Maharashtra State Board 12th Maths Solutions Chapter 7 Probability Distributions Ex 7.1

Question 1.
Let X represent the difference between a number of heads and the number of tails when a coin is tossed 6 times. What are the possible values of X?
Solution:
When a coin is tossed 6 times, the number of heads can be 0, 1, 2, 3, 4, 5, 6.
The corresponding number of tails will be 6, 5, 4, 3, 2, 1, 0.
∴ X can take values 0 – 6, 1 – 5, 2 – 4, 3 – 3, 4 – 2, 5 – 1, 6 – 0
i.e. -6, -4, -2, 0, 2, 4, 6.
∴ X = {-6, -4, -2, 0, 2, 4, 6}.

Question 2.
An urn contains 5 red and 2 black balls. Two balls are drawn at random. X denotes the number of black balls drawn. What are the possible values of X?
Solution:
The urn contains 5 red and 2 black balls.
If two balls are drawn from the urn, it contains either 0 or 1 or 2 black balls.
X can take values 0, 1, 2.
∴ X = {0, 1, 2}.

Question 3.
State which of the following are not the probability mass function of a random variable. Give reasons for your answer.

(i)

X	0	1	2
$P(X)$	0·4	0·4	0·2

(ii)

X	0	1	2	3	4
$P(X)$	0·1	0·5	0·2	–0·1	0·2

(iii)

X	0	1	2
$P(X)$	0·1	0·6	0·3

(iv)

Z	3	2	1	0	–1
$P(Z)$	0·3	0·2	0·4	0	0·05

(v)

Y	–1	0	1
$P(Y)$	0·6	0·1	0·2

(vi)

X	0	–1	–2
$P(X)$	0·3	0·4	0·3

Solution:
P.m.f. of random variable should satisfy the following conditions:
(a) $0 \leq p_i \leq 1$
(b) $\Sigma p_i = 1$.

(i)

X	0	1	2
$P(X)$	0·4	0·4	0·2

(a) Here $0 \leq p_i \leq 1$
(b) $\Sigma p_i = 0.4 + 0.4 + 0.2 = 1$
Hence, P(X) can be regarded as p.m.f. of the random variable X.

(ii)

X	0	1	2	3	4
$P(X)$	0·1	0·5	0·2	–0·1	0·2

P(X = 3) = -0.1, i.e. $P_i < 0$ which does not satisfy $0 \leq P_i \leq 1$
Hence, P(X) cannot be regarded as p.m.f. of the random variable X.

(iii)

X	0	1	2
$P(X)$	0·1	0·6	0·3

(a) Here $0 \leq p_i \leq 1$

(b) $\sum p_i = 0.1 + 0.6 + 0.3 = 1$
Hence, P(X) can be regarded as p.m.f. of the random variable X.

(iv)

Z	3	2	1	0	−1
$P(Z)$	0·3	0·2	0·4	0	0·05

Here $\sum p_i = 0.3 + 0.2 + 0.4 + 0 + 0.05 = 0.95 \neq 1$
Hence, P(Z) cannot be regarded as p.m.f. of the random variable Z.

(v)

Y	−1	0	1
$P(Y)$	0·6	0·1	0·2

Here $\sum p_i = 0.6 + 0.1 + 0.2 = 0.9 \neq 1$
Hence, P(Y) cannot be regarded as p.m.f. of the random variable Y.

(vi)

X	0	−1	−2
$P(X)$	0·3	0·4	0·3

(a) Here $0 \leq p_i \leq 1$
(b) $\sum p_i = 0.3 + 0.4 + 0.3 = 1$
Hence, P(X) can be regarded as p.m.f. of the random variable X.

Question 4.
Find the probability distribution of
(i) number of heads in two tosses of a coin.
(ii) number of tails in the simultaneous tosses of three coins.
(iii) number of heads in four tosses of a coin.
Solution:
(i) For two tosses of a coin the sample space is {HH, HT, TH, TT}
Let X denote the number of heads in two tosses of a coin.
Then X can take values 0, 1, 2.
∴ $P[X = 0] = P(0) = \frac{1}{4}$
$P[X = 1] = P(1) = \frac{2}{4} = \frac{1}{2}$
$P[X = 2] = P(2) = \frac{1}{4}$
∴ the required probability distribution is

$X = x$	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(ii) When three coins are tossed simultaneously, then the sample space is
{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}
Let X denotes the number of tails.
Then X can take the value 0, 1, 2, 3.
∴ $P[X = 0] = P(0) = \frac{1}{8}$
$P[X = 1] = P(1) = \frac{3}{8}$
$P[X = 2] = P(2) = \frac{3}{8}$
$P[X = 3] = P(3) = \frac{1}{8}$
∴ the required probability distribution is

$X = x$	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(iii) When a fair coin is tossed 4 times, then the sample space is
S = {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT}
∴ n(S) = 16
Let X denotes the number of heads.
Then X can take the value 0, 1, 2, 3, 4

When X = 0, then X = {TTTT}
∴ n(X) = 1
∴ P(X = 0) = $n(X)n(S)=116$
When X = 1, then
X = {HTTT, THTT, TTHT, TTTH}
∴ n(X) = 4
∴ P(X = 1) = $n(X)n(S)=416=14$
When X = 2, then
X = {HHTT, HTHT, HTTH, THHT, THTH, TTHH}
∴ n(X) = 6
∴ P(X = 2) = $n(X)n(S)=616=38$
When X = 3, then
X = {HHHT, HHTH, HTHH, THHH}
∴ n(X) = 4
∴ P(X = 3) = $n(X)n(S)=416=14$
When X = 4, then X = {HHHH}
∴ n(X) = 1
∴ P(X = 4) = $n(X)n(S)=116$
∴ the probability distribution of X is as follows:

x	0	1	2	3	4
$P(X=x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

Question 5.
Find the probability distribution of a number of successes in two tosses of a die, where success is defined as a number greater than 4 appearing on at least one die.
Solution:
When a die is tossed twice, the sample space s has 6 × 6 = 36 sample points.
∴ n(S) = 36
The trial will be a success if the number on at least one die is 5 or 6.
Let X denote the number of dice on which 5 or 6 appears.
Then X can take values 0, 1, 2.
When X = 0 i.e., 5 or 6 do not appear on any of the dice, then
X = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)}
∴ n(X) = 16.
∴ P(X = 0) = $n(X)n(S)=1636=49$
When X = 1, i.e. 5 or 6 appear on exactly one of the dice, then
X = {(1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (6, 1), (6, 2), (6, 3), (6, 4)}
∴ n(X) = 16
∴ P(X = 1) = $n(X)n(S)=1636=49$
When X = 2, i.e. 5 or 6 appear on both of the dice, then
X = {(5, 5), (5, 6), (6, 5), (6, 6)}
∴ n(X) = 4
∴ P(X = 2) = $n(X)n(S)=436=19$
∴ the required probability distribution is

$X=x$	0	1	2
$P(X=x)$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

Question 6.
From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.
Solution:
Here, the number of defective bulbs is the random variable.
Let the number of defective bulbs be denoted by X.
∴ X can take the value 0, 1, 2, 3, 4.
Since the draws are done with replacement, therefore the four draws are independent experiments.
Total number of bulbs is 30 which include 6 defectives.
∴ P(X = 0) = P(0) = P(all 4 non-defective bulbs)

= 2430×2430×2430×2430

= 256625

P(X = 1) = P (1) = P (1 defective and 3 non-defective bulbs)

$$=\frac{6}{30}\times\frac{24}{30}\times\frac{24}{30}\times\frac{24}{30}+\frac{24}{30}\times\frac{6}{30}\times\frac{24}{30}\times\frac{24}{30}+\frac{24}{30}\times\frac{24}{30}\times\frac{6}{30}\times\frac{24}{30}+\frac{24}{30}\times\frac{24}{30}\times\frac{24}{30}\times\frac{6}{30}$$

$$=\frac{256}{625}$$

P(X = 2) = P(2) = P(2 defective and 2 non-defective)

$$=\frac{6}{30}\times\frac{6}{30}\times\frac{24}{30}\times\frac{24}{30}+\frac{24}{30}\times\frac{6}{30}\times\frac{6}{30}\times\frac{24}{30}+\frac{24}{30}\times\frac{24}{30}\times\frac{6}{30}\times\frac{6}{30}+\frac{6}{30}\times\frac{24}{30}\times\frac{6}{30}\times\frac{24}{30}+\frac{6}{30}\times\frac{24}{30}\times\frac{24}{30}\times\frac{6}{30}+\frac{24}{30}\times\frac{6}{30}\times\frac{24}{30}\times\frac{6}{30}$$

$$=\frac{96}{625}$$

P(X = 3) = P(3) = P(3 defectives and 1 non-defective)

$$=\frac{6}{30}\times\frac{6}{30}\times\frac{6}{30}\times\frac{24}{30}+\frac{6}{30}\times\frac{6}{30}\times\frac{24}{30}\times\frac{6}{30}+\frac{6}{30}\times\frac{24}{30}\times\frac{6}{30}\times\frac{6}{30}+\frac{24}{30}\times\frac{6}{30}\times\frac{6}{30}\times\frac{6}{30}$$

$$=\frac{16}{625}$$

P(X = 4) = P(4) = P(all 4 defectives)

= 630×630×630×630

= 1625

∴ the required probability distribution is

$X=x$	0	1	2	3	4
$P(X=x)$	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

Question 7.
A coin is biased so that the head is 3 times as likely to occur as the tail. If the coin is tossed twice. Find the probability distribution of a number of tails.
Solution:
Given a biased coin such that heads is 3 times as likely as tails.
∴ P(H) = 34 and P(T) = 14
The coin is tossed twice.
Let X can be the random variable for the number of tails.
Then X can take the value 0, 1, 2.

