

$$x \frac{dv}{dx} = v^2 + 2v + 1 = (v + 1)^2$$

$$\therefore \frac{1}{(v + 1)^2} dv = \frac{1}{x} dx$$

Integrating, we get

$$\int (v + 1)^{-2} dv = \int \frac{1}{x} dx$$

$$\therefore \frac{(v + 1)^{-1}}{-1} = \log|x| + c_1$$

$$\therefore -\frac{1}{v + 1} = \log|x| + c_1$$

$$\therefore -\frac{1}{\frac{y}{x} + 1} = \log|x| + c_1$$

$$\therefore -\frac{x}{y + x} = \log|x| + c_1$$

$$\therefore \log|x| + \frac{x}{x + y} = -c_1$$

$$\therefore \log|x| + \frac{x}{x + y} = c, \text{ where } c = -c_1$$

This is the general solution.

Question 15.

$$(x^2 + y^2) dx - 2xy dy = 0.$$

Solution:

$$(x^2 + y^2) dx - 2xy dy = 0$$

$$\therefore 2xy dy = (x^2 + y^2) dx$$

$$\therefore \frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \dots\dots\dots(1)$$

$$\text{Put } y = vx$$

$$\therefore \frac{dy}{dx} = v + x du/dx$$

$$\therefore (1) \text{ becomes, } v + x \frac{du}{dx} = \frac{x^2 + v^2 x^2}{2x(vx)}$$

$$\therefore v + x \frac{du}{dx} = \frac{1 + v^2}{2v}$$

$$\therefore x \frac{dv}{dx} = \frac{1 + v^2}{2v} - v = \frac{1 + v^2 - 2v^2}{2v}$$

$$\therefore x \frac{dv}{dx} = \frac{1 - v^2}{2v}$$

$$\therefore \frac{2v}{1-v^2} dv = \frac{1}{x} dx$$

Integrating both sides, we get

$$\int \frac{2v}{1-v^2} dv = \int \frac{1}{x} dx$$

$$-\int \frac{-2v}{1-v^2} dv = \int \frac{1}{x} dx$$

$$\therefore -\log|1-v^2| = \log x + \log c_1 \dots \left[\because \frac{d}{dv}(1-v^2) = -2v \text{ and } \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c \right]$$

$$\therefore \log \left| \frac{1}{1-v^2} \right| = \log c_1 x$$

$$\therefore \log \left| \frac{1}{1 - \left(\frac{y^2}{x^2} \right)} \right| = \log c_1 x$$

$$\therefore \log \left| \frac{x^2}{x^2 - y^2} \right| = \log c_1 x$$

$$\therefore \frac{x^2}{x^2 - y^2} = c_1 x$$

$$\therefore x^2 - y^2 = \frac{1}{c_1} x$$

$$\therefore x^2 - y^2 = cx, \text{ where } c = \frac{1}{c_1}$$

This is the general solution.

Maharashtra State Board 12th Maths Solutions Chapter 6 Differential Equations Ex 6.5

Question 1.

Solve the following differential equations:

(i) $dy/dx + yx = x^3 - 3$

Solution:

$$dy/dx + yx = x^3 - 3 \dots\dots(1)$$

This is the linear differential equation of the form $dy/dx + P \cdot y = Q$, where $P = x$ and $Q = x^3 - 3$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x} dx}$$

$$= e^{\log x} = x$$

\therefore the solution of (1) is given by

$$y(\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c_1$$

$$\therefore y \cdot x = \int (x^3 - 3x) dx + c_1$$

$$\therefore xy = \int (x^4 - 3x) dx + c_1$$

$$\therefore xy = \frac{x^5}{5} - 3 \cdot \frac{x^2}{2} + c_1$$

$$\therefore \frac{x^2}{5} - \frac{3x^2}{2} - xy = c, \text{ where } c = -c_1$$

This is the general solution.

$$(ii) \cos^2 x \cdot dy/dx + y = \tan x$$

Solution:

$$\cos^2 x \cdot \frac{dy}{dx} + y = \tan x$$

$$\therefore \frac{dy}{dx} + \frac{1}{\cos^2 x} \cdot y = \frac{\tan x}{\cos^2 x}$$

$$\therefore \frac{dy}{dx} + \sec^2 x \cdot y = \tan x \cdot \sec^2 x \quad \dots (1)$$

This is the linear differential equation of the form

$$\frac{dy}{dx} + P \cdot y = Q, \text{ where } P = \sec^2 x \text{ and } Q = \tan x \cdot \sec^2 x$$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int \sec^2 x dx} = e^{\tan x}$$

\therefore the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$$

$$\therefore y \cdot e^{\tan x} = \int \tan x \cdot \sec^2 x \cdot e^{\tan x} dx + c$$

$$\text{Put } \tan x = t \quad \therefore \sec^2 x dx = dt$$

$$\therefore y \cdot e^{\tan x} = \int t \cdot e^t dt + c$$

$$\therefore y \cdot e^{\tan x} = t \int e^t dt - \left[\frac{d}{dt}(t) \int e^t dt \right] dt + c$$

$$= t \cdot e^t - \int 1 \cdot e^t dt + c$$

$$= t \cdot e^t - e^t + c$$

$$= e^t(t - 1) + c$$

$$\therefore y \cdot e^{\tan x} = e^{\tan x}(\tan x - 1) + c$$

This is the general solution.

$$(iii) (x + 2y^3) dy/dx = y$$

Solution:

$$(x + 2y^3) \frac{dy}{dx} = y$$

$$\therefore \frac{x + 2y^3}{y} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

$$\therefore \frac{x}{y} + 2y^2 = \frac{dx}{dy}$$

$$\therefore \frac{dx}{dy} - \frac{1}{y} \cdot x = 2y^2 \quad \dots (1)$$

This is the linear differential equation of the form

$$\frac{dx}{dy} + P \cdot x = Q, \text{ where } P = -\frac{1}{y} \text{ and } Q = 2y^2$$

$$\therefore \text{I.F.} = e^{\int P dy} = e^{\int -\frac{1}{y} dy}$$

$$= e^{-\log y} = e^{\log \left(\frac{1}{y}\right)} = \frac{1}{y}$$

\therefore the solution of (1) is given by

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dy + c$$

$$\therefore x \left(\frac{1}{y}\right) = \int 2y^2 \times \frac{1}{y} dy + c$$

$$\therefore \frac{x}{y} = 2 \int y dy + c$$

$$\therefore \frac{x}{y} = 2 \cdot \frac{y^2}{2} + c \quad \therefore x = y(c + y^2)$$

This is the general solution.

$$(iv) dy/dx + y \cdot \sec x = \tan x$$

Solution:

$$dy/dx + y \sec x = \tan x$$

$$\therefore dy/dx + (\sec x) \cdot y = \tan x \dots\dots(1)$$

This is the linear differential equation of the form

$$dy/dx + P \cdot y = Q, \text{ where } P = \sec x \text{ and } Q = \tan x$$

$$\therefore \text{I.F.} = e^{\int P dx}$$

$$= e^{\int \sec x dx}$$

$$= e^{\log(\sec x + \tan x)}$$

$$= \sec x + \tan x$$

\therefore the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$$

$$\therefore y(\sec x + \tan x) = \int \tan x (\sec x + \tan x) dx + c$$

$$\therefore (\sec x + \tan x) \cdot y = \int (\sec x \tan x + \tan^2 x) dx + c$$

$$\therefore (\sec x + \tan x) \cdot y = \int (\sec x \tan x + \sec 2x - 1) dx + c$$

$$\therefore (\sec x + \tan x) \cdot y = \sec x + \tan x - x + c$$

$$\therefore y(\sec x + \tan x) = \sec x + \tan x - x + c$$

This is the general solution.

$$(v) x \cdot dy/dx + 2y = x^2 \cdot \log x$$

Solution:

$$x \frac{dy}{dx} + 2y = x^2 \cdot \log x$$

$$\therefore \frac{dy}{dx} + \left(\frac{2}{x}\right) \cdot y = x \cdot \log x \quad \dots\dots(1)$$

This is the linear differential equation of the form

$$\frac{dy}{dx} + P \cdot y = Q, \text{ where } P = \frac{2}{x} \text{ and } Q = x \cdot \log x$$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2 \int \frac{1}{x} dx}$$

$$= e^{2 \log x} = e^{\log x^2} = x^2$$

\therefore the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$$

$$\therefore y \cdot x^2 = \int (x \log x) \cdot x^2 dx + c$$

$$\begin{aligned} \therefore x^2 \cdot y &= \int x^3 \cdot \log x dx + c \\ &= (\log x) \int x^3 dx - \int \left[\frac{d}{dx} (\log x) \int x^3 dx \right] dx + c \end{aligned}$$

$$= (\log x) \cdot \frac{x^4}{4} - \int \frac{1}{x} \cdot \frac{x^4}{4} dx + c$$

$$= \frac{1}{4} x^4 \log x - \frac{1}{4} \int x^3 dx + c$$

$$\therefore x^2 \cdot y = \frac{1}{4} x^4 \log x - \frac{1}{4} \cdot \frac{x^4}{4} + c$$

$$\therefore x^2 y = \frac{x^4 \log x}{4} - \frac{x^2}{16} + c$$

This is the general solution.

$$(vi) (x + y) dy/dx = 1$$

Solution:

$$(x + y) dy/dx = 1$$

$$\therefore dx dy = x + y$$

$$\therefore dx dy - x = y$$

$$\therefore dx dy + (-1)x = y \dots\dots(1)$$

This is the linear differential equation of the form

$$\frac{dx}{dy} + P \cdot x = Q, \text{ where } P = -1 \text{ and } Q = y$$

$$\therefore \text{I.F.} = e^{\int P dy} = e^{\int -1 dy} = e^{-y}$$

\therefore the solution of (1) is given by

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dy + c$$

$$\therefore x \cdot e^{-y} = \int y \cdot e^{-y} dy + c$$

$$\therefore e^{-y} \cdot x = y \int e^{-y} dy - \left[\frac{d}{dy} (y) \int e^{-y} dy \right] dy + c$$

$$= y \cdot \frac{e^{-y}}{-1} - \int 1 \cdot \frac{e^{-y}}{-1} dy + c$$

$$= -ye^{-y} + \int e^{-y} dy + c$$

$$\therefore e^{-y} \cdot x = -ye^{-y} + \frac{e^{-y}}{-1} + c$$

$$\therefore e^{-y} \cdot x + ye^{-y} + e^{-y} = c$$

$$\therefore e^{-y} (x + y + 1) = c$$

$$\therefore x + y + 1 = ce^y$$

This is the general solution.

$$(vii) (x + a) dy/dx - 3y = (x + a)^5$$

Solution:

$$(x + a) \frac{dy}{dx} - 3y = (x + a)^5$$

$$\therefore \frac{dy}{dx} - \frac{3y}{x+a} = (x+a)^4$$

$$\therefore \frac{dy}{dx} + \left(\frac{-3}{x+a} \right) y = (x+a)^4 \quad \dots\dots(1)$$

This is the linear differential equation of the form

$$\frac{dy}{dx} + P \cdot y = Q, \text{ where } P = \frac{-3}{x+a} \text{ and } Q = (x+a)^4$$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int \frac{-3}{x+a} dx} = e^{-3 \int \frac{1}{x+a} dx}$$

$$= e^{-3 \log|x+a|} = e^{\log(x+a)^{-3}}$$

$$= (x+a)^{-3} = \frac{1}{(x+a)^3}$$

\therefore the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$$

$$\therefore y \cdot \frac{1}{(x+a)^3} = \int (x+a)^4 \cdot \frac{1}{(x+a)^3} dx + c$$

$$\therefore \frac{y}{(x+a)^3} = \int (x+a) dx + c$$

$$\therefore \frac{y}{(x+a)^3} = \frac{(x+a)^2}{2} + c$$

$$\therefore 2y = (x+a)^5 + 2c(x+a)^3$$

This is the general solution.

$$(viii) dr + (2r \cot \theta + \sin 2\theta) d\theta = 0$$

Solution:

$$dr + (2r \cot \theta + \sin 2\theta) d\theta = 0$$

$$\therefore dr/d\theta + (2r \cot \theta + \sin 2\theta) = 0$$

$$\therefore dr/d\theta + (2 \cot \theta)r = -\sin 2\theta \dots\dots\dots(1)$$

This is the linear differential equation of the form $dr/d\theta + P \cdot r = Q$, where $P = 2 \cot \theta$ and $Q = -\sin 2\theta$

$$\therefore I.F. = e^{\int P d\theta} = e^{\int 2 \cot \theta d\theta}$$

$$= e^{2 \int \cot \theta d\theta} = e^{2 \log \sin \theta}$$

$$= e^{\log(\sin^2 \theta)} = \sin^2 \theta$$

\therefore the solution of (1) is given by

$$r \cdot (I.F.) = \int Q \cdot (I.F.) d\theta + c$$

$$\therefore r \cdot \sin^2 \theta = \int -\sin 2\theta \cdot \sin^2 \theta d\theta + c$$

$$\therefore r \sin^2 \theta = \int -2 \sin \theta \cos \theta \cdot \sin^2 \theta d\theta + c$$

$$\therefore r \sin^2 \theta = -2 \int \sin^3 \theta \cos \theta d\theta + c$$

$$\text{Put } \sin \theta = t \quad \therefore \cos \theta d\theta = dt$$

$$\therefore r \sin^2 \theta = -2 \int t^3 dt + c$$

$$\therefore r \sin^2 \theta = -2 \cdot \frac{t^4}{4} + c$$

$$\therefore r \sin^2 \theta = -\frac{1}{2} \sin^4 \theta + c$$

$$\therefore r \sin^2 \theta + \frac{\sin^4 \theta}{2} = c$$

This is the general solution.

