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Maharashtra State Board 12th Maths Solutions Chapter 1 Differentiation Ex 1.1

Question 1.
Differentiate the following w.r.t. x :
(i) $(x^3 – 2x – 1)^5$
Solution:
Method 1:
Let y = $(x^3 – 2x – 1)^5$
Put u = $x^3 – 2x – 1$. Then y = u^5

$$\therefore \frac{dy}{du} = \frac{d}{du}(u^5) = 5u^4$$
$$= 5(x^3 - 2x - 1)^4$$
$$\text{and } \frac{du}{dx} = \frac{d}{dx}(x^3 - 2x - 1)$$
$$= 3x^2 - 2 \times 1 - 0 = 3x^2 - 2$$
$$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 5(x^3 - 2x - 1)^4 (3x^2 - 2)$$
$$= 5(3x^2 - 2)(x^3 - 2x - 1)^4.$$

Method 2:
Let y = $(x^3 – 2x – 1)^5$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 2x - 1)^5$$
$$= 5(x^3 - 2x - 1)^4 \times \frac{d}{dx}(x^3 - 2x - 1)$$
$$= 5(x^3 - 2x - 1)^4 \times (3x^2 - 2 \times 1 - 0)$$
$$= 5(3x^2 - 2)(x^3 - 2x - 1)^4.$$

(ii) $\left(2x^{\frac{3}{2}}-3x^{\frac{4}{3}}-5\right)^{\frac{5}{2}}$
Solution:
Let y = $\left(2x^{\frac{3}{2}}-3x^{\frac{4}{3}}-5\right)^{\frac{5}{2}}$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{5}{2}}$$
$$= \frac{5}{2}\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{5}{2}-1} \times \frac{d}{dx}\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)$$
$$= \frac{5}{2}\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{3}{2}} \times \left(2 \times \frac{3}{2}x^{\frac{3}{2}-1} - 3 \times \frac{4}{3}x^{\frac{4}{3}-1} - 0\right)$$
$$= \frac{5}{2}\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{3}{2}}\left(3x^{\frac{1}{2}} - 4x^{\frac{1}{3}}\right)$$
$$= \frac{5}{2}\left(3\sqrt{x} - 4\sqrt[3]{x}\right)\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{3}{2}}.$$

(iii) x2+4x−7−−−−−−−−−√

Solution:

$$y=\sqrt{x^2+4x-7}\left[\sqrt{x}=\frac{1}{2\sqrt{x}}\right]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x^2+4x-7}}\cdot\frac{d}{dx}\left(x^2+4x-7\right)$$

$$=\frac{1}{2\sqrt{x^2+4x-7}}\left(\frac{d}{dx}x^2+\frac{d}{dx}4x-\frac{d}{dx}7\right)$$

$$=\frac{1}{2\sqrt{x^2+4x-7}}.(2x+4-0)$$

$$=\frac{2(x+2)}{2\sqrt{x^2+4x-7}}$$

$$=\frac{(x+2)}{\sqrt{x^2+4x-7}}.$$

(iv) x2+x2+1−−−−−√−−−−−−−−−−−√

Solution:

Let y = x2+x2+1−−−−−√−−−−−−−−−−−√

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2+\sqrt{x^2+1}\right)^{\frac{1}{2}}$$

$$=\frac{1}{2}\left(x^2+\sqrt{x^2+1}\right)^{-\frac{1}{2}}\cdot\frac{d}{dx}\left(x^2+\sqrt{x^2+1}\right)$$

$$=\frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}}\cdot\left[\frac{d}{dx}(x^2)+\frac{d}{dx}(\sqrt{x^2+1})\right]$$

$$=\frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}}\cdot\left[2x+\frac{1}{2\sqrt{x^2+1}}\cdot\frac{d}{dx}(x^2+1)\right]$$

$$=\frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}}\cdot\left[2x+\frac{1}{2\sqrt{x^2+1}}(2x+0)\right]$$

$$=\frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}}\cdot\left[2x+\frac{x}{\sqrt{x^2+1}}\right]$$

$$=\frac{1}{2\sqrt{x^2+\sqrt{x^2+1}}}\cdot\left[\frac{2x\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right]$$

$$=\frac{x(2\sqrt{x^2+1}+1)}{2\sqrt{x^2+1}\cdot\sqrt{x^2+\sqrt{x^2+1}}}.$$

(v) 35(2x2−7x−5)5√3

Solution:

Let y = 35(2x2−7x−5)5√3

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{3}{5}\frac{d}{dx}(2x^2-7x-5)^{-\frac{5}{3}}$$

$$=\frac{3}{5}\times\left(-\frac{5}{3}\right)(2x^2-7x-5)^{-\frac{5}{3}-1}\cdot\frac{d}{dx}(2x^2-7x-5)$$

$$=-(2x^2-7x-5)^{-\frac{8}{3}}\cdot(2\times 2x-7\times 1-0)$$

$$=-\frac{4x-7}{(2x^2-7x-5)^{\frac{8}{3}}}.$$

(vi) $\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^5$

Solution:

Let y = $\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^5$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^5$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\cdot\frac{d}{dx}\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\cdot\left[\frac{d}{dx}(3x-5)^{\frac{1}{2}}-\frac{d}{dx}(3x-5)^{-\frac{1}{2}}\right]$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\times$$

$$\left[\frac{1}{2}(3x-5)^{-\frac{1}{2}}\cdot\frac{d}{dx}(3x-5)-\left(-\frac{1}{2}\right)(3x-5)^{-\frac{3}{2}}\cdot\frac{d}{dx}(3x-5)\right]$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\times$$

$$\left[\frac{1}{2\sqrt{3x-5}}\cdot(3\times 1-0)+\frac{1}{2(3x-5)^{\frac{3}{2}}}\cdot(3\times 1-0)\right]$$

$$=5\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\cdot\left[\frac{3}{2\sqrt{3x-5}}+\frac{3}{2(3x-5)^{\frac{3}{2}}}\right]$$

$$=\frac{15}{2}\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4\cdot\left[\frac{3x-5+1}{(3x-5)^{\frac{3}{2}}}\right]$$

$$=\frac{15(3x-4)}{2(3x-5)^{\frac{3}{2}}}\left(\sqrt{3x-5}-\frac{1}{\sqrt{3x-5}}\right)^4.$$

Question 2.
Diffrentiate the following w.r.t. x
(i) $\cos(x^2 + a^2)$
Solution:
Let y = $\cos(x^2 + a^2)$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}[\cos(x^2 + a^2)]$
$= -\sin(x^2 + a^2)\cdot\frac{d}{dx}(x^2 + a^2)$
$= -\sin(x^2 + a^2)\cdot(2x + 0)$
$= -2x\sin(x^2 + a^2)$

(ii) $\sqrt{e^{(3x+2)}+5}$
Solution:
Let y = $\sqrt{e^{(3x+2)}+5}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[e^{(3x+2)}+5]^{\frac{1}{2}}$$

$$=\frac{1}{2}[e^{(3x+2)}+5]^{-\frac{1}{2}}\cdot\frac{d}{dx}[e^{(3x+2)}+5]$$

$$=\frac{1}{2\sqrt{e^{(3x+2)}+5}}\cdot[e^{(3x+2)}\cdot\frac{d}{dx}(3x+2)+0]$$

$$=\frac{1}{2\sqrt{e^{(3x+2)}+5}}\cdot[e^{(3x+2)}\cdot(3\times 1+0)]$$

$$=\frac{3e^{(3x+2)}}{2\sqrt{e^{(3x+2)}+5}}.$$

(iii) log[tan(x2)]
Solution:
Let y = log[tan(x2)]
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\log\left[\tan\left(\frac{x}{2}\right)\right]$$

$$=\frac{1}{\tan\left(\frac{x}{2}\right)}\cdot\frac{d}{dx}\left[\tan\left(\frac{x}{2}\right)\right]$$

$$=\frac{1}{\tan\left(\frac{x}{2}\right)}\cdot\sec^2\left(\frac{x}{2}\right)\cdot\frac{d}{dx}\left(\frac{x}{2}\right)$$

$$=\frac{\cos\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\cdot\frac{1}{\cos^2\left(\frac{x}{2}\right)}\cdot\frac{1}{2}\times 1$$

$$=\frac{1}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$=\frac{1}{\sin x}=\operatorname{cosec}x.$$

(iv) tanx−−√−−−−−−√
Solution:
Let y = tanx−−√−−−−−−√

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\sqrt{\tan\sqrt{x}})$$

$$=\frac{1}{2\sqrt{\tan\sqrt{x}}}\cdot\frac{d}{dx}(\tan\sqrt{x})$$

$$=\frac{1}{2\sqrt{\tan\sqrt{x}}}\times\sec^2\sqrt{x}\cdot\frac{d}{dx}(\sqrt{x})$$

$$=\frac{1}{2\sqrt{\tan\sqrt{x}}}\times\sec^2\sqrt{x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{\sec^2\sqrt{x}}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}.$$

(v) cot3[log (x3)]
Solution:
Let y = cot3[log (x3)]
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\cot(\log x^3)]^3$$

$$=3[\cot(\log x^3)]^2\cdot\frac{d}{dx}[\cot(\log x^3)]$$

$$=3\cot^2[\log(x^3)]\cdot[-\operatorname{cosec}^2(\log x^3)]\cdot\frac{d}{dx}(\log x^3)$$

$$=-3\cot^2[\log(x^3)]\cdot\operatorname{cosec}^2[\log(x^3)]\cdot 3\frac{d}{dx}(\log x)$$

$$=-3\cot^2[\log(x^3)]\cdot\operatorname{cosec}^2[\log(x^3)]\cdot 3\times\frac{1}{x}$$

$$=\frac{-9\operatorname{cosec}^2[\log(x^3)]\cdot\cot^2[\log(x^3)]}{x}.$$

(vi) 5sin3x+ 3
Solution:
Let y = 5sin3x+ 3
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(5^{\sin^3x+3})$$

$$=5^{\sin^3x+3}\cdot\log 5\cdot\frac{d}{dx}(\sin^3x+3)$$

$$=5^{\sin^3x+3}\cdot\log 5\cdot[3\sin^2x\cdot\frac{d}{dx}(\sin x)+0]$$

$$=5^{\sin^3x+3}\cdot\log 5\cdot[3\sin^2x\cos x]$$

$$=3\sin^2x\cos x\cdot 5^{\sin^3x+3}\cdot\log 5.$$

(vii) cosec (cosX−−−−−√)
Solution:
Let y = cosec (cosX−−−−−√)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\operatorname{cosec}(\sqrt{\cos x})]$$

$$= -\operatorname{cosec}(\sqrt{\cos x})\cdot\cot(\sqrt{\cos x})\cdot\frac{d}{dx}\sqrt{\cos x}$$

$$= -\operatorname{cosec}(\sqrt{\cos x})\cdot\cot(\sqrt{\cos x})\cdot\frac{1}{2\sqrt{\cos x}}\cdot\frac{d}{dx}(\cos x)$$

$$= -\operatorname{cosec}(\sqrt{\cos x})\cdot\cot(\sqrt{\cos x})\cdot\frac{1}{2\sqrt{\cos x}}\cdot(-\sin x)$$

$$=\frac{\sin x\cdot\operatorname{cosec}(\sqrt{\cos x})\cdot\cot(\sqrt{\cos x})}{2\sqrt{\cos x}}.$$

(viii) log[cos (x^3 – 5)]
Solution:
Let y = log[cos (x^3 – 5)]
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\{\log[\cos(x^3-5)]\}$$

$$=\frac{1}{\cos(x^3-5)}\cdot\frac{d}{dx}[\cos(x^3-5)]$$

$$=\frac{1}{\cos(x^3-5)}\cdot[-\sin(x^3-5)]\cdot\frac{d}{dx}(x^3-5)$$

$$= -\tan(x^3-5)\times(3x^2-0)$$

$$= -3x^2\tan(x^3-5).$$

(ix) $e^{3 \sin^2 x - 2 \cos^2 x}$
Solution:
Let y = $e^{3 \sin^2 x - 2 \cos^2 x}$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[e^{3\sin^2x-2\cos^2x}]$$

$$=e^{3\sin^2x-2\cos^2x}\cdot\frac{d}{dx}(3\sin^2x-2\cos^2x)$$

$$=e^{3\sin^2x-2\cos^2x}\cdot\left[3\frac{d}{dx}(\sin x)^2-2\frac{d}{dx}(\cos x)^2\right]$$

$$=e^{3\sin^2x-2\cos^2x}\cdot\left[3\times2\sin x\cdot\frac{d}{dx}(\sin x)-2\times2\cos x\cdot\frac{d}{dx}(\cos x)\right]$$

$$=e^{3\sin^2x-2\cos^2x}\cdot[6\sin x\cos x-4\cos x(-\sin x)]$$

$$=e^{3\sin^2x-2\cos^2x}\cdot(10\sin x\cos x)$$

$$=5(2\sin x\cos x)\cdot e^{3\sin^2x-2\cos^2x}$$

$$=5\sin 2x\cdot e^{3\sin^2x-2\cos^2x}.$$

(x) $\cos^2$[log (x^2+ 7)]
Solution:
Let y = $\cos^2$[log (x^2+ 7)]
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\{\cos[\log(x^2+7)]\}^2$$

$$=2\cos[\log(x^2+7)]\cdot\frac{d}{dx}\{\cos[\log(x^2+7)]\}$$

$$=2\cos[\log(x^2+7)]\cdot\{-\sin[\log(x^2+7)]\}\cdot\frac{d}{dx}[\log(x^2+7)]$$

$$=-2\sin[\log(x^2+7)]\cdot\cos[\log(x^2+7)]\times\frac{1}{x^2+7}\cdot\frac{d}{dx}(x^2+7)$$

$$=-\sin[2\log(x^2+7)]\times\frac{1}{x^2+7}\cdot(2x+0)$$

$$\ldots\ [\because\ 2\sin x\cdot\cos x=\sin 2x]$$

$$=\frac{-2x\cdot\sin[2\log(x^2+7)]}{x^2+7}.$$

(xi) tan[cos (sinx)]
Solution:
Let y = tan[cos (sinx)]
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\{\tan[\cos(\sin x)]\}$$

$$=\sec^2[\cos(\sin x)]\cdot\frac{d}{dx}[\cos(\sin x)]$$

$$=\sec^2[\cos(\sin x)]\cdot[-\sin(\sin x)]\cdot\frac{d}{dx}(\sin x)$$

$$=-\sec^2[\cos(\sin x)]\cdot\sin(\sin x)\cdot\cos x.$$

(xii) sec[tan (x^4 + 4)]
Solution:
Let y = sec[tan (x^4 + 4)]
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\{\sec[\tan(x^4+4)]\}$$

$$=\sec[\tan(x^4+4)]\cdot\tan[\tan(x^4+4)]\cdot\frac{d}{dx}[\tan(x^4+4)]$$

$$=\sec[\tan(x^4+4)]\cdot\tan[\tan(x^4+4)]\cdot\sec^2(x^4+4)\cdot\frac{d}{dx}(x^4+4)$$

= sec[tan(x^4 + 4)]·tan[tan(x^4 + 4)]·$\sec^2$(x^4 + 4)(4x^3 + 0)
= 4$x^3$$\sec^2$($x^4$ + 4)·sec[tan(x^4 + 4)]·tan[tan(x^4 + 4)].

(xiii) $e^{\log[(\log x)^2 - \log x^2]}$
Solution:
Let y = $e^{\log[(\log x)^2 - \log x^2]}$
= $(\log x)^2$ – log x^2 ...[∵ $e^{\log x}$ = x]
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[(\log x)^2-2\log x]$$

$$=\frac{d}{dx}(\log x)^2-2\frac{d}{dx}(\log x)$$

$$=2\log x\cdot\frac{d}{dx}(\log x)-2\times\frac{1}{x}$$

$$=2\log x\times\frac{1}{x}-\frac{2}{x}$$

$$=\frac{2\log x}{x}-\frac{2}{x}.$$

(xiv) $\sin\sqrt{\sin\sqrt{x}}$

Solution:

Let y = $\sin\sqrt{\sin\sqrt{x}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\sin\sqrt{\sin\sqrt{x}})$$

$$=\cos\sqrt{\sin\sqrt{x}}\cdot\frac{d}{dx}(\sqrt{\sin\sqrt{x}})$$

$$=\cos\sqrt{\sin\sqrt{x}}\times\frac{1}{2\sqrt{\sin\sqrt{x}}}\cdot\frac{d}{dx}(\sin\sqrt{x})$$

$$=\frac{\cos\sqrt{\sin\sqrt{x}}}{2\sqrt{\sin\sqrt{x}}}\times\cos\sqrt{x}\cdot\frac{d}{dx}(\sqrt{x})$$

$$=\frac{\cos\sqrt{\sin\sqrt{x}}\cdot\cos\sqrt{x}}{2\sqrt{\sin\sqrt{x}}}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{\cos\sqrt{\sin\sqrt{x}}\cdot\cos\sqrt{x}}{4\sqrt{x}\cdot\sqrt{\sin\sqrt{x}}}.$$

(xv) $\log[\sec(e^{x^2})]$

Solution:

Let y = $\log[\sec(e^{x^2})]$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\log(\sec e^{x^2})]$$

$$=\frac{1}{\sec(e^{x^2})}\cdot\frac{d}{dx}[\sec(e^{x^2})]$$

$$=\frac{1}{\sec(e^{x^2})}\cdot\sec(e^{x^2})\tan(e^{x^2})\cdot\frac{d}{dx}(e^{x^2})$$

$$=\tan(e^{x^2})\cdot e^{x^2}\cdot\frac{d}{dx}(x^2)$$

$$=\tan(e^{x^2})\cdot e^{x^2}\cdot 2x$$

$$=2x\cdot e^{x^2}\tan(e^{x^2}).$$

(xvi) $\log_{e^2}(\log x)$

Solution:

Let y = $\log_{e^2}(\log x) = \frac{\log(\log x)}{\log e^2}$

$$=\frac{\log(\log x)}{2\log e}=\frac{\log(\log x)}{2}\qquad\ldots\ [\because\ \log e=1]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}[\log(\log x)]$$

$$=\frac{1}{2}\times\frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$=\frac{1}{2\log x}\times\frac{1}{x}=\frac{1}{2x\log x}.$$

(xvii) $[\log\{\log(\log x)\}]^2$

Solution:

let y = $[\log\{\log(\log x)\}]^2$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\log\{\log(\log x)\}]^2$$

$$=2\cdot\log\{\log(\log x)\}\times\frac{d}{dx}[\log\{\log(\log x)\}]$$

$$=2\cdot\log\{\log(\log x)\}\times\frac{1}{\log(\log x)}\cdot\frac{d}{dx}[\log(\log x)]$$

$$=2\cdot\log\{\log(\log x)\}\times\frac{1}{\log(\log x)}\times\frac{1}{\log x}\times\frac{d}{dx}(\log x)$$

$$=2\cdot\log\{\log(\log x)\}\times\frac{1}{\log(\log x)}\times\frac{1}{\log x}\times\frac{1}{x}$$

$$=2\cdot\left[\frac{\log\{\log(\log x)\}}{x\cdot\log x\cdot\log(\log x)}\right].$$

(xviii) $\sin^2 x^2 - \cos^2 x^2$

Solution:

Let y = $\sin^2 x^2 - \cos^2 x^2$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\sin^2 x^2-\cos^2 x^2]$$

$$=\frac{d}{dx}(\sin x^2)^2-\frac{d}{dx}(\cos x^2)^2$$

$$=2\sin x^2\cdot\frac{d}{dx}(\sin x^2)-2\cos x^2\cdot\frac{d}{dx}(\cos x^2)$$

$$=2\sin x^2\cdot\cos x^2\cdot\frac{d}{dx}(x^2)-2\cos x^2\cdot(-\sin x^2)\cdot\frac{d}{dx}(x^2)$$

= $2\sin x^2\cdot\cos x^2 \times 2x + 2\sin x^2\cdot\cos x^2 \times 2x$

= $4x(2\sin x^2\cdot\cos x^2)$

= $4x\sin(2x^2)$.

Question 3.

Diffrentiate the following w.r.t. x

(i) $(x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4$

Solution:

Let y = $(x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4$

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}[(x^2 + 4x + 1)^3 + (x^3 - 5x - 2)^4]$

= $\frac{d}{dx}$ = $(x^2 + 4x + 1)^3 + \frac{d}{dx}(x^3 - 5x - 2)^4$

= $3(x^2 + 4x + 1)^2\cdot\frac{d}{dx}(x^2 + 4x + 1) + 4(x^3 - 5x - 2)^4\cdot\frac{d}{dx}(x^3 - 5x - 2)$

= $3(x^2 + 4x + 1)^3\cdot(2x + 4 \times 1 + 0) + 4(x^3 - 5x - 2)^3\cdot(3x^2 - 5 \times 1 - 0)$

= $6(x + 2)(x^2 + 4x + 1)^2 + 4(3x^2 - 5)(x^3 - 5x - 2)^3$.

(ii) $(1 + 4x)^5(3 + x - x^2)^8$

Solution:

Let y = $(1 + 4x)^5(3 + x - x^2)^8$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[(1+4x)^5(3+x-x^2)^8]$$

$$=(1+4x)^5\cdot\frac{d}{dx}(3+x-x^2)^8+(3+x-x^2)^8\cdot\frac{d}{dx}(1+4x)^5$$

$$=(1+4x)^5\times 8(3+x-x^2)^7\cdot\frac{d}{dx}(3+x-x^2)+(3+x-x^2)^8\times 5(1+4x)^4\cdot\frac{d}{dx}(1+4x)$$

= $8(1 + 4x)^5(3 + x - x^2)^7\cdot(0 + 1 - 2x) + 5(1 + 4x)^4(3 + x - x^2)^8\cdot(0 + 4 \times 1)$

= $8(1 - 2x)(1 + 4x)^5(3 + x - x^2)^7 + 20(1 + 4x)^4(3 + x - x^2)^8$.

(iii) $\frac{x}{\sqrt{7-3x}}$

Solution:

Let y = $\frac{x}{\sqrt{7-3x}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x}{\sqrt{7-3x}}\right)=\frac{\sqrt{7-3x}\cdot\frac{d}{dx}(x)-x\frac{d}{dx}(\sqrt{7-3x})}{(\sqrt{7-3x})^2}$$

$$=\frac{\sqrt{7-3x}\times 1-x\times\frac{1}{2\sqrt{7-3x}}\cdot\frac{d}{dx}(7-3x)}{7-3x}$$

$$=\frac{\sqrt{7-3x}-\frac{x}{2\sqrt{7-3x}}(0-3\times 1)}{7-3x}$$

$$=\frac{2(7-3x)+3x}{2(7-3x)^{\frac{3}{2}}}=\frac{14-6x+3x}{2(7-3x)^{\frac{3}{2}}}=\frac{14-3x}{2(7-3x)^{\frac{3}{2}}}$$

(iv) $(x^3-5)^5(x^3+3)^3$

Solution:

Let y = $(x^3-5)^5(x^3+3)^3$

Differentiating w.r.t. x, we get

$dydx = ddx[(x^3-5)^5(x^3+3)^3]$

$$=\frac{(x^3+3)^3\cdot\frac{d}{dx}(x^3-5)^5-(x^3-5)^5\cdot\frac{d}{dx}(x^3+3)^3}{[(x^3+3)^3]^2}$$

$$=\frac{(x^3+3)^3\times 5(x^3-5)^4\cdot\frac{d}{dx}(x^3-5)-(x^3-5)^5\times 3(x^3+3)^2\cdot\frac{d}{dx}(x^3+3)}{(x^3+3)^6}$$

$$=\frac{5(x^3+3)^3(x^3-5)^4\cdot(3x^2-0)-3(x^3-5)^5(x^3+3)^2\cdot(3x^2+0)}{(x^3+3)^6}$$

$$=\frac{3x^2(x^3+3)^2(x^3-5)^4[5(x^3+3)-3(x^3-5)]}{(x^3+3)^6}$$

$$=\frac{3x^2(x^3-5)^4(5x^3+15-3x^3+15)}{(x^3+3)^4}$$

$$=\frac{3x^2(x^3-5)^4(2x^3+30)}{(x^3+3)^4}$$

$$=\frac{6x^2(x^3+15)(x^3-5)^4}{(x^3+3)^4}$$

(v) $(1 + \sin^2 x)^2(1 + \cos^2 x)^3$

Solution:

Let y = $(1 + \sin^2 x)^2(1 + \cos^2 x)^3$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[(1+\sin^2 x)^2(1+\cos^2 x)^3]$$

$$=(1+\sin^2 x)^2\cdot\frac{d}{dx}(1+\cos^2 x)^3+(1+\cos^2 x)^3\frac{d}{dx}(1+\sin^2 x)^2$$

$$=(1+\sin^2 x)^2\times 3(1+\cos^2 x)^2\cdot\frac{d}{dx}(1+\cos^2 x)+(1+\cos^2 x)^3\times 2(1+\sin^2 x)\cdot\frac{d}{dx}(1+\sin^2 x)$$

$$=3(1+\sin^2 x)^2(1+\cos^2 x)^2\cdot\left[0+2\cos x\cdot\frac{d}{dx}(\cos x)\right]+2(1+\sin^2 x)(1+\cos^2 x)^3\cdot\left[0+2\sin x\cdot\frac{d}{dx}(\sin x)\right]$$

$= 3(1 + \sin^2 x)^2 (1 + \cos^2 x)^2\cdot[2\cos x(-\sin x)] + 2 (1 + \sin^2 x)(1 + \cos^2 x)^3\cdot[2\sin x\cdot\cos x]$

$= 3 (1 + \sin^2 x)^2 (1 + \cos^2 x)^2 (-\sin 2x) + 2(1 + \sin^2 x)(1 + \cos^2 x)^3(\sin 2x)$

$= \sin 2x (1 + \sin^2 x) (1 + \cos^2 x)^2 [-3(1 + \sin^2 x) + 2(1 + \cos^2 x)]$

$= \sin 2x (1 + \sin^2 x)(1 + \cos^2 x)^2(-3 – 3\sin^2 x + 2 + 2\cos^2 x)$

$= \sin 2x (1 + \sin^2 x)(1 + \cos^2 x)^2 [-1 – 3 \sin^2 x + 2 (1 – \sin^2 x)]$

= sin 2x(1 + sin2x)(1 + cos2x)2 (-1 – 3 sin2x + 2 – 2 sin2x)
= sin2x (1 + sin2x)(1 + cos2x)2(1 – 5 sin2x).