∴ P(X = 0) = P(HH) = 34×34=916

P(X = 1) = P(HT, TH) = 34×14+14×34=616=38

P(X = 2) = P(TT) = 14×14=116

∴ the required probability distribution is

$X=x$	0	1	2
$P(X=x)$	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

Question 8.
A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7
$P(X)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Determine:
(i) k
(ii) P(X < 3) (iii) P(X > 4)
Solution:
(i) Since P (x) is a probability distribution of x,
$\sum_{x=0}^{7} P(x) = 1$
⇒ P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) = 1
⇒ $0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$
⇒ $10k^2 + 9k - 1 = 0$
⇒ $10k^2 + 10k - k - 1 = 0$
⇒ 10k(k + 1) – 1(k + 1) = 0
⇒ (k + 1)(10k – 1) = 0
⇒ 10k – 1 = 0 ……..[∵ k ≠ -1]
⇒ k = $\frac{1}{10}$

(ii) P(X< 3) = P(0) + P(1) + P(2)
= 0 + k + 2k
= 3k
= 3($\frac{1}{10}$)
= $\frac{3}{10}$

(iii) P(0 < X < 3) = P (1) + P (2)
= k + 2k
= 3k
= 3($\frac{1}{10}$)
= $\frac{3}{10}$

Question 9.
Find expected value and variance of X for the following p.m.f.:

X	–2	–1	0	1	2
$P(X)$	0·2	0·3	0·1	0·15	0·25

Solution:
We construct the following table to calculate E(X) and V(X):

$X = x_i$	$p_i = P[X = x_i]$	$x_i \cdot p_i$	$x_i^2 \cdot p_i = x_i \times x_i \cdot p_i$
–2	0.2	–0.4	0.8
–1	0.3	–0.3	0.3
0	0.1	0	0
1	0.15	0.15	0.15
2	0.25	0.5	1
Total	1	–0.05	2.25

From the table,
$\Sigma x_i p_i$ = -0.05 and $\Sigma x_i^2 \cdot p_i$ = 2.25
∴ E(X) = $\Sigma x_i p_i$ = -0.05
and V(X) = $\Sigma x_i^2 + p_i - (\Sigma x_i + p_i)^2$
= 2.25 – $(-0.05)^2$
= 2.25 – 0.0025
= 2.2475
Hence, E(X) = -0.05 and V(X) = 2.2475.

Question 10.
Find expected value and variance of X, where X is the number obtained on the uppermost face when a fair die is thrown.
Solution:
If a die is tossed, then the sample space for the random variable X is
S = {1, 2, 3, 4, 5, 6}
∴ P(X) = $\frac{1}{6}$; X = 1, 2, 3, 4, 5, 6.

$$\therefore\ E(X)=\sum_{X\in S} X\cdot P(X)$$

$$=1\left(\frac{1}{6}\right)+2\left(\frac{1}{6}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)+5\left(\frac{1}{6}\right)+6\left(\frac{1}{6}\right)$$

$$=\frac{1}{6}(1+2+3+4+5+6)$$

$$=\frac{21}{6}=\frac{7}{2}=3.5$$

$$V(X)=E(X^2)-[E(X)]^2$$

$$=\sum_{X\in S} X^2\cdot P(X)-\left(\frac{7}{2}\right)^2$$

$$=\left[(1)^2\left(\frac{1}{6}\right)+(2)^2\left(\frac{1}{6}\right)+(3)^2\left(\frac{1}{6}\right)+(4)^2\left(\frac{1}{6}\right)+(5)^2\left(\frac{1}{6}\right)+(6)^2\left(\frac{1}{6}\right)\right]-\frac{49}{4}$$

$$=\frac{1}{6}(1+4+9+16+25+36)-\frac{49}{4}$$

$$=\frac{91}{6}-\frac{49}{4}=\frac{182-147}{12}=\frac{35}{12}=2.9167$$

Hence, E(X) = 3.5 and V(X) = 2.9167.

Question 11.
Find the mean number of heads in three tosses of a fair coin.
Solution:
When three coins are tossed the sample space is {HHH, HHT, THH, HTH, HTT, THT, TTH, TTT}
∴ n(S) = 8
Let X denote the number of heads when three coins are tossed.
Then X can take values 0, 1, 2, 3
P(X = 0) = P(0) = $\frac{1}{8}$
P(X = 1) = P(1) = $\frac{3}{8}$
P(X = 2) = P(2) = $\frac{3}{8}$
P(X = 3) = P(3) = $\frac{1}{8}$
∴ mean = E(X) = $\Sigma x_i P(x_i)$
= $0\times\frac{1}{8}+1\times\frac{3}{8}+2\times\frac{3}{8}+3\times\frac{1}{8}$
= $0+\frac{3}{8}+\frac{6}{8}+\frac{3}{8}$
= $\frac{12}{8}$
= 1.5

Question 12.
Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.
Solution:
When two dice are thrown, the sample space S has 6 × 6 = 36 sample points.
∴ n(S) = 36
Let X denote the number of sixes when two dice are thrown.
Then X can take values 0, 1, 2
When X = 0, then
X = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)}
∴ n(X) = 25
∴ P(X = 0) = $\frac{n(X)}{n(S)}=\frac{25}{36}$
When X = 1, then
X = {(1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)}
∴ n(X) = 10
∴ P(X = 1) = $\frac{n(X)}{n(S)}=\frac{10}{36}$
When X = 2, then X = {(6, 6)}
∴ n(X) = 1
∴ P(X = 2) = $\frac{n(X)}{n(S)}=\frac{1}{36}$

∴ E(X) = ΣxiP(xi)

= 0×2536+1×1036+2×136

= 0+1036+236

= 13

Question 13.
Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers. Find E(X).
Solution:
Two numbers are chosen from the first 6 positive integers.

∴ n(S) = 6C2=6×51×2 = 15

Let X denote the larger of the two numbers.
Then X can take values 2, 3, 4, 5, 6.
When X = 2, the other positive number which is less than 2 is 1.
∴ n(X) = 1

∴ P(X = 2) = P(2) = n(X)n(S)=115

When X = 3, the other positive number less than 3 can be 1 or 2 and hence can be chosen in 2 ways.
∴ n(X) = 2

P(X = 3) = P(3) = n(X)n(S)=215

Similarly, P(X = 4) = P(4) = 315

P(X = 5) = P(5) = 415

P(X = 6) = P(6) = 515

∴ E(X) = Σx_iP(x_i)

= 2×115+3×215+4×315+5×415+6×515

= 2+6+12+20+3015

= 7015

= 143

Question 14.
Let X denote the sum of numbers obtained when two fair dice are rolled. Find the standard deviation of X.
Solution:
If two fair dice are rolled then the sample space S of this experiment is
S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}
∴ n(S) = 36
Let X denote the sum of the numbers on uppermost faces.
Then X can take the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

Sum of Nos. (X)	Favourable events	No. of favourable cases	$P(X)$
2	(1, 1)	1	$\frac{1}{36}$
3	(1, 2), (2, 1)	2	$\frac{2}{36}$
4	(1, 3), (2, 2), (3, 1)	3	$\frac{3}{36}$
5	(1, 4), (2, 3), (3, 2), (4, 1)	4	$\frac{4}{36}$
6	(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)	5	$\frac{5}{36}$
7	(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)	6	$\frac{6}{36}$
8	(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)	5	$\frac{5}{36}$
9	(3, 6), (4, 5), (5, 4), (6, 3)	4	$\frac{4}{36}$
10	(4, 6), (5, 5), (6, 4)	3	$\frac{3}{36}$
11	(5, 6), (6, 5)	2	$\frac{2}{36}$
12	(6, 6)	1	$\frac{1}{36}$

$\therefore$ the probability distribution of X is given by

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$P[X = x_i]$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Expected value $= E(X) = \Sigma\, x_i \cdot P(x_i)$

$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) +$

$7\left(\frac{6}{36}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right)$

$= \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12)$

$= \frac{1}{36} \times 252 = 7.$

Also, $\Sigma x_i^2 \cdot P(x_i)$

$= 4 \times \frac{1}{36} + 9 \times \frac{2}{36} + 16 \times \frac{3}{36} + 25 \times \frac{4}{36} +$

$36 \times \frac{5}{36} + 49 \times \frac{6}{36} + 64 \times \frac{5}{36} + 81 \times \frac{4}{36} +$

$100 \times \frac{3}{36} + 121 \times \frac{2}{36} + 144 \times \frac{1}{36}$

$= \frac{1}{36}[4 + 18 + 48 + 100 + 180 + 294 + 320 + 324 +$

$300 + 242 + 144]$

$= \frac{1}{36}(1974) = 54.83$

$\therefore$ variance $= V(X) = \Sigma x_i^2 \cdot P(x_i) - [E(X)]^2$

$= 54 \cdot 83 - 49$

$= 5 \cdot 83$

$\therefore$ standard deviation $= \sqrt{V(X)}$

$= \sqrt{5.83} = 2.41.$

Question 15.
A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the student is recorded. What is the probability distribution of the random variable X? Find mean, variance, and standard deviation of X.
Solution:
Let X denote the age of the chosen student. Then X can take values 14, 15, 16, 17, 18, 19, 20, 21.
We make a frequency table to find the number of students with age X:

Age X	Tally mark	Number of students with age X
14	||	2
15	|	1
16	||	2
17	|||	3
18	|	1
19	||	2
20	|||	3
21	|	1
Total		15

The chances of any student selected are equally likely.
If there are m students with age X, then P(X) = $m15$
Using this, the following is the probability distribution of X:

$X = x$	14	15	16	17	18	19	20	21
$P(X = x)$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

$$\text{Mean} = E(X) = \Sigma x_i \cdot P(x_i)$$
$$= 14 \times \frac{2}{15} + 15 \times \frac{1}{15} + 16 \times \frac{2}{15} + 17 \times \frac{3}{15} + 18 \times \frac{1}{15} + 19 \times \frac{2}{15} + 20 \times \frac{3}{15} + 21 \times \frac{1}{15}$$
$$= \frac{1}{15}[28 + 15 + 32 + 51 + 18 + 38 + 60 + 21]$$
$$= \frac{1}{15} \times 263 = 17.53$$
$$\Sigma x_i^2 \cdot P(x_i) = 196 \times \frac{2}{15} + 225 \times \frac{1}{15} + 256 \times \frac{2}{15} + 289 \times \frac{3}{15} + 324 \times \frac{1}{15} + 361 \times \frac{2}{15} + 400 \times \frac{3}{15} + 441 \times \frac{1}{15}$$
$$= \frac{1}{15}(392 + 225 + 512 + 867 + 324 + 722 + 1200 + 44)$$
$$= \frac{1}{15}(4683) = 312.2$$

Variance = V(X) = $\Sigma x_i^2 \cdot P(x_i) - [E(X)]^2$
= $312.2 - (17.53)^2$
= 312.2 – 307.3
= 4.9
Standard deviation = $\sqrt{V(X)} = \sqrt{4.9} = 2.21$
Hence, mean = 17.53, variance = 4.9 and standard deviation = 2.21.