$$(ix) y dx + (x - y_2) dy = 0$$

Solution:

$$y dx + (x - y_2) dy = 0$$

$$\therefore y dx = -(x - y_2) dy$$

$$\therefore dx dy = -(x - y_2) y = -xy + y$$

$$\therefore dx dy + (1)y \cdot x = y \dots\dots\dots(1)$$

This is the linear differential equation of the form

$$\frac{dx}{dy} + P \cdot x = Q, \text{ where } P = \frac{1}{y} \text{ and } Q = y$$

$$\therefore I.F. = e^{\int P dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

\therefore the solution of (1) is given by

$$x \cdot (I.F.) = \int Q \cdot (I.F.) dy + c_1$$

$$\therefore xy = \int y \cdot y dy + c_1$$

$$\therefore xy = \int y^2 dy + c_1$$

$$\therefore xy = \frac{y^3}{3} + c_1$$

$$\therefore \frac{y^3}{3} = xy + c, \text{ where } c = -c_1$$

This is the general solution.

$$(x) (1-x_2) dy dx + 2xy = x(1-x_2)_2$$

Solution:

$$(1-x^2) \frac{dy}{dx} + 2xy = x(1-x^2)^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} + \left(\frac{2x}{1-x^2} \right) y = \frac{x}{(1-x^2)^{\frac{1}{2}}}$$

This is the linear differential equation of the form

$$\frac{dy}{dx} + P \cdot y = Q, \text{ where } P = \frac{2x}{1-x^2} \text{ and } Q = \frac{x}{(1-x^2)^{\frac{1}{2}}}$$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int \frac{2x}{1-x^2} dx}$$

$$= e^{-\int \frac{-2x}{1-x^2} dx} = e^{-\log|1-x^2|}$$

$$= e^{\log \left| \frac{1}{1-x^2} \right|} = \frac{1}{1-x^2}$$

\therefore the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$$

$$\therefore y \cdot \frac{1}{(1-x^2)} = \int \frac{x}{(1-x^2)^{\frac{1}{2}}} \cdot \frac{1}{1-x^2} dx + c$$

$$\therefore \frac{y}{(1-x^2)} = \int \frac{x}{(1-x^2)^{\frac{3}{2}}} dx + c$$

$$\text{Put } 1-x^2 = t \quad \therefore -2x dx = dt$$

$$\therefore x dx = -\frac{dt}{2}$$

$$\therefore \frac{y}{1-x^2} = \int \frac{1}{t^{\frac{3}{2}}} \cdot \frac{(-dt)}{2} + c$$

$$\therefore \frac{y}{1-x^2} = -\frac{1}{2} \int t^{-\frac{3}{2}} dt + c$$

$$\therefore \frac{y}{1-x^2} = -\frac{1}{2} \cdot \frac{t^{-\frac{1}{2}}}{(-1/2)} + c$$

$$\therefore \frac{y}{1-x^2} = \frac{1}{(1-x^2)^{\frac{1}{2}}} + c$$

$$\therefore y = \sqrt{1-x^2} + c(1-x^2)$$

This is the general solution.

$$(xi) (1+x^2) dy/dx + y = e^{\tan^{-1} x}$$

Solution:

$$(1+x^2) \frac{dy}{dx} + y = e^{\tan^{-1}x}$$

$$\therefore \frac{dy}{dx} + \frac{1}{1+x^2} \cdot y = \frac{e^{\tan^{-1}x}}{1+x^2} \quad \dots\dots\dots(1)$$

This is the linear differential equation of the form

$$\frac{dy}{dx} + P \cdot y = Q, \text{ where } P = \frac{1}{1+x^2} \text{ and } Q = \frac{e^{\tan^{-1}x}}{1+x^2}$$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{1+x^2} dx}$$

$$= e^{\tan^{-1}x}$$

\therefore the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$$

$$\therefore y \cdot e^{\tan^{-1}x} = \int \frac{e^{\tan^{-1}x}}{1+x^2} \cdot e^{\tan^{-1}x} dx + c$$

$$\therefore y \cdot e^{\tan^{-1}x} = \int (e^{\tan^{-1}x})' \cdot \frac{e^{\tan^{-1}x}}{1+x^2} dx + c$$

$$\text{Put } e^{\tan^{-1}x} = t \quad \therefore \frac{e^{\tan^{-1}x}}{1+x^2} dx = dt$$

$$\therefore y \cdot e^{\tan^{-1}x} = \int t dt + c$$

$$\therefore y \cdot e^{\tan^{-1}x} = \frac{t^2}{2} + c$$

$$\therefore y \cdot e^{\tan^{-1}x} = \frac{1}{2} (e^{\tan^{-1}x})^2 + c$$

$$\therefore y = \frac{1}{2} e^{\tan^{-1}x} + ce^{-\tan^{-1}x}$$

This is the general solution.

Question 2.

Find the equation of the curve which passes through the origin and has the slope $x + 3y - 1$ at any point (x, y) on it.

Solution:

Let $A(x, y)$ be the point on the curve $y = f(x)$.

Then slope of the tangent to the curve at the point A is dy/dx .

According to the given condition,

$$dy/dx = x + 3y - 1$$

$$\therefore dy/dx - 3y = x - 1 \quad \dots\dots\dots(1)$$

This is the linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = -3 \text{ and } Q = x - 1$$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int -3 dx} = e^{-3x}$$

\therefore the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$$

$$\therefore y \cdot e^{-3x} = \int (x - 1) \cdot e^{-3x} dx + c$$

$$\therefore e^{-3x} \cdot y = (x - 1) \int e^{-3x} dx - \int \left[\frac{d}{dx} (x - 1) \cdot \int e^{-3x} dx \right] dx + c_1$$

$$\therefore e^{-3x} \cdot y = (x - 1) \cdot \frac{e^{-3x}}{-3} - \int 1 \cdot \frac{e^{-3x}}{-3} dx + c_1$$

$$\therefore e^{-3x} \cdot y = -\frac{1}{3}(x - 1) \cdot e^{-3x} + \frac{1}{3} \int e^{-3x} dx + c_1$$

$$\therefore e^{-3x} \cdot y = -\frac{1}{3}(x - 1) e^{-3x} + \frac{1}{3} \cdot \frac{e^{-3x}}{-3} + c_1$$

$$\therefore e^{-3x} \cdot y = -\frac{1}{3}(x - 1) e^{-3x} - \frac{1}{9} e^{-3x} + c_1$$

$$\therefore 9y = -3(x - 1) - 1 + 9c_1 \cdot e^{3x}$$

$$\therefore 9y + 3(x - 1) + 1 = 9c_1 \cdot e^{3x}$$

$$\therefore 9y + 3x - 3 + 1 = 9c_1 \cdot e^{3x}$$

$$\therefore 3(x + 3y) = 2 + 9c_1 \cdot e^{3x}$$

$$\therefore 3(x + 3y) = 2 + c \cdot e^{3x}, \text{ where } c = 9c_1 \quad \dots (2)$$

This is the general equation of the curve.

But the required curve is passing through the origin (0, 0).

\therefore by putting $x = 0$ and $y = 0$ in (2), we get

$$0 = 2 + c$$

$$\therefore c = -2$$

\therefore from (2), the equation of the required curve is $3(x + 3y) = 2 - 2e^{3x}$ i.e. $3(x + 3y) = 2(1 - e^{3x})$.

Question 3.

Find the equation of the curve passing through the point $(32\sqrt{2}, 2 - \sqrt{2})$ having slope of the tangent to the curve at any point (x, y) is $-4x^9y$.

Solution:

Let $A(x, y)$ be the point on the curve $y = f(x)$.

Then the slope of the tangent to the curve at point A is $\frac{dy}{dx}$.

According to the given condition

$$\frac{dy}{dx} = -4x^9y$$

$$\therefore y dy = -4x^9 x dx$$

Integrating both sides, we get

$$\int y dy = -4x^{10} \int x dx$$

$$\therefore y_2^2 = -4x^{11} + c_1$$

$$\therefore 9y_2 = -4x_2 + 18c_1$$

$$\therefore 4x_2 + 9y_2 = c \text{ where } c = 18c_1$$

This is the general equation of the curve.

But the required curve is passing through the point $(32\sqrt{2}, 2 - \sqrt{2})$.

\therefore by putting $x = 32\sqrt{2}$ and $y = \sqrt{2}$ in (1), we get

$$4(32\sqrt{2})^2 + 9(2 - \sqrt{2})^2 = c$$

$$\therefore 18 + 18 = c$$

$$\therefore c = 36$$

\therefore from (1), the equation of the required curve is $4x^2 + 9y^2 = 36$.

Question 4.

The curve passes through the point $(0, 2)$. The sum of the coordinates of any point on the curve exceeds the slope of the tangent to the curve at any point by 5. Find the equation of the curve.

Solution:

Let A(x, y) be any point on the curve.

Then slope of the tangent to the curve at the point A is dy/dx .

According to the given condition

$$x + y = dy/dx + 5$$

$$\therefore dy/dx - y = x - 5 \dots\dots\dots(1)$$

This is the linear differential equation of the form

$dy/dx + P \cdot y = Q$, where $P = -1$ and $Q = x - 5$

$$\therefore I.F. = e^{\int P dx} = e^{\int -1 dx} = e^{-x}$$

\therefore the solution of (1) is given by

$$y \cdot (I.F.) = \int Q \cdot (I.F.) dx + c$$

$$\therefore y \cdot e^{-x} = \int (x - 5) e^{-x} dx + c$$

$$\therefore e^{-x} \cdot y = (x - 5) \int e^{-x} dx - \int \left[\frac{d}{dx} (x - 5) \int e^{-x} dx \right] dx + c$$

$$\therefore e^{-x} \cdot y = (x - 5) \cdot \frac{e^{-x}}{-1} - \int 1 \cdot \frac{e^{-x}}{-1} dx + c$$

$$\therefore e^{-x} \cdot y = -(x - 5) \cdot e^{-x} + \int e^{-x} dx + c$$

$$\therefore e^{-x} \cdot y = -(x - 5) e^{-x} + \frac{e^{-x}}{-1} + c$$

$$\therefore y = -(x - 5) - 1 + ce^x$$

$$\therefore y = -x + 5 - 1 + ce^x$$

$$\therefore y = 4 - x + ce^x$$

... (2)

This is the general equation of the curve.

But the required curve is passing through the point (0, 2).

\therefore by putting $x = 0$, $y = 2$ in (2), we get

$$2 = 4 - 0 + c$$

$$\therefore c = -2$$

\therefore from (2), the equation of the required curve is $y = 4 - x - 2e^x$.

Question 5.

If the slope of the tangent to the curve at each of its point is equal to the sum of abscissa and the product of the abscissa and ordinate of the point. Also, the curve passes through the point (0, 1). Find the equation of the curve.

Solution:

Let A(x, y) be the point on the curve $y = f(x)$.

Then slope of the tangent to the curve at the point A is dy/dx .

According to the given condition

$$dy/dx = x + xy$$

$$\therefore dy/dx - xy = x \dots\dots\dots(1)$$

This is the linear differential equation of the form

$dy/dx + Py = Q$, where $P = -x$ and $Q = x$

$$\therefore I.F. = e^{\int P dx} = e^{\int -x dx} = e^{-x^2/2}$$

\therefore the solution of (1) is given by

$$y \cdot (I.F.) = \int Q \cdot (I.F.) dx + c$$

$$\begin{aligned}\therefore y \cdot e^{-\frac{x^2}{2}} &= \int x \cdot e^{-\frac{x^2}{2}} dx + c \\ \therefore e^{-\frac{x^2}{2}} \cdot y &= \int x \cdot e^{-\frac{x^2}{2}} dx + c \\ \text{Put } -\frac{x^2}{2} &= t \quad \therefore -x dx = dt \\ \therefore x dx &= -dt \\ \therefore e^{-\frac{x^2}{2}} \cdot y &= \int e^t \cdot (-dt) + c \\ \therefore e^{-\frac{x^2}{2}} \cdot y &= - \int e^t dt + c \\ \therefore e^{-\frac{x^2}{2}} \cdot y &= -e^t + c \\ \therefore e^{-\frac{x^2}{2}} \cdot y &= -e^{-\frac{x^2}{2}} + c \\ \therefore y &= -1 + ce^{\frac{x^2}{2}} \\ \therefore 1 + y &= ce^{\frac{x^2}{2}} \quad \dots\dots(2)\end{aligned}$$

This is the general equation of the curve.

But the required curve is passing through the point (0, 1).

\therefore by putting $x = 0$ and $y = 1$ in (2), we get

$$1 + 1 = c$$

$$\therefore c = 2$$

\therefore from (2), the equation of the required curve is $1 + y = 2e^{\frac{x^2}{2}}$.

Maharashtra State Board 12th Maths Solutions Chapter 6 Differential Equations Ex 6.6

Question 1.

In a certain culture of bacteria, the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, find the number of times the bacteria are increased in 12 hours.

Solution:

Let x be the number of bacteria in the culture at time t .

Then the rate of increase is dx/dt which is proportional to x .

$$\therefore dx/dt \propto x$$

$$\therefore dx/dt = kx, \text{ where } k \text{ is a constant}$$

$$\therefore dx/x = k dt$$

On integrating, we get

$$\int dx/x = k \int dt + c$$

$$\therefore \log x = kt + c$$

Initially, i.e. when $t = 0$, let $x = x_0$

$$\log x_0 = k \times 0 + c$$

$$\therefore c = \log x_0$$

$$\therefore \log x = kt + \log x_0$$

$$\therefore \log x - \log x_0 = kt$$

$$\therefore \log(x/x_0) = kt \dots\dots(1)$$

Since the number doubles in 4 hours, i.e. when $t = 4$, $x = 2x_0$

$$\therefore \log\left(\frac{2x_0}{x_0}\right) = 4k \quad \therefore k = \frac{1}{4} \log 2$$

$$\therefore (1) \text{ becomes, } \log\left(\frac{x}{x_0}\right) = \frac{t}{4} \log 2$$

When $t = 12$, we get

$$\log\left(\frac{x}{x_0}\right) = \frac{12}{4} \log 2 = 3 \log 2$$

$$\therefore \log\left(\frac{x}{x_0}\right) = \log 8$$

$$\therefore \frac{x}{x_0} = 8 \quad \therefore x = 8x_0$$

\therefore the number of bacteria will be 8 times the original number in 12 hours.