(vi) cosx−−−−√+cosx−−√−−−−−−√

Solution:

Let y = cosx−−−−√+cosx−−√−−−−−−√

Differentiating w.r.t. x, we get

dydx = ddx[cosx−−−−√+cosx−−√−−−−−−√]

$$=\frac{d}{dx}(\cos x)^{\frac{1}{2}}+\frac{d}{dx}(\cos\sqrt{x})^{\frac{1}{2}}$$

$$=\frac{1}{2}(\cos x)^{-\frac{1}{2}}\cdot\frac{d}{dx}(\cos x)+\frac{1}{2}(\cos\sqrt{x})^{-\frac{1}{2}}\cdot\frac{d}{dx}(\cos\sqrt{x})$$

$$=\frac{1}{2\sqrt{\cos x}}\cdot(-\sin x)+\frac{1}{2\sqrt{\cos\sqrt{x}}}\times(-\sin\sqrt{x})\cdot\frac{d}{dx}(\sqrt{x})$$

$$=\frac{-\sin x}{2\sqrt{\cos x}}-\frac{\sin\sqrt{x}}{2\sqrt{\cos\sqrt{x}}}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{-\sin x}{2\sqrt{\cos x}}-\frac{\sin\sqrt{x}}{4\sqrt{x}\sqrt{\cos\sqrt{x}}}.$$

(vii) log(sec 3x+ tan 3x)
Solution:
Let y = log(sec 3x+ tan 3x)
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\log(\sec 3x+\tan 3x)]$$

$$=\frac{1}{\sec 3x+\tan 3x}\cdot\frac{d}{dx}(\sec 3x+\tan 3x)$$

$$=\frac{1}{\sec 3x+\tan 3x}\times\left[\frac{d}{dx}(\sec 3x)+\frac{d}{dx}(\tan 3x)\right]$$

$$=\frac{1}{\sec 3x+\tan 3x}\times\left[\sec 3x\tan 3x\cdot\frac{d}{dx}(3x)+\sec^2 3x\cdot\frac{d}{dx}(3x)\right]$$

$$=\frac{1}{\sec 3x+\tan 3x}\times[\sec 3x\tan 3x\times 3+\sec^2 3x\times 3]$$

$$=\frac{3\sec 3x(\tan 3x+\sec 3x)}{\sec 3x+\tan 3x}=3\sec 3x.$$

(viii) 1+sinx 1−sinx
Solution:

$$\text{Let } y=\frac{1+\sin x^\circ}{1-\sin x^\circ}=\frac{1+\sin\left(\frac{\pi x}{180}\right)}{1-\sin\left(\frac{\pi x}{180}\right)} \qquad \ldots\left[\because\ x^\circ=\left(\frac{\pi x}{180}\right)^c\right]$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{1+\sin\left(\frac{\pi x}{180}\right)}{1-\sin\left(\frac{\pi x}{180}\right)}\right)$$

$$=\frac{\left[1-\sin\left(\frac{\pi x}{180}\right)\right]\cdot\frac{d}{dx}\left[1+\sin\left(\frac{\pi x}{180}\right)\right]-\left[1+\sin\left(\frac{\pi x}{180}\right)\right]\cdot\frac{d}{dx}\left[1-\sin\left(\frac{\pi x}{180}\right)\right]}{\left[1-\sin\left(\frac{\pi x}{180}\right)\right]^2}$$

$$=\frac{\left[1-\sin\left(\frac{\pi x}{180}\right)\right]\cdot\left[0+\cos\left(\frac{\pi x}{180}\right)\cdot\frac{d}{dx}\left(\frac{\pi x}{180}\right)\right]-\left[1+\sin\left(\frac{\pi x}{180}\right)\right]\cdot\left[0-\cos\left(\frac{\pi x}{180}\right)\cdot\frac{d}{dx}\left(\frac{\pi x}{180}\right)\right]}{\left[1-\sin\left(\frac{\pi x}{180}\right)\right]^2}$$

$$=\frac{(1-\sin x^\circ)\left[(\cos x^\circ)\times\frac{\pi}{180}\times 1\right]-(1+\sin x^\circ)\left[(-\cos x^\circ)\times\frac{\pi}{180}\times 1\right]}{(1-\sin x^\circ)^2}$$

$$=\frac{\frac{\pi}{180}\cos x^\circ(1-\sin x^\circ+1+\sin x^\circ)}{(1-\sin x^\circ)^2}$$

$$=\frac{\pi\cos x^\circ}{90(1-\sin x^\circ)^2}.$$

(ix) $\cot\left(\frac{\log x}{2}\right)-\log\left(\frac{\cot x}{2}\right)$
Solution:
Let $y=\cot\left(\frac{\log x}{2}\right)-\log\left(\frac{\cot x}{2}\right)$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\cot\left(\frac{\log x}{2}\right)-\log\left(\frac{\cot x}{2}\right)\right]$$

$$=\frac{d}{dx}\left[\cot\left(\frac{\log x}{2}\right)\right]-\frac{d}{dx}\left[\log\left(\frac{\cot x}{2}\right)\right]$$

$$=-\operatorname{cosec}^2\left(\frac{\log x}{2}\right)\cdot\frac{d}{dx}\left(\frac{\log x}{2}\right)-\frac{1}{\left(\frac{\cot x}{2}\right)}\cdot\frac{d}{dx}\left(\frac{\cot x}{2}\right)$$

$$=-\operatorname{cosec}^2\left(\frac{\log x}{2}\right)\times\frac{1}{2}\times\frac{1}{x}-\frac{2}{\cot x}\times\frac{1}{2}\times(-\operatorname{cosec}^2 x)$$

$$=-\frac{\operatorname{cosec}^2\left(\frac{\log x}{2}\right)}{2x}+\tan x\cdot\operatorname{cosec}^2 x.$$

(x) $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$
Solution:

$$\text{Let } y=\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}=\frac{e^{2x}-\frac{1}{e^{2x}}}{e^{2x}+\frac{1}{e^{2x}}}=\frac{e^{4x}-1}{e^{4x}+1}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{e^{4x}-1}{e^{4x}+1}\right)$$

$$=\frac{(e^{4x}+1)\cdot\frac{d}{dx}(e^{4x}-1)-(e^{4x}-1)\cdot\frac{d}{dx}(e^{4x}+1)}{(e^{4x}+1)^2}$$

$$=\frac{(e^{4x}+1)\left[e^{4x}\cdot\frac{d}{dx}(4x)-0\right]-(e^{4x}-1)\left[e^{4x}\cdot\frac{d}{dx}(4x)+0\right]}{(e^{4x}+1)^2}$$

$$=\frac{(e^{4x}+1)\cdot e^{4x}\times 4-(e^{4x}-1)\cdot e^{4x}\times 4}{(e^{4x}+1)^2}$$

$$=\frac{4e^{4x}(e^{4x}+1-e^{4x}+1)}{(e^{4x}+1)^2}=\frac{8e^{4x}}{(e^{4x}+1)^2}.$$

(xi) $e^{x\surd+1}e^{x\surd-1}$
Solution:
let y = $e^{x\surd+1}e^{x\surd-1}$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{e^{\sqrt{x}}+1}{e^{\sqrt{x}}-1}\right)$$

$$=\frac{(e^{\sqrt{x}}-1)\frac{d}{dx}(e^{\sqrt{x}}+1)-(e^{\sqrt{x}}+1)\frac{d}{dx}(e^{\sqrt{x}}-1)}{(e^{\sqrt{x}}-1)^2}$$

$$=\frac{(e^{\sqrt{x}}-1)\left[e^{\sqrt{x}}\cdot\frac{d}{dx}(\sqrt{x})+0\right]-(e^{\sqrt{x}}+1)\left[e^{\sqrt{x}}\cdot\frac{d}{dx}(\sqrt{x})-0\right]}{(e^{\sqrt{x}}-1)^2}$$

$$=\frac{(e^{\sqrt{x}}-1)\left[e^{\sqrt{x}}\times\frac{1}{2\sqrt{x}}\right]-(e^{\sqrt{x}}+1)\left[e^{\sqrt{x}}\times\frac{1}{2\sqrt{x}}\right]}{(e^{\sqrt{x}}-1)^2}$$

$$=\frac{\frac{e^{\sqrt{x}}}{2\sqrt{x}}(e^{\sqrt{x}}-1-e^{\sqrt{x}}-1)}{(e^{\sqrt{x}}-1)^2}$$

$$=\frac{-e^{\sqrt{x}}}{\sqrt{x}(e^{\sqrt{x}}-1)^2}.$$

(xii) log[tan^{3}x·sin^{4}x·(x^2 + 7)^7]
Solution:
Let y = log [tan^{3}x·sin^{4}x·(x^2 + 7)^7]
= log tan^{3}x + log sin^{4}x + log (x^2 + 7)^7
= 3 log tan x + 4 log sin x + 7 log (x^2 + 7)
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[3\log\tan x+4\log\sin x+7\log(x^2+7)]$$

$$=3\frac{d}{dx}(\log\tan x)+4\frac{d}{dx}(\log\sin x)+7\frac{d}{dx}[\log(x^2+7)]$$

$$=3\times\frac{1}{\tan x}\cdot\frac{d}{dx}(\tan x)+4\times\frac{1}{\sin x}\cdot\frac{d}{dx}(\sin x)+$$

$$7\times\frac{1}{x^2+7}\cdot\frac{d}{dx}(x^2+7)$$

$$=3\times\frac{1}{\tan x}\cdot\sec^2 x+4\times\frac{1}{\sin x}\cdot\cos x+7\times\frac{1}{x^2+7}\cdot(2x+0)$$

$$=3\times\frac{\cos x}{\sin x}\times\frac{1}{\cos^2 x}+4\cot x+\frac{14x}{x^2+7}$$

$$=\frac{6}{2\sin x\cos x}+4\cot x+\frac{14x}{x^2+7}$$

$$=\frac{6}{\sin 2x}+4\cot x+\frac{14x}{x^2+7}$$

= 6cosec2x + 4 cotx + 14xx2+7

(xiii) log(1−cos3x1+cos3x−−−−−−√)

Solution:

$$\text{Let } y=\log\left(\sqrt{\frac{1-\cos 3x}{1+\cos 3x}}\right)$$

$$=\log\left(\sqrt{\frac{2\sin^2\left(\frac{3x}{2}\right)}{2\cos^2\left(\frac{3x}{2}\right)}}\right)$$

$$=\log\tan\left(\frac{3x}{2}\right)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\log\tan\left(\frac{3x}{2}\right)\right]$$

$$=\frac{1}{\tan\left(\frac{3x}{2}\right)}\times\frac{d}{dx}\left[\tan\left(\frac{3x}{2}\right)\right]$$

$$=\frac{1}{\tan\left(\frac{3x}{2}\right)}\times\sec^2\left(\frac{3x}{2}\right)\cdot\frac{d}{dx}\left(\frac{3x}{2}\right)$$

$$=\frac{\cos\left(\frac{3x}{2}\right)}{\sin\left(\frac{3x}{2}\right)}\times\frac{1}{\cos^2\left(\frac{3x}{2}\right)}\times\frac{3}{2}\times 1$$

$$=3\times\frac{1}{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)}$$

$$=3\times\frac{1}{\sin 3x}=3\operatorname{cosec}3x.$$

(xiv) log(⎷ 1+cos(5x2)1−cos(5x2))−−−−−−−−− √

Solution:

Using log(ab) = log a – log b

log a^b = b log a

$$y = \log\left(\sqrt{1+\cos\frac{5x}{2}}\right) - \log\left(\sqrt{1-\cos\left(\frac{5x}{2}\right)}\right)$$

$$y = \log\left(1+\cos\left(\frac{x}{2}\right)^{\frac{1}{2}} - \log\left(1-\cos\left(\frac{5x}{2}\right)\right)\right)^{\frac{1}{2}}$$

$$y = \frac{1}{2}\log\left[1+\cos\left(\frac{5x}{2}\right)\right] - \frac{1}{2}\log\left[\left(1-\cos\left(\frac{5x}{2}\right)\right]\right.$$

Differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2}\frac{1}{1+\cos\left(\frac{5x}{2}\right)}\frac{d}{dx}\left(1+\cos\frac{5x}{2}\right) - \frac{1}{2} \times \frac{1}{1-\cos\left(\frac{5x}{2}\right)}\frac{d}{dx}\left(1-\cos\frac{5x}{2}\right)$$

$$= \frac{1}{2\left(1+\cos\left(\frac{5x}{2}\right)\right)}\left(-\sin\left(\frac{5x}{2}\right).\frac{5}{2} - \frac{1}{2\left(1-\cos\left(\frac{5x}{2}\right)\right)}\left(\sin\left(\frac{5x}{2}\right).\frac{5}{2}\right.\right.$$

$$= \frac{-5\sin\left(\frac{5x}{2}\right)}{4\left(1+\cos\left(\frac{5x}{2}\right)\right)} - \frac{5\sin\left(\frac{5x}{2}\right)}{4\left(1-\cos\left(\frac{5x}{2}\right)\right)}$$

$$= \frac{-5}{4}\sin\left(\frac{5x}{2}\right)\left[\frac{1}{1+\cos\left(\frac{5x}{2}\right)} + \frac{1}{1-\cos\left(\frac{5x}{2}\right)}\right]$$

$$= \frac{\frac{-5}{2}\sin\left(\frac{5x}{2}\right)\left[1-\cos\left(\frac{5x}{2}\right)+1+\cos\frac{5x}{2}\right]}{\left[1-\cos^2\left(\frac{5x}{2}\right)\right]}$$

$$= \frac{-5}{4}\sin\left(\frac{5x}{2}\right) \times \frac{2}{\sin^2\left(\frac{5x}{2}\right)}$$ ...[∵ 1 – $\cos^2 x$ = $\sin^2 x$]

$$= \frac{-5}{4}\frac{1}{\sin\left(\frac{5x}{2}\right)}$$

$$-\frac{5}{2} \times \text{cosecx}$$

−52cosec(5x2)

(xv) log(1−sinx1+sinx−−−−−√)

Solution:

$$\text{Let } y = \log\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)$$

$$= \log\left(\sqrt{\frac{1-\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}}\right)$$

$$= \log\left(\sqrt{\frac{(1-\sin x)^2}{1-\sin^2 x}}\right)$$

$$= \log\left(\sqrt{\frac{(1-\sin x)^2}{\cos^2 x}}\right)$$

$$= \log\left(\frac{1-\sin x}{\cos x}\right)$$

$$= \log\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right)$$

$$= \log(\sec x - \tan x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\log(\sec x-\tan x)]$$

$$=\frac{1}{\sec x-\tan x}\cdot\frac{d}{dx}(\sec x-\tan x)$$

$$=\frac{1}{\sec x-\tan x}\times(\sec x\tan x-\sec^2 x)$$

$$=\frac{-\sec x(\sec x-\tan x)}{\sec x-\tan x}$$

$$=-\sec x.$$

(xvi) log[42x(x2+52x3−4√)32]

Solution:

$$\text{Let } y=\log\left[4^{2x}\left(\frac{x^2+5}{\sqrt{2x^3-4}}\right)^{\frac{3}{2}}\right]$$

$$=\log 4^{2x}+\log\left(\frac{x^2+5}{\sqrt{2x^3-4}}\right)^{\frac{3}{2}}$$

$$=2x\log 4+\frac{3}{2}\log\left(\frac{x^2+5}{\sqrt{2x^3-4}}\right)$$

$$=2x\log 4+\frac{3}{2}[\log(x^2+5)-\log(2x^3-4)^{\frac{1}{2}}]$$

$$=2x\log 4+\frac{3}{2}[\log(x^2+5)-\frac{1}{2}\log(2x^3-4)]$$

$$=2x\log 4+\frac{3}{2}\log(x^2+5)-\frac{3}{4}\log(2x^3-4)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[2x\log 4+\frac{3}{2}\log(x^2+5)-\frac{3}{4}\log(2x^3-4)]$$

$$=(2\log 4)\frac{d}{dx}(x)+\frac{3}{2}\frac{d}{dx}[\log(x^2+5)]-\frac{3}{4}\frac{d}{dx}[\log(2x^3-4)]$$

$$=(2\log 4)\times 1+\frac{3}{2}\times\frac{1}{x^2+5}\cdot\frac{d}{dx}(x^2+5)-$$

$$\frac{3}{4}\times\frac{1}{2x^3-4}\cdot\frac{d}{dx}(2x^3-4)$$

$$=2\log 4+\frac{3}{2(x^2+5)}\times(2x+0)-\frac{3}{4(2x^3-4)}\times(2\times 3x^2-0)$$

$$=2\log 4+\frac{3x}{x^2+5}-\frac{9x^2}{2(2x^3-4)}.$$

(xvii) log[ex2(5−4x)327−6x√3]

Solution:

Let y = $\log\left[\frac{e^{x^2}(5-4x)^{\frac{3}{2}}}{\sqrt[3]{7-6x}}\right]$

Using

log(A.B) = logA + logB

y = $\log e^{x^2} + \log\left(\frac{(5-4x)^{\frac{3}{2}}}{\sqrt[3]{7-6x}}\right)$

= $\log e^{x^2} + \log(5-4x)^{\frac{3}{2}} - \log\left(\sqrt[3]{7-6x}\right)$

= $x^2 \log e + \frac{3}{2}\log(5-4x) - \log(7-6x)^{\frac{1}{3}}$

= $x^2 + \frac{3}{2}\log(5-4x) - \frac{1}{3}\log(7-6x)$

Now,

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}x^2 + \frac{3}{2}\frac{d}{dx}\log(5-4x) - \frac{1}{3}\frac{d}{dx}\log(7-6x)$

= $2x + \frac{3}{2}\frac{1}{5-4x}(-4) - \frac{1}{3}\frac{1}{(7-6x)}x(-6)$

= $2x - \frac{6}{(5-4x)} + \frac{2}{(7-6x)}$

$2x - \frac{6}{5-4x} + \frac{2}{7-6x}.$

(xviii) log[$a_{cosx}(x_2-3)_3 logx$]

Solution:

$$\text{Let } y = \log\left[\frac{a^{\cos x}}{(x^2-3)^3 \log x}\right]$$

$$= \log a^{\cos x} - \log(x^2-3)^3 - \log(\log x)$$

$$= (\cos x)(\log a) - 3\log(x^2-3) - \log(\log x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[(\cos x)(\log a) - 3\log(x^2-3) - \log(\log x)]$$

$$= (\log a)\cdot\frac{d}{dx}(\cos x) - 3\frac{d}{dx}[\log(x^2-3)] - \frac{d}{dx}[\log(\log x)]$$

$$= (\log a)(-\sin x) - 3\times\frac{1}{x^2-3}\cdot\frac{d}{dx}(x^2-3) - \frac{1}{\log x}\cdot\frac{d}{dx}(\log x)$$

$$= -(\sin x)(\log a) - \frac{3}{x^2-3}\times(2x-0) - \frac{1}{\log x}\times\frac{1}{x}$$

$$= -(\sin x)(\log a) - \frac{6x}{x^2-3} - \frac{1}{x\log x}.$$

(xix) y= (25)log5(secx) − (16)log4(tanx)

Solution:

y = (25)log5(secx) − (16)log4(tanx)

= 52log5(secx) – 42log4(tanx)

= 5log5(sec5x) – 4log4(tan2x)

= sec2x – tan2x ... [∵ = x]

∴ y = 1

Differentiating w.r.t. x, we get

$dydx$ = ddx(1) = 0

(xx) $\frac{(x^2+2)^4}{\sqrt{x^2+5}}$

Solution:

Let y = $\frac{(x^2+2)^4}{\sqrt{x^2+5}}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{(x^2+2)^4}{\sqrt{x^2+5}}\right]$$

$$=\frac{\sqrt{x^2+5}\cdot\frac{d}{dx}(x^2+2)^4-(x^2+2)^4\cdot\frac{d}{dx}(\sqrt{x^2+5})}{(\sqrt{x^2+5})^2}$$

$$=\frac{\sqrt{x^2+5}\times 4(x^2+2)^3\cdot\frac{d}{dx}(x^2+2)-(x^2+2)^4\times\frac{1}{2\sqrt{x^2+5}}\cdot\frac{d}{dx}(x^2+5)}{x^2+5}$$

$$=\frac{\sqrt{x^2+5}\times 4(x^2+2)^3\cdot(2x+0)-\frac{(x^2+2)^4}{2\sqrt{x^2+5}}\times(2x+0)}{x^2+5}$$

$$=\frac{8x(x^2+5)(x^2+2)^3-x(x^2+2)^4}{(x^2+5)^{\frac{3}{2}}}$$

$$=\frac{x(x^2+2)^3[8(x^2+5)-(x^2+2)]}{(x^2+5)^{\frac{3}{2}}}$$

$$=\frac{x(x^2+2)^3(8x^2+40-x^2-2)}{(x^2+5)^{\frac{3}{2}}}$$

$$=\frac{x(x^2+2)^3(7x^2+38)}{(x^2+5)^{\frac{3}{2}}}.$$

Question 4.

A table of values of f, g, f ' and g' is given

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	1	6	−3	4
4	3	4	5	−6
6	5	2	−4	7

(i) If r(x) = f [g(x)] find r' (2).

Solution:

r(x) = f[g(x)]

∴ r'(x) = $\frac{d}{dx}$f[g(x)]

= f'[g(x)]·$\frac{d}{dx}$[g(x)]

= f'[g(x)·[g'(x)]

∴ r'(2) = f'[g(2)]·g'(2)

= f'(6)·g'(2) … [∵ g(x) = 6, when x = 2]

= -4 × 4 … [From the table]

= -16.

(ii) If R(x) = g[3 + f(x)] find R' (4).

Solution:

R(x) = g[3 + f(x)]

∴ R'(x) = $\frac{d}{dx}${g[3+f(x)]}

= g'[3 + f(x)]·$\frac{d}{dx}$[3 + f(x)]

= g'[3 +f(x)]·[0 + f'(x)]

= g'[3 + f(x)]·f'(x)

∴ R'(4) = g'[3 + f(4)]·f'(4)

= g'[3 + 3]·f'(4) … [∵ f(x) = 3, when x = 4]

= g'(6)·f'(4)

= 7 × 5 … [From the table]

= 35.