Question 16.
In a meeting, 70% of the member's favour and 30% oppose a certain proposal. A member is selected at random and we take X = 0 if he opposed and X = 1 if he is in favour. Find E(X) and Var(X).
Solution:
X takes values 0 and 1.
It is given that
P(X = 0) = P(0) = 30% = $\frac{30}{100}$ = 0.3
P(X = 1) = P(1) = 70% = $\frac{70}{100}$ = 0.7
∴ $E(X) = \Sigma x_i \cdot P(x_i) = 0 \times 0.3 + 1 \times 0.7 = 0.7$
Also, $\Sigma x_i^2 \cdot P(x_i) = 0 \times 0.3 + 1 \times 0.7 = 0.7$
∴ Variance = V(X) = $\Sigma x_i^2 \cdot P(x_i) - [E(X)]^2$
= $0.7 - (0.7)^2$
= 0.7 – 0.49
= 0.21
Hence, E(X) = 0.7 and Var(X) = 0.21.

Maharashtra State Board 12th Maths Solutions Chapter 7 Probability Distributions Ex 7.2

Question 1.
Verify which of the following is p.d.f. of r.v. X:
(i) f(x) = sin x, for 0 ≤ x ≤ $\frac{\pi}{2}$
(ii) f(x) = x, for 0 ≤ x ≤ 1 and -2 – x for 1 < x < 2
(iii) fix) = 2, for 0 ≤ x ≤ 1.
Solution:
f(x) is the p.d.f. of r.v. X if
(a) f(x) ≥ 0 for all x ∈ R and

(b) $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

(i) (a) $f(x) = \sin x \geqslant 0$ if $0 \leqslant x \leqslant \frac{\pi}{2}$

(b) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\infty} f(x)\,dx$

$= 0 + \int_{0}^{\pi/2} \sin x\,dx + 0$

$= [-\cos x]_{0}^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 0 + 1 = 1$

Hence, f(x) is the p.d.f. of X.

(ii) f(x) = x ≥ 0 if 0 ≤ x ≤ 1
For 1 < x < 2, -2 < -x < -1
-2 – 2 < -2 – x < -2 – 1
i.e. -4 < f(x) < -3 if 1 < x < 2
Hence, f(x) is not p.d.f. of X.

(iii) (a) f(x) = 2 ≥ 0 for 0 ≤ x ≤ 1

(b) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$

$= 0 + \int_{0}^{1} 2\,dx + 0 = [2x]_{0}^{1} = 2 - 0 = 2 \neq 1$

Hence, f(x) is not p.d.f. of X.

Question 2.
The following is the p.d.f. of r.v. X:
f(x) = $\frac{x}{8}$, for 0 < x < 4 and = 0 otherwise.
Find
(a) P(x < 1.5)
(b) P(1 < x < 2) (c) P(x > 2).
Solution:

(a) $P(x < 1.5) = \int_0^{1.5} f(x)\,dx = \int_0^{1.5} \frac{x}{8}\,dx$

$$= \frac{1}{8}\left[\frac{x^2}{2}\right]_0^{1.5} = \frac{(1.5)^2}{16} - 0 = \frac{\left(\frac{9}{4}\right)}{16} = \frac{9}{64}.$$

(b) $P(1 < x < 2) = \int_1^2 f(x)\,dx = \int_1^2 \frac{x}{8}\,dx$

$$= \frac{1}{8}\left[\frac{x^2}{2}\right]_1^2 = \frac{1}{8}\left[\frac{4}{2} - \frac{1}{2}\right] = \frac{3}{16}.$$

(c) $P(x > 2) = \int_2^4 f(x)\,dx = \int_2^4 \frac{x}{8}\,dx$

$$= \frac{1}{8}\left[\frac{x^2}{2}\right]_2^4 = \frac{1}{8}\left[\frac{16}{2} - \frac{4}{2}\right] = \frac{1}{8} \times 6 = \frac{3}{4}.$$

Question 3.
It is known that error in measurement of reaction temperature (in 0°C) in a certain experiment is continuous r.v. given by
f(x) = x23 for -1 < x < 2
= 0. otherwise.
(i) Verify whether f(x) is p.d.f. of r.v. X
(ii) Find P(0 < x ≤ 1)
(iii) Find the probability that X is negative.
Solution:

(i) $f(x) = \frac{x^2}{3} \geqslant 0$, for $-1 < x < 2$

Also, $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)dx$

$$= 0 + \int_{-1}^{2} \frac{x^2}{3}\,dx + 0 = \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{3}\left[\frac{8}{3} - \frac{(-1)}{3}\right] = \frac{1}{3}\left[\frac{9}{3}\right] = 1$$

$\therefore$ $f(x)$ is the p.d.f. of X.

(ii) $P(0 < x \leqslant 1) = \int_0^1 f(x)\,dx$

$$= \int_0^1 \frac{x^2}{3}\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}\left[\frac{1}{3} - 0\right] = \frac{1}{9}.$$

(iii) $P(x \text{ is negative})$

$$= P(-1 < x < 0) = \int_{-1}^{0} f(x)\,dx$$

$$= \int_{-1}^{0} \frac{x^2}{3}\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{0} = \frac{1}{3}\left[0 - \left(-\frac{1}{3}\right)\right] = \frac{1}{9}.$$

Question 4.
Find k if the following function represents p.d.f. of r.v. X
(i) f(x) = kx. for 0 < x < 2 and = 0 otherwise.
Also find P(14 < x < 32).
(ii) f(x) = kx(1 – x), for 0 < x < 1 and = 0 otherwise.
Also find P(14 < x < 12), P(x < 12).
Solution:
(i) Since, the function f is p.d.f. of X

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_{-\infty}^{0} f(x)dx + \int_{0}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx = 1$$

$$\therefore 0 + \int_{0}^{2} kx\,dx + 0 = 1 \qquad \therefore k\left[\frac{x^2}{2}\right]_0^2 = 1$$

$$\therefore k\left[\frac{4}{2} - 0\right] = 1$$

$$\therefore 2k = 1 \qquad \therefore k = \frac{1}{2}$$

$$P\left(\frac{1}{4} < x < \frac{3}{2}\right) = \int_{1/4}^{3/2} f(x)\,dx$$

$$= \int_{1/4}^{3/2} kx\,dx, \text{ where } k = \frac{1}{2}$$

$$= \frac{1}{2}\int_{1/4}^{3/2} x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_{1/4}^{3/2}$$

$$= \frac{1}{4}\left[\frac{9}{4} - \frac{1}{16}\right] = \frac{1}{4}\left[\frac{36-1}{16}\right] = \frac{35}{64}.$$

(ii) Since, the function f is the p.d.f. of X,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$$

$$\therefore 0 + \int_{0}^{1} kx(1-x)\,dx + 0 = 1$$

$$\therefore k\int_{0}^{1} (x - x^2)\,dx = 1$$

$$\therefore k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$\therefore k\left(\frac{1}{2} - \frac{1}{3} - 0\right) = 1$$

$$\therefore \frac{k}{6} = 1 \qquad \therefore k = 6.$$

$$P\left(\frac{1}{4}<x<\frac{1}{2}\right)=\int_{1/4}^{1/2} f(x)\,dx$$

$$=\int_{1/4}^{1/2} kx(1-x)\,dx$$

$$=k\int_{1/4}^{1/2}(x-x^2)dx$$

$$=6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{1/4}^{1/2} \quad \ldots [\because\ k=6]$$

$$=6\left[\left(\frac{1}{8}-\frac{1}{24}\right)-\left(\frac{1}{32}-\frac{1}{192}\right)\right]$$

$$=6\left[\frac{2}{24}-\frac{5}{192}\right]=6\left(\frac{11}{192}\right)$$

$$\therefore\ P\left(\frac{1}{4}<x<\frac{1}{2}\right)=\frac{11}{32}.$$

$$P\left(x<\frac{1}{2}\right)=\int_{-\infty}^{1/2} f(x)\,dx$$

$$=\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{1/2} f(x)\,dx$$

$$=0+\int_{0}^{1/2} kx(1-x)\,dx$$

$$=k\int_{0}^{1/2}(x-x^2)\,dx=k\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{0}^{1/2}$$

$$=k\left[\frac{1}{8}-\frac{1}{24}-0\right]=k\left(\frac{2}{24}\right)$$

$$=6\left(\frac{1}{12}\right) \quad \ldots [\because\ k=6]$$

$$\therefore\ P\left(X<\frac{1}{2}\right)=\frac{1}{2}.$$

Question 5.
Let X be the amount of time for which a book is taken out of the library by a randomly selected students and suppose X has p.d.f.
f(x) = 0.5x, for $0 \le x \le 2$ and = 0 otherwise.
Calculate:
(i) $P(x \le 1)$
(ii) $P(0.5 \le x \le 1.5)$
(iii) $P(x \ge 1.5)$.
Solution:
(i) $P(x \le 1)$