Question 2.

If the population of a country doubles in 60 years; in how many years will it be triple (treble) under the assumption that the rate of increase is proportional to the number of inhabitants?

[Given $\log 2 = 0.6912$, $\log 3 = 1.0986$]

Solution:

Let P be the population at time t years.

Then dP/dt , the rate of increase of population is proportional to P .

$$\therefore dP/dt \propto P$$

$$\therefore dP/dt = kP, \text{ where } k \text{ is a constant}$$

$$\therefore dP/P = k dt$$

On integrating, we get

$$\int dP/P = k \int dt + c$$

$$\therefore \log P = kt + c$$

Initially i.e. when $t = 0$, let $P = P_0$

$$\therefore \log P_0 = k \times 0 + c$$

$$\therefore c = \log P_0$$

$$\therefore \log P = kt + \log P_0$$

$$\therefore \log P - \log P_0 = kt$$

$$\therefore \log(P/P_0) = kt \dots\dots(1)$$

Since, the population doubles in 60 years, i.e. when $t = 60$, $P = 2P_0$

$$\therefore \log\left(\frac{2P_0}{P_0}\right) = 60k \quad \therefore k = \frac{1}{60} \log 2$$

$$\therefore (1) \text{ becomes, } \log\left(\frac{P}{P_0}\right) = \frac{t}{60} \log 2$$

When population becomes triple, i.e. when $P = 3P_0$, we

get

$$\log\left(\frac{3P_0}{P_0}\right) = \frac{t}{60} \log 2$$

$$\therefore \log 3 = \frac{t}{60} \log 2$$

$$\therefore t = 60 \left(\frac{\log 3}{\log 2} \right) = 60 \left(\frac{1.0986}{0.6912} \right)$$

$$= 60 \times 1.5894 = 95.364 \approx 95.4 \text{ years}$$

\therefore the population becomes triple in 95.4 years (approximately).

Question 3.

If a body cools from 80°C to 50°C at room temperature of 25°C in 30 minutes, find the temperature of the body after 1 hour.

Solution:

Let $\theta^\circ\text{C}$ be the temperature of the body at time t minutes.

The room temperature is given to be 25°C .

Then by Newton's law of cooling, $d\theta/dt$, the rate of change of temperature, is proportional to $(\theta - 25)$.

i.e. $\frac{d\theta}{dt} \propto (\theta - 25)$

$\therefore \frac{d\theta}{dt} = -k(\theta - 25)$, where $k > 0$

$\therefore \frac{d\theta}{\theta - 25} = -k dt$

On integrating, we get

$$\int \frac{1}{\theta - 25} d\theta = -k \int dt + c$$

$$\therefore \log(\theta - 25) = -kt + c$$

Initially, i.e. when $t = 0$, $\theta = 80$

$$\therefore \log(80 - 25) = -k \times 0 + c \quad \therefore c = \log 55$$

$$\therefore \log(\theta - 25) = -kt + \log 55$$

$$\therefore \log(\theta - 25) - \log 55 = -kt$$

$$\log\left(\frac{\theta - 25}{55}\right) = -kt \quad \dots\dots\dots(1)$$

Now, when $t = 30$, $\theta = 50$

$$\therefore \log\left(\frac{50 - 25}{55}\right) = -30k$$

$$\therefore k = -\frac{1}{30} \log\left(\frac{5}{11}\right)$$

$$\therefore (1) \text{ becomes, } \log\left(\frac{\theta - 25}{55}\right) = \frac{t}{30} \log\left(\frac{5}{11}\right)$$

When $t = 1$ hour = 60 minutes, then

$$\log\left(\frac{\theta - 25}{55}\right) = \frac{60}{30} \log\left(\frac{5}{11}\right) = 2 \log\left(\frac{5}{11}\right)$$

$$\therefore \log\left(\frac{\theta - 25}{55}\right) = \log\left(\frac{5}{11}\right)^2$$

$$\therefore \frac{\theta - 25}{55} = \left(\frac{5}{11}\right)^2 = \frac{25}{121}$$

$$\therefore \theta - 25 = 55 \times \frac{25}{121} = \frac{125}{11}$$

$$\therefore \theta = 25 + \frac{125}{11} = \frac{400}{11} = 36.36$$

\therefore the temperature of the body will be 36.36°C after 1 hour.

Question 4.

The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number doubles in 1 hour, find the number of bacteria after $2\frac{1}{2}$ hours. [Take $\sqrt{2} = 1.414$]

Solution:

Let x be the number of bacteria at time t .

Then the rate of increase is dx/dt which is proportional to x .

$$\therefore dx/dt \propto x$$

$$\therefore dx/dt = kx, \text{ where } k \text{ is a constant}$$

$$\therefore dx/x = k dt$$

On integrating, we get

$$\int dx/x = k \int dt + c$$

$$\therefore \log x = kt + c$$

Initially, i.e. when $t = 0$, $x = 1000$

$$\therefore \log 1000 = k \times 0 + c$$

$$\therefore c = \log 1000$$

$$\therefore \log x = kt + \log 1000$$

$$\therefore \log x - \log 1000 = kt$$

$$\therefore \log(x/1000) = kt \dots\dots\dots(1)$$

Now, when $t = 1$, $x = 2 \times 1000 = 2000$

$$\therefore \log(20001000) = k$$

$$\therefore k = \log 2$$

$$\therefore (1) \text{ becomes, } \log\left(\frac{x}{1000}\right) = t \log 2$$

$$\text{If } t = 2 \frac{1}{2} = \frac{5}{2}, \text{ then}$$

$$\log\left(\frac{x}{1000}\right) = \frac{5}{2} \log 2 = \log(2)^{\frac{5}{2}}$$

$$\therefore \left(\frac{x}{1000}\right) = (2)^{\frac{5}{2}} = 4\sqrt{2} = 4 \times 1.414 = 5.656$$

$$\therefore x = 5.656 \times 1000 = 5656$$

\therefore the number of bacteria after $2\frac{1}{2}$ hours = 5656.

Question 5.

The rate of disintegration of a radioactive element at any time t is proportional to its mass at that time. Find the time during which the original mass of 1.5 gm will disintegrate into its mass of 0.5 gm.

Solution:

Let m be the mass of the radioactive element at time t .

Then the rate of disintegration is dm/dt which is proportional to m .

$$\therefore \frac{dm}{dt} \propto m$$

$$\therefore \frac{dm}{dt} = -km, \text{ where } k > 0$$

$$\therefore \frac{dm}{m} = -k dt$$

On integrating, we get

$$\int \frac{1}{m} dm = -k \int dt + c$$

$$\therefore \log m = -kt + c$$

Initially, i.e. when $t = 0, m = 1.5$

$$\therefore \log(1.5) = -k \times 0 + c \quad \therefore c = \log\left(\frac{3}{2}\right)$$

$$\therefore \log m = -kt + \log\left(\frac{3}{2}\right)$$

$$\therefore \log m - \log\left(\frac{3}{2}\right) = -kt$$

$$\therefore \log\left(\frac{2m}{3}\right) = -kt$$

$$\text{When } m = 0.5 = \frac{1}{2}, \text{ then}$$

$$\log\left(\frac{2 \times \frac{1}{2}}{3}\right) = -kt$$

$$\therefore \log\left(\frac{1}{3}\right) = -kt$$

$$\therefore \log(3)^{-1} = -kt$$

$$\therefore -\log 3 = -kt$$

$$\therefore t = \frac{1}{k} \log 3$$

\therefore the original mass will disintegrate to 0.5 gm when $t = \frac{1}{k} \log 3$

Question 6.

The rate of decay of certain substances is directly proportional to the amount present at that instant. Initially, there is 25 gm of certain substance and two hours later it is found that 9 gm are left. Find the amount left after one more hour.

Solution:

Let x gm be the amount of the substance left at time t .

Then the rate of decay is dx/dt , which is proportional to x .

$$\therefore \frac{dx}{dt} \propto x$$

$$\therefore \frac{dx}{dt} = -kx, \text{ where } k > 0$$

$$\therefore \frac{1}{x} dx = -k dt$$

On integrating, we get

$$\int \frac{1}{x} dx = -k \int dt + c$$

$$\therefore \log x = -kt + c$$

Initially i.e. when $t = 0, x = 25$

$$\therefore \log 25 = -k \times 0 + c \quad \therefore c = \log 25$$

$$\therefore \log x = -kt + \log 25$$

$$\therefore \log x - \log 25 = -kt$$

$$\therefore \log\left(\frac{x}{25}\right) = -kt \quad \dots\dots(1)$$

Now, when $t = 2, x = 9$

$$\therefore \log\left(\frac{9}{25}\right) = -2k$$

$$\therefore -2k = \log\left(\frac{3}{5}\right)^2 = 2 \log\left(\frac{3}{5}\right)$$

$$\therefore k = -\log\left(\frac{3}{5}\right)$$

$$\therefore (1) \text{ becomes, } \log\left(\frac{x}{25}\right) = t \log\left(\frac{3}{5}\right)$$

When $t = 3$, then

$$\log\left(\frac{x}{25}\right) = 3 \log\left(\frac{3}{5}\right) = \log\left(\frac{3}{5}\right)^3$$

$$\therefore x/25 = 27/125$$

$$\therefore x = 275$$

\therefore the amount left after 3 hours 275 gm.

Question 7.

Find the population of a city at any time t , given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years, the population increased from 30,000 to 40,000.

Solution:

Let P be the population of the city at time t .

Then dP/dt , the rate of increase of population is proportional to P .

$$\therefore dP/dt \propto P$$

$$\therefore dP/dt = kP, \text{ where } k \text{ is a constant.}$$

$$\therefore dP/P = k dt$$

On integrating, we get

$$\int_1 P dP = k \int dt + c$$

$$\therefore \log P = kt + c$$

Initially, i.e. when $t = 0, P = 30000$

$$\therefore \log 30000 = k \times 0 + c$$

$$\therefore c = \log 30000$$

$$\therefore \log P = kt + \log 30000$$

$$\therefore \log P - \log 30000 = kt$$

$$\therefore \log(P/30000) = kt \dots\dots(1)$$

Now, when $t = 40, P = 40000$

$$\therefore \log\left(\frac{40000}{30000}\right) = k \times 40$$

$$\therefore k = \frac{1}{40} \log\left(\frac{4}{3}\right)$$

$$\therefore (1) \text{ becomes, } \log\left(\frac{P}{30000}\right) = \frac{t}{40} \log\left(\frac{4}{3}\right) = \log\left(\frac{4}{3}\right)^{\frac{t}{40}}$$

$$\therefore \frac{P}{30000} = \left(\frac{4}{3}\right)^{\frac{t}{40}}$$

$$\therefore P = 30000 \left(\frac{4}{3}\right)^{\frac{t}{40}}$$

\therefore the population of the city at time $t = 30000 \left(\frac{4}{3}\right)^{\frac{t}{40}}$.

Question 8.

A body cools according to Newton's law from 100°C to 60°C in 20 minutes. The temperature of the surroundings is 20°C . How long will it take to cool down to 30°C ?

Solution:

Let $\theta^\circ\text{C}$ be the temperature of the body at time t .

The temperature of the surrounding is given to be 20°C .

According to Newton's law of cooling

$$\frac{d\theta}{dt} \propto \theta - 20$$

$$\therefore \frac{d\theta}{dt} = -k(\theta - 20), \text{ where } k > 0$$

$$\therefore \frac{d\theta}{\theta - 20} = -k dt$$

On integrating, we get

$$\int \frac{1}{\theta - 20} d\theta = -k \int dt + c$$

$$\therefore \log(\theta - 20) = -kt + c$$

Initially, i.e. when $t = 0$, $\theta = 100$

$$\therefore \log(100 - 20) = -k \times 0 + c \quad \therefore c = \log 80$$

$$\therefore \log(\theta - 20) = -kt + \log 80$$

$$\therefore \log(\theta - 20) - \log 80 = -kt$$

$$\therefore \log\left(\frac{\theta - 20}{80}\right) = -kt \quad \dots\dots(1)$$

Now, when $t = 20$, $\theta = 60$

$$\therefore \log\left(\frac{60-20}{80}\right) = -k \times 20$$

$$\therefore \log\left(\frac{40}{80}\right) = -20k \quad \therefore k = -\frac{1}{20} \log\left(\frac{1}{2}\right)$$

\therefore (1) becomes, $\log\left(\frac{\theta-20}{80}\right) = \frac{t}{20} \log\left(\frac{1}{2}\right)$

When $\theta = 30$, then

$$\log\left(\frac{30 - 20}{80}\right) = \frac{t}{20} \log\left(\frac{1}{2}\right)$$

$$\therefore \log\left(\frac{1}{8}\right) = \log\left(\frac{1}{2}\right)^{\frac{t}{20}}$$

$$\therefore \left(\frac{1}{2}\right)^{\frac{t}{20}} = \frac{1}{8} = \left(\frac{1}{2}\right)^3$$

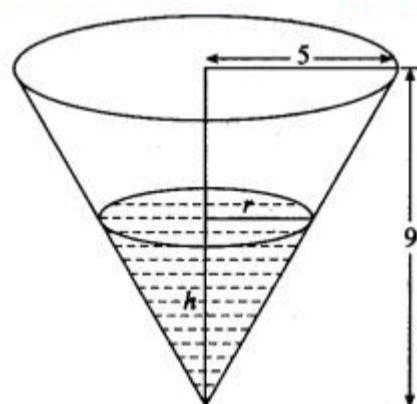
$$\therefore \frac{t}{20} = 3 \quad \therefore t = 60$$

∴ the body will cool down to 30°C in 60 minutes, i.e. in 1 hour.

Question 9.

A right circular cone has a height of 9 cm and a radius of the base of 5 cm. It is inverted and water is poured into it. If at any instant the water level rises at the rate of (πA) cm/sec, where A is the area of the water surface at that instant, show that the vessel will be full in 75 seconds.