(iii) If s(x) = f[9− f(x)] find s′ (4).
Solution:
s(x) = f[9− f(x)]
∴ s'(x) = $\frac{d}{dx}${f[9 – f(x)]}
= f'[9 – f(x)]·$\frac{d}{dx}$[0 – f(x)]
= f'[9 – f(x)]·[0 – f'(x)]
= -f'[9 – f(x)] – f'(x)
∴ s'(4) = -f'[9 – f(4)] – f'(4)
= -f'[9 – 3] – f'(4) … [∵ f(x) = 3, when x = 4]
= -f'(6) – f'(4)
= -(-4)(5) … [From the table]
= 20.

(iv) If S(x) = g[g(x)] find S′ (6)
Solution:
S(x) = g[g(x)]
∴ S'(x) = $\frac{d}{dx}$g[g(x)]
= g'[g(x)]·$\frac{d}{dx}$[g(x)]
= g'[g(x)]·g'(x)
∴ S ′(6) = g'[g'(6)]·g'(6)
= g'(2)·g'(6) … [∵ g (x) = 2, when x = 6]
= 4 × 7 … [From the table]
= 28.

Question 5.
Assume that f ′(3) = -1, g'(2) = 5, g(2) = 3 and y = f[g(x)] then $\left[\frac{dy}{dx}\right]_{x=2}$ = ?
Solution:
y = f[g(x)]
∴ $\frac{dy}{dx}$ = $\frac{d}{dx}${[g(x)]}

$$= f'[g(x)] \cdot \frac{d}{dx}[g(x)]$$

$$= f'[g(x)] \cdot g'(x)$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=2} = f'[g(2)] \cdot g'(2)$$

$$= f'(3) \cdot g'(2) \quad \ldots [\because g(2) = 3]$$

$$= -1 \times 5 \quad \ldots \text{(Given)}$$

$$= -5.$$

Question 6.
If h(x) = $\sqrt{4f(x)+3g(x)}$, f(1) = 4, g(1) = 3, f ′(1) = 3, g'(1) = 4 find h'(1).
Solution:

Given f(1) = 4, g(1) = 3, f '(1) = 3, g'(1) = 4 .....(1)

Now, $h(x)=\sqrt{4f(x)+3g(x)}$

$$\therefore\ h'(x)=\frac{d}{dx}[\sqrt{4f(x)+3g(x)}]$$

$$=\frac{1}{2\sqrt{4f(x)+3g(x)}}\cdot\frac{d}{dx}[4f(x)+3g(x)]$$

$$=\frac{1}{2\sqrt{4f(x)+3g(x)}}\times[4f'(x)+3g'(x)]$$

$$\therefore\ h'(1)=\frac{1}{2\sqrt{4f(1)+3g(1)}}\times[4f'(1)+3g'(1)]$$

$$=\frac{1}{2\sqrt{4\times4+3\times3}}\times[4\times3+3\times4]\quad\ldots\text{[By (1)]}$$

$$=\frac{1}{2\sqrt{25}}\times24$$

$$=\frac{1}{2\times5}\times24=\frac{12}{5}.$$

Question 7.
Find the x co-ordinates of all the points on the curve y = sin 2x – 2 sin x, 0 ≤ x < 2π where $\frac{dy}{dx}$ = 0.
Solution:
y = sin 2x – 2 sin x, 0 ≤ x < 2π

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}(\sin 2x-2\sin x)$$

$$=\frac{d}{dx}(\sin 2x)-2\frac{d}{dx}(\sin x)$$

$$=\cos 2x\cdot\frac{d}{dx}(2x)-2\cos x$$

= cos2x × 2 – 2cosx
= 2 (2 $\cos^2$x – 1) – 2 cosx
= 4 $\cos^2$x – 2 – 2 cos x
= 4 $\cos^2$x – 2 cos x – 2
If $\frac{dy}{dx}$ = 0, then 4 $\cos^2$x – 2 cos x – 2 = 0
∴ 4$\cos^2$x – 4cosx + 2cosx – 2 = 0
∴ 4 cosx (cosx – 1) + 2 (cosx – 1) = 0
∴ (cosx – 1)(4cosx + 2) = 0
∴ cosx – 1 = 0 or 4cosx + 2 = 0
∴ cos x = 1 or cos x = $-\frac{1}{2}$
∴ cos x = cos 0

$$\text{or}\quad \cos x=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)=\frac{\cos 2\pi}{3}$$

$$\text{or}\quad \cos x=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\frac{4\pi}{3}$$

$$\ldots[\because\ 0\leqslant x<2\pi]$$

$$\therefore\ x=0\quad\text{or}\quad x=\frac{2\pi}{3}\quad\text{or}\quad x=\frac{4\pi}{3}.$$

$$\therefore\ x=0\quad\text{or}\quad\frac{2\pi}{3}\quad\text{or}\quad\frac{4\pi}{3}.$$

Question 8.
Select the appropriate hint from the hint basket and fill up the blank spaces in the following paragraph. [Activity]
"Let f (x) = x^2 + 5 and g(x) = e^x + 3 then
f [g(x)] = _ _ _ _ _ _ _ _ and g [f(x)] =_ _ _ _ _ _ _ _.
Now f '(x) = _ _ _ _ _ _ _ _ and g'(x) = _ _ _ _ _ _ _ _.
The derivative off [g (x)] w. r. t. x in terms of f and g is _ _ _ _ _ _ _ _.

Therefore $\frac{d}{dx}$[f[g(x)]] = _ _ _ _ _ _ _ _ _ and [$\frac{d}{dx}$[f[g(x)]]]$_{x = 0}$ = _ _ _ _ _ _ _ _ _ _ _.
The derivative of g[f(x)] w. r. t. x in terms of f and g is _ _ _ _ _ _ _ _ _ _ _.
Therefore $\frac{d}{dx}$[g[f(x)]] = _ _ _ _ _ _ _ _ _ and [$\frac{d}{dx}$[g[f(x)]]]$_{x = 1}$ = _ _ _ _ _ _ _ _ _ _ _."
Hint basket : { f '[g(x)]·g'(x), $2e^{2x} + 6e^x$, 8, g'[f(x)]·f '(x), $2xe^{x^2 + 5}$, $-2e^6$, $e^{2x} + 6e^x + 14$, $e^{x^2 + 5} + 3$, 2x, e^x}
Solution:
f[g(x)] = $e^{2x} + 6e^x + 14$
g[f(x)] = $e^{x^2 + 5} + 3$
f'(x) = 2x, g'f(x) = e^x
The derivative of f[g(x)] w.r.t. x in terms of and g is f'[g(x)]·g'(x).
∴ $\frac{d}{dx}${f[g(x)]} = $2e^{2x} + 6e^x$ and $\frac{d}{dx}${f[g(x)]}$_{x = 0}$ = 8
The derivative of g[f(x)] w.r.t. x in terms of f and g is g'f(x)]·f'(x).
∴ $\frac{d}{dx}${g[(f(x)]} = $2xe^{x^2 + 5}$ and
$\frac{d}{dx}${g[(f(x)]}$_{x = -1}$ = $-2e^6$.

Maharashtra State Board 12th Maths Solutions Chapter 1 Differentiation Ex 1.2

Question 1.
Find the derivative of the function y = f(x) using the derivative of the inverse function x = f^{-1}(y) in the following
(i) y = $\sqrt{x}$
Solution:
y = $\sqrt{x}$ ... (1)
We have to find the inverse function of y = f(x), i.e. x in terms of y.
From (1),
y^2 = x ∴ x = y^2

$$\therefore\ x = f^{-1}(y) = y^2$$

$$\therefore\ \frac{dx}{dy} = \frac{d}{dy}(y^2) = 2y$$

$$= 2\sqrt{x} \qquad \ldots \text{ [By (1)]}$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{2\sqrt{x}}.$$

(ii) y = $\sqrt{2-\sqrt{x}}$
Solution:
y = $\sqrt{2-\sqrt{x}}$ ...(1)
We have to find the inverse function of y = f(x), i.e. x in terms of y.

From (1),

$$y^2 = 2 - \sqrt{x} \qquad \therefore \sqrt{x} = 2 - y^2$$

$$\therefore x = (2 - y^2)^2$$

$$\therefore x = f^{-1}(y) = (2 - y^2)^2$$

$$\therefore \frac{dx}{dy} = \frac{d}{dy}(2 - y^2)^2$$

$$= 2(2 - y^2) \cdot \frac{d}{dy}(2 - y^2)$$

$$= 2(2 - y^2) \cdot (0 - 2y)$$

$$= -4y(2 - y^2)$$

$$= -4\sqrt{2 - \sqrt{x}}(2 - 2 + \sqrt{x}) \qquad \ldots \text{[By (1)]}$$

$$= -4\sqrt{x}\sqrt{2 - \sqrt{x}}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = -\frac{1}{4\sqrt{x}\sqrt{2 - \sqrt{x}}}.$$

(iii) y = $\sqrt[3]{x-2}$

Solution:

y = $\sqrt[3]{x-2}$ ....(1)

We have to find the inverse function of y = f(x), i.e. x in terms of y.

From (1),

$$y^3 = x - 2 \qquad \therefore x = y^3 + 2$$

$$\therefore x = f^{-1}(y) = y^3 + 2$$

$$\therefore \frac{dx}{dy} = \frac{d}{dy}(y^3 + 2)$$

$$= 3y^2 + 0 = 3y^2$$

$$= 3(\sqrt[3]{(x-2)})^2 \qquad \ldots \text{[By (1)]}$$

$$= 3(x-2)^{\frac{2}{3}} = 3 \cdot (\sqrt[3]{(x-2)^2})$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{3\sqrt[3]{(x-2)^2}}, \; x > 2.$$

(iv) y = log (2x – 1)

Solution:

y = log (2x – 1) ...(1)

We have to find the inverse function of y = f(x), i.e. x in terms of y.

From (1),

$$2x-1=e^{y} \qquad \therefore\ 2x=e^{y}+1$$

$$\therefore\ x=f^{-1}(y)=\frac{1}{2}(e^{y}+1)$$

$$\therefore\ \frac{dx}{dy}=\frac{1}{2}\frac{d}{dy}(e^{y}+1)$$

$$=\frac{1}{2}(e^{y}+0)=\frac{1}{2}e^{y}$$

$$=\frac{1}{2}e^{\log(2x-1)} \qquad \ldots\ [\text{By (1)}]$$

$$=\frac{1}{2}(2x-1) \qquad \ldots\ [\because\ e^{\log x}=x]$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{\left(\frac{dx}{dy}\right)}=\frac{2}{2x-1}.$$

(v) y = 2x + 3
Solution:
y = 2x + 3 ....(1)
We have to find the inverse function of y = f(x), i.e. x in terms of y.
From (1),

$$2x=y-3 \qquad \therefore\ x=\frac{y-3}{2}$$

$$\therefore\ x=f^{-1}(y)=\frac{y-3}{2}$$

$$\therefore\ \frac{dx}{dy}=\frac{1}{2}\frac{d}{dy}(y-3)$$

$$=\frac{1}{2}(1-0)=\frac{1}{2}$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{\left(\frac{dx}{dy}\right)}=\frac{1}{\left(\frac{1}{2}\right)}=2.$$

(vi) y = e^x – 3
Solution:
y = e^x – 3 ....(1)
We have to find the inverse function of y = f(x), i.e. x in terms of y.
From (1),
e^x = y + 3
∴ x = log(y + 3)
∴ x = $f^{-1}(y)$ = log(y + 3)

$$\therefore\ \frac{dx}{dy}=\frac{d}{dy}[\log(y+3)]$$

$$=\frac{1}{y+3}\cdot\frac{d}{dy}(y+3)$$

$$=\frac{1}{y+3}\cdot(1+0)=\frac{1}{y+3}$$

$$=\frac{1}{e^{x}-3+3} \qquad \ldots\ [\text{By (1)}]$$

$$=\frac{1}{e^{x}}$$

$$\therefore\ \frac{dy}{dx}=\frac{1}{\left(\frac{dx}{dy}\right)}=\frac{1}{\left(\frac{1}{e^{x}}\right)}=e^{x}.$$

(vii) $y = e^{2x-3}$
Solution:
$y = e^{2x-3}$ ....(1)
We have to find the inverse function of y = f(x), i.e. x in terms of y.
From (1),
$2x - 3 = \log y \therefore 2x = \log y + 3$

$$\therefore x = f^{-1}(y) = \frac{1}{2}(\log y + 3)$$

$$\therefore \frac{dx}{dy} = \frac{1}{2}\frac{d}{dy}(\log y + 3)$$

$$= \frac{1}{2}\left(\frac{1}{y} + 0\right) = \frac{1}{2y}$$

$$= \frac{1}{2e^{2x-3}} \quad \ldots \text{ [By (1)]}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{\left(\frac{1}{2e^{2x-3}}\right)} = 2e^{2x-3}.$$

(viii) $y = \log_2\left(\frac{x}{2}\right)$
Solution:
$y = \log_2\left(\frac{x}{2}\right)$ ...(1)
We have to find the inverse function of y = f(x), i.e. x in terms of y.
From (1),
$\frac{x}{2} = 2^y \therefore x = 2 \cdot 2^y = 2^{y+1}$
$\therefore x = f^{-1}(y) = 2^{y+1}$

$$\therefore \frac{dx}{dy} = \frac{d}{dy}(2^{y+1})$$

$$= 2^{y+1} \cdot \log 2 \cdot \frac{d}{dy}(y+1)$$

$$= 2^{y+1} \cdot \log 2 \cdot (1 + 0)$$

$$= 2^{y+1} \cdot \log 2 = 2^{\log_2\left(\frac{x}{2}\right)+1} \cdot \log 2 \quad \ldots \text{ [By (1)]}$$

$$= 2^{\log_2\left(\frac{x}{2}\right) + \log_2 2} \cdot \log 2$$

$$= 2^{\log_2\left(\frac{x}{2} \times 2\right)} \cdot \log 2 = 2^{\log_2 x} \cdot \log 2$$

$$= x \log 2 \quad \ldots \left[\because a^{\log_a x} = x\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{x \log 2}.$$

Question 2.
Find the derivative of the inverse function of the following
(i) $y = x^2 \cdot e^x$
Solution:
$y = x^2 \cdot e^x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x^2\cdot e^x)$$

$$=x^2\frac{d}{dx}(e^x)+e^x\frac{d}{dx}(x^2)$$

$$=x^2\cdot e^x+e^x\times 2x$$

$$=xe^x(x+2)$$

The derivative of inverse function of $y=f(x)$ is given by

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{xe^x(x+2)}.$$

(ii) y = x cos x
Solution:
y = x cos x
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x\cos x)$$

$$=x\frac{d}{dx}(\cos x)+\cos x\frac{d}{dx}(x)$$

$$=x(-\sin x)+\cos x\times 1$$

$$=\cos x-x\sin x$$

The derivative of inverse function of $y=f(x)$ is given by

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{\cos x-x\sin x}.$$

(iii) $y = x\cdot 7^x$
Solution:
$y = x\cdot 7^x$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x\cdot 7^x)$$

$$=x\frac{d}{dx}(7^x)+7^x\frac{d}{dx}(x)$$

$$=x\cdot 7^x\log 7+7^x\times 1$$

$$=7^x(x\log 7+1)$$

The derivative of inverse function of $y=f(x)$ is given by

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{7^x(x\log 7+1)}.$$

(iv) $y = x^2 + \log x$
Solution:
$y = x^2 + \log x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x^2+\log x)$$
$$=\frac{d}{dx}(x^2)+\frac{d}{dx}(\log x)$$
$$=2x+\frac{1}{x}=\frac{2x^2+1}{x}$$

The derivative of inverse function of $y=f(x)$ is given by

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{\left(\frac{2x^2+1}{x}\right)}=\frac{x}{2x^2+1}.$$

(v) y = x logx
Solution:
y = x logx
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(x\log x)$$
$$=x\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(x)$$
$$=x\times\frac{1}{x}+(\log x)\times 1$$
$$=1+\log x$$

The derivative of inverse function of $y=f(x)$ is given by

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{1+\log x}.$$

Question 3.
Find the derivative of the inverse of the following functions, and also fid their value at the points indicated against them.
(i) $y = x^5 + 2x^3 + 3x$, at x = 1
Solution:
$y = x^5 + 2x^3 + 3x$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}(x^5 + 2x^3 + 3x)$
$= 5x^4 + 2 \times 3x^2 + 3 \times 1$
$= 5x^4 + 6x^2 + 3$
The derivative of inverse function of y = f(x) is given by

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{5x^4+6x^2+3}$$

At $x=1$, $\frac{dx}{dy}=\frac{1}{(5x^4+6x^2+3)_{\text{at } x=1}}$

$$=\frac{1}{5(1)^4+6(1)^2+3}$$
$$=\frac{1}{5+6+3}=\frac{1}{14}.$$

(ii) $y = e^x + 3x + 2$, at x = 0
Solution:
$y = e^x + 3x + 2$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}(e^x + 3x + 2)$
The derivative of inverse function of y = f(x) is given by

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{e^x+3}$$

At $x=0$, $\frac{dx}{dy}=\frac{1}{(e^x+3)_{\text{at } x=0}}$

$$=\frac{1}{e^0+3}=\frac{1}{1+3}=\frac{1}{4}.$$

(iii) $y = 3x^2 + 2 \log x^3$, at $x = 1$
Solution:
$y = 3x^2 + 2 \log x^3$
$= 3x^2 + 6 \log x$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(3x^2+6\log x)$$

$$=3\frac{d}{dx}(x^2)+6\frac{d}{dx}(\log x)$$

$$=3\times 2x+6\times\frac{1}{x}=6x+\frac{6}{x}$$

$$=\frac{6x^2+6}{x}$$

The derivative of inverse function of $y = f(x)$ is given by

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{\left(\frac{6x^2+6}{x}\right)}$$

$$=\frac{x}{6x^2+6}$$

At $x=1$, $\frac{dx}{dy}=\left(\frac{x}{6x^2+6}\right)_{\text{at } x=1}$

$$=\frac{1}{6(1)^2+6}=\frac{1}{12}.$$

(iv) $y = \sin (x - 2) + x^2$, at $x = 2$
Solution:
$y = \sin (x - 2) + x^2$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\sin(x-2)+x^2]$$

$$=\frac{d}{dx}[\sin(x-2)]+\frac{d}{dx}(x^2)$$

$$=\cos(x-2).\frac{d}{dx}(x-2)+2x$$

$$=\cos(x-2).(1-0)+2x$$

$$=\cos(x-2)+2x$$

The derivative of inverse function of $y=f(x)$ is given by

$$\frac{dx}{dy}=\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{\cos(x-2)+2x}$$

At $x=2$, $\frac{dx}{dy}=\left(\frac{1}{[\cos(x-2)+2x]}\right)_{\text{at } x=2}$

$$=\frac{1}{\cos 0+2(2)}=\frac{1}{1+4}=\frac{1}{5}.$$

Question 4.
If $f(x) = x^3 + x - 2$, find $(f^{-1})'(0)$.
Question is modified.
If $f(x) = x^3 + x - 2$, find $(f^{-1})'(-2)$.
Solution:
$f(x) = x^3 + x - 2$ ....(1)
Differentiating w.r.t. x, we get

$$f'(x) = \frac{d}{dx}(x^3 + x - 2)$$

$$= 3x^2 + 1 - 0 = 3x^2 + 1$$

We know that

$$(f^{-1})'(y) = \frac{1}{f'(x)} \quad \dots (2)$$

From (1), $y = f(x) = -2$, when $x = 0$

$$\therefore \text{ from (2), } (f^{-1})'(-2) = \frac{1}{f'(0)} = \frac{1}{(3x^2+1)_{\text{at } x=0}}$$

$$= \frac{1}{3(0)+1} = 1.$$

Question 5.
Using derivative prove
(i) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$
Solution:
let $f(x) = \tan^{-1}x + \cot^{-1}x$
Differentiating w.r.t. x, we get

$$f'(x) = \frac{d}{dx}(\tan^{-1}x + \cot^{-1}x)$$

$$= \frac{d}{dx}(\tan^{-1}x) + \frac{d}{dx}(\cot^{-1}x)$$

$$= \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0$$

Since, $f'(x) = 0$, $f(x)$ is a constant function.
Let $f(x) = k$.
For any value of x, $f(x) = k$
Let $x = 0$.
Then $f(0) = k$ ....(2)
From (1), $f(0) = \tan^{-1}(0) + \cot^{-1}(0)$
$= 0 + \frac{\pi}{2} = \frac{\pi}{2}$

$$\therefore k = \frac{\pi}{2} \quad \dots \text{[By (2)]}$$

$$\therefore f(x) = k = \frac{\pi}{2}$$

$$\text{Hence, } \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}. \quad \dots \text{[By (1)]}$$

(ii) $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}$ . . . [for $|x| \geq 1$]
Solution:
Let $f(x) = \sec^{-1}x + \text{cosec}^{-1}x$ for $|x| \geq 1$ ....(1)
Differentiating w.r.t. x, we get

$$f'(x) = \frac{d}{dx}(\sec^{-1}x + \text{cosec}^{-1}x)$$

$$= \frac{d}{dx}(\sec^{-1}x) + \frac{d}{dx}(\text{cosec}^{-1}x)$$

$$= \frac{1}{x\sqrt{x^2-1}} - \frac{1}{x\sqrt{x^2-1}} = 0.$$

Since, $f'(x) = 0$, $f(x)$ is a constant function.
Let $f(x) = k$.
For any value of x, $f(x) = k$, where $|x| > 1$

Let x = 2.
Then, f(2) = k ......(2)

From (1), $f(2) = \sec^{-1}(2) + \operatorname{cosec}^{-1}(2) = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$

$\therefore\ k = \frac{\pi}{2}$ ... [By (2)]

$\therefore\ f(x) = k = \frac{\pi}{2}$

Hence, $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$. ... [By (1)]

Question 6.
Diffrentiate the following w. r. t. x.
(i) $\tan^{-1}(\log x)$
Solution:
Let y = $\tan^{-1}(\log x)$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(\log x)]$$

$$= \frac{1}{1+(\log x)^2} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{1+(\log x)^2} \times \frac{1}{x}$$

$$= \frac{1}{x[1+(\log x)^2]}.$$

(ii) $\operatorname{cosec}^{-1}(e^{-x})$
Solution:
Let y = $\operatorname{cosec}^{-1}(e^{-x})$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\operatorname{cosec}^{-1}(e^{-x})]$$

$$= \frac{-1}{e^{-x}\sqrt{(e^{-x})^2-1}} \cdot \frac{d}{dx}(e^{-x})$$

$$= \frac{-1}{e^{-x}\sqrt{e^{-2x}-1}} \times e^{-x} \cdot \frac{d}{dx}(-x)$$

$$= \frac{-1}{\sqrt{e^{-2x}-1}} \times -1$$

$$= \frac{1}{\sqrt{\frac{1}{e^{2x}}-1}}$$

$$= \frac{e^x}{\sqrt{1-e^{2x}}}.$$

(iii) $\cot^{-1}(x^3)$
Solution:
Let y = $\cot^{-1}(x^3)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\cot^{-1}(x^3)]$$

$$=\frac{-1}{1+(x^3)^2}\cdot\frac{d}{dx}(x^3)$$

$$=\frac{-1}{1+x^6}\times 3x^2$$

$$=\frac{-3x^2}{1+x^6}.$$

(iv) $\cot^{-1}(4^x$

Solution:

Let y = $\cot^{-1}(4^x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\cot^{-1}(4^x)]$$

$$=\frac{-1}{1+(4^x)^2}\cdot\frac{d}{dx}(4^x)$$

$$=\frac{-1}{1+4^{2x}}\times 4^x\log 4$$

$$=-\frac{4^x\log 4}{1+4^{2x}}.$$

(v) $\tan^{-1}(\sqrt{x})$

Solution:

Let y = $\tan^{-1}(\sqrt{x})$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\tan^{-1}(\sqrt{x})]$$

$$=\frac{1}{1+(\sqrt{x})^2}\cdot\frac{d}{dx}(\sqrt{x})$$

$$=\frac{1}{1+x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{2\sqrt{x}(1+x)}.$$

(vi) $\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$

Solution:

Let y = $\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)\right]$$

$$=\frac{1}{\sqrt{1-\left(\sqrt{\frac{1+x^2}{2}}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{\frac{1+x^2}{2}}\right)$$

$$=\frac{1}{\sqrt{1-\left(\frac{1+x^2}{2}\right)}}\times\frac{1}{\sqrt{2}}\frac{d}{dx}(\sqrt{1+x^2})$$

$$=\frac{\sqrt{2}}{\sqrt{2-1-x^2}}\times\frac{1}{\sqrt{2}}\times\frac{1}{2\sqrt{1+x^2}}\cdot\frac{d}{dx}(1+x^2)$$

$$=\frac{1}{\sqrt{1-x^2}}\times\frac{1}{2\sqrt{1+x^2}}\cdot(0+2x)$$

$$=\frac{x}{\sqrt{(1-x^2)(1+x^2)}}=\frac{x}{\sqrt{1-x^4}}.$$

(vii) $\cos^{-1}(1 - x^2)$
Solution:
Let y = $\cos^{-1}(1 - x^2)$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\cos^{-1}(1-x^2)]$$

$$=\frac{-1}{\sqrt{1-(1-x^2)^2}}\cdot\frac{d}{dx}(1-x^2)$$

$$=\frac{-1}{\sqrt{1-(1-2x^2+x^4)}}\cdot(0-2x)$$

$$=\frac{2x}{\sqrt{2x^2-x^4}}$$

$$=\frac{2x}{x\sqrt{2-x^2}}=\frac{2}{\sqrt{2-x^2}}.$$

(viii) $\sin^{-1}(x^{\frac{3}{2}})$
Solution:
Let y = $\sin^{-1}(x^{\frac{3}{2}})$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}\left(x^{\frac{3}{2}}\right)\right]$$

$$=\frac{1}{\sqrt{1-\left(x^{\frac{3}{2}}\right)^2}}\cdot\frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$

$$=\frac{1}{\sqrt{1-x^3}}\times\frac{3}{2}x^{\frac{1}{2}}$$

$$=\frac{3\sqrt{x}}{2\sqrt{1-x^3}}.$$

(ix) $\cos^3[\cos^{-1}(x^3)]$
Solution:

Let $y = \cos^3[\cos^{-1}(x^3)]$
$= [\cos(\cos^{-1}x^3)]^3$
$= (x^3)^3 = x^9$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}(x^9) = 9x^8$.