$$f(x) = 0.5\,x, \quad 0 \le x \le 2$$
$$= 0, \quad \text{otherwise}$$

$$P(X \le 1) = \int_0^1 0.5\,x\,dx$$

$$= 0.5\int_0^1 x\,dx$$

$$= 0.5\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$$

$$\therefore\ P(X \le 1) = \frac{1}{4}$$

(ii) $P(0.5 \le x \le 1.5)$

f(x) = 0.5 x , $0 \leq x \leq 2$

= 0 , otherwise

P $(0.5 \leq X \leq 1.5)$

$$= \int_{0.5}^{1.5} 0.5\text{ x}$$

$$= 0.5 \int_{0.5}^{1.5} \text{x dx}$$

$$= \frac{1}{2} \times \left[\frac{x^2}{2}\right]_{0.5}^{1.5}$$

$$= \frac{1}{2} \times \frac{1}{2}\left[(1.5)^2 - (0.5)^2\right]$$

$$= \frac{1}{2} \times \frac{1}{2}[2.25 - 0.25]$$

$$= \frac{1}{4} \times 2 = \frac{1}{2}$$

$$P\ (0.5 \leq X \leq 1.5) = \frac{1}{2}$$

(iii) $P(x \geq 1.5)$

f(x) = 0.5 x , $0 \leq x \leq 2$

= 0 , otherwise

$P(x \geq 1.5)$

$$= \int_{1.5}^{2} f(x)dx$$

$$= \int_{1.5}^{2} 0.5x dx$$

$$= 0.5\left[\frac{x^2}{2}\right]_{1.5}^{2}$$

$$= 0.5\left[\frac{4}{2} - \frac{2.25}{2}\right]$$

$$= 0.5 \times \frac{1.75}{2}$$

$$= \frac{0.875}{2}$$

$$= \frac{0.875}{2} \times \frac{1000}{1000}$$

$$= \frac{875 \div 125}{2000 \div 125}$$

$$= \frac{7}{16}$$

Question 6.
Suppose that X is waiting time in minutes for a bus and its p.d.f. is given by f(x) = 15, for $0 \leq x \leq 5$ and = 0 otherwise. Find the probability that
(i) waiting time is between 1 and 3
(ii) waiting time is more than 4 minutes.
Solution:

(i) Required probability = P(1 < X < 3)

$$= \int_1^3 f(x)dx = \int_1^3 \frac{1}{5}\,dx$$

$$= \frac{1}{5}\int_1^3 1dx = \frac{1}{5}[x]_1^3$$

$$= \frac{1}{5}[3-1] = \frac{2}{5}.$$

(ii) Required probability = P(X > 4)

$$= \int_4^\infty f(x)dx = \int_4^5 f(x)dx + \int_5^\infty f(x)dx$$

$$= \int_4^5 \frac{1}{5}\,dx + 0$$

$$= \frac{1}{5}\int_4^5 1dx = \frac{1}{5}[x]_4^5$$

$$= \frac{1}{5}[5-4] = \frac{1}{5}.$$

Question 7.
Suppose the error involved in making a certain measurement is a continuous r.v. X with p.d.f.
$f(x) = k(4 - x^2)$, $-2 \le x \le 2$ and 0 otherwise.
Compute:
(i) P(X > 0)
(ii) P(-1 < X < 1)
(iii) P(-0.5 < X or X > 0.5).
Solution:
Since, f is the p.d.f. of X,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore\ \int_{-\infty}^{-2} f(x)\,dx + \int_{-2}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx = 1$$

$$\therefore\ 0 + \int_{-2}^{2} k(4-x^2)\,dx + 0 = 1$$

$$\therefore\ k\int_{-2}^{2}(4-x^2)\,dx = 1$$

$$\therefore\ k\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 1$$

$$\therefore\ k\left[\left(8-\frac{8}{3}\right) - \left(-8+\frac{8}{3}\right)\right] = 1$$

$$\therefore\ k\left(\frac{16}{3}+\frac{16}{3}\right) = 1$$

$$\therefore\ k\left(\frac{32}{3}\right) = 1 \qquad \therefore\ k = \frac{3}{32}$$

(i) $P(X > 0) = \int_0^\infty f(x)\,dx$

$$= \int_0^2 f(x)\,dx + \int_2^\infty f(x)\,dx$$

$$= \int_0^2 k(4-x^2)\,dx + 0$$

$$= k\int_0^2 (4-x^2)\,dx$$

$$= \frac{3}{32}\left[4x - \frac{x^3}{3}\right]_0^2 \qquad \ldots\left[\because\ k = \frac{3}{32}\right]$$

$$= \frac{3}{32}\left[8 - \frac{8}{3}\right] = \frac{3}{32}\times\frac{16}{3} = \frac{1}{2}$$

(ii) $P(-1 < X < 1) = \int_{-1}^{1} f(x)\,dx$

$$= \int_{-1}^{1} k(4-x^2)\,dx$$

$$= k\int_{-1}^{1} (4-x^2)\,dx$$

$$= \frac{3}{32}\left[4x - \frac{x^3}{3}\right]_{-1}^{1} \quad \ldots \left[\because k = \frac{3}{32}\right]$$

$$= \frac{3}{32}\left[\left(4-\frac{1}{3}\right) - \left(-4+\frac{1}{3}\right)\right]$$

$$= \frac{3}{32}\left(\frac{11}{3}+\frac{11}{3}\right) = \frac{3}{32}\left(\frac{22}{3}\right) = \frac{11}{16}.$$

(iii) $P(X < -0.5 \text{ or } X > 0.5)$

$$= P(X < -0.5) + P(X > 0.5)$$

$$= \int_{-\infty}^{-0.5} f(x)\,dx + \int_{0.5}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{-2} f(x)\,dx + \int_{-2}^{-0.5} f(x)\,dx + \int_{0.5}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx$$

$$= 0 + \int_{-2}^{-1/2} k(4-x^2)dx + \int_{1/2}^{2} k(4-x^2)dx + 0$$

$$= k\int_{-2}^{-1/2} (4-x^2)dx + k\int_{1/2}^{2} (4-x^2)dx$$

$$= \frac{3}{32}\left[4x - \frac{x^3}{3}\right]_{-2}^{-1/2} + \frac{3}{32}\left[4x - \frac{x^3}{3}\right]_{1/2}^{2} \ldots \left[\because k = \frac{3}{32}\right]$$

$$= \frac{3}{32}\left[\left(-2+\frac{1}{24}\right) - \left(-8+\frac{8}{3}\right)\right] + \frac{3}{32}\left[\left(8-\frac{8}{3}\right) - \left(2-\frac{1}{24}\right)\right]$$

$$= \frac{3}{32}\left(\frac{-47}{24}+\frac{16}{3}\right) + \frac{3}{32}\left(\frac{16}{3}-\frac{47}{24}\right)$$

$$=\frac{3}{32}\left(\frac{-47}{24}+\frac{16}{3}+\frac{16}{3}-\frac{47}{24}\right)$$

$$=\frac{3}{32}\left(\frac{-47+128+128-47}{24}\right)$$

$$=\frac{3}{32}\left(\frac{162}{24}\right)=\frac{81}{128}=0.6328.$$

Alternative Method :

$$P(X<-0.5 \text{ or } X>0.5)$$

$$=1-P(-0.5 \leqslant X \leqslant 0.5)$$

$$=1-\int_{-0.5}^{0.5} f(x)\,dx$$

$$=1-\int_{-1/2}^{1/2} k(4-x^2)\,dx$$

$$=1-k\int_{-1/2}^{1/2}(4-x^2)\,dx$$

$$=1-\frac{3}{32}\left[4x-\frac{x^3}{3}\right]_{-1/2}^{1/2} \quad \ldots\left[\because k=\frac{3}{32}\right]$$

$$=1-\frac{3}{32}\left[\left(2-\frac{1}{24}\right)-\left(-2+\frac{1}{24}\right)\right]$$

$$=1-\frac{3}{32}\left(2-\frac{1}{24}+2-\frac{1}{24}\right)$$

$$=1-\frac{3}{32}\left(4-\frac{1}{12}\right)$$

$$=1-\frac{3}{32}\times\frac{47}{12}=1-\frac{47}{128}$$

$$=\frac{128-47}{128}=\frac{81}{128}=0.6328.$$

Question 8.
The following is the p.d.f. of continuous r.v. X
f(x) = x8, for 0 < x < 4 and = 0 otherwise.
(i) Find expression for c.d.f. of X.
(ii) Find F(x) at x = 0.5, 1.7 and 5.
Solution:
(i) Let F(x) be the c.d.f. of X

$$\text{Then } F(x)=\int_{-\infty}^{x} f(x)\,dx=\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{x} f(x)\,dx$$

$$=0+\int_0^x \frac{x}{8}\,dx$$

$$=\frac{1}{8}\left[\frac{x^2}{2}\right]_0^x=\frac{1}{8}\left[\frac{x^2}{2}-0\right]=\frac{x^2}{16}$$

$$\therefore\ F(x)=\frac{x^2}{16}.$$

(ii) $F(0.5)=F\left(\frac{1}{2}\right)=\frac{\left(\frac{1}{2}\right)^2}{16}=\frac{1}{64}$

$$F(1.7)=\frac{(1.7)^2}{16}=\frac{2.89}{16}=0.18$$

$$f(x)=\frac{x}{8}, \text{ for } 0<x<4 \text{ and } 5>4$$

$$\therefore\ F(5)=1.$$

Question 9.