Solution:



Let r be the radius of the water surface and h be the height of the water at time t .

∴ area of the water surface $A = \pi r^2$ sq cm.

Since height of the right circular cone is 9 cm and radius of the base is 5 cm.

$rh = 59$

$$\therefore r = 59h$$

∴ area of water surface, i.e. $A = \pi(59h)_2$

$$\therefore A = 25\pi h_2 s_1 \dots \dots \dots (1)$$

The water level, i.e. the rate of change of h is dh/dt rises at the rate of (πA) cm/sec.

$$\therefore \frac{dh}{dt} = \frac{\pi}{A} = \frac{\pi \times 81}{25\pi h^2}$$

$$\therefore \frac{dh}{dt} = \frac{81}{25h^2}$$

$$\therefore h^2 dh = \frac{81}{25} dt$$

On integrating, we get

$$\int h^2 dh = \frac{81}{25} \int dt + c$$

$$\therefore \frac{h^3}{3} = \frac{81}{25} t + c$$

Initially, i.e. when $t = 0, h = 0$

$$\therefore 0 = 0 + c \quad \therefore c = 0$$

$$\therefore \frac{h^3}{3} = \frac{81}{25} t$$

When the vessel will be full, $h = 9$

$$\therefore \frac{(9)^3}{3} = \frac{81}{25} t$$

$$\therefore t = \frac{81 \times 9 \times 25}{3 \times 81} = 75$$

Hence, the vessel will be full in 75 seconds.

Question 10.

Assume that a spherical raindrop evaporates at a rate proportional to its surface area. If its radius originally is 3 mm and 1 hour later has been reduced to 2 mm, find an expression for the radius of the raindrop at any time t .

Solution:

Let r be the radius, V be the volume and S be the surface area of the spherical raindrop at time t .

Then $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

The rate at which the raindrop evaporates is dV/dt which is proportional to the surface area.

$$\therefore dV/dt \propto S$$

$$\therefore dV/dt = -kS, \text{ where } k > 0 \dots\dots(1)$$

$$\text{Now, } V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2$$

$$\therefore \frac{dV}{dt} = \frac{4\pi}{3} \times 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\therefore (1) \text{ becomes, } 4\pi r^2 \frac{dr}{dt} = -k(4\pi r^2)$$

$$\therefore \frac{dr}{dt} = -k$$

$$\therefore dr = -k dt$$

On integrating, we get

$$\int dr = -k \int dt + c$$

$$\therefore r = -kt + c$$

Initially, i.e. when $t = 0, r = 3$

$$\therefore 3 = -k \times 0 + c$$

$$\therefore c = 3$$

$$\therefore r = -kt + 3$$

When $t = 1, r = 2$

$$\therefore 2 = -k \times 1 + 3$$

$$\therefore k = 1$$

$$\therefore r = -t + 3$$

$$\therefore r = 3 - t, \text{ where } 0 \leq t \leq 3.$$

This is the required expression for the radius of the raindrop at any time t .

Question 11.

The rate of growth of the population of a city at any time t is proportional to the size of the population. For a certain city, it is found that the constant of proportionality is 0.04. Find the population of the city after 25 years, if the initial population is 10,000. [Take $e = 2.7182$]

Solution:

Let P be the population of the city at time t.

Then the rate of growth of population is dP/dt which is proportional to P.

$$\therefore dP/dt \propto P$$

$$\therefore dP/dt = kP, \text{ where } k = 0.04$$

$$\therefore dP/dt = (0.04)P$$

$$\therefore \frac{dP}{P} = (0.04)dt$$

On integrating, we get

$$\int \frac{dP}{P} = (0.04) \int dt + c$$

$$\therefore \log P = (0.04)t + c$$

Initially, i.e., when $t = 0$, $P = 10000$

$$\therefore \log 10000 = (0.04) \times 0 + c$$

$$\therefore c = \log 10000$$

$$\therefore \log P = (0.04)t + \log 10000$$

$$\therefore \log P - \log 10000 = (0.04)t$$

$$\therefore \log(P/10000) = (0.04)t$$

When $t = 25$, then

$$\therefore \log(P/10000) = 0.04 \times 25 = 1$$

$$\therefore \log(P/10000) = \log e \dots [\because \log e = 1]$$

$$\therefore P/10000 = e = 2.7182$$

$$\therefore P = 2.7182 \times 10000 = 27182$$

\therefore the population of the city after 25 years will be 27,182.

Question 12.

Radium decomposes at a rate proportional to the amount present at any time. If p percent of the amount disappears in one year, what percent of the amount of radium will be left after 2 years?

Solution:

Let x be the amount of the radium at time t.

Then the rate of decomposition is dx/dt which is proportional to x.

$$\therefore \frac{dx}{dt} \propto x$$

$$\therefore \frac{dx}{dt} = -kx, \text{ where } k > 0$$

$$\therefore \frac{1}{x} dx = -k dt$$

On integrating, we get

$$\int \frac{1}{x} dx = -k \int dt$$

$$\therefore \log x = -kt + c$$

Let the original amount be x_0 , i.e. $x = x_0$ when $t = 0$.

$$\therefore \log x_0 = -k \times 0 + c \quad \therefore c = \log x_0$$

$$\therefore \log x = -kt + \log x_0$$

$$\therefore \log x - \log x_0 = -kt$$

$$\therefore \log\left(\frac{x}{x_0}\right) = -kt \quad \dots (1)$$

But p% of the amount disappears in one year,

$$\therefore \text{when } t = 1, x = x_0 - p\% \text{ of } x_0, \text{ i.e. } x = x_0 - \frac{px_0}{100}$$

$$\therefore \log\left(\frac{x_0 - \frac{px_0}{100}}{x_0}\right) = -k \times 1$$

$$\therefore k = -\log\left(1 - \frac{p}{100}\right) = -\log\left(\frac{100-p}{100}\right)$$

$$\therefore (1) \text{ becomes, } \log\left(\frac{x}{x_0}\right) = t \log\left(\frac{100-p}{100}\right)$$

When $t = 2$, then |

$$\log\left(\frac{x}{x_0}\right) = 2 \log\left(\frac{100-p}{100}\right) = \log\left(\frac{100-p}{100}\right)^2$$

$$\therefore \frac{x}{x_0} = \left(\frac{100-p}{100}\right)^2$$

$$\therefore x = \left(\frac{100-p}{100}\right)^2 x_0 = \left(1 - \frac{p}{100}\right)^2 x_0$$

$$\therefore \% \text{ left after 2 years} = \frac{100 \times \left(1 - \frac{p}{100}\right)^2 x_0}{x_0}$$

$$= 100 \left(1 - \frac{p}{100}\right)^2 = \left[10 \left(1 - \frac{p}{100}\right)\right]^2$$

$$= \left(10 - \frac{p}{10}\right)^2$$

Hence, $(10 - p/10)^2$ % of the amount will be left after 2 years.

Maharashtra State Board 12th Maths Solutions Chapter 6 Differential Equations Miscellaneous Exercise 6

(I) Choose the correct option from the given alternatives:

Question 1.

The order and degree of the differential equation $1 + (dy/dx)^2 - \sqrt{d^2y/dx^2} = 0$ are respectively.....

- (a) 2, 1
- (b) 1, 2
- (c) 3, 2
- (d) 2, 3

Answer:

- (d) 2, 3

Question 2.

The differential equation of $y = c_2 + cx$ is.....

- (a) $x^4(dy/dx)^2 - x dy/dx = y$
- (b) $dy/dx^2 + x dy/dx + y = 0$
- (c) $x^3(dy/dx)^2 + x dy/dx = y$
- (d) $d^2y/dx^2 + dy/dx - y = 0$

Answer:

(a) $x^2 + dy/dx)^2 - x dy/dx = y$

Question 3.

$x^2 + y^2 = a^2$ is a solution of

(a) $d^2y/dx^2 + dy/dx - y = 0$

(b) $y = x^2 + (dy/dx)^2 - \sqrt{1 + a^2 y}$

(c) $y = x dy/dx + a^2 + (dy/dx)^2 - \sqrt{1 + a^2 y}$

(d) $d^2y/dx^2 = (x+1) dy/dx$

Answer:

(c) $y = x dy/dx + a^2 + (dy/dx)^2 - \sqrt{1 + a^2 y}$

Hint : $x^2 + y^2 = a^2 \quad \therefore 2x + 2y \frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore x \frac{dy}{dx} + a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= x \left(-\frac{x}{y}\right) + a \sqrt{1 + \frac{x^2}{y^2}} = -\frac{x^2}{y} + a \times \frac{a}{y}$$

$$= \frac{a^2 - x^2}{y} = \frac{y^2}{y} = y.$$

Question 4.

The differential equation of all circles having their centres on the line $y = 5$ and touching the X-axis is

(a) $y^2(1+dy/dx)^2 = 25$

(b) $(y-5)^2[1+(dy/dx)^2] = 25$

(c) $(y-5)^2 + [1+(dy/dx)^2] = 25$

(d) $(y-5)^2[1-(dy/dx)^2] = 25$

Answer:

(b) $(y-5)^2[1+(dy/dx)^2] = 25$

Hint : Equation of the circle is

$$(x-h)^2 + (y-5)^2 = 5^2 \quad \dots\dots\dots (1)$$

$$\therefore 2(x-h) + 2(y-5) \frac{dy}{dx} = 0$$

$$\therefore (x-h)^2 = (y-5)^2 \left(\frac{dy}{dx}\right)^2$$

$$\therefore 25 - (y-5)^2 = (y-5)^2 \left(\frac{dy}{dx}\right)^2 \quad \dots\dots [By(1)]$$

$$\therefore (y-5)^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right] = 25$$

Question 5.

The differential equation $y dy/dx + x = 0$ represents family of

(a) circles

(b) parabolas

(c) ellipses

(d) hyperbolas

Answer:

(a) circles

Hint:

$$y dy/dx + x = 0$$

$$\therefore \int y dy + \int x dx = c$$

$$\therefore y^2/2 + x^2/2 = c$$

$\therefore x^2 + y^2 = 2c$ which is a circle.

Question 6.

The solution of $x \cdot dy/dx = \tan^{-1}x$ is.....

- (a) $x^2 \tan^{-1}x + c = 0$
- (b) $x \tan^{-1}x + c = 0$
- (c) $x - \tan^{-1}x = c$
- (d) $y = x^2 \tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x) + c$

Answer:

- (d) $y = x^2 \tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x) + c$

$$\text{Hint : } \frac{1}{x} \frac{dy}{dx} = \tan^{-1}x \quad \therefore dy = x \tan^{-1}x dx$$

$$\therefore \int dy = \int (\tan^{-1}x) \cdot x dx$$

$$\therefore y = (\tan^{-1}x) \cdot \frac{x^2}{2} - \int \frac{d}{dx}(\tan^{-1}x) \cdot \frac{x^2}{2} dx + c$$

$$= \frac{x^2 \tan^{-1}x}{2} - \int \frac{1}{1+x^2} \times \frac{1}{2} x^2 dx + c$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) dx + c$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2} (x - \tan^{-1}x) + c.]$$

Question 7.

The solution of $(x+y)^2 dy/dx = 1$ is.....

- (a) $x = \tan^{-1}(x+y) + c$
- (b) $y \tan^{-1}(xy) = c$
- (c) $y = \tan^{-1}(x+y) + c$
- (d) $y + \tan^{-1}(x+y) = c$

Answer:

- (c) $y = \tan^{-1}(x+y) + c$

$$\text{Hint : } (x+y)^2 \frac{dy}{dx} = 1$$

$$\text{Put } x+y = u \quad \therefore 1 + \frac{dy}{dx} = \frac{du}{dx}$$

$$\therefore u^2 \left(\frac{du}{dx} - 1 \right) = 1$$

$$\therefore u^2 \frac{du}{dx} = u^2 + 1$$

$$\therefore \int \frac{u^2}{u^2+1} du = \int dx$$

$$\therefore \int \frac{(u^2+1)-1}{u^2+1} du = \int dx$$

$$\therefore \int \left(1 - \frac{1}{u^2+1}\right) du = \int dx$$

$$\therefore u - \tan^{-1}u = x + c$$

$$\therefore x + y - \tan^{-1}(x+y) = x + c$$

$$\therefore y = \tan^{-1}(x+y) + c]$$

Question 8.

The Solution of $dy/dx = y + x^2 - y^2/2$ is.....

- (a) $\sin^{-1}(yx) = 2 \log |x| + c$
- (b) $\sin^{-1}(yx) = \log |x| + c$
- (c) $\sin(xy) = \log |x| + c$
- (d) $\sin(yx) = \log |y| + c$

Answer:

(b) $\sin^{-1}(yx) = \log|x| + c$

Hint: $\frac{dy}{dx} = \frac{y + \sqrt{x^2 - y^2}}{x}$

Put $y = vx \quad \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\therefore v + x \frac{dv}{dx} = \frac{vx + \sqrt{x^2 - v^2 x^2}}{x} = v + \sqrt{1 - v^2}$$

$$\therefore x \frac{dv}{dx} = \sqrt{1 - v^2}$$

$$\therefore \int \frac{1}{\sqrt{1 - v^2}} dv = \int \frac{1}{x} dx$$

$$\therefore \sin^{-1}v = \log|x| + c$$

$$\therefore \sin^{-1}\left(\frac{y}{x}\right) = \log|x| + c$$

Question 9.

The solution of $dy/dx + y = \cos x - \sin x$ is.....

- (a) $y e^x = \cos x + c$
- (b) $y e^x + e^x \cos x = c$
- (c) $y e^x = e^x \cos x + c$
- (d) $y^2 e^x = e^x \cos x + c$

Answer:

- (c) $y e^x = e^x \cos x + c$

Hint:

$$dy/dx + y = \cos x - \sin x$$

$$I.F. = e^{\int 1 dx} = e^x$$

∴ the solution is $y \cdot e^x = \int (\cos x - \sin x) e^x + c$

$$\therefore y \cdot e^x = e^x \cos x + c$$

Question 10.