(x) $\sin^4[\sin^{-1}(\sqrt{x})]$
Solution:
Let $y = \sin^4[\sin^{-1}(\sqrt{x})]$
$= \{\sin[\sin^{-1}(\sqrt{x})]\}^8$
$= (\sqrt{x})^4 = x^2$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}(x^2) = 2x$.

Question 7.
Diffrentiate the following w. r. t. x.
(i) $\cot^{-1}[\cot (e^{x^2})]$
Solution:
Let $y = \cot^{-1}[\cot (e^{x^2})] = e^{x^2}$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{x^2}) = e^{x^2} \cdot \frac{d}{dx}(x^2)$$
$$= e^{x^2} \times 2x = 2xe^{x^2}.$$

(ii) $\text{cosec}^{-1}\left(\frac{1}{\cos(5^x)}\right)$
Solution:

$$\text{Let } y = \text{cosec}^{-1}\left[\frac{1}{\cos(5^x)}\right]$$
$$= \text{cosec}^{-1}[\sec(5^x)]$$
$$= \text{cosec}^{-1}\left[\text{cosec}\left(\frac{\pi}{2} - 5^x\right)\right]$$
$$= \frac{\pi}{2} - 5^x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 5^x\right)$$
$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}(5^x)$$
$$= 0 - 5^x \cdot \log 5$$
$$= -5^x \cdot \log 5.$$

(iii) cos-1(1+cosx2−−−−−√)

Solution:

$$\text{Let } y=\cos^{-1}\left(\sqrt{\frac{1+\cos x}{2}}\right)$$

$$=\cos^{-1}\left(\sqrt{\frac{2\cos^2\left(\frac{x}{2}\right)}{2}}\right)$$

$$=\cos^{-1}\left[\cos\left(\frac{x}{2}\right)\right]$$

$$=\frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x}{2}\right)=\frac{1}{2}\frac{d}{dx}(x)$$

$$=\frac{1}{2}\times 1=\frac{1}{2}.$$

(iv) cos-1(1−cos(x2)2−−−−−−−√)

Solution:

$$\text{Let } y=\cos^{-1}\left(\sqrt{\frac{1-\cos(x^2)}{2}}\right)$$

$$=\cos^{-1}\left(\sqrt{\frac{2\sin^2\left(\frac{x^2}{2}\right)}{2}}\right)$$

$$=\cos^{-1}\left[\sin\left(\frac{x^2}{2}\right)\right]$$

$$=\cos^{-1}\left[\cos\left(\frac{\pi}{2}-\frac{x^2}{2}\right)\right]$$

$$=\frac{\pi}{2}-\frac{x^2}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{\pi}{2}-\frac{x^2}{2}\right)$$

$$=\frac{d}{dx}\left(\frac{\pi}{2}\right)-\frac{1}{2}\frac{d}{dx}(x^2)$$

$$=0-\frac{1}{2}\times 2x=-x.$$

(v) tan-1(1−tan(x2)1+tan(x2))

Solution:

$$\text{Let } y = \tan^{-1}\left[\frac{1 - \tan\left(\frac{x}{2}\right)}{1 + \tan\left(\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\tan\left(\frac{\pi}{4}\right) - \tan\left(\frac{x}{2}\right)}{1 + \tan\left(\frac{\pi}{4}\right)\cdot\tan\left(\frac{x}{2}\right)}\right] \quad \ldots \left[\because \ \tan\frac{\pi}{4} = 1\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right]$$

$$= \frac{\pi}{4} - \frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{4}\right) - \frac{1}{2}\frac{d}{dx}(x)$$

$$= 0 - \frac{1}{2} \times 1 = -\frac{1}{2}.$$

(vi) cosec-1(14cos32x−3cos2x)
Solution:

$$\text{Let } y = \operatorname{cosec}^{-1}\left(\frac{1}{4\cos^3 2x - 3\cos 2x}\right)$$

$$= \operatorname{cosec}^{-1}\left(\frac{1}{\cos 6x}\right) \quad \ldots \ [\because \ \cos 3x = 4\cos^3 x - 3\cos x]$$

$$= \operatorname{cosec}^{-1}(\sec 6x)$$

$$= \operatorname{cosec}^{-1}\left[\operatorname{cosec}\left(\frac{\pi}{2} - 6x\right)\right]$$

$$= \frac{\pi}{2} - 6x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 6x\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - 6\frac{d}{dx}(x)$$

$$= 0 - 6 \times 1 = -6.$$

(vii) tan-1(1+cos(x3)sin(x3))
Solution:

$$\text{Let } y = \tan^{-1}\left[\frac{1+\cos\left(\frac{x}{3}\right)}{\sin\left(\frac{x}{3}\right)}\right]$$

$$= \tan^{-1}\left[\frac{2\cos^2\left(\frac{x}{6}\right)}{2\sin\left(\frac{x}{6}\right)\cos\left(\frac{x}{6}\right)}\right]$$

$$= \tan^{-1}\left[\cot\left(\frac{x}{6}\right)\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{6}\right)\right]$$

$$= \frac{\pi}{2}-\frac{x}{6}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{6}\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{1}{6}\frac{d}{dx}(x)$$

$$= 0 - \frac{1}{6}\times 1 = -\frac{1}{6}.$$

(viii) cot-1(sin3x1+cos3x)
Solution:
Let y = cot-1(sin3x1+cos3x)

$$= \cot^{-1}\left[\frac{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)}{2\cos^2\left(\frac{3x}{2}\right)}\right]$$

$$= \cot^{-1}\left[\tan\left(\frac{3x}{2}\right)\right]$$

$$= \cot^{-1}\left[\cot\left(\frac{\pi}{2}-\frac{3x}{2}\right)\right]$$

$$= \frac{\pi}{2}-\frac{3x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-\frac{3x}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{3}{2}\frac{d}{dx}(x)$$

$$= 0 - \frac{3}{2}\times 1 = -\frac{3}{2}.$$

(ix) tan-1(cos7x1+sin7x)
Solution:

$$\text{Let } y = \tan^{-1}\left(\frac{\cos 7x}{1+\sin 7x}\right)$$

$$= \tan^{-1}\left[\frac{\sin\left(\frac{\pi}{2}-7x\right)}{1+\cos\left(\frac{\pi}{2}-7x\right)}\right]$$

$$= \tan^{-1}\left[\frac{2\sin\left(\frac{\pi}{4}-\frac{7x}{2}\right)\cdot\cos\left(\frac{\pi}{4}-\frac{7x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{7x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{7x}{2}\right)\right]$$

$$= \frac{\pi}{4}-\frac{7x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}-\frac{7x}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{4}\right)-\frac{7}{2}\frac{d}{dx}(x)$$

$$= 0-\frac{7}{2}\times 1 = -\frac{7}{2}.$$

(x) $\tan^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

Solution:

Let y = $\tan^{-1}\left(\sqrt{\frac{1+\cos x}{1-\cos x}}\right)$

$$= \tan^{-1}\left[\sqrt{\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)}}\right]$$

$$= \tan^{-1}\left[\cot\left(\frac{x}{2}\right)\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]$$

$$= \frac{\pi}{2}-\frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right)-\frac{1}{2}\frac{d}{dx}(x)$$

$$= 0-\frac{1}{2}\times 1 = -\frac{1}{2}.$$

(xi) $\tan^{-1}$(cosec x + cot x)

Solution:

Let y = $\tan^{-1}$(cosec x + cot x)

$$= \tan^{-1}\left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right)$$

$$= \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$$

$$= \tan^{-1}\left[\frac{2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left[\cot\left(\frac{x}{2}\right)\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right]$$

$$= \frac{\pi}{2} - \frac{x}{2}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - \frac{x}{2}\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{1}{2}\frac{d}{dx}(x)$$

$$= 0 - \frac{1}{2} \times 1 = -\frac{1}{2}.$$

(xii) cot-1(⎛ ⎜ 1+sin(4x3)√+1−sin(4x3)√1+sin(4x3)√−1−sin(4x3)√⎞ ⎠ ⎟
Solution:

$$\text{Let } y = \cot^{-1}\left[\frac{\sqrt{1+\sin\left(\frac{4x}{3}\right)} + \sqrt{1-\sin\left(\frac{4x}{3}\right)}}{\sqrt{1+\sin\left(\frac{4x}{3}\right)} - \sqrt{1-\sin\left(\frac{4x}{3}\right)}}\right]$$

$$1+\sin\left(\frac{4x}{3}\right) = 1+\cos\left(\frac{\pi}{2} - \frac{4x}{3}\right) = 2\cos^2\left(\frac{\pi}{4} - \frac{2x}{3}\right)$$

$$\therefore \sqrt{1+\sin\left(\frac{4x}{3}\right)} = \sqrt{2}\cos\left(\frac{\pi}{4} - \frac{2x}{3}\right)$$

$$\text{Also, } 1-\sin\left(\frac{4x}{3}\right) = 1-\cos\left(\frac{\pi}{2} - \frac{4x}{3}\right) = 2\sin^2\left(\frac{\pi}{4} - \frac{2x}{3}\right)$$

$$\therefore \sqrt{1-\sin\left(\frac{4x}{3}\right)} = \sqrt{2}\sin\left(\frac{\pi}{4} - \frac{2x}{3}\right)$$

$$\therefore \frac{\sqrt{1+\sin\left(\frac{4x}{3}\right)} + \sqrt{1-\sin\left(\frac{4x}{3}\right)}}{\sqrt{1+\sin\left(\frac{4x}{3}\right)} - \sqrt{1-\sin\left(\frac{4x}{3}\right)}}$$

$$= \frac{\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)+\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)}{\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)-\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)}$$

$$= \frac{\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)+\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)}{\cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)-\sin\left(\frac{\pi}{4}-\frac{2x}{3}\right)}$$

$$= \frac{1+\tan\left(\frac{\pi}{4}-\frac{2x}{3}\right)}{1-\tan\left(\frac{\pi}{4}-\frac{2x}{3}\right)} \quad \ldots \left[\text{Dividing by } \cos\left(\frac{\pi}{4}-\frac{2x}{3}\right)\right]$$

$$= \frac{\tan\frac{\pi}{4}+\tan\left(\frac{\pi}{4}-\frac{2x}{3}\right)}{1-\tan\frac{\pi}{4}\cdot\tan\left(\frac{\pi}{4}-\frac{2x}{3}\right)} \quad \ldots \left[\because\ \tan\frac{\pi}{4}=1\right]$$

$$= \tan\left[\frac{\pi}{4}+\frac{\pi}{4}-\frac{2x}{3}\right] = \tan\left(\frac{\pi}{2}-\frac{2x}{3}\right)$$

$$= \cot\left(\frac{2x}{3}\right)$$

$$\therefore\ y = \cot^{-1}\left[\cot\left(\frac{2x}{3}\right)\right] = \frac{2x}{3}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x}{3}\right) = \frac{2}{3}\frac{d}{dx}(x)$$

$$= \frac{2}{3}\times 1 = \frac{2}{3}.$$

Question 8.

$$\sin^{-1}\left(\frac{4\sin x+5\cos x}{\sqrt{41}}\right)$$

(i)

Solution:

$$\text{Let } y = \sin^{-1}\left(\frac{4\sin x+5\cos x}{\sqrt{41}}\right)$$

$$= \sin^{-1}\left[(\sin x)\left(\frac{4}{\sqrt{41}}\right)+(\cos x)\left(\frac{5}{\sqrt{41}}\right)\right]$$

Since, $\left(\frac{4}{\sqrt{41}}\right)^2+\left(\frac{5}{\sqrt{41}}\right)^2 = \frac{16}{41}+\frac{25}{41} = 1,$

we can write, $\frac{4}{\sqrt{41}} = \cos\alpha$ and $\frac{5}{\sqrt{41}} = \sin\alpha.$

$$\therefore\ y = \sin^{-1}(\sin x\cos\alpha+\cos x\sin\alpha)$$

$$= \sin^{-1}[\sin(x+\alpha)]$$

$$= x+\alpha, \quad \text{where } \alpha \text{ is a constant}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x+\alpha)$$

$$= \frac{d}{dx}(x)+\frac{d}{dx}(\alpha)$$

$$= 1+0 = 1.$$

$$\cos^{-1}\left(\frac{\sqrt{3}\cos x-\sin x}{2}\right)$$
(ii)
Solution:

$$\text{Let } y=\cos^{-1}\left(\frac{\sqrt{3}\cos x-\sin x}{2}\right)$$

$$=\cos^{-1}\left[(\cos x)\left(\frac{\sqrt{3}}{2}\right)-(\sin x)\left(\frac{1}{2}\right)\right]$$

$$=\cos^{-1}\left(\cos x\cos\frac{\pi}{6}-\sin x\sin\frac{\pi}{6}\right)$$

$$\ldots\left[\because\ \cos\frac{\pi}{6}=\frac{\sqrt{3}}{2},\ \sin\frac{\pi}{6}=\frac{1}{2}\right]$$

$$=\cos^{-1}\left[\cos\left(x+\frac{\pi}{6}\right)\right]$$

$$=x+\frac{\pi}{6}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(x+\frac{\pi}{6}\right)$$

$$=\frac{d}{dx}(x)+\frac{d}{dx}\left(\frac{\pi}{6}\right)$$

$$=1+0=1.$$

$$\sin^{-1}\left(\frac{\cos\sqrt{x}+\sin\sqrt{x}}{\sqrt{2}}\right)$$
(iii)
Solution:

y = $\sin^{-1}\left(\frac{\cos\sqrt{x}+\sin\sqrt{x}}{\sqrt{2}}\right)$

= $\sin^{-1}\left(\frac{1}{\sqrt{2}}\cos\sqrt{x}+\frac{1}{\sqrt{2}}\sin\sqrt{x}\right)$

Put,

$\frac{1}{\sqrt{2}}$ = sinx

$\frac{1}{\sqrt{2}}$ = cosα

Also,

$\sin^2\alpha + \cos^2\alpha = \left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2 = 1$

And,

tanα = 1

∴ α = $\tan^{-1}1$

y = $\sin^{-1}(\sin\alpha.\cos\sqrt{x}+\cos\alpha.\sin(x)$

= $\sin^{-1}(\sin(\alpha+\sqrt{x}))$

y = $\alpha+\sqrt{x}$

y = $\tan^{-1}(1)+\sqrt{x}$

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=\frac{d}{dx}(\tan^{-1}+\sqrt{x})$

= $0+\frac{1}{2\sqrt{x}}$

= $\frac{1}{2\sqrt{x}}$.

(iv) $\cos^{-1}\left(\frac{3\cos 3x-4\sin 3x}{5}\right)$

Solution:

Let $y=\cos^{-1}\left(\frac{3\cos 3x-4\sin 3x}{5}\right)$

$=\cos^{-1}\left[(\cos 3x)\left(\frac{3}{5}\right)-(\sin 3x)\left(\frac{4}{5}\right)\right]$

Since, $\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2=\frac{9}{25}+\frac{16}{25}=1,$

we can write, $\frac{3}{5}=\cos\alpha$ and $\frac{4}{5}=\sin\alpha$.

$\therefore\ y=\cos^{-1}(\cos 3x\cos\alpha-\sin 3x\sin\alpha)$

$=\cos^{-1}[\cos(3x+\alpha)]$

$=3x+\alpha,$ where α is a constant.

Differentiating w.r.t. x, we get

$\frac{dy}{dx}=\frac{d}{dx}(3x+\alpha)$

$=3\frac{d}{dx}(x)+\frac{d}{dx}(\alpha)$

$=3\times 1+0=3.$

$$\cos^{-1}\left(\frac{3\cos(e^x)+2\sin(e^x)}{\sqrt{13}}\right)$$

(v)

Solution:

$$y = \cos^{-1}\left(\frac{3\cos(e^x)+2\sin(e^x)}{\sqrt{13}}\right)$$

$$= \cos^{-1}\left(\cos(e^x).\ \frac{3}{\sqrt{13}}+\sin(e^x)\frac{2}{\sqrt{13}}\right)$$

Put,

$$\frac{3}{\sqrt{13}} = \cos x$$

$$\frac{2}{\sqrt{3}} = \sin x$$

Also,

$$\sin^2\alpha + \cos^2\alpha = \frac{9}{13}+\frac{4}{13} = 1$$

And,

$$\tan\alpha = \frac{\sin x}{\cos\alpha} = \frac{2}{3}$$

$$\therefore \alpha = \tan^{-1}\left(\frac{2}{3}\right)$$

$y = \cos^{-1}(\cos e^x . \cos\alpha + \sin e^x . \cos\alpha)$

$y = \cos^{-1}(\cos e^x - \alpha))$ $\quad \because \cos^{-1}x . (\cos x) = x$

$y = e^x - \alpha$

$$= e^x = \tan^{-1}\left(\frac{2}{3}\right)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^x - \tan^{-1}\left(\frac{2}{3}\right)\right)$$

$= e^x - 0$

$= e^x$.

$$\operatorname{cosec}^{-1}\left(\frac{10}{6\sin(2^x)-8\cos(2^x)}\right)$$

(vi)

Solution:

$$\text{Let } y = \operatorname{cosec}^{-1}\left[\frac{10}{6\sin(2^x)-8\cos(2^x)}\right]$$

$$= \sin^{-1}\left[\frac{6\sin(2^x)-8\cos(2^x)}{10}\right]$$

$$\ldots\left[\because \operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \sin^{-1}\left[\{\sin(2^x)\}\left(\frac{6}{10}\right)-\{\cos(2^x)\}\left(\frac{8}{10}\right)\right]$$

$$\text{Since, } \left(\frac{6}{10}\right)^2+\left(\frac{8}{10}\right)^2 = \frac{36}{100}+\frac{64}{100} = 1,$$

$$\text{we can write, } \frac{6}{10} = \cos\alpha \text{ and } \frac{8}{10} = \sin\alpha.$$

$y = \sin^{-1}[\sin(2^x)\cdot\cos\alpha - \cos(2^x)\cdot\sin\alpha]$

$= \sin^{-}[\sin(2^x - \alpha)]$

$= 2^x - \alpha$, where α is a constant

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = \frac{d}{dx}(2^x - \alpha)$
$= \frac{d}{dx}(2^x) - \frac{d}{dx}(\alpha)$
$= 2^x \cdot \log 2 - 0$
$= 2^x \cdot \log 2$

Question 9.
Diffrentiate the following w. r. t. x.
(i) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$
Solution:

$$\text{Let } y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Put $x = \tan\theta$. Then $\theta = \tan^{-1} x$

$$\therefore\ y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$$
$$= 2\theta = 2\tan^{-1} x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1} x) = 2\frac{d}{dx}(\tan^{-1} x)$$
$$= 2 \times \frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

(ii) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$
Solution:

$$\text{Let } y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Put $x = \tan\theta$. Then $\theta = \tan^{-1} x$

$$\therefore\ y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta)$$
$$= 2\theta = 2\tan^{-1} x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1} x) = 2\frac{d}{dx}(\tan^{-1} x)$$
$$= 2 \times \frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

(iii) $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$
Solution:

$$\text{Let } y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Put $x = \tan\theta$. Then $\theta = \tan^{-1} x$

$$\therefore\ y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\cos 2\theta)$$

$$= \sin^{-1}\left[\sin\left(\frac{\pi}{2} - 2\theta\right)\right] = \frac{\pi}{2} - 2\theta$$

$$= \frac{\pi}{2} - 2\tan^{-1} x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 2\tan^{-1} x\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - 2\frac{d}{dx}(\tan^{-1} x)$$

$$= 0 - 2 \times \frac{1}{1+x^2} = \frac{-2}{1+x^2}.$$

(iv) sin-1(2x1−x2−−−−−√)

Solution:

$$\text{Let } y = \sin^{-1}(2x\sqrt{1-x^2})$$

Put $x = \sin\theta$. Then $\theta = \sin^{-1} x$

$$\therefore\ y = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$
$$= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta)$$
$$= 2\theta = 2\sin^{-1} x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\sin^{-1} x) = 2\frac{d}{dx}(\sin^{-1} x)$$

$$= 2 \times \frac{1}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}.$$

We can also put $x = \cos\theta$. Then $\theta = \cos^{-1} x$

$$\therefore\ y = \sin^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})$$
$$= \sin^{-1}(2\cos\theta\sin\theta) = \sin^{-1}(\sin 2\theta)$$
$$= 2\theta = 2\cos^{-1} x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(2\cos^{-1} x) = 2\frac{d}{dx}(\cos^{-1} x)$$

$$= 2 \times \frac{-1}{\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

Hence, $\dfrac{dy}{dx} = \pm\dfrac{2}{\sqrt{1-x^2}}.$

(v) cos-1(3x – 4x3)

Solution:

$$\text{Let } y = \cos^{-1}(3x - 4x^3)$$

Put $x = \sin\theta$. Then $\theta = \sin^{-1}x$

$$\therefore\ y = \cos^{-1}(3\sin\theta - 4\sin^3\theta)$$

$$= \cos^{-1}(\sin 3\theta) = \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 3\theta\right)\right]$$

$$= \frac{\pi}{2} - 3\theta = \frac{\pi}{2} - 3\sin^{-1}x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 3\sin^{-1}x\right)$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - 3\frac{d}{dx}(\sin^{-1}x)$$

$$= 0 - 3 \times \frac{1}{\sqrt{1-x^2}} = \frac{-3}{\sqrt{1-x^2}}.$$

(vi) $\cos^{-1}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$

Solution:

$$\text{Let } y = \cos^{-1}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

$$= \cos^{-1}\left[\frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}}\right]$$

$$= \cos^{-1}\left(\frac{e^{2x} - 1}{e^{2x} + 1}\right)$$

Put $e^x = \tan\theta$. Then $\theta = \tan^{-1}(e^x)$

$$\therefore\ y = \cos^{-1}\left(\frac{\tan^2\theta - 1}{\tan^2\theta + 1}\right) = \cos^{-1}\left[-\left(\frac{1 - \tan^2\theta}{1 + \tan^2\theta}\right)\right]$$

$$= \cos^{-1}(-\cos 2\theta) = \cos^{-1}[\cos(\pi - 2\theta)]$$

$$= \pi - 2\theta = \pi - 2\tan^{-1}(e^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\pi - 2\tan^{-1}(e^x)]$$

$$= \frac{d}{dx}(\pi) - 2\frac{d}{dx}[\tan^{-1}(e^x)]$$

$$= 0 - 2 \times \frac{1}{1 + (e^x)^2} \cdot \frac{d}{dx}(e^x)$$

$$= \frac{-2}{1 + e^{2x}} \times e^x = -\frac{2e^x}{1 + e^{2x}}.$$

(vii) $\cos^{-1}\left(\frac{1 - 9^x}{1 + 9^x}\right)$

Solution:

$$\text{Let } y=\cos^{-1}\left(\frac{1-9^x}{1+9^x}\right)=\cos^{-1}\left[\frac{1-(3^x)^2}{1+(3^x)^2}\right]$$