Given the p.d.f. of a continuous random r.v. X, $f(x) = \frac{x^2}{3}$, for $-1 < x < 2$ and = 0 otherwise. Determine c.d.f. of X and hence find $P(X < 1)$; $P(X < -2)$, $P(X > 0)$, $P(1 < X < 2)$.

Solution:

Let $F(x)$ be the c.d.f. of X

Then $F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{-1} f(x)dx + \int_{-1}^{x} f(x)dx$

$$= 0 + \int_{-1}^{x} \frac{x^2}{3} dx = \frac{1}{3}\int_{-1}^{x} x^2 dx$$

$$= \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{x} = \frac{1}{3}\left[\frac{x^3}{3} - \left(-\frac{1}{3}\right)\right]$$

$$\therefore\ F(x) = \frac{x^3+1}{9}$$

(i) $P(x < 1) = F(1) - F(-1)$

$$= \left[\frac{1^3+1}{9}\right] - \left[\frac{(-1)^3+1}{9}\right] = \frac{2-0}{9} = \frac{2}{9}$$

(ii) $f(x) = \frac{x^2}{3}$ for $-1 < x < 2$ and $-2 < -1$

$\therefore\ F(-2) = 0$ i.e. $P(x < -2) = 0$

(iii) $P(X > 0) = 1 - P(X \leqslant 0)$

$$= 1 - [F(0) - F(-1)]$$

$$= 1 - \left[\left(\frac{0^3+1}{9}\right) - \left(\frac{(-1)^3+1}{9}\right)\right]$$

$$= 1 - \left(\frac{1}{9} - 0\right) = 1 - \frac{1}{9} = \frac{8}{9}.$$

(iv) $P(1 < x < 2) = F(2) - F(1)$

$$= \left(\frac{2^3+1}{9}\right) - \left(\frac{1^3+1}{9}\right) = 1 - \frac{2}{9} = \frac{7}{9}.$$

Question 10.

If a r.v. X has p.d.f.

$f(x) = \frac{c}{x}$ for $1 < x < 3$, $c > 0$. Find c, E(X), Var (X).

Solution:

Since f(x) is p.d.f of r.v. X

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_{-\infty}^{1} f(x)dx + \int_{1}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$$

$$\therefore\ 0 + \int_{1}^{3} f(x)\,dx + 0 = 1$$

$$\therefore \int_{1}^{3} \frac{c}{x}\,dx = 1 \qquad \therefore\ c\int_{1}^{3} \frac{1}{x}\,dx = 1$$

$$\therefore\ c[\log x]_1^3 = 1 \qquad \therefore\ c[\log 3 - \log 1] = 1$$

$$\therefore\ c = \frac{1}{\log\ 3} \qquad \ldots\ [\because \log 1 = 0]$$

$$E(X) = \int_{-\infty}^{\infty} x\,f(x)dx = \int_{-\infty}^{1} xf(x)dx + \int_{1}^{3} xf(x)dx + \int_{3}^{\infty} x\,f(x)dx$$

$$= 0 + \int_{1}^{3} x\,f(x)dx + 0 = \int_{1}^{3} x\cdot\frac{c}{x}\,dx$$

$$= c\int_{1}^{3} 1dx, \text{ where } c = \frac{1}{\log\ 3}$$

$$= \frac{1}{\log\ 3}\,[x]_1^3 = \frac{1}{\log\ 3}\,[3-1] = \frac{2}{\log\ 3}$$

$$\text{Consider, } \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-\infty}^{1} x^2 f(x)dx + \int_{1}^{3} x^2 f(x)dx + \int_{3}^{\infty} x^2 f(x)dx$$

$$= 0 + \int_{1}^{3} x^2 f(x)dx + 0 = \int_{1}^{3} x^2\cdot\frac{c}{x}\,dx$$

$$= c\int_{1}^{3} x\ dx, \text{ where } c = \frac{1}{\log\ 3}$$

$$= \frac{1}{\log\ 3}\int_{1}^{3} x\ dx = \frac{1}{\log\ 3}\left[\frac{x^2}{2}\right]_1^3$$

$$= \frac{1}{\log 3}\left[\frac{9}{2} - \frac{1}{2}\right] = \frac{4}{\log 3}$$

$$\text{Now, Var }(X) = \int_{-\infty}^{\infty} x^2\,f(x)\,dx - [E(X)]^2$$

$$= \frac{4}{\log\ 3} - \left(\frac{2}{\log\ 3}\right)^2$$

$$= \frac{4}{\log 3} - \frac{4}{(\log 3)^2}$$

$$= \frac{4(\log\ 3) - 4}{(\log 3)^2} = \frac{4\,[\log 3 - 1]}{(\log\ 3)^2}$$

Hence, $c = \dfrac{1}{\log\ 3}$, $E(X) = \dfrac{2}{\log\ 3}$ and Var $(X) = \dfrac{4[\log\ 3 - 1]}{(\log\ 3)^2}$.

Maharashtra State Board 12th Maths Solutions Chapter 7 Probability Distributions Miscellaneous Exercise 7

(I) Choose the correct option from the given alternatives:

Question 1.
P.d.f. of a c.r.v. X is f(x) = 6x(1 – x), for 0 ≤ x ≤ 1 and = 0, otherwise (elsewhere) If P(X < a) = P(X > a), then a =
(a) 1
(b) 12
(c) 13
(d) 14
Answer:
(b) 12

Question 2.
If the p.d.f. of a c.r.v. X is f(x) = $3(1 – 2x^2)$, for 0 < x < 1 and = 0, otherwise (elsewhere), then the c.d.f. of X is F(x) =
(a) $2x – 3x^2$
(b) $3x – 4x^3$
(c) $3x – 2x^3$
(d) $2x^3 – 3x$
Answer:
(c) $3x – 2x^3$

Question 3.
If the p.d.f. of a c.r.v. X is f(x) = $x^2 18$, for -3 < x < 3 and = 0, otherwise, then P(|X| < 1) =
(a) 127
(b) 128
(c) 129
(d) 126
Answer:
(a) 127

Question 4.
If p.m.f. of a d.r.v. X takes values 0, 1, 2, 3, … which probability P(X = x) = k(x +1) . 5-x, where k is a constant, then P(X = 0) =
(a) 725
(b) 1625
(c) 1825
(d) 1925
Answer:

(b) 1625

$$\textbf{Hint}: k\left[\frac{1}{5^0}+\frac{2}{5^1}+\frac{3}{5^2}+\ldots\right]=1$$

$$\text{Let } S=\frac{k}{5^0}+\frac{2k}{5^1}+\frac{3k}{5^2}+\ldots\ldots\ldots$$

$$\text{i.e. } S=k+\frac{2k}{5}+\frac{3k}{5^2}+\ldots\ldots\ldots$$

$$\therefore\ \frac{1}{5}S=\frac{k}{5}+\frac{2k}{5^2}+\frac{3k}{5^3}+\ldots\ldots\ldots$$

$$\therefore\ S-\frac{1}{5}S=k+\frac{k}{5}+\frac{k}{5^2}+\frac{k}{5^3}+\ldots\ldots\ldots$$

$$\therefore\ \frac{4}{5}S=k\left[1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\ldots\ldots\ldots\right]$$

$$=k\left[\frac{1}{1-\frac{1}{5}}\right]=\frac{5k}{4}$$

$$\therefore\ S=\frac{25k}{16}=1 \qquad \therefore\ k=\frac{16}{25}$$

$$\therefore\ P(X=0)=k(0+1)5^0=k=\frac{16}{25}.$$

Question 5.
If p.m.f. of a d.r.v. X is P(X = x) = (5Cx)25, for x = 0, 1, 2, 3, 4, 5 and = 0, otherwise. If a = P(X ≤ 2) and b = P(X ≥ 3), then
(a) a < b
(b) a > b
(c) a = b
(d) a + b
Answer:
(c) a = b

Question 6.
If p.m.f. of a d.r.v. X is P(X = x) = xn(n+1), for x = 1, 2, 3, ……, n and = 0, otherwise, then E(X) =
(a) n1+12
(b) n3+16
(c) n2+15
(d) n1+13
Answer:
(b) n3+16

Question 7.
If p.m.f. of a d.r.v. X is P(x) = cx3, for x = 1, 2, 3 and = 0, otherwise (elsewhere), then E(X) =
(a) 343297
(b) 294251
(c) 297294
(d) 294297
Answer:
(b) 294251

Question 8.
If the d.r.v. X has the following probability distribution:

X	−2	−1	0	1	2	3
$P(X=x)$	0·1	k	0·2	$2k$	0·3	k

then P(X = -1) =
(a) 110
(b) 210
(c) 310
(d) 410

Answer:
(a) 110

Question 9.
If the d.r.v. X has the following probability distribution:

X	1	2	3	4	5	6	7
$P(X = x)$	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

then k =
(a) 17
(b) 18
(c) 19
(d) 110
Answer:
(d) 110

Question 10.
Find the expected value of X for the following p.m.f.

X	–2	–1	0	1	2
$P(X)$	0·3	0·4	0·2	0·15	0·25

(a) 0.85
(b) -0.35
(c) 0.15
(d) -0.15
Answer:
(b) -0.35

(II) Solve the following:

Question 1.
Identify the random variable as either discrete or continuous in each of the following. If the random variable is discrete, list its possible values:
(i) An economist is interested in the number of unemployed graduates in the town of population 1 lakh.
(ii) Amount of syrup prescribed by a physician.
(iii) The person on a high protein diet is interesting to gain weight in a week.
(iv) 20 white rats are available for an experiment. Twelve rats are males. A scientist randomly selects 5 rats, the number of female rats selected on a specific day.
(v) A highway-safety group is interested in studying the speed (in km/hr) of a car at a checkpoint.
Solution:
(i) Let X = number of unemployed graduates in a town.
Since the population of the town is 1 lakh, X takes the finite values.
∴ random variable X is discrete.
Range = {0, 1, 2, …, 99999, 100000}.