The integrating factor of linear differential equation $x dy/dx + 2y = x^2 \log x$ is.....

- (a) $1/x$
- (b) k
- (c) $1/n_2$
- (d) x^2

Answer:

- (d) x^2

Hint:

$$I.F. = e^{\int 2/x dx}$$

$$= e^{2 \log x}$$

$$= x^2$$

Question 11.

The solution of the differential equation $dy/dx = \sec x - y \tan x$ is.....

- (a) $y \sec x + \tan x = c$
- (b) $y \sec x = \tan x + c$
- (c) $\sec x + y \tan x = c$
- (d) $\sec x = y \tan x + c$

Answer:

- (b) $y \sec x = \tan x + c$

Hint:

$$dy/dx = \sec x - y \tan x$$

$$\therefore dy/dx + y \tan x = \sec x$$

$$I.F. = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$

∴ the solution is

$$y \cdot \sec x = \int \sec x \cdot \sec x dx + c$$

$$\therefore y \sec x = \tan x + c$$

Question 12.

The particular solution of $dy/dx = xe^y - x$, when $x = y = 0$ is.....

- (a) $ex-y = x + 1$
- (b) $ex+y = x + 1$
- (c) $ex + ey = x + 1$
- (d) $ey-x = x - 1$

Answer:

- (a) $ex-y = x + 1$

$$\text{Hint : } \frac{dy}{dx} = xe^{y-x} = \frac{xe^y}{e^x}$$

$$\therefore \int e^{-y} dy = \int xe^{-x} dx$$

$$\therefore e^{-y} = x \cdot \frac{e^{-x}}{-1} - \int 1 \cdot \frac{e^{-x}}{-1} dx + c$$

$$\therefore -e^{-y} = -xe^{-x} + \frac{e^{-x}}{-1} + c$$

$$\therefore e^{-y} = \frac{x}{e^x} + \frac{1}{e^x} - c$$

$$\therefore e^{x-y} = x + 1 - ce^x$$

When $x = y = 0$, we get

$$1 = 1 - c \quad \therefore c = 0$$

\therefore particular solution is

$$e^{x-y} = x + 1$$

Question 13.

$x_2a_2 - y_2b_2 = 1$ is a solution of.....

- (a) $d_2ydx_2 + yx + (dydx)_2 = 0$
- (b) $xyd_2ydx_2 + 2(dydx)_2 - ydydx = 0$
- (c) $yd_2ydx_2 + 2(dydx)_2 + y = 0$
- (d) $xydydx + yd_2ydx_2 = 0$

Answer:

- (b) $xyd_2ydx_2 + 2(dydx)_2 - ydydx = 0$

$$\text{Hint : } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots (1)$$

$$\therefore \frac{1}{a^2} \times 2x - \frac{1}{b^2} \times 2y \frac{dy}{dx} = 0$$

$$\therefore \frac{x}{a^2} - \frac{y}{b^2} \frac{dy}{dx} = 0 \quad \dots (2)$$

$$\text{and } \frac{1}{a^2} \times 1 - \frac{1}{b^2} \left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] = 0 \quad \dots (3)$$

Equations (1), (2) and (3) are consistent

$$\therefore \begin{vmatrix} x^2 & -y^2 & 1 \\ x & -y \frac{dy}{dx} & 0 \\ 1 & -\left[y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \right] & 0 \end{vmatrix} = 0$$

$$\therefore xy \frac{d^2y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 - y \frac{dy}{dx} = 0$$

Question 14.

The decay rate of certain substances is directly proportional to the amount present at that instant. Initially, there are 27 grams of substance and 3 hours later it is found that 8 grams left. The amount left after one more hour is.....

- (a) 5.23 grams
- (b) 5.13 grams
- (c) 5.1 grams

(d) 5 grams

Answer:

(b) 513 grams

Question 15.

If the surrounding air is kept at 20°C and the body cools from 80°C to 70°C in 5 minutes, the temperature of the body after 15 minutes will be.....

(a) 51.7°C

(b) 54.7°C

(c) 52.7°C

(d) 50.7°C

Answer:

(b) 54.7°C

(II) Solve the following:

Question 1.

Determine the order and degree of the following differential equations:

(i) $d_2ydx_2 + 5dydx + y = x^3$

Solution:

The given D.E. is $d_2ydx_2 + 5dydx + y = x^3$ This D.E. has highest order derivative d_2ydx_2 with power 1.

∴ the given D.E. is of order 2 and degree 1.

(ii) $(d_3ydx_3)^2 = 1 + dydx \dots \sqrt{5}$

Solution:

The given D.E. is $(d_3ydx_3)^2 = 1 + dydx \dots \sqrt{5}$

$(d_3ydx_3)^2 \times 5 = 1 + dydx$

$(d_3ydx_3)^{10} = 1 + dydx$

This D.E. has highest order derivative d_3ydx_3 with power 10.

∴ the given D.E. is of order 3 and degree 10.

(iii) $1 + (dydx)^2 \dots \sqrt{3} = d_2ydx_2$

Solution:

The given D.E. is $1 + (dydx)^2 \dots \sqrt{3} = d_2ydx_2$

On cubing both sides, we get

$1 + (dydx)^2 = (d_2ydx_2)^3$

This D.E. has highest order derivative d_2ydx_2 with power 3.

∴ the given D.E. is of order 2 and degree 3.

(iv) $dydx = 3y + 1 + 5(dydx)^2 \dots \sqrt{4}$

Solution:

The given D.E. is

$$\frac{dy}{dx} = 3y + \sqrt[4]{1 + 5\left(\frac{dy}{dx}\right)^2}$$

$$\therefore \frac{dy}{dx} - 3y = \sqrt[4]{1 + 5\left(\frac{dy}{dx}\right)^2}$$

$$\therefore \left(\frac{dy}{dx} - 3y\right)^4 = 1 + 5\left(\frac{dy}{dx}\right)^2$$

This D.E. has the highest order derivative $dydx$ with power 4.

∴ the given D.E. is of order 1 and degree 4.

(v) $d_4ydx_4 + \sin(dydx) = 0$

Solution:

The given D.E. is $d_4ydx_4 + \sin(dydx) = 0$ This D.E. has highest order derivative d_4ydx_4 .

∴ order = 4

Since this D.E. cannot be expressed as a polynomial in differential coefficient, the degree is not defined.

Question 2.

In each of the following examples verify that the given function is a solution of the differential equation.

(i) $x^2 + y^2 = r^2; x dy/dx + r^2 - (dy/dx)^2 = y$

Solution:

$$x^2 + y^2 = r^2 \dots\dots (1)$$

Differentiating both sides w.r.t. x, we get

$$2x + 2y \frac{dy}{dx} = 0$$

$$\therefore 2y \frac{dy}{dx} = -2x \quad \therefore \frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore x \frac{dy}{dx} + r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

$$= x \left(-\frac{x}{y}\right) + r \sqrt{1 + \left(-\frac{x}{y}\right)^2}$$

$$= -\frac{x^2}{y} + r \sqrt{1 + \frac{x^2}{y^2}}$$

$$= -\frac{x^2}{y} + r \sqrt{\frac{y^2 + x^2}{y^2}}$$

$$= -\frac{x^2}{y} + r \sqrt{\frac{r^2}{y^2}} \quad \dots\dots [By (1)]$$

$$= -\frac{x^2}{y} + \frac{r^2}{y} = \frac{r^2 - x^2}{y}$$

$$\therefore \frac{y^2}{y} = y$$

Hence, $x^2 + y^2 = r^2$ is a solution of the D.E.

$$x dy/dx + r^2 - (dy/dx)^2 = y$$

(ii) $y = e^{ax} \sin bx; d^2y/dx^2 - 2ad y/dx + (a^2 + b^2)y = 0$

Solution:

$$y = e^{ax} \sin bx$$

$$\begin{aligned}\therefore \frac{dy}{dx} &= \frac{d}{dx}(e^{ax} \sin bx) \\ &= e^{ax} \cdot \frac{d}{dx}(\sin bx) + \sin bx \cdot \frac{d}{dx}(e^{ax}) \\ &= e^{ax} \times \cos bx \cdot \frac{d}{dx}(bx) + \sin bx \cdot e^{ax} \cdot \frac{d}{dx}(ax) \\ &= e^{ax} \cos bx \times b + e^{ax} \sin bx \times a \\ &= e^{ax}(b \cos bx + a \sin bx)\end{aligned}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}[e^{ax}(b \cos bx + a \sin bx)]$$

$$\begin{aligned}&= e^{ax} \cdot \frac{d}{dx}(b \cos bx + a \sin bx) + \\ &\quad (b \cos bx + a \sin bx) \cdot \frac{d}{dx}(e^{ax}) \\ &= e^{ax} \left[b(-\sin bx) \cdot \frac{d}{dx}(bx) + a \cos bx \cdot \frac{d}{dx}(bx) \right] + \\ &\quad (b \cos bx + a \sin bx) \cdot e^{ax} \cdot \frac{d}{dx}(ax) \\ &= e^{ax}[-b \sin bx \times b + a \cos bx \times b] + \\ &\quad (b \cos bx + a \sin bx) \cdot e^{ax} \times a \\ &= e^{ax}(-b^2 \sin bx + ab \cos bx + ab \cos bx + a^2 \sin bx) \\ &= e^{ax}[(a^2 - b^2) \sin bx + 2ab \cos bx]\end{aligned}$$

$$\begin{aligned}\therefore \frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y &= e^{ax}[(a^2 - b^2)] \sin bx + 2ab \cos bx] - \\ &\quad 2a \cdot e^{ax}(b \cos bx + a \sin bx) + (a^2 + b^2) \cdot e^{ax} \sin bx \\ &= e^{ax}[(a^2 - b^2) \sin bx + 2ab \cos bx - 2ab \cos bx - \\ &\quad 2a^2 \sin bx + (a^2 + b^2) \sin bx] \\ &= e^{ax}[(a^2 - b^2) \sin bx - (a^2 - b^2) \sin bx] \\ &= e^{ax} \times 0 = 0\end{aligned}$$

Hence, $y = e^{ax} \sin bx$ is a solution of the D.E.

$$\frac{d^2y}{dx^2} - 2a \frac{dy}{dx} + (a^2 + b^2)y = 0.$$

$$(iii) y = 3 \cos(\log x) + 4 \sin(\log x); x^2 d_2 y dx_2 + x dy dx + y = 0$$

Solution:

$$y = 3 \cos(\log x) + 4 \sin(\log x) \dots (1)$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}\frac{dy}{dx} &= 3 \frac{d}{dx}[\cos(\log x)] + 4 \frac{d}{dx}[\sin(\log x)] \\ &= 3[-\sin(\log x)] \frac{d}{dx}(\log x) + 4 \cos(\log x) \frac{d}{dx}(\log x) \\ &= -3 \sin(\log x) \times \frac{1}{x} + 4 \cos(\log x) \times \frac{1}{x} \\ \therefore x \frac{dy}{dx} &= -3 \sin(\log x) + 4 \cos(\log x)\end{aligned}$$

Differentiating again w.r.t. x , we get,

$$x \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x) = -3 \frac{d}{dx}[\sin(\log x)] + 4 \frac{d}{dx}[\cos(\log x)]$$

$$\begin{aligned}\therefore x \frac{d^2y}{dx^2} + \frac{dy}{dx} \times 1 &= -3 \cos(\log x) \cdot \frac{d}{dx}(\log x) + \\ &4[-\sin(\log x)] \cdot \frac{d}{dx}(\log x)\end{aligned}$$

$$\therefore x \frac{d^2y}{dx^2} + \frac{dy}{dx} = -3 \cos(\log x) \times \frac{1}{x} - 4 \sin(\log x) \times \frac{1}{x}$$

$$\therefore x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = -[3 \cos(\log x) + 4 \sin(\log x)] = -y$$

... [By (1)]

$$\therefore x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0$$

Hence, $y = 3 \cos(\log x) + 4 \sin(\log x)$ is a solution of the

$$\text{D.E. } x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0.$$

(iv) $xy = ae^x + be^{-x} + x^2; x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 2 = xy + 2$

Solution:

$$xy = ae^x + be^{-x} + x^2$$

$$\therefore xy - x^2 = ae^x + be^{-x} \quad \dots (1)$$

Differentiating both sides w.r.t. x , we get

$$x \frac{dy}{dx} + y \cdot \frac{d}{dx}(x) - 2x = ae^x + be^{-x} \times (-1)$$

$$\therefore x \frac{dy}{dx} + y - 2x = ae^x - be^{-x}$$

Differentiating again w.r.t. x , we get

$$x \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x) + \frac{dy}{dx} - 2 \times 1 = ae^x - be^{-x}(-1)$$

$$\therefore x \frac{d^2y}{dx^2} + \frac{dy}{dx} \times 1 + \frac{dy}{dx} - 2 = ae^x + be^{-x}$$

$$\therefore x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 2 = xy - x^2 \quad \dots [By (1)]$$

$$\therefore x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + x^2 = xy + 2$$

Hence, $xy = ae^x + be^{-x} + x^2$ is a solution of the D.E.

$$x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + x^2 = xy + 2$$

(v) $x^2 = 2y^2 \log y, x^2 + y^2 = xy \frac{dxdy}{dx}$

Solution:

$$x_2 = 2y_2 \log y \dots\dots(1)$$

Differentiating both sides w.r.t. y, we get

$$\begin{aligned} 2x \frac{dx}{dy} &= 2 \frac{d}{dy}(y^2 \log y) \\ &= 2 \left[y^2 \frac{d}{dy}(\log y) + (\log y) \cdot \frac{d}{dy}(y^2) \right] \\ &= 2 \left[y^2 \times \frac{1}{y} + (\log y) \times 2y \right] \end{aligned}$$

$$\therefore x \frac{dx}{dy} = y + 2y \log y$$

$$\therefore xy \frac{dx}{dy} = y^2 + 2y^2 \log y$$

$$= y^2 + x^2 \quad \dots\dots [By(1)]$$

$$\therefore x_2 + y_2 = xy \frac{dx}{dy}$$

Hence, $x_2 = 2y_2 \log y$ is a solution of the D.E.

$$x_2 + y_2 = xy \frac{dx}{dy}$$

Question 3.