Put $3^x=\tan\theta$. Then $\theta=\tan^{-1}(3^x)$

$$\therefore\ y=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}(\cos 2\theta)$$

$$=2\theta=2\tan^{-1}(3^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[2\tan^{-1}(3^x)]=2\frac{d}{dx}[\tan^{-1}(3^x)]$$

$$=2\times\frac{1}{1+(3^x)^2}\cdot\frac{d}{dx}(3^x)$$

$$=\frac{2}{1+3^{2x}}\times 3^x\log 3$$

$$=\frac{2\cdot 3^x\log 3}{1+3^{2x}}.$$

(viii) sin-1(4x+121+24x)

Solution:

$$\text{Let } y=\sin^{-1}\left(\frac{4^{x+\frac{1}{2}}}{1+2^{4x}}\right)$$

$$=\sin^{-1}\left[\frac{4^x\cdot 4^{\frac{1}{2}}}{1+(2^2)^{2x}}\right]$$

$$=\sin^{-1}\left(\frac{2\cdot 4^x}{1+4^{2x}}\right)$$

Put $4^x=\tan\theta$, Then $\theta=\tan^{-1}(4^x)$

$$\therefore\ y=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)=\sin^{-1}(\sin 2\theta)$$

$$=2\theta=2\tan^{-1}(4^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[2\tan^{-1}(4^x)]=2\frac{d}{dx}[\tan^{-1}(4^x)]$$

$$=2\times\frac{1}{1+(4^x)^2}\cdot\frac{d}{dx}(4^x)$$

$$=\frac{2}{1+4^{2x}}\times 4^x\log 4$$

$$=\frac{2\cdot 4^x\log 4}{1+4^{2x}}.$$

Note : The answer can also be written as :

$$\frac{dy}{dx}=\frac{4^{\frac{1}{2}}\cdot 4^x\log 4}{1+4^{2x}}=\frac{4^{x+\frac{1}{2}}\cdot\log 4}{1+4^{2x}}.$$

(ix) sin-1(1−25x21+25x2)

Solution:

$$\text{Let } y = \sin^{-1}\left(\frac{1-25x^2}{1+25x^2}\right) = \sin^{-1}\left[\frac{1-(5x)^2}{1+(5x)^2}\right]$$

Put $5x = \tan\theta$. Then $\theta = \tan^{-1}(5x)$

$$\therefore\ y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\cos 2\theta)$$

$$= \sin^{-1}\left[\sin\left(\frac{\pi}{2} - 2\theta\right)\right] = \frac{\pi}{2} - 2\theta$$

$$= \frac{\pi}{2} - 2\tan^{-1}(5x)$$

Differentiating w.r.t. x, we get

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}\left[\frac{\pi}{2} - 2\tan^{-1}(5x)\right]$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - 2\frac{d}{dx}[\tan^{-1}(5x)]$$

$$= 0 - 2 \times \frac{1}{1+(5x)^2} \cdot \frac{d}{dx}(5x)$$

$$= \frac{-2}{1+25x^2} \times 5 = \frac{-10}{1+25x^2}.$$

(x) $\sin^{-1}\left(\frac{1-x^3}{1+x^3}\right)$

Solution:

$$\text{Let } y = \sin^{-1}\left(\frac{1-x^3}{1+x^3}\right)$$

$$= \sin^{-1}\left[\frac{1-\left(x^{\frac{3}{2}}\right)^2}{1+\left(x^{\frac{3}{2}}\right)^2}\right]$$

Put $x^{\frac{3}{2}} = \tan\theta$. Then $\theta = \tan^{-1}\left(x^{\frac{3}{2}}\right)$

$$\therefore\ y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\cos 2\theta)$$

$$= \sin^{-1}\left[\sin\left(\frac{\pi}{2} - 2\theta\right)\right] = \frac{\pi}{2} - 2\theta$$

$$= \frac{\pi}{2} - 2\tan^{-1}(x^{\frac{3}{2}})$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\pi}{2} - 2\tan^{-1}(x^{\frac{3}{2}})\right]$$

$$= \frac{d}{dx}\left(\frac{\pi}{2}\right) - 2\frac{d}{dx}[\tan^{-1}(x^{\frac{3}{2}})]$$

$$= 0 - 2 \times \frac{1}{1+\left(x^{\frac{3}{2}}\right)^2} \cdot \frac{d}{dx}(x^{\frac{3}{2}})$$

$$= -\frac{2}{1+x^3} \times \frac{3}{2}x^{\frac{1}{2}}$$

$$= -\frac{3\sqrt{x}}{1+x^3}.$$

(xi) $\tan^{-1}\left(\frac{2x^{\frac{5}{2}}}{1-x^5}\right)$

$$\text{Let } y = \tan^{-1}\left(\frac{2x^{\frac{5}{2}}}{1-x^5}\right)$$

Put $x^{\frac{5}{2}} = \tan\theta$. Then $\theta = \tan^{-1}(x^{\frac{5}{2}})$

$$\therefore\ y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta)$$

$$= 2\theta = 2\tan^{-1}(x^{\frac{5}{2}})$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[2\tan^{-1}(x^{\frac{5}{2}})]$$

$$= 2\frac{d}{dx}[\tan^{-1}(x^{\frac{5}{2}})]$$

$$= 2 \times \frac{1}{1+\left(x^{\frac{5}{2}}\right)^2} \cdot \frac{d}{dx}(x^{\frac{5}{2}})$$

$$= \frac{2}{1+x^5} \times \frac{5}{2}x^{\frac{3}{2}}$$

$$= \frac{5x\sqrt{x}}{1+x^5}.$$

(xii) $\cot^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

Solution:

Let y = $\cot^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

$$= \tan^{-1}\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) \quad \ldots \left[\because\ \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \tan^{-1}\left(\frac{1+\sqrt{x}}{1-1\times\sqrt{x}}\right)$$

$$= \tan^{-1}(1) + \tan^{-1}(\sqrt{x})$$

$$\ldots \left[\because\ \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}x + \tan^{-1}y\right]$$

$$= \frac{\pi}{4} + \tan^{-1}(\sqrt{x})$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\pi}{4} + \tan^{-1}(\sqrt{x})\right]$$

$$= \frac{d}{dx}\left(\frac{\pi}{4}\right) + \frac{d}{dx}[\tan^{-1}(\sqrt{x})]$$

$$= 0 + \frac{1}{1+(\sqrt{x})^2} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{1+x} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}(1+x)}.$$

Question 10.

Diffrentiate the following w. r. t. x.

(i) $\tan^{-1}\left(\frac{8x}{1-15x^2}\right)$

Solution:

Let y = $\tan^{-1}\left(\frac{8x}{1-15x^2}\right)$

$$= \tan^{-1}\left[\frac{5x+3x}{1-(5x)(3x)}\right]$$

$$= \tan^{-1}(5x) + \tan^{-1}(3x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(5x) + \tan^{-1}(3x)]$$

$$= \frac{d}{dx}[\tan^{-1}(5x)] + \frac{d}{dx}[\tan^{-1}(3x)]$$

$$= \frac{1}{1+(5x)^2} \cdot \frac{d}{dx}(5x) + \frac{1}{1+(3x)^2} \cdot \frac{d}{dx}(3x)$$

$$= \frac{1}{1+25x^2} \times 5 + \frac{1}{1+9x^2} \times 3$$

$$= \frac{5}{1+25x^2} + \frac{3}{1+9x^2}.$$

(ii) $\cot^{-1}\left(\frac{1+35x^2}{2x}\right)$

Solution:

Let y = $\cot^{-1}\left(\frac{1+35x^2}{2x}\right)$

$$= \tan^{-1}\left(\frac{2x}{1+35x^2}\right) \quad \ldots \left[\because \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \tan^{-1}\left[\frac{7x-5x}{1+(7x)(5x)}\right]$$

$$= \tan^{-1}(7x) - \tan^{-1}(5x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(7x) - \tan^{-1}(5x)]$$

$$= \frac{d}{dx}[\tan^{-1}(7x)] - \frac{d}{dx}[\tan^{-1}(5x)]$$

$$= \frac{1}{1+(7x)^2} \cdot \frac{d}{dx}(7x) - \frac{1}{1+(5x)^2} \cdot \frac{d}{dx}(5x)$$

$$= \frac{1}{1+49x^2} \times 7 - \frac{1}{1+25x^2} \times 5$$

$$= \frac{7}{1+49x^2} - \frac{5}{1+25x^2}.$$

(iii) $\tan^{-1}\left(\frac{2\sqrt{x}}{1+3x}\right)$

Solution:

Let y = tan-1(2x√1+3x)

$$= \tan^{-1}\left[\frac{3\sqrt{x}-\sqrt{x}}{1+(3\sqrt{x})(\sqrt{x})}\right]$$

$$= \tan^{-1}(3\sqrt{x}) - \tan^{-1}(\sqrt{x})$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(3\sqrt{x}) - \tan^{-1}(\sqrt{x})]$$

$$= \frac{d}{dx}[\tan^{-1}(3\sqrt{x})] - \frac{d}{dx}[\tan^{-1}(\sqrt{x})]$$

$$= \frac{1}{1+(3\sqrt{x})^2}\cdot\frac{d}{dx}(3\sqrt{x}) - \frac{1}{1+(\sqrt{x})^2}\cdot\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{1+9x}\times 3\times\frac{1}{2\sqrt{x}} - \frac{1}{1+x}\times\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}\left[\frac{3}{1+9x} - \frac{1}{1+x}\right].$$

(iv) tan-1(2x+21−3(4x))

Solution:

Let y = tan-1(2x+21−3(4x))

$$= \tan^{-1}\left[\frac{2^2\cdot 2^x}{1-3(4^x)}\right]$$

$$= \tan^{-1}\left[\frac{4\cdot 2^x}{1-3(4^x)}\right]$$

$$= \tan^{-1}\left[\frac{3\cdot 2^x + 2^x}{1-(3\cdot 2^x\times 2^x)}\right]$$

$$= \tan^{-1}(3\cdot 2^x) + \tan^{-1}(2^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(3\cdot 2^x) + \tan^{-1}(2^x)]$$

$$= \frac{d}{dx}[\tan^{-1}(3\cdot 2^x)] + \frac{d}{dx}[\tan^{-1}(2^x)]$$

$$= \frac{1}{1+(3\cdot 2^x)^2}\cdot\frac{d}{dx}(3\cdot 2^x) + \frac{1}{1+(2^x)^2}\cdot\frac{d}{dx}(2^x)$$

$$= \frac{1}{1+9(2^{2x})}\times 3\times 2^x\log 2 + \frac{1}{1+2^{2x}}\times 2^x\log 2$$

$$= 2^x\log 2\left[\frac{3}{1+9(2^{2x})} + \frac{1}{1+2^{2x}}\right].$$

(v) tan-1(2x1+22x+1)

Solution:

Let y = $\tan^{-1}\left(\frac{2^x}{1+2^{2x+1}}\right)$

$$= \tan^{-1}\left[\frac{2\cdot 2^x - 2^x}{1+(2\cdot 2^x)(2^x)}\right]$$

$$= \tan^{-1}(2\cdot 2^x) - \tan^{-1}(2^x)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(2\cdot 2^x) - \tan^{-1}(2^x)]$$

$$= \frac{d}{dx}[\tan^{-1}(2\cdot 2^x)] - \frac{d}{dx}[\tan^{-1}(2^x)]$$

$$= \frac{1}{1+(2\cdot 2^x)^2}\cdot\frac{d}{dx}(2\cdot 2^x) - \frac{1}{1+(2^x)^2}\cdot\frac{d}{dx}(2^x)$$

$$= \frac{1}{1+4(2^{2x})} \times 2 \times 2^x \log 2 - \frac{1}{1+2^{2x}} \times 2^x \log 2$$

$$= 2^x \log 2\left[\frac{2}{1+4(2^{2x})} - \frac{1}{1+2^{2x}}\right].$$

(vi) $\cot^{-1}\left(\frac{a^2-6x^2}{5ax}\right)$

Solution:

Let y = $\cot^{-1}\left(\frac{a^2-6x^2}{5ax}\right)$

$$= \tan^{-1}\left(\frac{5ax}{a^2-6x^2}\right) \quad \ldots \left[\because \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \tan^{-1}\left[\frac{5\left(\frac{x}{a}\right)}{1-6\left(\frac{x}{a}\right)^2}\right] \quad \ldots \text{[Dividing by } a^2\text{]}$$

$$= \tan^{-1}\left[\frac{3\left(\frac{x}{a}\right)+2\left(\frac{x}{a}\right)}{1-3\left(\frac{x}{a}\right)\times 2\left(\frac{x}{a}\right)}\right]$$

$$= \tan^{-1}\left(\frac{3x}{a}\right) + \tan^{-1}\left(\frac{2x}{a}\right)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{3x}{a}\right) + \tan^{-1}\left(\frac{2x}{a}\right)\right]$$

$$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{3x}{a}\right)\right] + \frac{d}{dx}\left[\tan^{-1}\left(\frac{2x}{a}\right)\right]$$

$$= \frac{1}{1+\left(\frac{3x}{a}\right)^2}\cdot\frac{d}{dx}\left(\frac{3x}{a}\right) + \frac{1}{1+\left(\frac{2x}{a}\right)^2}\cdot\frac{d}{dx}\left(\frac{2x}{a}\right)$$

$$= \frac{1}{1+\left(\frac{9x^2}{a^2}\right)} \times \frac{3}{a} \times 1 + \frac{1}{1+\left(\frac{4x^2}{a^2}\right)} \times \frac{2}{a} \times 1$$

$$= \frac{a^2}{a^2+9x^2} \times \frac{3}{a} + \frac{a^2}{a^2+4x^2} \times \frac{2}{a}$$

$$= \frac{3a}{a^2+9x^2} + \frac{2a}{a^2+4x^2}.$$

(vii) $\tan^{-1}\left(\frac{a+b\tan x}{b-a\tan x}\right)$

Solution:

Let y = $\tan^{-1}\left(\frac{a+b\tan x}{b-a\tan x}\right)$

$$= \tan^{-1}\left[\frac{\frac{a}{b}+\tan x}{1-\frac{a}{b}\cdot\tan x}\right]$$

$$= \tan^{-1}\left(\frac{a}{b}\right)+\tan^{-1}(\tan x)$$

$$= \tan^{-1}\left(\frac{a}{b}\right)+x$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\tan^{-1}\left(\frac{a}{b}\right)+x\right]$$

$$=\frac{d}{dx}\left[\tan^{-1}\left(\frac{a}{b}\right)\right]+\frac{d}{dx}(x)$$

$$=0+1=1.$$

(viii) $\tan^{-1}\left(\frac{5-x}{6x^2-5x-3}\right)$

Solution:

Let y = $\tan^{-1}\left(\frac{5-x}{6x^2-5x-3}\right)$

= $\tan^{-1}\left[\frac{5-x}{1+(6x^2-5x-4)}\right]$

$$= \tan^{-1}\left[\frac{(2x+1)-(3x-4)}{1+(2x+1)(3x-4)}\right]$$

$$= \tan^{-1}(2x+1)-\tan^{-1}(3x-4)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}[\tan^{-1}(2x+1)]-\frac{d}{dx}[\tan^{-1}(3x-4)]$$

$$=\frac{1}{1+(2x+1)^2}\cdot\frac{d}{dx}(2x+1)-\frac{1}{1+(3x-4)^2}\cdot\frac{d}{dx}(3x-4)$$

$$=\frac{1}{1+(2x+1)^2}\cdot(2\times 1+0)-\frac{1}{1+(3x-4)^2}\cdot(3\times 1-0)$$

$$=\frac{2}{1+(2x+1)^2}-\frac{3}{1+(3x-4)^2}.$$

(ix) $\cot^{-1}\left(\frac{4-x-2x^2}{3x+2}\right)$

Solution:

Let y = $\cot^{-1}\left(\frac{4-x-2x^2}{3x+2}\right)$

$$= \tan^{-1}\left(\frac{3x+2}{4-x-2x^2}\right) \dots \left[\because \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \tan^{-1}\left[\frac{3x+2}{1-(2x^2+x-3)}\right]$$

$$= \tan^{-1}\left[\frac{(2x+3)+(x-1)}{1-(2x+3)(x-1)}\right]$$

$$= \tan^{-1}(2x+3) + \tan^{-1}(x-1)$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(2x+3) + \tan^{-1}(x-1)]$$

$$= \frac{d}{dx}[\tan^{-1}(2x+3)] + \frac{d}{dx}[\tan^{-1}(x-1)]$$

$$= \frac{1}{1+(2x+3)^2} \cdot \frac{d}{dx}(2x+3) + \frac{1}{1+(x-1)^2} \cdot \frac{d}{dx}(x-1)$$

$$= \frac{1}{1+(2x+3)^2} \cdot (2 \times 1 + 0) + \frac{1}{1+(x-1)^2} \cdot (1-0)$$

$$= \frac{2}{1+(2x+3)^2} + \frac{1}{1+(x-1)^2}.$$

Maharashtra State Board 12th Maths Solutions Chapter 1 Differentiation Ex 1.4

Question 1.
Find $\frac{dy}{dx}$ if
(i) $x = at^2$, $y = 2at$
Solution:
$x = at^2$, $y = 2at$
Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(at^2) = a\frac{d}{dt}(t^2)$$

$$= a \times 2t = 2at$$

$$\text{and } \frac{dy}{dt} = \frac{d}{dt}(2at) = 2a\frac{d}{dt}(t)$$

$$= 2a \times 1 = 2a$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{2a}{2at} = \frac{1}{t}.$$

(ii) x = a cot θ, y = b cosec θ
Solution:
x = a cot θ, y = b cosec θ
Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta}=a\frac{d}{d\theta}(\cot\theta)=a(-\operatorname{cosec}^2\theta)$$
$$=-a\operatorname{cosec}^2\theta$$
$$\text{and }\frac{dy}{d\theta}=b\frac{d}{d\theta}(\operatorname{cosec}\theta)=b(-\operatorname{cosec}\theta\cot\theta)$$
$$=-b\operatorname{cosec}\theta\cot\theta$$
$$\therefore\ \frac{dy}{dx}=\frac{(dy/d\theta)}{(dx/d\theta)}=\frac{-b\operatorname{cosec}\theta\cot\theta}{-a\operatorname{cosec}^2\theta}$$
$$=\frac{b}{a}\cdot\frac{\cot\theta}{\operatorname{cosec}\theta}=\frac{b}{a}\times\frac{\cos\theta}{\sin\theta}\times\sin\theta$$
$$=\left(\frac{b}{a}\right)\cos\theta.$$

(iii) x = $\sqrt{a^2+m^2}$, y = log (a^2 + m^2)
Solution:
x = $\sqrt{a^2+m^2}$, y = log (a^2 + m^2)
Differentiating x and y w.r.t. m, we get
$\frac{dx}{dm}=\frac{d}{dm}\left(\sqrt{a^2+m^2}\right)$

$$=\frac{1}{2\sqrt{a^2+m^2}}\cdot\frac{d}{dm}(a^2+m^2)$$
$$=\frac{1}{2\sqrt{a^2+m^2}}\times(0+2m)=\frac{m}{\sqrt{a^2+m^2}}$$
$$\text{and }\frac{dy}{dm}=\frac{d}{dm}[\log(a^2+m^2)]$$
$$=\frac{1}{a^2+m^2}\cdot\frac{d}{dm}(a^2+m^2)$$
$$=\frac{1}{a^2+m^2}\times(0+2m)=\frac{2m}{a^2+m^2}$$
$$\therefore\ \frac{dy}{dx}=\frac{(dy/dm)}{(dx/dm)}=\frac{\left(\frac{2m}{a^2+m^2}\right)}{\left(\frac{m}{\sqrt{a^2+m^2}}\right)}$$
$$=\frac{2}{\sqrt{a^2+m^2}}.$$

(iv) x = sin θ, y = tan θ
Solution:
x = sin θ, y = tan θ
Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(\sin\theta)=\cos\theta$$
$$\text{and }\frac{dy}{d\theta}=\frac{d}{d\theta}(\tan\theta)=\sec^2\theta$$
$$\frac{dy}{dx}=\frac{(dy/d\theta)}{(dx/d\theta)}=\frac{\sec^2\theta}{\cos\theta}$$
$$=\sec^3\theta.$$

(v) x = a(1 – cos θ), y = b(θ – sin θ)
Solution:
x = a(1 – cos θ), y = b(θ – sin θ)
Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta}=a\frac{d}{d\theta}(1-\cos\theta)$$
$$=a[0-(-\sin\theta)]=a\sin\theta$$
$$\text{and } \frac{dy}{d\theta}=b\frac{d}{d\theta}(\theta-\sin\theta)$$
$$=b(1-\cos\theta)$$
$$\therefore \frac{dy}{dx}=\frac{(dy/d\theta)}{(dx/d\theta)}=\frac{b(1-\cos\theta)}{a\sin\theta}$$
$$=\frac{b\times 2\sin^2(\theta/2)}{a\times 2\sin(\theta/2)\cos(\theta/2)}=\left(\frac{b}{a}\right)\tan\left(\frac{\theta}{2}\right)$$

(vi) x = $\left(t+\frac{1}{t}\right)^a$, y = $a^{t+\frac{1}{t}}$, where a > 0, a ≠ 1 and t ≠ 0
Solution:
x = $\left(t+\frac{1}{t}\right)^a$, y = $a^{t+\frac{1}{t}}$ ....(1)
Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}\left(t+\frac{1}{t}\right)^a=a\left(t+\frac{1}{t}\right)^{a-1}\cdot\frac{d}{dt}\left(t+\frac{1}{t}\right)$$
$$=a\left(t+\frac{1}{t}\right)^{a-1}\cdot\left(1-\frac{1}{t^2}\right)$$
$$\text{and } \frac{dy}{dt}=\frac{d}{dt}\left[a^{\left(t+\frac{1}{t}\right)}\right]$$
$$=a^{\left(t+\frac{1}{t}\right)}\cdot\log a\cdot\frac{d}{dt}\left(t+\frac{1}{t}\right)$$
$$=a^{\left(t+\frac{1}{t}\right)}\cdot\log a\cdot\left(1-\frac{1}{t^2}\right)$$
$$\therefore \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{a^{\left(t+\frac{1}{t}\right)}\cdot\log a\cdot\left(1-\frac{1}{t^2}\right)}{a\left(t+\frac{1}{t}\right)^{a-1}\cdot\left(1-\frac{1}{t^2}\right)}$$
$$=\frac{a^{t+\frac{1}{t}}\cdot\log a\cdot\left(t+\frac{1}{t}\right)}{a\cdot\left(t+\frac{1}{t}\right)^a}$$
$$=\frac{y\cdot\log a\cdot\left(\frac{t^2+1}{t}\right)}{ax} \qquad \dots \text{[By (1)]}$$
$$=\frac{y(t^2+1)\log a}{axt}.$$

(vii) x = $\cos^{-1}\left(\frac{2t}{1+t^2}\right)$, y = $\sec^{-1}\left(\sqrt{1+t^2}\right)$
Solution:
x = $\cos^{-1}\left(\frac{2t}{1+t^2}\right)$, y = $\sec^{-1}\left(\sqrt{1+t^2}\right)$
Put t = tan θ Then θ = $\tan^{-1}t$

$$\therefore\ x=\cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right),\ y=\sec^{-1}\left(\sqrt{1+\tan^2\theta}\right)$$

$$\therefore\ x=\cos^{-1}(\sin 2\theta),\ y=\sec^{-1}\left(\sqrt{\sec^2\theta}\right)$$

$$\therefore\ x=\cos^{-1}\left[\cos\left(\frac{\pi}{2}-2\theta\right)\right],\ y=\sec^{-1}(\sec\theta)$$

$$\therefore\ x=\frac{\pi}{2}-2\theta,\quad y=\theta$$

$$\therefore\ x=\frac{\pi}{2}-2\tan^{-1}t,\ y=\tan^{-1}t$$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}\left(\frac{\pi}{2}\right)-2\frac{d}{dt}(\tan^{-1}t)$$

$$=0-2\times\frac{1}{1+t^2}=\frac{-2}{1+t^2}$$

and $\frac{dy}{dt}=\frac{d}{dt}(\tan^{-1}t)=\frac{1}{1+t^2}$

$$\therefore\ \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{\left(\frac{1}{1+t^2}\right)}{\left(\frac{-2}{1+t^2}\right)}$$

$$=-\frac{1}{2}.$$

(viii) $x = \cos^{-1}(4t^3 - 3t)$, $y = \tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$

Solution:

$x = \cos^{-1}(4t^3 - 3t)$, $y = \tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$

Put $t = \cos\theta$. Then $\theta = \cos^{-1}t$

$x = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$$y = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)$$

$$\therefore\ x = \cos^{-1}(\cos 3\theta),\ y = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) = \tan^{-1}(\tan\theta)$$

$$\therefore\ x = 3\theta \quad \text{and} \quad y = \theta$$

$$\therefore\ x = 3\cos^{-1}t \quad \text{and} \quad y = \cos^{-1}t$$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = 3\frac{d}{dt}(\cos^{-1}t)$$

$$= 3 \times \frac{-1}{\sqrt{1-t^2}} = \frac{-3}{\sqrt{1-t^2}}$$

and $\frac{dy}{dt} = \frac{d}{dt}(\cos^{-1}t) = \frac{-1}{\sqrt{1-t^2}}$

$$\therefore\ \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\left(\frac{-1}{\sqrt{1-t^2}}\right)}{\left(\frac{-3}{\sqrt{1-t^2}}\right)}$$

$$= \frac{1}{3}.$$

Alternative Method :

$$x = \cos^{-1}(4t^3 - 3t),\ t = \tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$$

Put $t = \cos\theta$.