(ii) Let X = amount of syrup prescribed by a physician.
Then X takes uncountable infinite values.
∴ random variable X is continuous.

(iii) Let X = gain of weight in a week
Then X takes uncountable infinite values
∴ random variable X is continuous.

(iv) Let X = number of female rats selected on a specific day.
Since the total number of rats is 20 which includes 12 males and 8 females, X takes the finite values.
∴ random variable X is discrete.
Range = {0, 1, 2, 3, 4, 5}

(v) Let X = speed of .the car in km/hr.
Then X takes uncountable infinite values
∴ random variable X is continuous.

Question 2.
The probability distribution of discrete r.v. X is as follows:

$X = x$	1	2	3	4	5	6
$P(X = x)$	k	$2k$	$3k$	$4k$	$5k$	$6k$

(i) Determine the value of k.
(ii) Find P(X ≤ 4), P(2 < X < 4), P(X ≥ 3).
Solution:
(i) Since P (x) is a probability distribution of x,

$$\Sigma_{x=0}^{6} P(x) = 1$$

∴ P (0) + P (1) + P(2) + P(3) + P(4) + P(5) + P (6) = 1

∴ k + 2k + 3k + 4k + 5k + 6k = 1

∴ 21k = 1

∴ k = $\frac{1}{21}$

(ii)

P(X ≤ 4)

= P (0) + P (1) + P(2) + P(3) + P(4)

=k + 2k + 3k + 4k

= 10k

= $10\left(\frac{1}{21}\right)$

= $\frac{10}{21}$

P(2< X < 4)

= P(3)

= 3k

= $3\left(\frac{1}{21}\right)$

= $\frac{1}{7}$

P(X ≥ 3)

= P (1) + P(2) + P(3)

= k + 2k + 3k

= 6k

= $6\left(\frac{1}{21}\right)$

= $\frac{2}{7}$

Question 3.
The following is the probability distribution of X:

$X = x$	−3	−2	−1	0	1	2	3
$P(X = x)$	0·05	0·1	0·15	0·20	0·25	0·15	0.1

Find the probability that

(i) X is positive
(ii) X is non-negative
(iii) X is odd
(iv) X is even.
Solution:
(i) P(X is positive) = P(X = 1) + P(X = 2) + P(X = 3)
= 0.25 + 0.15 + 0.1
= 0.50

(ii) P(X is non-negative)
= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)
= 0.20 + 0.25 + 0.15 + 0.1
= 0.70

(iii) P(X is odd)
= P(X = -3) + P(X = -1) + P(X = 1) + P(X = 3)
= 0.05 + 0.15 + 0.25 + 0.1
= 0.55

(iv) P(X is even)
= P(X = -2) + P(X = 0) + P(X = 2)
= 0.10 + 0.20 + 0.15
= 0.45.

Question 4.
The p.m.f. of a r.v. X is given by P(X = x) = x = $\frac{{}^5C_x}{2^5}$, for x = 0, 1, 2, 3, 4, 5 and = 0, otherwise. Then show that P(X ≤ 2) = P(X ≥ 3).
Solution:
P(X ≤ 2) = P(X = 0) + P(X = 1) + P(X = 2)
= $\frac{{}^5C_0}{2^5}+\frac{{}^5C_1}{2^5}+\frac{{}^5C_2}{2^5}$
= $\frac{{}^5C_5}{2^5}+\frac{{}^5C_4}{2^5}+\frac{{}^5C_3}{2^5}$ ………[latex]{ }^{n} \mathrm{C}_{r}={ }^{n} \mathrm{C}_{n-r}[/latex]
= P(X = 5) + P(X = 4) + P(X = 3)
= P(X ≥ 3)
∴ P(X ≤ 2) = P(X ≥ 3).

Question 5.
In the p.m.f. of r.v. X

x	1	2	3	4	5
$P(X)$	$\frac{1}{20}$	$\frac{3}{20}$	a	$2a$	$\frac{1}{20}$

Find a and obtain c.d.f. of X.
Solution:
For p.m.f. of a r.v. X
$\sum_{i=1}^{5} P(X=x)=1$
∴ P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) = 1

$$\therefore \frac{1}{20}+\frac{3}{20}+a+2a+\frac{1}{20}=1$$

$$\therefore 3a=1-\frac{5}{20}=1-\frac{1}{4}=\frac{3}{4} \qquad \therefore a=\frac{1}{4}$$

∴ the p.m.f. of the r.v. X is

$X=x$	1	2	3	4	5
$P(X=x)$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{5}{20}$	$\frac{10}{20}$	$\frac{1}{20}$

Let F(x) be the c.d.f. of X.
Then F(x) = P(X ≤ x)
∴ F(1) = P(X ≤ 1) = P(X = 1) = $\frac{1}{20}$
F(2) = P(X ≤ 2) = P(X = 1) + P (X = 2)
$=\frac{1}{20}+\frac{3}{20}=\frac{4}{20}=\frac{1}{5}$
P(3) = P(X ≤ 3) = P(X = 1) + P(X = 2) + P(X = 3)
$=\frac{1}{20}+\frac{3}{20}+\frac{5}{20}=\frac{9}{20}$
F(4) = P(X ≤ 4) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)

=120+320+520+1020=1920

$F(5) = P(X \leq 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

=120+320+520+1020+120=2020=1

Hence, the c.d.f. of the random variable X is as follows:

x_i	1	2	3	4	5
$F(x_i)$	$\frac{1}{20}$	$\frac{1}{5}$	$\frac{9}{20}$	$\frac{19}{20}$	1

Question 6.
A fair coin is tossed 4 times. Let X denote the number of heads obtained. Write down the probability distribution of X. Also, find the formula for p.m.f. of X.
Solution:
When a fair coin is tossed 4 times then the sample space is
S = {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT}
∴ n(S) = 16
X denotes the number of heads.
∴ X can take the value 0, 1, 2, 3, 4
When X = 0, then X = {TTTT}
∴ n (X) = 1
∴ P(X = 0) = n(X)n(S)=116=4C016
When X = 1, then
X = {HTTT, THTT, TTHT, TTTH}
∴ n(X) = 4
∴ P(X = 1) = n(X)n(S)=416=4C116
When X = 2, then
X = {HHTT, HTHT, HTTH, THHT, THTH, TTHH}
∴ n(X) = 6
∴ P(X = 2) = n(X)n(S)=616=4C216
When X = 3, then
X = {HHHT, HHTH, HTHH, THHH}
∴ n(X) = 4
∴ P(X = 3) = n(X)n(S)=416=4C316
When X = 4, then X = {HHHH}
∴ n(X) = 1
∴ P(X = 4) = n(X)n(S)=116=4C416
∴ the probability distribution of X is as follows:

x	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Also, the formula for p.m.f. of X is
P(x) = 4Cx16, x = 0, 1, 2, 3, 4 and = 0, otherwise.

Question 7.
Find the probability distribution of the number of successes in two tosses of a die, where success is defined as
(i) number greater than 4
(ii) six appear on at least one die.
Solution:
When a die is tossed two times, we obtain (6 × 6) = 36 number of observations.
Let X be the random variable, which represents the number of successes.
Here, success refers to the number greater than 4.
P(X = 0) = P(number less than or equal to 4 on both the tosses)
= 46×46=1636=49
P(X = 1) = P(number less than or equal to 4 on first toss and greater than 4 on second toss) + P(number greater than 4 on first toss and less than or equal to 4 on second toss)
= 46×26+46×26
= 836+836
= 1636
= 49
P(X = 2) = P(number greater than 4 on both the tosses)

= 26×26=436=19

Thus, the probability distribution is as follows:

X	0	1	2
P(X)	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

(ii) Here, success means six appears on at least one die.

P(Y = 0) = P(six appears on none of the dice) = 56×56=2536

P(Y = 1) = P(six appears on none of the dice x six appears on at least one of the dice) + P(six appears on none of the dice x six appears on at least one of the dice)

= 16×56+16×56=536+536=1036

P(Y = 2) = P(six appears on at least one of the dice) = 16×16=136

Thus, the required probability distribution is as follows:

Y	0	1	2
P(Y)	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

Question 8.
A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7
$P(X)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Determine:
(i) k
(ii) P(X > 6)
(iii) P(0 < X < 3).

Question 9.
The following is the c.d.f. of a r.v. X:

X	–3	–2	–1	0	1	2	3	4
$F(X)$	0·1	0·3	0·5	0·65	0·75	0·85	0·9	1

Find
(i) p.m.f. of X
(ii) P(-1 ≤ X ≤ 2)
(iii) P(X ≤ X > 0).