Obtain the differential equation by eliminating the arbitrary constants from the following equations:

$$(i) y_2 = a(b - x)(b + x)$$

Solution:

$$y_2 = a(b - x)(b + x) = a(b^2 - x^2)$$

Differentiating both sides w.r.t. x, we get

$$2y \frac{dy}{dx} = a(0 - 2x) = -2ax$$

$$\therefore y \frac{dy}{dx} = -ax \dots\dots(1)$$

Differentiating again w.r.t. x, we get

$$y \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{dy}{dx} = -a \times 1$$

$$\therefore y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = -a$$

$$\therefore xy \frac{d^2y}{dx^2} + x \frac{dy}{dx} = -ax$$

$$\therefore xy \frac{d^2y}{dx^2} + x \frac{dy}{dx} = y \frac{dy}{dx} \quad \dots\dots [By(1)]$$

$$\therefore xy \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y \frac{dy}{dx} = 0$$

This is the required D.E.

$$(ii) y = a \sin(x + b)$$

Solution:

$$y = a \sin(x + b)$$

$$\therefore \frac{dy}{dx} = a \frac{d}{dx} [\sin(x + b)]$$

$$= a \cos(x + b) \cdot \frac{d}{dx}(x + b)$$

$$= a \cos(x + b) \times (1 + 0)$$

$$= a \cos(x + b)$$

$$\text{and } \frac{d^2y}{dx^2} = a \frac{d}{dx} [\cos(x + b)]$$

$$= a [-\sin(x + b)] \cdot \frac{d}{dx}(x + b)$$

$$= -a \sin(x + b) \times (1 + 0)$$

$$\therefore \frac{d^2y}{dx^2} = -y \quad \dots\dots [By(1)]$$

$$\therefore \frac{d^2y}{dx^2} + y = 0$$

This is the required D.E.

$$(iii) (y - a)^2 = b(x + 4)$$

Solution:

$$(y - a)^2 = b(x + 4) \dots\dots(1)$$

Differentiating both sides w.r.t. x, we get

$$2(y-a) \cdot ddx(y-a) = bddx(x+4)$$

$$\therefore 2(y-a) \cdot \left(\frac{dy}{dx} - 0 \right) = b(1+0)$$

$$\therefore 2(y-a) \frac{dy}{dx} = b$$

$$\therefore 2(y-a) \frac{dy}{dx} = \frac{(y-a)^2}{x+4} \quad \dots\dots [\text{By (1)}]$$

$$2(x+4) \frac{dy}{dx} = y - a$$

Differentiating again w.r.t. x, we get

$$2 \left[(x+4) \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x+4) \right] = \frac{dy}{dx} - 0$$

$$\therefore 2 \left[(x+4) \frac{d^2y}{dx^2} + \frac{dy}{dx} \times (1+0) \right] = \frac{dy}{dx}$$

$$\therefore 2(x+4) \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\therefore 2(x+4) \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

$$(iv) y = a\cos(\log x) + b\sin(\log x) \dots\dots \checkmark$$

Solution:

$$y = a\cos(\log x) + b\sin(\log x) \dots\dots \checkmark$$

$$\therefore y_2 = a \cos (\log x) + b \sin (\log x) \dots\dots(1)$$

Differentiating both sides w.r.t. x, we get

$$\begin{aligned}
 2y \frac{dy}{dx} &= a \frac{d}{dx} [\cos(\log x)] + b \frac{d}{dx} [\sin(\log x)] \\
 &= a [-\sin(\log x)] \cdot \frac{d}{dx} (\log x) + b \cos(\log x) \cdot \frac{d}{dx} (\log x) \\
 &= -a \sin(\log x) \times \frac{1}{x} + b \cos(\log x) \times \frac{1}{x} \\
 \therefore 2xy \frac{dy}{dx} &= -a \sin(\log x) + b \cos(\log x)
 \end{aligned}$$

Differentiating again w.r.t. x , we get

$$\begin{aligned}
 2 \left[xy \cdot \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \cdot \frac{d}{dx} (xy) \right] &= -a \frac{d}{dx} [\sin(\log x)] + b \frac{d}{dx} [\cos(\log x)] \\
 \therefore 2 \left[xy \frac{d^2y}{dx^2} + \frac{dy}{dx} \left(x \frac{dy}{dx} + y \times 1 \right) \right] &= -a \cos(\log x) \cdot \frac{d}{dx} (\log x) + b [-\sin(\log x)] \cdot \frac{d}{dx} (\log x) \\
 \therefore 2xy \frac{d^2y}{dx^2} + 2x \left(\frac{dy}{dx} \right)^2 + 2y \frac{dy}{dx} &= -a \cos(\log x) \times \frac{1}{x} - b \sin(\log x) \times \frac{1}{x} \\
 \therefore 2x^2y \frac{d^2y}{dx^2} + 2x^2 \left(\frac{dy}{dx} \right)^2 + 2xy \frac{dy}{dx} &= -[a \cos(\log x) + b \sin(\log x)] = -y^2 \quad \dots \text{ [By (1)]} \\
 \therefore 2x^2y \frac{d^2y}{dx^2} + 2x^2 \left(\frac{dy}{dx} \right)^2 + 2xy \frac{dy}{dx} + y^2 &= 0
 \end{aligned}$$

(v) $y = Ae^{3x+1} + Be^{-3x+1}$

Solution:

$$y = Ae^{3x+1} + Be^{-3x+1} \dots \text{ (1)}$$

Differentiating twice w.r.t. x , we get

$$\begin{aligned}
 \frac{dy}{dx} &= Ae^{3x+1} \cdot \frac{d}{dx} (3x+1) + Be^{-3x+1} \cdot \frac{d}{dx} (-3x+1) \\
 &= Ae^{3x+1} \times (3+1+0) + Be^{-3x+1} \times (-3 \times 1+0) \\
 &= 3Ae^{3x+1} - 3Be^{-3x+1}
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \frac{d^2y}{dx^2} &= 3Ae^{3x+1} \cdot \frac{d}{dx} (3x+1) - 3Be^{-3x+1} \cdot \frac{d}{dx} (-3x+1) \\
 &= 3Ae^{3x+1} \times (3 \times 1+0) - 3Be^{-3x+1} \times (-3 \times 1+0) \\
 &= 9Ae^{3x+1} - 9Be^{-3x+1} \\
 &= 9(Ae^{3x+1} + Be^{-3x+1}) \\
 &= 9y \quad \dots \text{ [By (1)]}
 \end{aligned}$$

$$\therefore \frac{d^2y}{dx^2} - 9y = 0$$

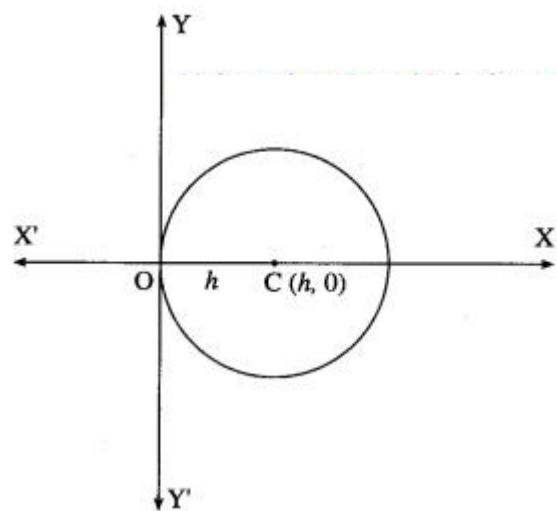
This is the required D.E.

Question 4.

Form the differential equation of:

- (i) all circles which pass through the origin and whose centres lie on X-axis.

Solution:



Let C(h, 0) be the centre of the circle which pass through the origin. Then radius of the circle is h.

\therefore equation of the circle is $(x - h)^2 + (y - 0)^2 = h^2$

$$\therefore x^2 - 2hx + h^2 + y^2 = h^2$$

$$\therefore x^2 + y^2 = 2hx \dots\dots(1)$$

Differentiating both sides w.r.t. x, we get

$$2x + 2y \frac{dy}{dx} = 2h$$

Substituting the value of $2h$ in equation (1), we get

$$x^2 + y^2 = (2x + 2y \frac{dy}{dx}) x$$

$$\therefore x^2 + y^2 = 2x^2 + 2xy \frac{dy}{dx}$$

$$\therefore 2xy \frac{dy}{dx} + x^2 - y^2 = 0$$

This is the required D.E.

(ii) all parabolas which have $4b$ as latus rectum and whose axis is parallel to Y-axis.

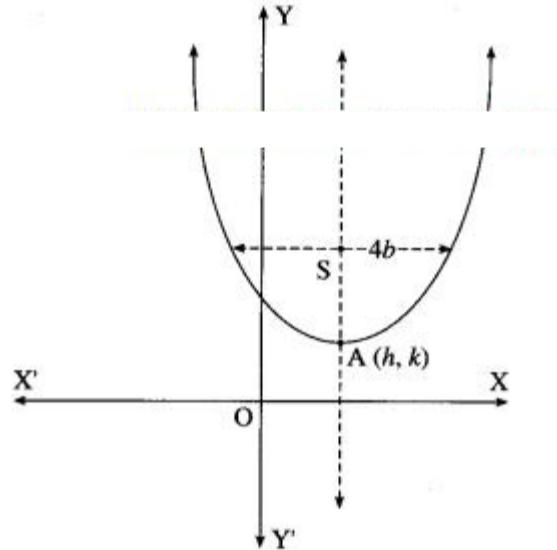
Solution:

Let A(h, k) be the vertex of the parabola which has $4b$ as latus rectum and whose axis is parallel to the Y-axis.

Then equation of the parabola is

$$(x - h)^2 = 4b(y - k) \dots\dots (1)$$

where h and k are arbitrary constants.



Differentiating both sides of (1) w.r.t. x, we get

$$2(x - h) \cdot d\frac{d}{dx}(x - h) = 4b \cdot d\frac{d}{dx}(y - k)$$

$$\therefore 2(x - h) \cdot (1 - 0) = 4b(dy \frac{d}{dx} - 0)$$

$$\therefore (x - h) = 2b \frac{dy}{dx}$$

Differentiating again w.r.t. x, we get

$$1 - 0 = 2b \frac{d^2y}{dx^2}$$

$$\therefore 2b \frac{d^2y}{dx^2} - 1 = 0$$

This is the required D.E.

(iii) an ellipse whose major axis is twice its minor axis.

Solution:

Let $2a$ and $2b$ be lengths of the major axis and minor axis of the ellipse.

Then $2a = 2(2b)$

$$\therefore a = 2b$$

\therefore equation of the ellipse is

$$x^2 a^2 + y^2 b^2 = 1$$

$$\therefore x^2 (2b)^2 + y^2 b^2 = 1$$

$$\therefore x^2 4b^2 + y^2 b^2 = 1$$

$$\therefore x^2 + 4y^2 = 4b^2$$

Differentiating w.r.t. x, we get

$$2x + 4 \times 2y \frac{dy}{dx} = 0$$

$$\therefore x + 4y \frac{dy}{dx} = 0$$

This is the required D.E.

(iv) all the lines which are normal to the line $3x + 2y + 7 = 0$.

Solution:

Slope of the line $3x - 2y + 7 = 0$ is $-3/2$.

\therefore slope of normal to this line is $-2/3$

Then the equation of the normal is

$$y = -2/3x + k, \text{ where } k \text{ is an arbitrary constant.}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -2/3 + 0$$

$$\therefore 3\frac{dy}{dx} + 2 = 0$$

This is the required D.E.

(v) the hyperbola whose length of transverse and conjugate axes are half of that of the given hyperbola $x^2/16 - y^2/36 = k$.

Solution:

The equation of the hyperbola is $x^2/16 - y^2/36 = k$

$$\text{i.e., } x^2/16k - y^2/36k = 1$$

Comparing this equation with $x^2/a^2 - y^2/b^2 = 1$, we get

$$a^2 = 16k, b^2 = 36k$$

$$\therefore a = 4\sqrt{k}, b = 6\sqrt{k}$$

$$\therefore l(\text{transverse axis}) = 2a = 8\sqrt{k}$$

$$\text{and } l(\text{conjugate axis}) = 2b = 12\sqrt{k}$$

Let $2A$ and $2B$ be the lengths of the transverse and conjugate axes of the required hyperbola.

Then according to the given condition

$$2A = a = 4\sqrt{k} \text{ and } 2B = b = 6\sqrt{k}$$

$$\therefore A = 2\sqrt{k} \text{ and } B = 3\sqrt{k}$$

\therefore equation of the required hyperbola is

$$x^2/A^2 - y^2/B^2 = 1$$

$$\text{i.e., } x^2/4k - y^2/9k = 1$$

$\therefore 9x^2 - 4y^2 = 36k$, where k is an arbitrary constant.

Differentiating w.r.t. x, we get

$$9 \times 2x - 4 \times 2y \frac{dy}{dx} = 0$$

$$\therefore 9x - 4y \frac{dy}{dx} = 0$$

This is the required D.E.

Question 5.

Solve the following differential equations:

(i) $\log(dy/dx) = 2x + 3y$

Solution:

$$\log \left(\frac{dy}{dx} \right) = 2x + 3y$$

$$\therefore \frac{dy}{dx} = e^{2x+3y} = e^{2x} \cdot e^{3y}$$

$$\therefore \frac{1}{e^{3y}} dy = e^{2x} dx$$

Integrating both sides, we get

$$\int e^{-3y} dy = \int e^{2x} dx$$

$$\therefore \int e^{-3y} dy = \int e^{2x} dx$$

$$\therefore \frac{e^{-3y}}{-3} = \frac{e^{2x}}{2} + c_1$$

$$\therefore 2e^{-3y} = -3e^{2x} + 6c_1$$

$$\therefore 2e^{-3y} + 3e^{2x} = c, \text{ where } c = 6c_1$$

This is the general solution.