Then $x = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$,

$$y = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)$$

$$\therefore\ x = \cos^{-1}(\cos 3\theta),\ y = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) = \tan^{-1}(\tan\theta)$$

$$\therefore\ x = 3\theta,\ y = \theta$$

$$\therefore\ x = 3y$$

$$\therefore\ y = \frac{1}{3}x$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{3}\frac{d}{dx}(x)$$

$$= \frac{1}{3} \times 1 = \frac{1}{3}.$$

Question 2.
Find $\frac{dy}{dx}$, if
(i) $x = \operatorname{cosec}^2\theta$, $y = \cot^3\theta$ at $\theta = \frac{\pi}{6}$
Solution:
$x = \operatorname{cosec}^2\theta$, $y = \cot^3\theta$

Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(\operatorname{cosec}\theta)^2=2\operatorname{cosec}\theta\cdot\frac{d}{d\theta}(\operatorname{cosec}\theta)$$
$$=2\operatorname{cosec}\theta(-\operatorname{cosec}\theta\cot\theta)$$
$$=-2\operatorname{cosec}^2\theta\cot\theta$$
$$\text{and }\frac{dy}{d\theta}=\frac{d}{d\theta}(\cot\theta)^3=3\cot^2\theta\cdot\frac{d}{d\theta}(\cot\theta)$$
$$=3\cot^2\theta\cdot(-\operatorname{cosec}^2\theta)$$
$$=-3\cot^2\theta\cdot\operatorname{cosec}^2\theta$$
$$\therefore\ \frac{dy}{dx}=\frac{(dy/d\theta)}{(dx/d\theta)}=\frac{-3\cot^2\theta\cdot\operatorname{cosec}^2\theta}{-2\operatorname{cosec}^2\theta\cdot\cot\theta}$$
$$=\frac{3}{2}\cot\theta$$
$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }\theta=\frac{\pi}{6}}=\frac{3}{2}\cot\frac{\pi}{6}=\frac{3\sqrt{3}}{2}.$$

(ii) x = a cos3θ, y = a sin3θ at θ = π3
Solution:
x = a cos3θ, y = a sin3θ
Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta}=a\frac{d}{d\theta}(\cos\theta)^3$$
$$=a\times3\cos^2\theta\cdot\frac{d}{d\theta}(\cos\theta)$$
$$=3a\cos^2\theta(-\sin\theta)=-3a\cos^2\theta\sin\theta$$
$$\text{and }\frac{dy}{d\theta}=a\frac{d}{d\theta}(\sin\theta)^3$$
$$=a\times3\sin^2\theta\cdot\frac{d}{d\theta}(\sin\theta)$$
$$=3a\sin^2\theta\cos\theta$$
$$\therefore\ \frac{dy}{dx}=\frac{(dy/d\theta)}{(dx/d\theta)}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$
$$=-\tan\theta$$
$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }\theta=\frac{\pi}{3}}=-\tan\frac{\pi}{3}=-\sqrt{3}.$$

(iii) x = t2 + t + 1, y = sin(πt2) + cos(πt2) at t = 1
Solution:
x = t2 + t + 1, y = sin(πt2) + cos(πt2)

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}(t^2+t+1)$$
$$=2t+1+0=2t+1$$
$$\text{and } \frac{dy}{dt}=\frac{d}{dt}\left[\sin\left(\frac{\pi t}{2}\right)\right]+\frac{d}{dt}\left[\cos\left(\frac{\pi t}{2}\right)\right]$$
$$=\cos\left(\frac{\pi t}{2}\right)\cdot\frac{d}{dt}\left(\frac{\pi t}{2}\right)+\left[-\sin\left(\frac{\pi t}{2}\right)\right]\cdot\frac{d}{dt}\left(\frac{\pi t}{2}\right)$$
$$=\cos\left(\frac{\pi t}{2}\right)\times\frac{\pi}{2}\times 1-\sin\left(\frac{\pi t}{2}\right)\times\frac{\pi}{2}\times 1$$
$$=\frac{\pi}{2}\left[\cos\left(\frac{\pi t}{2}\right)-\sin\left(\frac{\pi t}{2}\right)\right]$$
$$\therefore\ \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{\frac{\pi}{2}\left[\cos\left(\frac{\pi t}{2}\right)-\sin\left(\frac{\pi t}{2}\right)\right]}{2t+1}.$$
$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at } t=1}=\frac{\frac{\pi}{2}\left[\cos\frac{\pi}{2}-\sin\frac{\pi}{2}\right]}{2(1)+1}$$
$$=\frac{\frac{\pi}{2}(0-1)}{3}=-\frac{\pi}{6}.$$

(iv) x = 2 cos t + cos 2t, y = 2 sin t – sin 2t at t = $\pi 4$

Solution:

x = 2 cos t + cos 2t, y = 2 sin t – sin 2t

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}(2\cos t+\cos 2t)$$

$$=2\frac{d}{dt}(\cos t)+\frac{d}{dt}(\cos 2t)$$

$$=2(-\sin t)+(-\sin 2t)\cdot\frac{d}{dt}(2t)$$

$$=-2\sin t-\sin 2t\times 2\times 1$$

$$=-2\sin t-2\sin 2t$$

and $\dfrac{dy}{dt}=\dfrac{d}{dt}(2\sin t-\sin 2t)$

$$=2\frac{d}{dt}(\sin t)-\frac{d}{dt}(\sin 2t)$$

$$=2\cos t-\cos 2t\cdot\frac{d}{dt}(2t)$$

$$=2\cos t-\cos 2t\times 2\times 1$$

$$=2\cos t-2\cos 2t$$

$$\therefore\ \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{2\cos t-2\cos 2t}{-2\sin t-2\sin 2t}$$

$$=\frac{\cos t-\cos 2t}{-\sin t-\sin 2t}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at } t=\frac{\pi}{4}}=\frac{\cos\frac{\pi}{4}-\cos\frac{\pi}{2}}{-\sin\frac{\pi}{4}-\sin\frac{\pi}{2}}$$

$$=\frac{\frac{1}{\sqrt{2}}-0}{-\frac{1}{\sqrt{2}}-1}=\frac{-1}{1+\sqrt{2}}$$

$$=\frac{-1}{1+\sqrt{2}}\times\frac{1-\sqrt{2}}{1-\sqrt{2}}$$

$$=\frac{-(1-\sqrt{2})}{1-2}=1-\sqrt{2}.$$

(v) x = t + 2 sin(πt), y = 3t – cos(πt) at t = 12
Solution:
x = t + 2 sin(πt), y = 3t – cos(πt)
Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}[t + 2\sin(\pi t)]$$

$$= \frac{d}{dt}(t) + 2 \cdot \frac{d}{dt}[\sin(\pi t)]$$

$$= 1 + 2 \times \cos(\pi t) \cdot \frac{d}{dx}(\pi t)$$

$$= 1 + 2\cos(\pi t) \times \pi \times 1$$

$$= 1 + 2\pi\cos(\pi t)$$

and $\dfrac{dy}{dt} = \dfrac{d}{dt}[3t - \cos(\pi t)]$

$$= 3\frac{d}{dt}(t) - \frac{d}{dt}[\cos(\pi t)]$$

$$= 3 \times 1 - [-\sin(\pi t)] \cdot \frac{d}{dt}(\pi t)$$

$$= 3 + \sin(\pi t) \times \pi \times 1$$

$$= 3 + \pi\sin(\pi t)$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{3 + \pi\sin(\pi t)}{1 + 2\pi\cos(\pi t)}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } t=\frac{1}{2}} = \frac{3 + \pi\sin\left(\frac{\pi}{2}\right)}{1 + 2\pi\cos\left(\frac{\pi}{2}\right)}$$

$$= \frac{3 + \pi \times 1}{1 + 2\pi(0)} = 3 + \pi.$$

Question 3.
(i) If x = $asec\theta-tan\theta----------\surd$, y = $asec\theta+tan\theta----------\surd$, then show that $dydx=-yx$
Solution:
x = $asec\theta-tan\theta----------\surd$, y = $asec\theta+tan\theta----------\surd$

$\therefore \frac{x}{a} = \sqrt{\sec\theta - \tan\theta}, \frac{y}{a} = \sqrt{\sec\theta + \tan\theta}$

$$\therefore \sec\theta - \tan\theta = \frac{x^2}{a^2} \quad \dots (1)$$

$$\sec\theta + \tan\theta = \frac{y^2}{a^2} \quad \dots (2)$$

Adding (1) and (2), we get

$$2\sec\theta = \frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{x^2+y^2}{a^2}$$

$$\therefore \sec\theta = \frac{x^2+y^2}{2a^2}$$

Subtracting (1) from (2), we get

$$2\tan\theta = \frac{y^2}{a^2} - \frac{x^2}{a^2} = \frac{y^2-x^2}{a^2}$$

$$\therefore \tan\theta = \frac{y^2-x^2}{2a^2}$$

$\therefore \sec^2\theta - \tan^2\theta = 1$ gives,

$$\left(\frac{x^2+y^2}{2a^2}\right)^2 - \left(\frac{y^2-x^2}{2a^2}\right)^2 = 1$$

$$\therefore (x^2+y^2)^2 - (y^2-x^2)^2 = 4a^4$$

$$\therefore (x^4+2x^2y^2+y^4) - (y^4-2x^2y^2+x^4) = 4a^4$$

$$\therefore 4x^2y^2 = 4a^4$$

$$\therefore x^2y^2 = a^4$$

Differentiating both sides w.r.t. x, we get

$$x^2 \cdot \frac{d}{dx}(y^2) + y^2 \cdot \frac{d}{dx}(x^2) = 0$$

$$\therefore x^2 \times 2y\frac{dy}{dx} + y^2 \times 2x = 0$$

$$\therefore 2x^2y\frac{dy}{dx} = -2xy^2$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x}.$$

(ii) If x = $e^{\sin 3t}$, y = $e^{\cos 3t}$, then show that $\frac{dy}{dx} = -\frac{y \log x}{x \log y}$

Solution:

x = $e^{\sin 3t}$, y = $e^{\cos 3t}$

log x = log $e^{\sin 3t}$, log y = log $e^{\cos 3t}$

log x = (sin 3t)(log e), log y = (cos 3t)(log e)

log x = sin 3t, log y = cos 3t ….. (1) [∵ log e = 1]

Differentiating both sides w.r.t. t, we get

$$\frac{1}{x}\cdot\frac{dx}{dt}=\frac{d}{dt}(\sin 3t)=\cos 3t\cdot\frac{d}{dt}(3t)$$

$$=\cos 3t\times 3=3\cos 3t$$

$$\text{and } \frac{1}{y}\cdot\frac{dy}{dt}=\frac{d}{dt}(\cos 3t)=-\sin 3t\cdot\frac{d}{dx}(3t)$$

$$=-\sin 3t\times 3=-3\sin 3t$$

$$\therefore\ \frac{dx}{dt}=3x\cos 3t \text{ and } \frac{dy}{dt}=-3y\sin 3t$$

$$\therefore\ \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{-3y\sin 3t}{3x\cos 3t}.$$

$$=\frac{-y\sin 3t}{x\cos 3t}=\frac{-y\log x}{x\log y}. \qquad \text{... [By (1)]}$$

(iii) If x = $\frac{t+1}{t-1}$, y = $\frac{1-t}{t+1}$, then show that $y^2 - \frac{dy}{dx} = 0$.
Solution:
x = $\frac{t+1}{t-1}$, y = $\frac{1-t}{t+1}$

$$\therefore\ y=\frac{1}{\left(\frac{t+1}{1-t}\right)}=\frac{-1}{\left(\frac{t+1}{t-1}\right)}$$

$$\therefore\ y=-\frac{1}{x}$$

$$\therefore\ xy=-1 \qquad \text{... (1)}$$

Differentiating both sides w.r.t. *t*, we get

$$x\frac{dy}{dx}+y\cdot\frac{d}{dx}(x)=0$$

$$\therefore\ x\frac{dy}{dx}+y\times 1=0$$

$$\therefore\ -\frac{1}{y}\frac{dy}{dx}+y=0$$

$$\therefore\ -\frac{dy}{dx}+y^2=0$$

$$\therefore\ y^2-\frac{dy}{dx}=0.$$

(iv) If x = a $\cos^3$t, y = a $\sin^3$t, then show that $\frac{dy}{dx}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}$
Solution:
x = a $\cos^3$t, y = a $\sin^3$t
Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=a\frac{d}{dt}(\cos t)^3=a\cdot 3(\cos t)^2\frac{d}{dt}(\cos t)$$

$$=3a\cos^2 t(-\sin t)=-3a\cos^2 t\sin t$$

$$\text{and } \frac{dy}{dt}=a\frac{d}{dt}(\sin t)^3=a\cdot 3(\sin t)^2\frac{d}{dt}(\sin t)$$

$$=3a\sin^2 t\cdot\cos t$$

$$\therefore\ \frac{dy}{dx}=\left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right)=\frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t}=-\frac{\sin t}{\cos t} \qquad \text{... (1)}$$

$$\text{Now, } x=a\cos^3 t \qquad \therefore\ \cos^3 t=\frac{x}{a}$$

$$\therefore\ \cos t=\left(\frac{x}{a}\right)^{1/3}$$

$y = a\sin^3 t \quad \therefore\ \sin^3 t = \frac{y}{a} \quad \therefore\ \sin t = \left(\frac{y}{a}\right)^{1/3}$

$\therefore$ from (1), $\frac{dy}{dx} = -\frac{y^{1/3}/a^{1/3}}{x^{1/3}/a^{1/3}} = -\left(\frac{y}{x}\right)^{1/3}.$

Alternative Method :

$x = a\cos^3 t,\ y = a\sin^3 t$

$\therefore\ \cos^3 t = \frac{x}{a},\ \sin^3 t = \frac{y}{a}$

$\therefore\ \cos t = \left(\frac{x}{a}\right)^{1/3},\ \sin t = \left(\frac{y}{a}\right)^{1/3}$

$\therefore\ \cos^2 t + \sin^2 t = 1$ gives

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1$$

$\therefore\ x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating both sides w.r.t. x, we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\cdot\frac{dy}{dx} = 0$$

$\therefore\ \frac{2}{3}y^{-1/3}\frac{dy}{dx} = -\frac{2}{3}x^{-1/3}$

$\therefore\ \frac{dy}{dx} = -\left(\frac{x}{y}\right)^{-1/3} = -\left(\frac{y}{x}\right)^{1/3}.$

(v) If $x = 2\cos^4(t + 3)$, $y = 3\sin^4(t + 3)$, show that $\frac{dy}{dx} = -\sqrt{\frac{3y}{2x}}$

Solution:

$x = 2\cos^4(t + 3)$, $y = 3\sin^4(t + 3)$

$\therefore\ \cos^4(t+3) = \frac{x}{2},\ \sin^4(t+3) = \frac{y}{3}$

$\therefore\ \cos^2(t+3) = \sqrt{\frac{x}{2}},\ \sin^2(t+3) = \sqrt{\frac{y}{3}}$

$\because\ \cos^2(t+3) + \sin^2(t+3) = 1$

$\therefore\ \sqrt{\frac{x}{2}} + \sqrt{\frac{y}{3}} = 1$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{2}}\frac{d}{dx}(\sqrt{x}) + \frac{1}{\sqrt{3}}\frac{d}{dx}(\sqrt{y}) = 0$$

$\therefore\ \frac{1}{\sqrt{2}} \times \frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{3}} \times \frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx} = 0$

$\therefore\ \frac{1}{2\sqrt{3}\cdot\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{2}\cdot\sqrt{x}}$

$\therefore\ \frac{dy}{dx} = -\frac{\sqrt{3}\cdot\sqrt{y}}{\sqrt{2}\cdot\sqrt{x}} = -\sqrt{\frac{3y}{2x}}.$

(vi) If $x = \log(1 + t^2)$, $y = t - \tan^{-1}t$, show that $\frac{dy}{dx} = \frac{\sqrt{e^x - 1}}{2}$

Solution:

$x = \log(1 + t^2)$, $y = t - \tan^{-1}t$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}[\log(1+t^2)]=\frac{1}{1+t^2}\cdot\frac{d}{dt}(1+t^2)$$

$$=\frac{1}{1+t^2}\times(0+2t)=\frac{2t}{1+t^2}$$

and $\dfrac{dy}{dt}=\dfrac{d}{dt}(t)-\dfrac{d}{dt}(\tan^{-1}t)$

$$=1-\frac{1}{1+t^2}=\frac{1+t^2-1}{1+t^2}=\frac{t^2}{1+t^2}$$

$$\therefore\ \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{\left(\dfrac{t^2}{1+t^2}\right)}{\left(\dfrac{2t}{1+t^2}\right)}=\frac{t}{2}$$

Now, $x=\log(1+t^2)$

$\therefore\ 1+t^2=e^x$

$\therefore\ t^2=e^x-1$

$\therefore\ t=\sqrt{e^x-1}$

$\therefore\ \dfrac{dy}{dx}=\dfrac{\sqrt{e^x-1}}{2}$.

(vii) If x = $\sin^{-1}(e^t)$, y = $\sqrt{1-e^{2t}}$, show that sin x + $\frac{dy}{dx}$ = 0
Solution:
x = $\sin^{-1}(e^t)$, y = $\sqrt{1-e^{2t}}$
Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}[\sin^{-1}(e^t)]$$

$$=\frac{1}{\sqrt{1-(e^t)^2}}\cdot\frac{d}{dt}(e^t)$$

$$=\frac{1}{\sqrt{1-e^{2t}}}\times e^t=\frac{e^t}{\sqrt{1-e^{2t}}}$$

and $\dfrac{dy}{dt}=\dfrac{d}{dt}(\sqrt{1-e^{2t}})$

$$=\frac{1}{2\sqrt{1-e^{2t}}}\cdot\frac{d}{dt}(1-e^{2t})$$

$$=\frac{1}{2\sqrt{1-e^{2t}}}\times(0-e^{2t}\times 2)$$

$$=\frac{-e^{2t}}{\sqrt{1-e^{2t}}}$$

$$\therefore\ \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{\left(\dfrac{-e^{2t}}{\sqrt{1-e^{2t}}}\right)}{\left(\dfrac{e^t}{\sqrt{1-e^{2t}}}\right)}$$

$$=-e^t$$

$$=-\sin x \qquad \ldots\ [\because\ x=\sin^{-1}(e^t)]$$

$$\therefore\ \sin x+\frac{dy}{dx}=0.$$

(viii) If x = $\frac{2bt}{1+t^2}$, y = $a\left(\frac{1-t^2}{1+t^2}\right)$, show that $\frac{dx}{dy}=-\frac{b^2y}{a^2x}$
Solution:

$x = \frac{2bt}{1+t^2}$, $y = a\left(\frac{1-t^2}{1+t^2}\right)$

Put $t = \tan\theta$.

Then $x = b\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$, $y = a\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\therefore\ x = b\sin 2\theta,\ \ y = a\cos 2\theta$

$\therefore\ \frac{x}{b} = \sin 2\theta,\ \frac{y}{a} = \cos 2\theta$

$\therefore\ \left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = \sin^2 2\theta + \cos^2 2\theta$

$\therefore\ \frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

Differentiating both sides w.r.t. y, we get

$\frac{1}{b^2} \times 2x\frac{dx}{dy} + \frac{1}{a^2} \times 2y = 0$

$\therefore\ \frac{2x}{b^2}\frac{dx}{dy} = \frac{-2y}{a^2}$

$\therefore\ \frac{dx}{dy} = -\frac{b^2y}{a^2x}.$

Question 4.
(i) Differentiate x sin x w.r.t tan x.
Solution:
Let u = x sinx and v = tan x
Then we want to find $\frac{du}{dv}$
Differentiating u and v w.r.t. x, we get

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx}(x\sin x)\\ &= x\frac{d}{dx}(\sin x) + (\sin x)\cdot\frac{a}{dx}(x)\\ &= x\cos x + (\sin x)\times 1\\ &= x\cos x + \sin x\end{aligned}$$

and $\frac{dv}{dx} = \frac{d}{dx}(\tan x) = \sec^2 x$

$\therefore\ \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{x\cos x + \sin x}{\sec^2 x}.$

(ii) Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ w.r.t $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$
Solution:
Let u = $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and v = $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$
Then we want to find $\frac{du}{dv}$

Put $x = \tan\theta$. Then $\theta = \tan^{-1} x$.

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

$$= 2\theta = 2\tan^{-1} x$$

$$\therefore \frac{du}{dx} = 2\frac{d}{dx}(\tan^{-1}x) = 2 \times \frac{1}{1+x^2} = \frac{2}{1+x^2}$$

Also, $v = \cos^{-1}\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$

$$= 2\theta = 2\tan^{-1} x$$

$$\therefore \frac{dv}{dx} = 2\frac{d}{dx}(\tan^{-1}x) = 2 \times \frac{1}{1+x^2} = \frac{2}{1+x^2}$$

$$\therefore \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{\left(\frac{2}{1+x^2}\right)}{\left(\frac{2}{1+x^2}\right)} = 1.$$

Alternative Method :

Let $u = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ and $v = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

Then we want to find $\dfrac{du}{dv}$

Put $x = \tan\theta$.

Then $u = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$

and $v = \cos^{-1}\left(\dfrac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta$

$\therefore u = v$

Differentiating both sides w.r.t. v, we get

$$\frac{du}{dv} = 1.$$

(iii) Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ w.r.t $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$

Solution:

Let $u=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ and

$v=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$. Then we want to find $\frac{du}{dv}$.