Solution:
(i) From the given table
F(-3) = 0.1, F(-2) = 0.3, F(-1) = 0.5
F(0) = 0.65, f(1) = 0.75, F(2) = 0.85
F(3) = 0.9, F(4) = 1
P(X = -3) = F(-3) = 0.1
P(X = -2) = F(-2) – F(-3) = 0.3 – 0.1 = 0.2
P(X = -1) = F(-1) – F(-2) = 0.5 – 0.3 = 0.2
P(X = 0) = F(0) – F(-1) = 0.65 – 0.5 = 0.15
P(X = 1) = F(1) – F(0) = 0.75 – 0.65 = 0.1
P(X = 2) = F(2) – F(1) = 0.85 – 0.75 = 0.1
P(X = 3) = F(3) – F(2) = 0.9 – 0.85 = 0.1
P(X = 4) = F(4) – F(3) = 1 – 0.9 = 0.1
∴ the p.m.f of X is as follows:

$X = x$	-3	-2	-1	0	1	2	3	4
$P(X = x)$	0.1	0.2	0.2	0.15	0.1	0.1	0.05	0.1

(ii) P(-1 ≤ X ≤ 2) = P(X = -1) + P(X = 0) + P(X = 1) + P(X = 2)
= 0.2 + 0.15 + 0.1 + 0.1
= 0.55

(iii) (X ≤ 3) ∩ (X > 0)
= { -3, -2, -1, 0, 1, 2, 3} n {1, 2, 3, 4}

= {1, 2, 3}

$\therefore\ P[(X \leqslant 3) \cap (X > 0]$

$= P(X = 1) + P(X = 2) + P(X = 3)$

$\therefore\ P[(X \leqslant 3)/X > 0] = \frac{P[(X \leqslant 3) \cap (X > 0)]}{P(X > 0)}$

$= \frac{P(X = 1) + P(X = 2) + P(X = 3)}{P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)}$

$= \frac{0.1 + 0.1 + 0.05}{0.1 + 0.1 + 0.05 + 0.1}$

$= \frac{0.25}{0.35} = \frac{5}{7}.$

Question 10.
Find the expected value, variance, and standard deviation of the random variable whose p.m.f's are given below:

(i)

$X = x$	1	2	3
$P(X = x)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

(ii)

$X = x$	−1	0	1
$P(X = x)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

(iii)

$X = x$	1	2	3	...	n
$P(X = x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	...	$\frac{1}{n}$

(iv)

$X = x$	0	1	2	3	4	5
$P(X = x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

Solution:
(i) We construct the following table to find the expected value, variance, and standard deviation:

x_i	$P(x_i)$	$x_i \cdot P(x_i)$	$x_i^2 \cdot P(x_i) = x_i \times x_i \cdot P(x_i)$
1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
2	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{8}{5}$
3	$\frac{2}{5}$	$\frac{6}{5}$	$\frac{18}{5}$
Total	1	$\frac{11}{5}$	$\frac{27}{5}$

From the table,

$\Sigma\, x_i \cdot P(x_i) = \frac{11}{5},\ \Sigma\, x_i^2 \cdot P(x_i) = \frac{27}{5}$

Expected value $= E(X) = \Sigma\, x_i \cdot P(x_i)$

$= \frac{11}{5} = 2.2$

Variance $= V(x) = \Sigma\, x_i^2 \cdot P(x_i) - [\Sigma\, x_i \cdot P(x_i)]^2$

$= \frac{27}{5} - \left(\frac{11}{5}\right)^2 = \frac{27}{5} - \frac{121}{25}$

$= \frac{14}{25} = 0.56$

Standard deviation $= \sqrt{V(X)} = \sqrt{0.56} = 0.7483.$

(ii) We construct the following table to find the expected value, variance, and standard deviation:

x_i	$P(x_i)$	$x_i \cdot P(x_i)$	$x_i^2 \cdot P(x_i) = x_i \times x_i \cdot P(x_i)$
-1	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$
0	$\frac{2}{5}$	0	0
1	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
Total	1	$\frac{1}{5}$	$\frac{3}{5}$

From the table,

$\sum x_i \cdot P(x_i) = \frac{1}{5}$,

$\sum x_i^2 \cdot P(x_i) = \frac{3}{5}$

Expected value = $E(X) = \sum x_i \cdot P(x_i) = \frac{1}{5} = 0.2$

Variance = $V(x) = \sum x_i^2 \cdot P(x_i) - [\sum x_i \cdot P(x_i)]^2$

$= \frac{3}{5} - \left(\frac{1}{5}\right)^2$

$= \frac{3}{5} - \frac{1}{25}$

$= \frac{15}{25} - \frac{1}{25}$

$= \frac{14}{25}$

$= 0.56$

Standard deviation = $\sqrt{V(X)}$

$= \sqrt{0.56}$

$= 0.7483$

(iii) We construct the following table to find the expected value, variance, and standard deviation:

x_i	$P(x_i)$	$x_i \cdot P(x_i)$	$x_i^2 \cdot P(x_i) = x_i \times x_i \cdot P(x_i)$
1	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$
2	$\frac{1}{n}$	$\frac{2}{n}$	$\frac{2^2}{n}$
3	$\frac{1}{n}$	$\frac{3}{n}$	$\frac{3^2}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n}$	$\frac{n}{n}$	$\frac{n^2}{n}$

From the table,

$$\Sigma x_i \cdot P(x_i) = \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots + \frac{n}{n}$$

$$= \frac{1}{n}(1 + 2 + 3 + \dots + n)$$

$$= \frac{1}{n} \sum_{r=1}^{n} r = \frac{1}{n} \times \frac{n(n+1)}{2}$$

$$= \frac{n+1}{2}$$

$$\Sigma\, x_i^2 \cdot P(x_i) = \frac{1}{n} + \frac{2^2}{n} + \frac{3^2}{n} + \ldots + \frac{n^2}{n}$$

$$= \frac{1}{n}(1^2 + 2^2 + 3^2 + \ldots + n^2)$$

$$= \frac{1}{n} \sum_{r=1}^{n} r^2 = \frac{1}{n} \times \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6}$$

$\therefore$ expected value $= E(X) = \Sigma\, x_i \cdot P(x_i)$

$$= \frac{n+1}{2}$$

Variance $= V(X) = \Sigma\, x_i^2 \cdot P(x_i) - [\Sigma\, x_i \cdot P(x_i)]^2$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$$

$$= \frac{n+1}{2}\left[\frac{4n+2-3n-3}{6}\right]$$

$$= \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

Standard deviation $= \sqrt{V(X)} = \sqrt{\frac{n^2-1}{12}}$

$$= \frac{\sqrt{n^2-1}}{2\sqrt{3}}.$$

(iv) We construct the following table to find the expected value, variance, and standard deviation:

x_i	$P(x_i)$	$x_i \cdot P(x_i)$	$x_i^2 \cdot P(x_i) = x_i \times x_i \cdot P(x_i)$
0	$\frac{1}{32}$	0	0
1	$\frac{5}{32}$	$\frac{5}{32}$	$\frac{5}{32}$
2	$\frac{10}{32}$	$\frac{20}{32}$	$\frac{40}{32}$
3	$\frac{10}{32}$	$\frac{30}{32}$	$\frac{90}{32}$
4	$\frac{5}{32}$	$\frac{20}{32}$	$\frac{80}{32}$
5	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{25}{32}$
Total	$\frac{32}{32}$	$\frac{80}{32}$	$\frac{240}{32}$

From the table,

$\sum x_i \cdot P(x_i) = \frac{80}{32} = \frac{5}{2}$,

$\sum x_i^2 \cdot P(x_i) = \frac{240}{32} = \frac{15}{2}$

Expected value = E (X) = $\sum x_i \cdot P(x_i)$

$= \frac{5}{2} = 2.5$

Variance = V(x) = $\sum x_i^2 \cdot P(x_i) - [\sum x_i \cdot P(x_i)]^2$

$= \frac{15}{2} - \left(\frac{5}{2}\right)^2$

$= \frac{15}{2} - \frac{25}{4}$

$= \frac{30}{4} - \frac{25}{4}$

$= \frac{5}{4}$

= 1.25

Standard deviation = $\sqrt{V(X)} = \sqrt{1.25}$

= 1.118

Question 11.
A player tosses two wins. He wins ₹ 10 if 2 heads appear, ₹ 5 if 1 head appears and ₹ 2 if no head appears. Find the expected winning amount and variance of the winning amount.
Solution:
When a coin is tossed twice, the sample space is
S = {HH, HT, TH, HH}
Let X denote the amount he wins.
Then X takes values 10, 5, 2.
P(X = 10) = P(2 heads appear) = 14
P(X = 5) = P(1 head appears) = 24 = 12
P(X = 2) = P(no head appears) = 14
We construct the following table to calculate the mean and the variance of X:

x_i	$P(x_i)$	$x_i P(x_i)$	$x_i^2 P(x_i)$
10	$\frac{1}{4}$	$\frac{5}{2}$	25
5	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{25}{2}$
2	$\frac{1}{4}$	$\frac{1}{2}$	1
Total	1	5.5	38.5

From the table $\Sigma x_i \cdot P(x_i) = 5.5$, $\Sigma x2i \cdot P(xi) = 38.5$
$E(X) = \Sigma x_i \cdot P(x_i) = 5.5$
$Var(X) = \Sigma x2i \cdot P(xi) – [E(X)]^2$
$= 38.5 – (5.5)^2$
= 38.5 – 30.25
= 8.25
∴ Hence, expected winning amount = ₹ 5.5 and variance of winning amount = ₹ 8.25.

Question 12.
Let the p.m.f. of r.v. X be P(x) = 3-x10, for x = -1, 0, 1, 2 and = 0, otherwise.
Calculate E(X) and Var(X).

Solution:
P(X) = 3-x10
X takes values -1, 0, 1, 2
P(X = -1) = P(-1) = 3+110=410
P(X = 0) = P(0) = 3-010=310
P(X = 1) = P(1) = 3-110=210
P(X = 2) = P(2) = 3-210=110
We construct the following table to calculate the mean and variance of X:

x_i	$P(x_i)$	$x_i P(x_i)$	$x_i^2 P(x_i)$
-1	$\frac{4}{10}$	$-\frac{4}{10}$	$\frac{4}{10}$
0	$\frac{3}{10}$	0	0
1	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{2}{10}$
2	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{4}{10}$
Total	1	0	1

From the table
$\Sigma x_i P(x_i) = 0$ and $\Sigma x_i 2 \cdot P(x_i) = 1$
$E(X) = \Sigma x_i P(x_i) = 0$
$Var(X) = \Sigma x_i 2 \cdot P(x_i) – [E(X)]2$
= 1 – 0
= 1
Hence, E(X) = 0, Var (X) = 1.