(ii) $dy/dx = x^2y + y$

Solution:

$$\frac{dy}{dx} = x^2y + y$$

$$\therefore \frac{dy}{dx} = y(x^2 + 1)$$

$$\therefore \frac{1}{y} dy = (x^2 + 1) dx$$

Integrating both sides, we get

$$\int \frac{1}{y} dy = \int (x^2 + 1) dx$$

$$\therefore \log|y| = \frac{x^3}{3} + x + c$$

This is the general solution.

(iii) $dy/dx = 2y - x^2y + x$

Solution:

$$\frac{dy}{dx} = \frac{2y-x}{2y+x} \quad \dots\dots (1)$$

Put $y = vx \quad \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$

$\therefore (1)$ becomes, $v + x \frac{dv}{dx} = \frac{2vx-x}{2vx+x}$

$$\therefore v + x \frac{dv}{dx} = \frac{2v-1}{2v+1}$$

$$\therefore x \frac{dv}{dx} = \frac{2v-1}{2v+1} - v = \frac{2v-1-2v^2-v}{2v+1}$$

$$\therefore x \frac{dv}{dx} = -\left(\frac{2v^2-v+1}{2v+1}\right)$$

$$\therefore \frac{2v+1}{2v^2-v+1} dv = -\frac{1}{x} dx$$

Integrating both sides, we get

$$\int \frac{2v+1}{2v^2-v+1} dv = - \int \frac{1}{x} dx$$

$$\therefore \int \frac{\frac{1}{2}(4v-1)+\frac{3}{2}}{2v^2-v+1} dv = - \int \frac{1}{x} dx$$

$$\therefore \frac{1}{2} \int \frac{4v-1}{2v^2-v+1} dv + \frac{3}{2} \int \frac{1}{2v^2-v+1} dv = - \int \frac{1}{x} dx$$

$$\therefore \frac{1}{2} \int \frac{4v-1}{2v^2-v+1} dv + \frac{3}{4} \int \frac{1}{v^2-\frac{1}{2}v+\frac{1}{2}} dv \equiv - \int \frac{1}{x} dx$$

$$\therefore \frac{1}{2} \int \frac{4v-1}{2v^2-v+1} dv + \frac{3}{4} \int \frac{1}{\left(v^2-\frac{1}{2}v+\frac{1}{16}\right)+\frac{7}{16}} dv = - \int \frac{1}{x} dx$$

$$\therefore \frac{1}{2} \int \frac{4v-1}{2v^2-v+1} dv + \frac{3}{4} \int \frac{1}{\left(v-\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2} dv = - \int \frac{1}{x} dx$$

$$\therefore \frac{1}{2} \log|2v^2-v+1| + \frac{3}{4} \times \frac{1}{\left(\frac{\sqrt{7}}{4}\right)} \tan^{-1} \left[\frac{v-\frac{1}{4}}{\left(\frac{\sqrt{7}}{4}\right)} \right] = - \log|x| + c_1$$

$$\dots \left[\because \frac{d}{dv}(2v^2-v+1) = 4v-1 \text{ and } \int \frac{f'(v)}{f(v)} dv = \log|f(v)| + c \right]$$

$$\therefore \frac{1}{2} \log \left| 2\left(\frac{y^2}{x^2}\right) - \frac{y}{x} + 1 \right| + \frac{3}{\sqrt{7}} \tan^{-1} \left[\frac{4\left(\frac{y}{x}\right)-1}{\sqrt{7}} \right] = - \log|x| + c_1$$

$$\therefore \frac{1}{2} \log \left| \frac{2y^2-xy+x^2}{x^2} \right| + \frac{3}{\sqrt{7}} \tan^{-1} \left(\frac{4y-x}{\sqrt{7}x} \right) = - \log|x| + c_1$$

$$\therefore \log \left| \frac{x^2-xy+2y^2}{x^2} \right| + \frac{6}{\sqrt{7}} \tan^{-1} \left(\frac{4y-x}{\sqrt{7}x} \right) = - 2 \log|x| + 2c_1$$

$$\therefore \log|x^2-xy+2y^2| - \log x^2 + \frac{6}{\sqrt{7}} \tan^{-1} \left(\frac{4y-x}{\sqrt{7}x} \right) = - \log x^2 + c, \text{ where } c = 2c_1$$

$$\therefore \log|x^2-xy+2y^2| + \frac{6}{\sqrt{7}} \tan^{-1} \left(\frac{4y-x}{\sqrt{7}x} \right) = c.$$

This is the general solution.

(iv) $x dy = (x+y+1) dx$

Solution:

$$x \ dy = (x + y + 1) \ dx$$

$$\therefore \frac{dy}{dx} = \frac{x+y+1}{x} = \frac{x+1}{x} + \frac{y}{x}$$

$$\therefore \frac{dy}{dx} - \frac{1}{x} \cdot y = \frac{x+1}{x} \quad \dots\dots(1)$$

This is the linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = -\frac{1}{x} \text{ and } Q = \frac{x+1}{x}$$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int -\frac{1}{x} dx}$$

$$= e^{-\log x} = e^{\log\left(\frac{1}{x}\right)} = \frac{1}{x}$$

\therefore the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$$

$$\therefore y \cdot \frac{1}{x} = \int \frac{x+1}{x} \times \frac{1}{x} dx + c$$

$$\therefore \frac{y}{x} = \int \frac{x+1}{x^2} dx + c$$

$$\therefore \frac{y}{x} = \int \left(\frac{1}{x} + \frac{1}{x^2} \right) dx + c$$

$$\therefore \frac{y}{x} = \int \frac{1}{x} dx + \int x^{-2} dx + c$$

$$\therefore \frac{y}{x} = \log|x| + \frac{x}{-1} + c$$

$$\therefore y = x \log x - 1 + cx.$$

This is the general solution.

$$(v) dy/dx + y \cot x = x^2 \cot x + 2x$$

Solution:

$$dy/dx + y \cot x = x \cot x + 2x \dots\dots(1)$$

This is the linear differential equation of the form

$$dy/dx + Py = Q, \text{ where } P = \cot x \text{ and } Q = x^2 \cot x + 2x$$

$$\therefore \text{I.F.} = e^{\int P dx}$$

$$= e^{\int \cot x dx}$$

$$= e^{\log(\sin x)}$$

$$= \sin x$$

\therefore the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$$

$$\therefore y \sin x = \int (x^2 \cot x + 2x) \sin x dx + c$$

$$\therefore y \sin x = \int (x^2 \cot x \cdot \sin x + 2x \sin x) dx + c$$

$$\therefore y \sin x = \int x^2 \cos x dx + 2 \int x \sin x dx + c$$

$$\therefore y \sin x = x^2 \int \cos x dx - \int [ddx(x^2)] \int \cos x dx + 2 \int x \sin x dx + c$$

$$\therefore y \sin x = x^2 (\sin x) - 2x(\sin x) dx + 2 \int x \sin x dx + c$$

$$\therefore y \sin x = x^2 \sin x - 2 \int x \sin x dx + 2 \int x \sin x dx + c$$

$$\therefore y \sin x = x^2 \sin x + c$$

$$\therefore y = x^2 + c \operatorname{cosec} x$$

This is the general solution.

$$(vi) y \log y = (\log y_2 - x) dy/dx$$

Solution:

$$y \log y = (\log y^2 - x) \frac{dy}{dx}$$

$$\therefore \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{\log y^2 - x}{y \log y}$$

$$\therefore \frac{dx}{dy} = \frac{2 \log y - x}{y \log y}$$

$$\therefore \frac{dy}{dx} = \frac{2}{y} - \frac{x}{y \log y}$$

$$\therefore \frac{dx}{dy} + \frac{x}{y \log y} = \frac{2}{y} \quad \dots (1)$$

This is the linear differential equation of the form

$$\frac{dx}{dy} + Px = Q, \text{ where } P = \frac{1}{y \log y} \text{ and } Q = \frac{2}{y}$$

$$\therefore \text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{y \log y} dy}$$

$$= e^{\int \frac{(1/y)}{\log y} dy} = e^{\log |\log y|} = \log y$$

\therefore the solution of (1) is given by

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dy + c$$

$$\therefore x \cdot \log y = \int \frac{2}{y} \cdot \log y dy + c$$

$$\therefore (\log y) \cdot x = 2 \int \frac{\log y}{y} dy + c$$

$$\text{Put } \log y = t \quad \therefore \frac{1}{y} dy = dt$$

$$\therefore (\log y) \cdot x = 2 \int t dt + c$$

$$\therefore x \log y = 2 \cdot \frac{t^2}{2} + c$$

$$\therefore x \log y = (\log y)^2 + c$$

This is the general solution.

(vii) $4 \frac{dx}{dy} + 8x = 5e^{-3y}$

Solution:

$$4 \frac{dx}{dy} + 8x = 5e^{-3y}$$

$$\therefore \frac{dx}{dy} + 2x = \frac{5}{4}e^{-3y} \quad \dots\dots (1)$$

This is the linear differential equation of the form

$$\frac{dx}{dy} + Px = Q, \text{ where } P = 2, \text{ and } Q = \frac{5}{4}e^{-3y}$$

$$\therefore \text{I.F.} = e^{\int P dy} = e^{\int 2 dy} = e^{2y}$$

\therefore the solution of (1) is given by

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dy + c_1$$

$$\therefore x \cdot e^{2y} = \int \frac{5}{4}e^{-3y} \cdot e^{2y} dy + c_1$$

$$\therefore x \cdot e^{2y} = \frac{5}{4} \int e^{-y} dy + c_1$$

$$\therefore xe^{2y} = \frac{5}{4} \cdot \frac{e^{-y}}{-1} + c_1$$

$$\therefore 4xe^{2y} = -5e^{-y} + 4c_1$$

$$\therefore 4xe^{2y} + 5e^{-y} = c, \text{ where } c = 4c_1$$

This is the general solution.

Question 6.

Find the particular solution of the following differential equations:

(i) $y(1 + \log x) = (\log x^x) \frac{dy}{dx}$, when $y(e) = e^2$

Solution:

$$y(1 + \log x) = (\log x^x) \frac{dy}{dx}$$

$$y(1 + \log x) - (\log x^x) \frac{dy}{dx} = 0$$

$$y(1 + \log x) \frac{dx}{dy} - x \log x = 0$$

$$\therefore \frac{1 + \log x}{x \log x} dx - \frac{dy}{y} = 0$$

Integrating both sides, we get

$$\therefore \int \frac{1 + \log x}{x \log x} dx - \frac{dy}{y} = c_1 \quad \dots\dots (1)$$

Put $x \log x = t$

$$\text{Then } \left[x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x) \right] dx = dt$$

$$\therefore \left[\frac{x}{x} + (\log x)(1) \right] dx = dt$$

$$\therefore \int \frac{1 + \log x}{x \log x} dx = \int \frac{dt}{t} = \log|t| = \log|x \log x|$$

\therefore from (1), the general solution is

$$\log|x \log x| - \log|y| = \log c, \text{ where } c_1 = \log c$$

$$\therefore \log \left| \frac{x \log x}{y} \right| = \log c$$

$$\therefore \frac{x \log x}{y} = c$$

$$\therefore x \log x = cy$$

This is the general solution.

$$\text{Now, } y = e^2, \text{ when } x = e$$

$$\therefore e \log e = c \cdot e^2$$

$$\therefore 1 = c \cdot e$$

$$\therefore c = \frac{1}{e}$$

$$\therefore \text{the particular solution is } x \log x = \left(\frac{1}{e} \right) y$$

$$\therefore y = ex \log x.$$

$$(ii) (x + 2y^2) dy/dx = y, \text{ when } x = 2, y = 1$$

Solution:

$$(x + 2y^2) \frac{dy}{dx} = y$$

$$\therefore \frac{dy}{dx} = \frac{x + 2y^2}{y} = \frac{x}{y} + 2y$$

$$\therefore \frac{dx}{dy} - \frac{1}{y} \cdot x = 2y \quad \dots (1)$$

This is the linear differential equation of the form

$$\frac{dx}{dy} + Px = Q, \text{ where } P = -\frac{1}{y} \text{ and } Q = 2y.$$

$$\therefore \text{I.F.} = e^{\int P dy} = e^{\int -\frac{1}{y} dy}$$

$$= e^{-\log y} = e^{\log \left(\frac{1}{y} \right)} = \frac{1}{y}$$

\therefore the solution of (1) is given by

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dy + c$$

$$\therefore x \times \frac{1}{y} = \int 2y \times \frac{1}{y} dy + c$$

$$\therefore \frac{x}{y} = 2 \int 1 dy + c$$

$$\therefore \frac{x}{y} = 2y + c$$

$$\therefore x = 2y^2 + cy$$

This is the general solution.

When $x = 2, y = 1$, we have

$$2 = 2(1)^2 + c(1)$$

$$\therefore c = 0$$

\therefore the particular solution is $x = 2y^2$.