Put $x=\cos\theta$. Then $\theta=\cos^{-1}x$.

$$\therefore\ u=\tan^{-1}\left(\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right)=\tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$=\tan^{-1}(\cot\theta)=\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\theta\right)\right]$$

$$=\frac{\pi}{2}-\theta=\frac{\pi}{2}-\cos^{-1}x$$

$$\therefore\ \frac{du}{dx}=\frac{d}{dx}\left(\frac{\pi}{2}\right)-\frac{d}{dx}(\cos^{-1}x)$$

$$=0-\frac{-1}{\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x^2}}$$

$$v=\sec^{-1}\left(\frac{1}{2x^2-1}\right)=\cos^{-1}(2x^2-1)$$

$$=\cos^{-1}(2\cos^2\theta-1)=\cos^{-1}(\cos 2\theta)$$

$$=2\theta=2\cos^{-1}x$$

$$\therefore\ \frac{dv}{dx}=2\cdot\frac{d}{dx}(\cos^{-1}x)=\frac{-2}{\sqrt{1-x^2}}$$

$$\therefore\ \frac{du}{dv}=\frac{du/dx}{dv/dx}=\frac{1}{\sqrt{1-x^2}}\times\frac{\sqrt{1-x^2}}{-2}=-\frac{1}{2}.$$

(iv) Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ w.r.t. $\tan^{-1}x$
Solution:
Let u = $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ and v = $\tan^{-1}x$
Then we want to find $\frac{du}{dv}$
Put x = tan θ. Then θ = $\tan^{-1}x$.

$$\therefore\ u=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)=\cos^{-1}(\cos 2\theta)=2\theta$$

$$\therefore\ u=2\tan^{-1}x$$

$$\therefore\ \frac{du}{dx}=2\cdot\frac{d}{dx}(\tan^{-1}x)=2\times\frac{1}{1+x^2}$$

$$=\frac{2}{1+x^2}$$

Also, $v=\tan^{-1}x$

$$\therefore\ \frac{dv}{dx}=\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

$$\therefore\ \frac{du}{dv}=\frac{(du/dx)}{(dv/dx)}=\frac{\left(\frac{2}{1+x^2}\right)}{\left(\frac{1}{1+x^2}\right)}=2.$$

(v) Differentiate 3^x w.r.t. $\log_x 3$.
Solution:
Let u = 3^x and v = $\log_x 3$.
Then we want to find $\frac{du}{dv}$
Differentiating u and v w.r.t. x, we get

$\frac{du}{dx}=\frac{d}{dx}(3^x)=3^x\cdot\log 3$

and $\frac{dv}{dx}=\frac{d}{dx}(\log_x 3)=\frac{d}{dx}\left(\frac{\log 3}{\log x}\right)$

$$=\log 3\cdot\frac{d}{dx}(\log x)^{-1}$$

$$=(\log 3)(-1)(\log x)^{-2}\cdot\frac{d}{dx}(\log x)$$

$$=\frac{-\log 3}{(\log x)^2}\times\frac{1}{x}=\frac{-\log 3}{x(\log x)^2}$$

$$\therefore\ \frac{du}{dv}=\frac{(du/dx)}{(dv/dx)}=\frac{3^x\cdot\log 3}{\left[\frac{-\log 3}{x(\log x)^2}\right]}$$

$$=-x(\log x)^2\cdot 3^x.$$

(vi) Differentiate $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ w.r.t. $\sec^{-1}x$.

Solution:

Let u = $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$ and v = $\sec^{-1}x$

Then we want to find $\frac{du}{dv}$.

Differentiating u and v w.r.t. x, we get

$$\frac{du}{dx}=\frac{d}{dx}\left[\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)\right]$$

$$\frac{\cos x}{1+\sin x}=\frac{\sin\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}$$

$$=\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cdot\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}$$

$$=\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)$$

$$\therefore\ \frac{du}{dx}=\frac{d}{dx}\left[\tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}\right]$$

$$=\frac{d}{dx}\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{d}{dx}\left(\frac{\pi}{4}\right)-\frac{1}{2}\frac{d}{dx}(x)$$

$$=0-\frac{1}{2}\times 1=-\frac{1}{2}$$

and $\frac{dv}{dx}=\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$

$$\therefore\ \frac{du}{dv}=\frac{(du/dx)}{(dv/dx)}=\frac{\left(-\frac{1}{2}\right)}{\left(\frac{1}{x\sqrt{x^2-1}}\right)}=-\frac{x\sqrt{x^2-1}}{2}.$$

(vii) Differentiate x^x w.r.t. $x^{\sin x}$.

Solution:

Let u = x^x and v = $x^{\sin x}$

Then we want to find $\frac{du}{dx}$.

Take, u = x^x

log u = log x^x = x log x

Differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx}=\frac{d}{dx}(x\log x)$$

$$=x\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(x)$$

$$=x\times\frac{1}{x}+(\log x)\times 1$$

$$\therefore\ \frac{du}{dx}=u(1+\log x)=x^x(1+\log x)$$

Also, $v=x^{\sin x}$

$$\therefore\ \log v=\log x^{\sin x}=(\sin x)(\log x)$$

$$\frac{1}{v}\cdot\frac{dv}{dx}=\frac{d}{dx}[(\sin x)(\log x)]$$

$$=(\sin x)\cdot\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(\sin x)$$

$$=(\sin x)\times\frac{1}{x}+(\log x)(\cos x)$$

$$\therefore\ \frac{dv}{dx}=v\left[\frac{\sin x}{x}+(\log x)(\cos x)\right]$$

$$=x^{\sin x}\left[\frac{\sin x}{x}+(\log x)(\cos x)\right]$$

$$\therefore\ \frac{du}{dv}=\frac{(du/dx)}{(dv/dx)}=\frac{x^x(1+\log x)}{x^{\sin x}\left[\frac{\sin x}{x}+(\log x)(\cos x)\right]}$$

$$=\frac{x^x(1+\log x)\times x}{x^{\sin x}[\sin x+x\cos x\cdot\log x]}$$

$$=\frac{(1+\log x)\cdot x^{x+1-\sin x}}{\sin x+x\cos x\cdot\log x}.$$

(viii) Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$
Solution:
Let u = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and v = $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$
Then we want to find $\frac{du}{dv}$
u = $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$
Put x = tan θ. Then θ = $\tan^{-1}x$ and

and $v=\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$.

Then we want to find $\frac{du}{dv}$.

$$u=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

Put $x=\tan\theta$. Then $\theta=\tan^{-1}x$ and

$$\frac{\sqrt{1+x^2}-1}{x}=\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}$$

$$=\frac{\sec\theta-1}{\tan\theta}=\frac{\frac{1}{\cos\theta}-1}{\left(\frac{\sin\theta}{\cos\theta}\right)}$$

$$=\frac{1-\cos\theta}{\sin\theta}=\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}$$

$$=\tan\left(\frac{\theta}{2}\right)$$

$$\therefore\ u=\tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right]=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x$$

$$\therefore\ \frac{du}{dx}=\frac{1}{2}\frac{d}{dx}(\tan^{-1}x)$$

$$=\frac{1}{2}\times\frac{1}{1+x^2}=\frac{1}{2(1+x^2)}$$

$$v=\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$$

Put $x=\sin\theta$. Then $\theta=\sin^{-1}x$ and

$$\frac{2x\sqrt{1-x^2}}{1-2x^2}=\frac{2\sin\theta\sqrt{1-\sin^2\theta}}{1-2\sin^2\theta}$$

$$=\frac{2\sin\theta\cos\theta}{1-2\sin^2\theta}=\frac{\sin2\theta}{\cos2\theta}=\tan2\theta$$

$$\therefore\ v=\tan^{-1}(\tan2\theta)=2\theta=2\sin^{-1}x$$

$$\therefore\ \frac{dv}{dx}=2\frac{d}{dx}(\sin^{-1}x)$$

$$=2\times\frac{1}{\sqrt{1-x^2}}=\frac{2}{\sqrt{1-x^2}}$$

$$\therefore\ \frac{du}{dv}=\frac{(du/dx)}{(dv/dx)}=\frac{\left[\frac{1}{2(1+x^2)}\right]}{\left(\frac{2}{\sqrt{1-x^2}}\right)}$$

$$=\frac{1}{2(1+x^2)}\times\frac{\sqrt{1-x^2}}{2}=\frac{\sqrt{1-x^2}}{4(1+x^2)}.$$

Maharashtra State Board 12th Maths Solutions Chapter 1 Differentiation Ex 1.5

Question 1.
Find the second order derivatives of the following:
(i) $2x^5 - 4x^3 - \frac{2}{x^2} - 9$
Solution:
Let $y = 2x^5 - 4x^3 - \frac{2}{x^2} - 9$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}\left(2x^5 - 4x^3 - \frac{2}{x^2} - 9\right)$$
$$= 2\frac{d}{dx}(x^5) - 4\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^{-2}) - \frac{d}{dx}(9)$$
$$= 2 \times 5x^4 - 4 \times 3x^2 - 2(-2)x^{-3} - 0$$
$$= 10x^4 - 12x^2 + 4x^{-3}$$
$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}(10x^4 - 12x^2 + 4x^{-3})$$
$$= 10\frac{d}{dx}(x^4) - 12\frac{d}{dx}(x^2) + 4\frac{d}{dx}(x^{-3})$$
$$= 10 \times 4x^3 - 12 \times 2x + 4(-3)x^{-4}$$
$$= 40x^3 - 24x - \frac{12}{x^4}.$$

(ii) $e^{2x} \cdot \tan x$
Solution:
Let $y = e^{2x} \cdot \tan x$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(e^{2x} \cdot \tan x)$$
$$= e^{2x} \cdot \frac{d}{dx}(\tan x) + \tan x \cdot \frac{d}{dx}(e^{2x})$$
$$= e^{2x} \times \sec^2 x + \tan x \times e^{2x} \cdot \frac{d}{dx}(2x)$$
$$= e^{2x} \cdot \sec^2 x + e^{2x} \cdot \tan x \times 2$$
$$= e^{2x}(\sec^2 x + 2\tan x)$$
$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}[e^{2x}(\sec^2 x + 2\tan x)]$$
$$= e^{2x} \cdot \frac{d}{dx}(\sec^2 x + 2\tan x) + (\sec^2 x + 2\tan x)\frac{d}{dx}(e^{2x})$$
$$= e^{2x}\left[\frac{d}{dx}(\sec x)^2 + 2\frac{d}{dx}(\tan x)\right] + (\sec^2 x + 2\tan x) \times e^{2x} \cdot \frac{d}{dx}(2x)$$
$$= e^{2x}[2\sec x \cdot \frac{d}{dx}(\sec x) + 2\sec^2 x] + (\sec^2 x + 2\tan x)e^{2x} \times 2$$
$$= e^{2x}(2\sec x \cdot \sec x \tan x + 2\sec^2 x) + 2e^{2x}(\sec^2 x + 2\tan x)$$
$$= 2e^{2x}(\sec^2 x \tan x + \sec^2 x + \sec^2 x + 2\tan x)$$
$$= 2e^{2x}[\sec^2 x(\tan x + 1) + 1 + \tan^2 x + 2\tan x]$$
$$= 2e^{2x}[\sec^2 x(1 + \tan x) + (1 + \tan x)^2]$$
$$= 2e^{2x}[(1 + \tan x)(\sec^2 x + 1 + \tan x)]$$
$$= 2e^{2x}[(1 + \tan x)(1 + \tan^2 x + 1 + \tan x)]$$
$$= 2e^{2x}(1 + \tan x)(2 + \tan x + \tan^2 x).$$

(iii) $e^{4x} . \cos 5x$
Solution:
Let y = $e^{4x} . \cos 5x$

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(e^{4x} \cdot \cos 5x)$$
$$= e^{4x} \cdot \frac{d}{dx}(\cos 5x) + \cos 5x \cdot \frac{d}{dx}(e^{4x})$$
$$= e^{4x} \cdot (-\sin 5x) \cdot \frac{d}{dx}(5x) + \cos 5x \times e^{4x} \cdot \frac{d}{dx}(4x)$$
$$= -e^{4x} \cdot \sin 5x \times 5 + e^{4x} \cos 5x \times 4$$
$$= e^{4x}(4\cos 5x - 5\sin 5x)$$
$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}[e^{4x}(4\cos 5x - 5\sin 5x)]$$
$$= e^{4x}\frac{d}{dx}(4\cos 5x - 5\sin 5x) + (4\cos 5x - 5\sin 5x) \cdot \frac{d}{dx}(e^{4x})$$
$$= e^{4x}\left[4(-\sin 5x) \cdot \frac{d}{dx}(5x) - 5\cos 5x \cdot \frac{d}{dx}(5x)\right] + (4\cos 5x - 5\sin 5x) \times e^{4x} \cdot \frac{d}{dx}(4x)$$
$$= e^{4x}[-4\sin 5x \times 5 - 5\cos 5x \times 5] + (4\cos 5x - 5\sin 5x)e^{4x} \times 4$$
$$= e^{4x}(-20\sin 5x - 25\cos 5x + 16\cos 5x - 20\sin 5x)$$
$$= e^{4x}(-9\cos 5x - 40\sin 5x)$$
$$= -e^{4x}(9\cos 5x + 40\sin 5x).$$

(iv) $x^3 . \log x$
Solution:
Let y = $x^3 . \log x$

$$\text{Then, } \frac{dy}{dx} = \frac{d}{dx}(x^3 \cdot \log x)$$
$$= x^3\frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x^3)$$
$$= x^3 \times \frac{1}{x} + (\log x) \times 3x^2$$
$$= x^2 + 3x^2 \log x$$
$$= x^2(1 + 3\log x)$$
$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}[x^2(1 + 3\log x)]$$
$$= x^2 \cdot \frac{d}{dx}(1 + 3\log x) + (1 + 3\log x) \cdot \frac{d}{dx}(x^2)$$
$$= x^2\left(0 + 3 \times \frac{1}{x}\right) + (1 + 3\log x) \times 2x$$
$$= 3x + 2x + 6x\log x$$
$$= 5x + 6x\log x = x(5 + 6\log x).$$

(v) log(log x)
Solution:
Let y = log(log x)

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}[\log(\log x)]$$

$$= \frac{1}{\log x} \cdot \frac{d}{dx}(\log x)$$

$$= \frac{1}{\log x} \times \frac{1}{x} = \frac{1}{x \log x}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}(x \log x)^{-1}$$

$$= (-1)(x \log x)^{-2} \cdot \frac{d}{dx}(x \log x)$$

$$= \frac{-1}{(x \log x)^2} \cdot \left[x \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x) \right]$$

$$= \frac{-1}{(x \log x)^2} \cdot \left[x \times \frac{1}{x} + (\log x) \times 1 \right]$$

$$= -\frac{1 + \log x}{(x \log x)^2}.$$

(vi) x^x
Solution:
$y = x^x$
$\log y = \log x^x = x \log x$
Differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(x \log x)$$

$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{d}{dx}(\log x) + (\log x) \cdot \frac{d}{dx}(x)$$

$$= \frac{x}{x} + (\log x)(1) = 1 + \log x$$

$$\therefore \frac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x)$$

$$\therefore \frac{d}{dx}(x^x) = x^x(1 + \log x) \qquad \ldots (1)$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}[x^x(1 + \log x)]$$

$$= x^x \cdot \frac{d}{dx}(1 + \log x) + (1 + \log x) \cdot \frac{d}{dx}(x^x)$$

$$= x^x \left(0 + \frac{1}{x} \right) + (1 + \log x) \cdot x^x(1 + \log x) \qquad \ldots \text{[By (1)]}$$

$$= x^{x-1} + x^x(1 + \log x)^2.$$

Question 2.
Find $\frac{d^2y}{dx^2}$ of the following:
(i) x = a(θ – sin θ), y = a (1 – cos θ)
Solution:
x = a(θ – sin θ), y = a (1 – cos θ)
Differentiating x and y w.r.t. θ, we get

$\frac{dx}{d\theta}=a\frac{d}{d\theta}(\theta-\sin\theta)$ = a(1 – cos θ) …….(1)

$$\text{and } \frac{dy}{d\theta}=a\frac{d}{d\theta}(1-\cos\theta)$$
$$=a[0-(-\sin\theta)]=a\sin\theta$$
$$\therefore \frac{dy}{dx}=\frac{(dy/d\theta)}{(dx/d\theta)}=\frac{a\sin\theta}{a(1-\cos\theta)}$$
$$=\frac{2\sin\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\theta}{2}\right)}=\cot\left(\frac{\theta}{2}\right)$$
$$\text{and } \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\cot\left(\frac{\theta}{2}\right)\right]$$
$$=\frac{d}{d\theta}\left[\cot\left(\frac{\theta}{2}\right)\right]\cdot\frac{d\theta}{dx}$$
$$=-\operatorname{cosec}^2\left(\frac{\theta}{2}\right)\cdot\frac{d}{d\theta}\left(\frac{\theta}{2}\right)\times\frac{1}{\left(\frac{dx}{d\theta}\right)}$$
$$=-\operatorname{cosec}^2\left(\frac{\theta}{2}\right)\times\frac{1}{2}\times\frac{1}{a(1-\cos\theta)} \quad \ldots \text{[By (1)]}$$
$$=-\frac{1}{2a}\operatorname{cosec}^2\left(\frac{\theta}{2}\right)\times\frac{1}{2\sin^2\left(\frac{\theta}{2}\right)}$$
$$=-\frac{1}{4a}\cdot\operatorname{cosec}^4\left(\frac{\theta}{2}\right).$$

(ii) $x = 2at^2$, $y = 4at$
Solution:
$x = 2at^2$, $y = 4at$
Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}(2at^2)=2a\cdot\frac{d}{dt}(t^2)$$
$$=2a\times 2t=4at \quad \ldots (1)$$
$$\text{and } \frac{dy}{dt}=\frac{d}{dt}(4at)=4a\frac{d}{dt}(t)$$
$$=4a\times 1=4a$$
$$\therefore \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{4a}{4at}=\frac{1}{t}$$
$$\text{and } \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{t}\right)=\frac{d}{dt}(t^{-1})\times\frac{dt}{dx}$$
$$=-1(t)^{-2}\times\frac{1}{\left(\frac{dx}{dt}\right)}$$
$$=-\frac{1}{t^2}\times\frac{1}{4at} \quad \ldots \text{[By (1)]}$$
$$=-\frac{1}{4at^3}.$$

(iii) $x = \sin\theta$, $y = \sin^3\theta$ at $\theta = \frac{\pi}{2}$
Solution:
$x = \sin\theta$, $y = \sin^3\theta$
Differentiating x and y w.r.t. θ, we get,

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(\sin\theta)=\cos\theta \quad \ldots (1)$$

and $$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin\theta)^3=3(\sin\theta)^2\cdot\frac{d}{d\theta}(\sin\theta)$$

$$=3\sin^2\theta\cdot\cos\theta$$

$$\therefore \frac{dy}{dx}=\frac{(dy/d\theta)}{(dx/d\theta)}=\frac{3\sin^2\theta\cos\theta}{\cos\theta}$$

$$=3\sin^2\theta$$

and $$\frac{d^2y}{dx^2}=3\frac{d}{dx}(\sin\theta)^2$$

$$=3\frac{d}{d\theta}(\sin\theta)^2\times\frac{d\theta}{dx}$$

$$=3\times 2\sin\theta\cdot\frac{d}{d\theta}(\sin\theta)\times\frac{1}{\left(\frac{dx}{d\theta}\right)}$$

$$=6\sin\theta\cdot\cos\theta\times\frac{1}{\cos\theta} \quad \ldots \text{[By (1)]}$$

$$=6\sin\theta$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{\text{at }\theta=\frac{\pi}{2}}=6\sin\frac{\pi}{2}$$

$$=6\times 1=6.$$

Alternative Method :

$x=\sin\theta,\ y=\sin^3\theta$

$\therefore\ y=x^3$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(x^3)=3x^2$$

$$\therefore \frac{d^2y}{dx^2}=3\frac{d}{dx}(x^2)=3\times 2x=6x$$

If $\theta=\frac{\pi}{2}$, then $x=\sin\frac{\pi}{2}=1$

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{\text{at }\theta=\frac{\pi}{2}}=\left(\frac{d^2y}{dx^2}\right)_{\text{at }x=1}=6(1)=6.$$

(iv) x = a cos θ, y = b sin θ at θ = $\pi 4$
Solution:
x = a cos θ, y = b sin θ
Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(a\cos\theta)=a\frac{d}{d\theta}(\cos\theta)$$

$$=a(-\sin\theta)=-a\sin\theta \quad \ldots (1)$$

and $\frac{dy}{d\theta} = \frac{d}{d\theta}(b\sin\theta) = b\frac{d}{dx}(\sin\theta)$

$= b\cos\theta$

$\therefore \frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{b\cos\theta}{-a\sin\theta}$

$= \left(-\frac{b}{a}\right)\cot\theta$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\left(-\frac{b}{a}\right)\cot\theta\right]$

$= -\frac{b}{a}\cdot\frac{d}{d\theta}(\cot\theta) \times \frac{d\theta}{dx}$

$= \left(-\frac{b}{a}\right)(-\operatorname{cosec}^2\theta) \times \frac{1}{\left(\frac{dx}{d\theta}\right)}$

Question 3.
(i) If x = at^2 and y = 2at, then show that $xy\frac{d^2y}{dx^2}+a=0$
Solution:
x = at^2, y = 2at ....(1)
Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt}=\frac{d}{dt}(at^2)=a\frac{d}{dt}(t^2)$$

$$=a\times 2t=2at \quad \ldots (2)$$

and $\frac{dy}{dt}=\frac{d}{dt}(2at)=2a\frac{d}{dt}(t)$

$$=2a\times 1=2a$$

$$\therefore \frac{dy}{dx}=\frac{(dy/dt)}{(dx/dt)}=\frac{2a}{2at}=\frac{1}{t}$$

and $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{t}\right)=\frac{d}{dt}(t^{-1})\cdot\frac{dt}{dx}$

$$=(-1)t^{-2}\cdot\frac{1}{\left(\frac{dx}{dt}\right)}=\frac{-1}{t^2}\times\frac{1}{2at} \quad \ldots \text{[By (2)]}$$

$$=-\frac{1}{2at^3}.$$

$$\therefore \frac{d^2y}{dx^2}=-\frac{1}{(at^2)(2at)}\times a$$

$$=-\frac{a}{xy} \quad \ldots \text{[By (1)]}$$

$$\therefore xy\frac{d^2y}{dx^2}=-a$$

$$\therefore xy\frac{d^2y}{dx^2}+a=0.$$

(ii) If y = $e^{m\tan^{-1}x}$, show that $(1+x^2)\frac{d^2y}{dx^2}+(2x-m)\frac{dy}{dx}=0$

Solution:

y = $e^{m\tan^{-1}x}$ ……..(1)

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(e^{m\tan^{-1}x})$$

$$=e^{m\tan^{-1}x}\cdot\frac{d}{dx}(m\tan^{-1}x)$$

$$=e^{m\tan^{-1}x}\times m\times\frac{1}{1+x^2}$$

$$\therefore (1+x^2)\frac{dy}{dx}=my \quad \ldots \text{[By (1)]}$$

Differentiating again w.r.t. x, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{d}{dx}(1+x^2)=m\frac{dy}{dx}$$

$$\therefore (1+x^2)\frac{d^2y}{dx^2}+\frac{dy}{dx}(0+2x)=m\frac{dy}{dx}$$

$$\therefore (1+x^2)\frac{d^2y}{dx^2}+2x\cdot\frac{dy}{dx}=m\frac{dy}{dx}.$$

$$\therefore (1+x^2)\frac{d^2y}{dx^2}+(2x-m)\frac{dy}{dx}=0.$$

(iii) If x = cos t, y = e^{mt}, show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0$

Solution:

x = cos t, y = e^{mt}

$\therefore\ t = \cos^{-1} x$ and $y = e^{m \cos^{-1} x}$ ... (1)

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}(e^{m \cos^{-1} x})$$

$$= e^{m \cos^{-1} x} \cdot \frac{d}{dx}(m \cos^{-1} x)$$

$$= e^{m \cos^{-1} x} \times m \times \frac{-1}{\sqrt{1-x^2}}$$

$$\therefore\ \sqrt{1-x^2} \cdot \frac{dy}{dx} = -my \quad \text{... [By (1)]}$$

$$\therefore\ (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2 y^2$$

Differentiating again w.r.t. x, we get

$$(1-x^2) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2 \cdot \frac{d}{dx}(1-x^2) = m^2 \cdot \frac{d}{dx}(y^2)$$

$$\therefore\ (1-x^2) \cdot 2\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 (0-2x) = m^2 \times 2y\frac{dy}{dx}$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = m^2 y$$

$$\therefore\ (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2 y = 0.$$

(iv) If y = x + tan x, show that $\cos^2 x \cdot \frac{d^2y}{dx^2} - 2y + 2x = 0$
Solution:
y = x + tan x