Question 13.
Suppose the error involved in making a certain measurement is a continuous r.v. X with p.d.f.
$f(x) = k(4 – x2)$, $-2 \leq x \leq 2$ and = 0 otherwise.
Compute
(i) $P(X > 0)$
(ii) $P(-1 < X < 1)$
(iii) $P(X < -0.5$ or $X > 0.5)$.
Solution:
(i) $P(X > 0)$

Since, f is the p.d.f. of X,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\therefore \int_{-\infty}^{-2} f(x)dx + \int_{-2}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx = 1$$

$$\therefore 0 + \int_{-2}^{2} k\left(4 - x^2\right)dx = 1$$

$$\therefore k \int_{-2}^{2} \left(4 - x^2\right)dx = 1$$

$$\therefore k\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 1$$

$$\therefore k\left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right] = 1$$

$$\therefore k\left(\frac{16}{3} + \frac{16}{3}\right) = 1$$

$$\therefore k\left(\frac{32}{3}\right) = 1$$

$$\therefore k = \frac{3}{32}$$

P(x > 0)

$$= \int_{0}^{\infty} f(x)dx$$

$$= \int_{0}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx$$

$$= \int_{0}^{2} k\left(4 - x^2\right)dx + 0$$

$$= k\int_{0}^{2} \left(4 - x^2\right)dx$$

$$= \frac{3}{32}\left[4x - \frac{x^3}{3}\right]_{0}^{2} \text{ ....} \left[\because k = \frac{3}{32}\right]$$

$$= \frac{3}{32}\left[8 - \frac{8}{3}\right] = \frac{3}{32} \times \frac{16}{3} = \frac{1}{2}$$

(ii) P(-1 < X < 1)

Since, f is the p.d.f. of X,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\therefore \int_{-\infty}^{-2} f(x)dx + \int_{-2}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx = 1$$

$$\therefore 0 + \int_{-2}^{2} k\left(4 - x^2\right)dx = 1$$

$$\therefore k \int_{-2}^{2} \left(4 - x^2\right)dx = 1$$

$$\therefore k\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 1$$

$$\therefore k\left[\left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)\right] = 1$$

$$\therefore k\left(\frac{16}{3} + \frac{16}{3}\right) = 1$$

$$\therefore k\left(\frac{32}{3}\right) = 1$$

$$\therefore k = \frac{3}{32}$$

P(–1 < x < 1)

$$= \int_{-1}^{1} f(x)dx$$

$$= \int_{-1}^{1} k\left(4 - x^2\right)dx$$

$$= k \int_{-1}^{1} \left(4 - x^2\right)dx$$

$$= \frac{3}{32}\left[4x - \frac{x^3}{3}\right]_{-1}^{1} \quad \text{.......}\left[\because k = \frac{3}{32}\right]$$

$$= \frac{3}{32}\left[\left(4 - \frac{1}{3}\right) - \left(-4 + \frac{1}{3}\right)\right]$$

$$= \frac{3}{32}\left(\frac{11}{3} + \frac{11}{3}\right)$$

$$= \frac{3}{32}\left(\frac{22}{3}\right)$$

$$= \frac{11}{6}$$

(iii) P(X < -0.5 or X > 0.5)

Since, f is the p.d.f. of X,

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\therefore \int_{-\infty}^{-2} f(x)dx + \int_{-2}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx = 1$$

$$\therefore 0 + \int_{-2}^{2} k(4-x^2)dx = 1$$

$$\therefore k\int_{-2}^{2} (4-x^2)dx = 1$$

$$\therefore k\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = 1$$

$$\therefore k\left[\left(8-\frac{8}{3}\right) - \left(-8+\frac{8}{3}\right)\right] = 1$$

$$\therefore k\left(\frac{16}{3}+\frac{16}{3}\right) = 1$$

$$\therefore k\left(\frac{32}{3}\right) = 1$$

$$\therefore k = \frac{3}{32}$$

P (–0·5 < x or x > 0·5)

= P (x < –0·5) + P (x > – 0·5)

$$= \int_{-\infty}^{-0.5} f(x)dx + \int_{0.5}^{\infty} f(x)dx$$

$$= \int_{-\infty}^{-2} f(x)dx + \int_{-2}^{-0.5} f(x)dx + \int_{0.5}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx$$

$$= 0 + \frac{\int_{-2}^{-1}}{2} k(4-x^2)dx + \int_{\frac{1}{2}}^{2} k(4-x^2)dx + 0$$

$$= k\,\frac{\int_{-2}^{1}}{2}(4-x^2)dx + \int_{\frac{1}{2}}^{2}(4-x^2)dx$$

$$= \frac{3}{32}\left[4x-\frac{x^3}{3}\right]_{-2}^{-\frac{1}{2}} + \frac{3}{32}\left[4x-\frac{x^3}{3}\right]_{\frac{1}{2}}^{2} \quad ....[\because k = \frac{3}{32}]$$

$$= \frac{3}{32}\left[\left(-2+\frac{1}{24}\right)-\left(-8+\frac{8}{3}\right)\right] + \frac{3}{32}\left[\left(8-\frac{8}{3}\right)-\left(2-\frac{1}{24}\right)\right]$$

$$= \frac{3}{32}\left(\frac{-47}{24}+\frac{16}{3}\right) + \frac{3}{32}\left(\frac{16}{3}-\frac{47}{24}\right)$$

$$= \frac{3}{32}\left(\frac{-47}{24}+\frac{16}{3}+\frac{16}{3}-\frac{47}{24}\right)$$

$$= \frac{3}{32}\left(\frac{-47+128+128-47}{24}\right)$$

$$= \frac{3}{32}\left(\frac{162}{24}\right) = \frac{81}{128}$$

= 0.6328.

Alternative Method :

P (x< – 0·5 or x > 0·5)

$= 1 - P(-0.5 \le x \le 0.5)$

$= 1 - \int_{-0.5}^{0.5} f(x)dx$

$= 1 - \int_{-\frac{1}{2}}^{\frac{1}{2}} k(4-x^2)dx$

$= 1 - k\int_{-\frac{1}{2}}^{\frac{1}{2}} (4-x^2)dx$

$= 1 - \frac{3}{32}\left[4x - \frac{x^3}{3}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ ......[$\because$ k = $\frac{3}{32}$]

$= 1 - \frac{3}{32}\left[\left(2 - \frac{1}{24}\right) - \left(-2 + \frac{1}{24}\right)\right]$

$= 1 - \frac{3}{32}\left(2 - \frac{1}{24} + 2 - \frac{1}{24}\right)$

$= 1 - \frac{3}{32}\left(4 - \frac{1}{12}\right)$

$= 1 - \frac{3}{32} \times \frac{47}{12} = 1 - \frac{47}{128}$

$= \frac{128 - 47}{128} = \frac{81}{128}$

= 0.6328

Question 14.
The p.d.f. of a continuous r.v. X is given by f(x) = 12a, for 0 < x < 2a and = 0, otherwise. Show that P(X < a2) = P(X > 3a2)
Solution:

$$P\left(X < \frac{a}{2}\right) = \int_{-\infty}^{a/2} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{a/2} f(x)\,dx$$

$$= 0 + \int_{0}^{a/2} \frac{1}{2a}\,dx$$

$$= \frac{1}{2a}\int_{0}^{a/2} 1\,dx = \frac{1}{2a}\Big[x\Big]_{0}^{a/2}$$

$$= \frac{1}{2a}\left[\frac{a}{2} - 0\right] = \frac{1}{4} \qquad \ldots (1)$$

$$P\left(X > \frac{3a}{2}\right) = \int_{3a/2}^{\infty} f(x)\,dx$$

$$= \int_{3a/2}^{2a} f(x)\,dx + \int_{2a}^{\infty} f(x)\,dx$$

$$= \int_{3a/2}^{2a} \frac{1}{2a}\,dx + 0$$

$$= \frac{1}{2a}\int_{3a/2}^{2a} 1\,dx = \frac{1}{2a}\Big[x\Big]_{3a/2}^{2a}$$

$$= \frac{1}{2a}\left[2a - \frac{3a}{2}\right] = \frac{1}{2a}\left(\frac{a}{2}\right) = \frac{1}{4} \qquad \ldots (2)$$

From (1) and (2), we get

$$P\left(X < \frac{a}{2}\right) = P\left(X > \frac{3a}{2}\right).$$

Question 15.
The p.d.f. of r.v. X is given by f(x) = kx√, for 0 < x < 4 and = 0, otherwise. Determine k. Determine c.d.f. of X and hence find P(X ≤ 2) and P(X ≤ 1).
Solution:
Since f is p.d.f. of the r.v. X,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\therefore \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx = 1$$

$$\therefore 0 + \int_{0}^{4} \frac{k}{\sqrt{x}}\,dx + 0 = 1$$

$$\therefore k\int_{0}^{4} x^{-\frac{1}{2}}\,dx = 1$$

$$\therefore k\left[\frac{x^{\frac{1}{2}}}{1/2}\right]_0^4 = 1 \qquad \therefore 2k[2-0] = 1$$

$$\therefore 4k = 1 \qquad \therefore k = \frac{1}{4}.$$

Let $F(X)$ be the c.d.f. of X.

$$\therefore F(X) = P(X \leqslant x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} \frac{k}{\sqrt{x}}\,dx$$

$$= k\int_{0}^{x} x^{-\frac{1}{2}}\,dx = k\left[\frac{x^{\frac{1}{2}}}{1/2}\right]_0^x$$

$$= 2k\sqrt{x} = 2\left(\frac{1}{4}\right)\sqrt{x} \qquad \ldots\left[\because k = \frac{1}{4}\right]$$

$$\therefore F(X) = \frac{\sqrt{x}}{2}$$

$$P(X \leqslant 2) = F(2) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$P(X \leqslant 1) = F(1) = \frac{\sqrt{1}}{2} = \frac{1}{2}.$$

Hence, $k = \frac{1}{4}$, $P(X \leqslant 2) = \frac{1}{\sqrt{2}}$, $P(X \leqslant 1) = \frac{1}{2}$.