(iii) $dy/dx - 3y \cot x = \sin 2x$, when $y(\pi/2) = 2$

Solution:

$$dy/dx - 3y \cot x = \sin 2x$$

$$dy/dx = (3 \cot x)y + \sin 2x \dots\dots(1)$$

This is the linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = -3 \cot x \text{ and } Q = \sin 2x$$

$$\therefore \text{I.F.} = e^{\int P dx} = e^{\int -3 \cot x dx}$$

$$= e^{-3 \log \sin x} = e^{\log (\sin x)^{-3}}$$

$$= (\sin x)^{-3} = \frac{1}{\sin^3 x}$$

\therefore the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$$

$$\therefore y \times \frac{1}{\sin^3 x} = \int \sin 2x \times \frac{1}{\sin 3x} dx + c$$

$$\therefore y \operatorname{cosec}^3 x = \int 2 \sin x \cos x \times \frac{1}{\sin^3 x} dx + c$$

$$\therefore y \operatorname{cosec}^3 x = 2 \int \frac{\cos x}{\sin^2 x} dx + c$$

$$\text{Put } \sin x = t \quad \therefore \cos x dx = dt$$

$$\therefore y \operatorname{cosec}^3 x = 2 \int \frac{1}{t^2} dt + c$$

$$\therefore y \operatorname{cosec}^3 x = 2 \int t^{-2} dt + c$$

$$\therefore y \operatorname{cosec}^3 x = 2 \left[\frac{t^{-1}}{-1} \right] + c$$

$$\therefore y \operatorname{cosec}^3 x = \frac{-2}{\sin x} + c$$

$$\therefore y \operatorname{cosec}^3 x + 2 \operatorname{cosec} x = c$$

This is the general solution.

$$\text{Now, } y\left(\frac{\pi}{2}\right) = 2, \text{ i.e. } y = 2, \text{ when } x = \frac{\pi}{2}$$

$$\therefore 2 \operatorname{cosec}^3 \frac{\pi}{2} + 2 \operatorname{cosec} \frac{\pi}{2} = c$$

$$\therefore 2(1)^3 + 2(1) = c \quad \therefore c = 4$$

\therefore the particular solution is

$$y \operatorname{cosec}^3 x + 2 \operatorname{cosec} x = 4$$

$$\therefore y \operatorname{cosec}^2 x + 2 = 4 \sin x.$$

(iv) $(x+y) dy + (x-y) dx = 0$; when $x = 1 = y$

Solution:

$$(x+y)dy + (x-y)dx = 0$$

$$\therefore (x+y)dy = -(x-y)dx$$

$$\therefore \frac{dy}{dx} = \frac{y-x}{y+x} \quad \dots (1)$$

$$\text{Put } y = vx \quad \therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\therefore (1) \text{ becomes, } v + x \frac{dv}{dx} = \frac{vx - x}{vx + x} = \frac{v-1}{v+1}$$

$$\therefore x \frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v-1-v^2-v}{v+1}$$

$$\therefore x \frac{dv}{dx} = -\left(\frac{1+v^2}{1+v}\right)$$

$$\therefore \frac{1+v}{1+v^2} dv = -\frac{1}{x} dx$$

$$\int \frac{1+v}{1+v^2} dv = - \int \frac{1}{x} dx$$

$$\therefore \int \left(\frac{1}{1+v^2} + \frac{v}{1+v^2} \right) dv = - \int \frac{1}{x} dx$$

$$\therefore \int \frac{1}{1+v^2} dv + \frac{1}{2} \int \frac{2v}{1+v^2} dv = - \int \frac{1}{x} dx$$

$$\therefore \tan^{-1} v + \frac{1}{2} \log |1+v^2| = -\log x + c$$

$$\dots \left[\because \frac{d}{dv}(1+v^2) = 2v \text{ and } \int \frac{f'(v)}{f(v)} dv = \log |f(v)| + c \right]$$

$$\therefore \tan^{-1} \left(\frac{y}{x} \right) + \frac{1}{2} \log \left| 1 + \frac{y^2}{x^2} \right| = -\log |x| + c$$

$$\therefore \tan^{-1} \left(\frac{y}{x} \right) + \frac{1}{2} \log \left| \frac{x^2 + y^2}{x^2} \right| = -\log |x| + c$$

$$\therefore \tan^{-1} \left(\frac{y}{x} \right) + \frac{1}{2} \log |x^2 + y^2| - \frac{1}{2} \log |x^2| = -\log |x| + c$$

$$\therefore \tan^{-1} \left(\frac{y}{x} \right) + \log \sqrt{x^2 + y^2} - \log |x| = -\log |x| + c$$

$$\therefore \tan^{-1} \left(\frac{y}{x} \right) + \log \sqrt{x^2 + y^2} = c$$

This is the general solution.

When $x = 1 = y$, we have

$$\tan^{-1} \left(\frac{1}{1} \right) + \log \sqrt{1^2 + 1^2} = c$$

$$\therefore \tan^{-1} \left(\tan \frac{\pi}{4} \right) + \log \sqrt{2} = c$$

$$\therefore c = \frac{\pi}{4} + \log \sqrt{2}$$

\therefore the particular solution is

$$\therefore \tan^{-1} \left(\frac{y}{x} \right) + \log \sqrt{x^2 + y^2} = \frac{\pi}{4} + \log \sqrt{2}.$$

$$(v) 2e_{xy}dx + (y - 2xe_{xy})dy = 0, \text{ when } y(0) = 1$$

Solution:

$$(1 + 2e^{x/y})dx + 2e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$$

$$\therefore (1 + 2e^{x/y})dx = -2e^{x/y}\left(1 - \frac{x}{y}\right)dy$$

$$\therefore (1 + 2e^{x/y})dx = 2e^{x/y}\left(\frac{x}{y} - 1\right)dy$$

$$\therefore \frac{dy}{dx} = \frac{2e^{x/y}\left(\frac{x}{y} - 1\right)}{1 + 2e^{x/y}} \quad \dots\dots (1)$$

$$\text{Put } x = vy \quad \therefore \frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$\therefore (1) \text{ becomes, } v + y\frac{dv}{dy} = \frac{2e^v(v-1)}{1+2e^v}$$

$$\begin{aligned} \therefore y\frac{dv}{dy} &= \frac{2e^v(v-1)}{1+2e^v} - v \\ &= \frac{2ve^v - 2e^v - v - 2ve^v}{1+2e^v} \end{aligned}$$

$$\therefore \left(\frac{1+2e^v}{v+2e^v}\right)dv \equiv -\frac{1}{y}dy$$

Integrating both sides, we get

$$\int \frac{1+2e^v}{v+2e^v} dv = - \int \frac{1}{y} dy$$

$$\therefore \log|v+2e^v| = -\log y + \log c$$

$$\dots \left[\because \frac{d}{dv}(v+2e^v) = 1+2e^v \text{ and } \int \frac{f'(v)}{f(v)} dv = \log|f(v)| + c \right]$$

$$\therefore \log|v+2e^v| + \log y = \log c$$

$$\therefore \log|y(v+2e^v)| = \log c$$

$$\therefore y(v+2e^v) = c$$

$$\therefore y\left(\frac{x}{y} + 2e^{x/y}\right) = c$$

$$\therefore x + 2ye^{x/y} = c$$

This is the general solution.

Now, $y(0) = 1$, i.e. when $x = 0$, $y = 1$

$$\therefore 0 + 2(1)e^0 = c \quad \therefore c = 2$$

\therefore the particular solution is $x + 2ye^{x/y} = 2$.

Question 7.

Show that the general solution of differential equation $dy/dx + y_2 + y + 1/x_2 + x + 1 = 0$ is given by $(x + y + 1) = c(1 - x - y - 2xy)$.

Solution:

$$\frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0$$

$$\therefore \frac{dy}{dx} = -\left(\frac{y^2 + y + 1}{x^2 + x + 1}\right)$$

$$\therefore \frac{1}{y^2 + y + 1} dy = -\frac{1}{x^2 + x + 1} dx$$

Integrating both sides, we get

$$\int \frac{1}{y^2 + y + 1} dy = - \int \frac{1}{x^2 + x + 1} dx$$

$$\therefore \int \frac{1}{(y^2 + y + \frac{1}{4}) + \frac{3}{4}} dy = - \int \frac{1}{(x^2 + x + \frac{1}{4}) + \frac{3}{4}} dx$$

$$\therefore \int \frac{1}{(y + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} dy = - \int \frac{1}{(x + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} dx$$

$$\therefore \frac{1}{(\frac{\sqrt{3}}{2})} \tan^{-1} \left[\frac{y + \frac{1}{2}}{(\frac{\sqrt{3}}{2})} \right] = - \frac{1}{(\frac{\sqrt{3}}{2})} \tan^{-1} \left[\frac{x + \frac{1}{2}}{(\frac{\sqrt{3}}{2})} \right] + c_1$$

$$\therefore \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2y+1}{\sqrt{3}} \right) + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) = c_1$$

$$\therefore \frac{2}{\sqrt{3}} \tan^{-1} \left[\frac{\left(\frac{2y+1}{\sqrt{3}} \right) + \left(\frac{2x+1}{\sqrt{3}} \right)}{1 - \left(\frac{2y+1}{\sqrt{3}} \right) \left(\frac{2x+1}{\sqrt{3}} \right)} \right] = c_1$$

$$\therefore \tan^{-1} \left[\frac{\left(\frac{2y+1+2x+1}{\sqrt{3}} \right)}{\left(\frac{3-4xy-2y-2x-1}{3} \right)} \right] = \frac{\sqrt{3}}{2} c_1$$

$$\therefore \frac{(2y+2x+2)\sqrt{3}}{2-2x-2y-4xy} = \tan \left(\frac{\sqrt{3}}{2} c_1 \right)$$

$$\therefore \frac{x+y+1}{1-x-y-2xy} = \frac{1}{\sqrt{3}} \tan \left(\frac{\sqrt{3}}{2} c_1 \right) = c,$$

$$\text{where } c = \frac{1}{\sqrt{3}} \tan \left(\frac{\sqrt{3}}{2} c_1 \right)$$

$$\therefore (x+y+1) = c(1-x-y-2xy)$$

This is the general solution.

Question 8.

The normal lines to a given curve at each point (x, y) on the curve pass through $(2, 0)$. The curve passes through $(2, 3)$. Find the equation of the curve.

Solution:

Let $P(x, y)$ be a point on the curve $y = f(x)$.

Then slope of the normal to the curve is $-1(\frac{dy}{dx})$

\therefore equation of the normal is

$$Y-y = -\frac{1}{\left(\frac{dy}{dx}\right)}(X-x)$$

$$\therefore (Y-y)\frac{dy}{dx} = -(X-x)$$

$$\therefore (Y-y)\frac{dy}{dx} + (X-x) = 0$$

Since, this normal passes through (2, 0), we get

$$(0-y)\frac{dy}{dx} + (2-x) = 0$$

$$\therefore -y\frac{dy}{dx} = x-2 \quad \therefore -y dy = (x-2)dx$$

Integrating both sides, we get

$$-\int y dy = \int (x-2)dx$$

$$\therefore -\frac{y^2}{2} = \frac{x^2}{2} - 2x + c_1$$

$$\therefore \frac{x^2}{2} + \frac{y^2}{2} - 2x + c_1 = 0$$

$$\therefore x^2 + y^2 - 4x + 2c_1 = 0$$

$$\therefore x^2 + y^2 = 4x - 2c_1$$

$$\therefore x^2 + y^2 = 4x + c, \text{ where } c = -2c_1$$

This is the general equation of the curve.

Since, the required curve passed through the point (2, 3), we get

$$2^2 + 3^2 = 4(2) + c$$

$$\therefore c = 5$$

\therefore equation of the required curve is $x^2 + y^2 = 4x + 5$.

Question 9.

The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after t seconds.

Solution:

Let r be the radius and V be the volume of the spherical balloon at any time t.

Then the rate of change in volume of the spherical balloon is dV/dt which is a constant.

$$\therefore \frac{dV}{dt} = k', \text{ where } k' \text{ is a constant.}$$

$$\text{But } V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dt} = \frac{4\pi}{3} \times 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\therefore 4\pi r^2 \frac{dr}{dt} = k'$$

$$\therefore r^2 \frac{dr}{dt} = \frac{k'}{4\pi} = k, \text{ where } k = \frac{k'}{4\pi}$$

$$\therefore r^2 dr = k dt$$

Integrating both sides, we get

$$\int r^2 dr = k \int dt$$

$$\therefore \frac{r^3}{3} = kt + c$$

Initially the radius is 3 units

i.e. $r = 3$, when $t = 0$

$$\therefore \frac{3^3}{3} = k(0) + c$$

$$\therefore c = 9$$

$$\therefore \frac{x^3}{3} = kt + 9$$

Also, $r = 6$, when $t = 3$

$$\therefore \frac{(6)^3}{3} = k(3) + 9$$

$$\therefore 3k = 72 - 9 = 63 \quad \therefore k = 21$$

$$\therefore \frac{r^3}{3} = 21t + 9$$

$$\therefore r^3 = 63t + 27$$

$$\therefore r = (63t + 27)^{\frac{1}{3}}$$

Hence, the radius of the spherical balloon after t seconds is $(63t+27)^{\frac{1}{3}}$ units.

Question 10.

A person's assets start reducing in such a way that the rate of reduction of assets is proportional to the square root of the assets existing at that moment. If the assets at the beginning are ₹ 10 lakhs and they dwindle down to ₹ 10,000 after 2 years, show that the person will be bankrupt in 229 years from the start.

Solution:

Let x be the assets of the person at time t years.

Then the rate of reduction is dx/dt which is proportional to \sqrt{x} .

$$\therefore dx/dt \propto \sqrt{x}$$

$$\therefore dx/dt = -k\sqrt{x}, \text{ where } k > 0$$

$$\therefore dx/\sqrt{x} = -k dt$$

Integrating both sides, we get

$$\int x^{-\frac{1}{2}} dx = -k \int dt$$

$$\therefore x^{\frac{1}{2}}(12) = -kt + c$$

$$\therefore 2\sqrt{x} = -kt + c$$

At the beginning, i.e. at $t = 0$, $x = 10,00,000$

$$2\sqrt{10,00,000} = -k(0) + c$$

$$\therefore c = 2000$$

$$\therefore 2\sqrt{x} = -kt + 2000 \dots\dots(1)$$

Also, when $t = 2$, $x = 10,000$

$$\therefore 2\sqrt{10000} = -k \times 2 + 2000$$

$$\therefore 2k = 1800$$

$$\therefore k = 900$$

∴ (1) becomes,

$$\therefore 2\sqrt{x} = -900t + 2000$$

When the person will be bankrupt, $x = 0$

$$\therefore 0 = -900t + 2000$$

$$\therefore 900t = 2000$$

$$\therefore t = 200/9 = 229$$

Hence, the person will be bankrupt in 229 years.