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}(x + \tan x)$$

$$= 1 + \sec^2 x$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}(1 + \sec^2 x)$$

$$= \frac{d}{dx}(1) + \frac{d}{dx}(\sec x)^2$$

$$= 0 + 2\sec x \cdot \frac{d}{dx}(\sec x)$$

$$= 2\sec x \cdot \sec x \tan x$$

$$= 2\sec^2 x \tan x$$

$$\therefore\ \cos^2 x \cdot \frac{d^2y}{dx^2} - 2y + 2x$$

$$= \cos^2 x\,(2\sec^2 x \tan x) - 2(x + \tan x) + 2x$$

$$= \cos^2 x \times \frac{2}{\cos^2 x} \times \tan x - 2x - 2\tan x + 2x$$

$$= 2\tan x - 2\tan x.$$

$$\therefore\ \cos^2 x \cdot \frac{d^2y}{dx^2} - 2y + 2x = 0.$$

(v) If y = e^{ax} . sin (bx), show that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.
Solution:
y = e^{ax} . sin (bx) ....(1)

$$\therefore \frac{dy}{dx}=\frac{d}{dx}[e^{ax}\cdot\sin(bx)]$$

$$=e^{ax}\cdot\frac{d}{dx}[\sin(bx)]+\sin(bx)\cdot\frac{d}{dx}(e^{ax})$$

$$=e^{ax}\cdot\cos(bx)\cdot\frac{d}{dx}(bx)+\sin(bx)\times e^{ax}\cdot\frac{d}{dx}(ax)$$

$$=e^{ax}\cdot\cos(bx)\times b+e^{ax}\cdot\sin(bx)\times a$$

$$\therefore\ y_1=e^{ax}[b\cos(bx)+a\sin(bx)] \qquad \dots (2)$$

Differentiating again w.r.t. x, we get

$$\frac{dy_1}{dx}=\frac{d}{dx}[e^{ax}\{b\cos(bx)+a\sin(bx)\}]$$

$$=e^{ax}\cdot\frac{d}{dx}[b\cos(bx)+a\sin(bx)]+$$

$$[b\cos(bx)+a\sin(bx)]\cdot\frac{d}{dx}(e^{ax})$$

$$=e^{ax}\cdot[b\{-\sin(bx)\}\cdot\frac{d}{dx}(bx)+a\cos(bx)\cdot\frac{d}{dx}(bx)]+$$

$$[b\cos(bx)+a\sin(bx)]\times e^{ax}\cdot\frac{d}{dx}(ax)$$

$$=e^{ax}[-b\sin(bx)\times b+a\cos(bx)\times b]+$$

$$[b\cos(bx)+a\sin(bx)]e^{ax}\times a$$

$$=e^{ax}[-b^2\sin(bx)+ab\cos(bx)+ab\cos(bx)+a^2\sin(bx)]$$

$$\therefore\ y_2=e^{ax}[-b^2\sin(bx)+2ab\cos(bx)+a^2\sin(bx)] \quad \dots (3)$$

$$\therefore\ y_2-2ay_1+(a^2+b^2)y$$

$$=e^{ax}[-b^2\sin(bx)+2ab\cos(bx)+a^2\sin(bx)]-$$

$$2a\cdot e^{ax}[b\cos(bx)+a\sin(bx)]+$$

$$(a^2+b^2)e^{ax}\sin(bx) \quad \dots \text{[By (1), (2) and (3)]}$$

$$=e^{ax}[-b^2\sin bx+2ab\cos(bx)+a^2\sin(bx)-$$

$$2ab\cos(bx)-2a^2\sin(bx)+a^2\sin(bx)+b^2\sin(bx)]$$

$$=e^{ax}\times 0$$

$$\therefore\ y_2-2ay_1+(a^2+b^2)y=0.$$

(vi) If $\sec^{-1}\left(\frac{7x^3-5y^3}{7x^3+5y^3}\right)=m$, show that $\frac{d^2y}{dx^2}=0$

Solution:

Solution : $\sec^{-1}\left(\frac{7x^3-5y^3}{7x^3+5y^3}\right)=m$

$$\therefore\ \frac{7x^3-5y^3}{7x^3+5y^3}=\sec m=k \qquad \dots \text{(Say)}$$

$$\therefore\ 7x^3-5y^3=7kx^3+5ky^3$$

$$\therefore\ (5k+5)y^3=(7-7k)x^2$$

$$\therefore\ \frac{y^3}{x^3}=\frac{7-7k}{5k+5}$$

$$\therefore \frac{y}{x}=\left(\frac{7-7k}{5k+5}\right)^{\frac{1}{3}}=p, \text{ where } p \text{ is a constant.}$$

$$\therefore \frac{d}{dx}\left(\frac{y}{x}\right)=\frac{d}{dx}(p)$$

$$\therefore \frac{x\frac{dy}{dx}-y\frac{d}{dx}(x)}{x^2}=0$$

$$\therefore x\frac{dy}{dx}-y\times 1=0$$

$$\therefore x\frac{dy}{dx}=y$$

$$\therefore \frac{dy}{dx}=\frac{y}{x} \qquad \ldots (1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$=\frac{x\frac{dy}{dx}-y\frac{d}{dx}(x)}{x^2}$$

$$=\frac{x\left(\frac{y}{x}\right)-y\times 1}{x^2} \qquad \ldots [\text{By } (1)]$$

$$=\frac{y-y}{x^2}=\frac{0}{x^2}=0$$

Note : $\frac{dy}{dx}=\frac{y}{x}$, where $\frac{y}{x}=p$,

$$\therefore \frac{dy}{dx}=p, \text{ where } p \text{ is a constant.}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}(p)=0.$$

(vii) If $2y = \sqrt{x+1}+\sqrt{x-1}$, show that $4(x^2 - 1)y_2 + 4xy_1 - y = 0$.

Solution:

$2y = \sqrt{x+1}+\sqrt{x-1}$ …… (1)

Differentiating both sides w.r.t. x, we get

$$\therefore\ 2\frac{dy}{dx}=\frac{d}{dx}(\sqrt{x+1})+\frac{d}{dx}(\sqrt{x-1})$$

$$=\frac{1}{2\sqrt{x+1}}(1+0)+\frac{1}{2\sqrt{x-1}}(1-0)$$

$$\therefore\ 2\frac{dy}{dx}=\frac{1}{2\sqrt{x+1}}+\frac{1}{2\sqrt{x-1}}$$

$$=\frac{\sqrt{x-1}+\sqrt{x+1}}{2\sqrt{x+1}\cdot\sqrt{x-1}}$$

$$=\frac{2y}{2\sqrt{x^2-1}} \qquad \text{... [By (1)]}$$

$$\therefore\ 2\sqrt{x^2-1}\frac{dy}{dx}=y$$

$$\therefore\ 4(x^2-1)\cdot\left(\frac{dy}{dx}\right)^2=y^2$$

Differentiating both sides w.r.t. x, we get

$$4(x^2-1)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2\cdot\frac{d}{dx}[4(x^2-1)]=2y\frac{dy}{dx}$$

$$\therefore\ 4(x^2-1)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\cdot 4(2x)=2y\frac{dy}{dx}$$

Cancelling $2\frac{dy}{dx}$ on both sides, we get

$$4(x^2-1)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}=y$$

$$\therefore\ 4(x^2-1)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}-y=0$$

$$\therefore\ 4(x^2-1)y_2+4xy_1-y=0.$$

(viii) If y = $\log(x+\sqrt{x^2+a^2})^m$, show that $(x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$

Solution:

y = $\log(x+\sqrt{x^2+a^2})^m$ = $m\log(x+\sqrt{x^2+a^2})$

$$\therefore \frac{dy}{dx}=m\frac{d}{dx}[\log(x+\sqrt{x^2+a^2})]$$

$$=m\times\frac{1}{x+\sqrt{x^2+a^2}}\cdot\frac{d}{dx}(x+\sqrt{x^2+a^2})$$

$$=\frac{m}{x+\sqrt{x^2+a^2}}\times\left[1+\frac{1}{2\sqrt{x^2+a^2}}\cdot\frac{d}{dx}(x^2+a^2)\right]$$

$$=\frac{m}{x+\sqrt{x^2+a^2}}\times\left[1+\frac{1}{2\sqrt{x^2+a^2}}\times(2x+0)\right]$$

$$=\frac{m}{x+\sqrt{x^2+a^2}}\times\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$$

$$\therefore \frac{dy}{dx}=\frac{m}{\sqrt{x^2+a^2}}$$

$$\therefore \sqrt{x^2+a^2}\frac{dy}{dx}=m$$

$$\therefore (x^2+a^2)\left(\frac{dy}{dx}\right)^2=m^2$$

Differentiating both sides w.r.t. x, we get

$$(x^2+a^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2\cdot\frac{d}{dx}(x^2+a^2)=\frac{d}{dx}(m^2)$$

$$\therefore (x^2+a^2)\times 2\frac{dy}{dx}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)^2\times(2x+0)=0$$

$$\therefore (x^2+a^2)\cdot 2\frac{dy}{dx}\frac{d^2y}{dx^2}+2x\left(\frac{dy}{dx}\right)^2=0$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$(x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0.$$

(ix) If y = sin(m $\cos^{-1}x$), then show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$

Solution:

y = sin(m $\cos^{-1}x$)

$\sin^{-1}y = m\cos^{-1}x$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx}=m\times\frac{-1}{\sqrt{1-x^2}}$$

$$\therefore\ \sqrt{1-x^2}\cdot\frac{dy}{dx}=-m\sqrt{1-y^2}$$

$$\therefore\ (1-x^2)\left(\frac{dy}{dx}\right)^2=m^2(1-y^2)$$

$$\therefore\ (1-x^2)\left(\frac{dy}{dx}\right)^2=m^2-m^2y^2$$

Differentiating both sides w.r.t. x, we get

$$(1-x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2\cdot\frac{d}{dx}(1-x^2)=0-m^2\cdot\frac{d}{dx}(y^2)$$

$$\therefore\ (1-x^2)\cdot 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2=-2m^2y\frac{dy}{dx}$$

Cancelling $2\frac{dy}{dx}$ throughout, we get

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=-m^2y$$

$$\therefore\ (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0.$$

(x) If y = log(log 2x), show that $xy_2 + y_1(1 + xy_1) = 0$.

Solution:

y = log(log 2x)

$$\therefore\ \frac{dy}{dx}=\frac{d}{dx}[\log(\log 2x)]$$

$$=\frac{1}{\log 2x}\cdot\frac{d}{dx}(\log 2x)$$

$$= \frac{1}{\log 2x} \times \frac{1}{2x} \cdot \frac{d}{dx}(2x)$$

$$= \frac{1}{\log 2x} \times \frac{1}{2x} \times 2$$

$$\therefore \frac{dy}{dx} = \frac{1}{x \log 2x}$$

$$\therefore (\log 2x) \cdot \frac{dy}{dx} = \frac{1}{x} \quad \ldots (1)$$

Differentiating both sides w.r.t. x, we get

$$(\log 2x) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(\log 2x) = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\therefore (\log 2x) \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1}{2x} \cdot \frac{d}{dx}(2x) = -\frac{1}{x^2}$$

$$\therefore (\log 2x) \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1}{2x} \times 2 = -\frac{1}{x^2}$$

$$\therefore (\log 2x) \cdot \frac{d^2y}{dx^2} + \frac{1}{x} \cdot \frac{dy}{dx} = -\frac{1}{x} \cdot \frac{1}{x}$$

$$\therefore (\log 2x) \cdot \frac{d^2y}{dx^2} + \left[(\log 2x) \cdot \frac{dy}{dx}\right] \cdot \frac{dy}{dx} = -\frac{1}{x}\left[(\log 2x) \cdot \frac{dy}{dx}\right]$$

$$\ldots [\text{By } (1)]$$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -\frac{1}{x}\frac{dy}{dx}$$

$$\therefore x\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = -\frac{dy}{dx}$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 = 0$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(1 + x\frac{uy}{dx}\right) = 0$$

$$\therefore xy_2 + y_1(1 + xy_1) = 0.$$

(xi) If $x^2 + 6xy + y^2 = 10$, show that $\frac{d^2y}{dx^2} = \frac{80}{(3x+y)^3}$

Solution:

$x^2 + 6xy + y^2 = 10$ …… (1)

Differentiating both sides w.r.t. x, we get

$$2x + 6\left[x\frac{dy}{dx} + y \cdot \frac{d}{dx}(x)\right] + 2y\frac{dy}{dx} = 0$$

$$\therefore 2x + 6x\frac{dy}{dx} + 6y \times 1 + 2y\frac{dy}{dx} = 0$$

$$\therefore (6x + 2y)\frac{dy}{dx} = -2x - 6y$$

$$\therefore \frac{dy}{dx} = \frac{-2(x+3y)}{2(3x+y)} = -\left(\frac{x+3y}{3x+y}\right) \quad \ldots (2)$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x+3y}{3x+y}\right)$$

$$= -\left[\frac{(3x+y)\cdot\frac{d}{dx}(x+3y) - (x+3y)\cdot\frac{d}{dx}(3x+y)}{(3x+y)^2}\right]$$

$$= -\left[\frac{(3x+y)\left(1+3\frac{dy}{dx}\right) - (x+3y)\left(3+\frac{dy}{dx}\right)}{(3x+y)^2}\right]$$

$$= \frac{1}{(3x+y)^2}\left[-(3x+y)\left\{1-\frac{3(x+3y)}{3x+y}\right\} + (x+3y)\left(3-\frac{x+3y}{3x+y}\right)\right] \quad \ldots \text{[By (2)]}$$

$$= \frac{1}{(3x+y)^2}\left[-(3x+y)\left(\frac{3x+y-3x-9y}{3x+y}\right) + (x+3y)\left(\frac{9x+3y-x-3y}{3x+y}\right)\right]$$

$$= \frac{1}{(3x+y)^2}\left[8y + \frac{(x+3y)(8x)}{3x+y}\right]$$

$$= \frac{1}{(3x+y)^2}\left[\frac{8y(3x+y)+(x+3y)8x}{3x+y}\right]$$

$$= \frac{24xy+8y^2+8x^2+24xy}{(3x+y)^3}$$

$$= \frac{8x^2+48xy+8y^2}{(3x+y)^3} = \frac{8(x^2+6xy+y^2)}{(3x+y)^3}$$

$$= \frac{8(10)}{(3x+y)^3} \quad \ldots \text{[By (1)]}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{80}{(3x+y)^3}.$$

(xii) If x = a sin t – b cos t, y = a cos t + b sin t, Show that $\frac{d^2y}{dx^2} = -\frac{x^2+y^2}{y^3}$

Solution:

x = a sin t – b cos t, y = a cos t + b sin t

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = a\frac{d}{dx}(\sin t) - b\frac{d}{dt}(\cos t)$$

$$= a\cos t - b(-\sin t) = a\cos t + b\sin t$$

and $\frac{dy}{dt} = a\frac{d}{dx}(\cos t) - b\frac{d}{dt}(\sin t)$

$$= a(-\sin t) + b\cos t = -a\sin t + b\cos t$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{-a\sin t + b\cos t}{a\cos t + b\sin t}$$

$$= -\left(\frac{a\sin t - b\cos t}{a\cos t + b\sin t}\right)$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y} \quad \ldots (1)$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x}{y}\right) = -\left[\frac{y\frac{d}{dx}(x) - x\frac{dy}{dx}}{y^2}\right]$$

$$= -\left[\frac{y \times 1 - x\left(-\frac{x}{y}\right)}{y^2}\right] \quad \ldots [\text{By } (1)]$$

$$= -\left[\frac{y^2 + x^2}{y^3}\right]$$

$$\therefore \frac{d^2y}{dx^2} = -\frac{x^2 + y^2}{y^3}.$$

Question 4.
Find the nth derivative of the following:
(i) $(ax + b)^m$
Solution:
Let $y = (ax + b)^m$

Then $\frac{dy}{dx} = \frac{d}{dx}(ax + b)^m$

$$= m(ax + b)^{m-1} \cdot \frac{d}{dx}(ax + b)$$

$$= m(ax + b)^{m-1} \times (a \times 1 + 0)$$

$$= am(ax + b)^{m-1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}[am(ax + b)^{m-1}]$$

$$= am\frac{d}{dx}(ax + b)^{m-1}$$

$$= am(m-1)(ax+b)^{m-2} \cdot \frac{d}{dx}(ax+b)$$
$$= am(m-1)(ax+b)^{m-2} \times (a \times 1 + 0)$$
$$= a^2m(m-1)(ax+b)^{m-2}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}[a^2m(m-1)(ax+b)^{m-2}]$$
$$= a^2m(m-1)\frac{d}{dx}(ax+b)^{m-2}$$
$$= a^2m(m-1)(m-2)(ax+b)^{m-3}\frac{d}{dx}(ax+b)$$
$$= a^2m(m-1)(m-2)(ax+b)^{m-3} \times (a \times 1 + 0)$$
$$= a^3m(m-1)(m-2)(ax+b)^{m-3}$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n} = a^nm(m-1)(m-2)\ \dots\ (m-n+1)(ax+b)^{m-n}$$

Case (i) : If $m > 0$, $m > n$, then

$$\frac{d^ny}{dx^n} = \frac{a^n \cdot m(m-1)(m-2)\dots(m-n+1)(m-n)\dots 3\cdot 2\cdot 1}{(m-n)(m-n-1)\dots 3\cdot 2\cdot 1} \times (ax+b)^{m-n}$$

$$\therefore\ \frac{d^ny}{dx^n} = \frac{a^n \cdot m!\,(ax+b)^{m-n}}{(m-n)!},\ \text{if } m > 0,\ m > n.$$

Case (ii) : If $m > 0$ and $m < n$, then its m^{th} order derivative is a constant and every derivatives after m^{th} order are zero.

$$\therefore\ \frac{d^ny}{dx^n} = 0,\ \text{if } m > 0,\ m < n.$$

Case (iii) : If $m > 0$, $m = n$, then

$$\frac{d^ny}{dx^n} = a^n \cdot n(n-1)(n-2)\dots(n-n+1)(ax+b)^{n-n}$$
$$= a^n \cdot n(n-1)(n-2)\dots 1 \cdot (ax+b)^0$$
$$\therefore\ \frac{d^ny}{dx^n} = a^n \cdot n!,\ \text{if } m > 0,\ m = n.$$

(ii) $1x$
Solution:

Let y = 1x

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$= \frac{(-1)^1 1!}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{1}{x^2}\right) = -1\frac{d}{dx}(x^{-2})$$

$$= (-1)(-2)x^{-3} = \frac{(-1)^2 \cdot 1 \cdot 2}{x^3}$$

$$= \frac{(-1)^2 2!}{x^3}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{(-1)^2 \cdot 2!}{x^3}\right] = (-1)^2 \cdot 2! \frac{d}{dx}(x^{-3})$$

$$= (-1)^2 \cdot 2! \cdot (-3)x^{-4}$$

$$= \frac{(-1)^3 \times 3 \cdot 2!}{x^4} = \frac{(-1)^3 \cdot 3!}{x^4}$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n} = \frac{(-1)^n \cdot n!}{x^{n+1}}.$$

(iii) e^{ax+b}
Solution:
Let y = e^{ax+b}

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(e^{ax+b}) = e^{ax+b} \cdot \frac{d}{dx}(ax+b)$$

$$= e^{ax+b} \times (a \times 1 + 0) = ae^{ax+b}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(ae^{ax+b}) = a \cdot \frac{d}{dx}(e^{ax+b})$$

$$= ae^{ax+b} \cdot \frac{d}{dx}(ax+b)$$

$$= ae^{ax+b} \times (a \times 1 + 0) = a^2 \cdot e^{ax+b}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}[a^2 e^{ax+b}] = a^2 \frac{d}{dx}(e^{ax+b})$$

$$= a^2 e^{ax+b} \cdot \frac{d}{dx}(ax+b)$$

$$= a^2 e^{ax+b} \times (a \times 1 + 0) = a^3 \cdot e^{ax+b}$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n} = a^n \cdot e^{ax+b}.$$

(iv) a^{px+q}
Solution:

Let y = apx+q

Then $\frac{dy}{dx} = \frac{d}{dx}(a^{px+q}) = a^{px+q} \log a \cdot \frac{d}{dx}(px+q)$

$= a^{px+q} \cdot \log a \cdot (p \times 1 + 0) = p \log a \cdot a^{px+q}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}[p \log a \cdot a^{px+q}]$

$= p \log a \cdot \frac{d}{dx}(a^{px+q})$

$= p \log a \cdot a^{px+q} \cdot \log a \cdot \frac{d}{dx}(px+q)$

$= p \log a \cdot a^{px+q} \cdot \log a \times (p \times 1 + 0)$

$= p^2 \cdot (\log a)^2 \cdot a^{px+q}$

$\frac{d^3y}{dx^3} = \frac{d}{dx}[p^2 \cdot (\log a)^2 \cdot a^{px+q}]$

$= p^2 \cdot (\log a)^2 \cdot \frac{d}{dx}(a^{px+q})$

$= p^2 \cdot (\log a)^2 \cdot a^{px+q} \cdot \log a \cdot \frac{d}{dx}(px+q)$

$= p^2 \cdot (\log a)^3 \cdot a^{px+q} \times (p \times 1 + 0)$

$= p^3 \cdot (\log a)^3 \cdot a^{px+q}$

In general, the n^{th} order derivative is given by

$\frac{d^ny}{dx^n} = p^n \cdot (\log a)^n \cdot a^{px+q}$.

(v) log(ax + b)
Solution:
Let y = log(ax + b)

Then $\frac{dy}{dx}=\frac{d}{dx}[\log(ax+b)]$

$$=\frac{1}{ax+b}\cdot\frac{d}{dx}(ax+b)$$

$$=\frac{1}{ax+b}\times(a\times 1+0)=\frac{a}{ax+b}$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{a}{ax+b}\right)=a\frac{d}{dx}(ax+b)^{-1}$$

$$=a(-1)(ax+b)^{-2}\cdot\frac{d}{dx}(ax+b)$$

$$=\frac{(-1)a}{(ax+b)^2}\times(a\times 1+0)$$

$$=\frac{(-1)a^2}{(ax+b)^2}$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left[\frac{(-1)^1a^2}{(ax+b)^2}\right]=(-1)^1a^2\cdot\frac{d}{dx}(ax+b)^{-2}$$

$$=(-1)^1a^2\cdot(-2)(ax+b)^{-3}\cdot\frac{d}{dx}(ax+b)$$

$$=\frac{(-1)^2\cdot 1\cdot 2\cdot a^2}{(ax+b)^3}\times(a\times 1+0)$$

$$=\frac{(-1)^2\cdot 2!\,a^3}{(ax+b)^3}$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n}=\frac{(-1)^{n-1}\cdot(n-1)!\,a^n}{(ax+b)^n}.$$

(vi) cos x
Solution:
Let y = cos x

Then $\frac{dy}{dx}=\frac{d}{dx}(\cos x)=-\sin x$

$$=\cos\left(\frac{\pi}{2}+x\right)$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(-\sin x)=-\cos x$$

$$=\cos(\pi+x)=\cos\left(\frac{2\pi}{2}+x\right)$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}(-\cos x)=-\frac{d}{dx}(\cos x)$$

$$=-(-\sin x)=\sin x$$

$$=\cos\left(\frac{3\pi}{2}+x\right)$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n}=\cos\left(\frac{n\pi}{2}+x\right).$$

(vii) sin(ax + b)
Solution:
Let y = sin(ax + b)

Then $\frac{dy}{dx}=\frac{d}{dx}[\sin(ax+b)]$

$$=\cos(ax+b)\cdot\frac{d}{dx}(ax+b)$$

$$=\cos(ax+b)\times(a\times1+0)$$

$$=a\sin\left[\frac{\pi}{2}+(ax+b)\right]$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}[a\cos(ax+b)]$$

$$=a\frac{d}{dx}[\cos(ax+b)]$$

$$=a[-\sin(ax+b)]\cdot\frac{d}{dx}(ax+b)$$

$$=a[-\sin(ax+b)]\times(a\times1+0)$$

$$=a^2\cdot\sin[\pi+(ax+b)]$$

$$=a^2\cdot\sin\left[\frac{2\pi}{2}+(ax+b)\right]$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}[-a^2\sin(ax+b)]$$

$$=-a^2\frac{d}{dx}[\sin(ax+b)]$$

$$=-a^2\cdot\cos(ax+b)\cdot\frac{d}{dx}(ax+b)$$

$$=-a^2\cdot\cos(ax+b)\times(a\times1+0)$$

$$=a^3\cdot\sin\left[\frac{3\pi}{2}+(ax+b)\right]$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n}=a^n\cdot\sin\left[\frac{n\pi}{2}+(ax+b)\right].$$

(viii) cos(3 – 2x)
Solution:

Let y = cos (3 – 2x)

Then $\frac{dy}{dx} = \frac{d}{dx}[\cos(3 - 2x)]$

$= \cos(3 - 2x).\ \frac{d}{dx}(3 - 2x)$

= cos (3 – 2x) x (a x 1 + 0)

$= a\cos\left[\frac{\pi}{2} + (3 - 2x)\right]$

$\frac{d^2y}{dx^2} = \frac{d}{dx}[a\cos(3 - 2x)]$

$= a\frac{d}{dx}[\cos(3 - 2x)]$

$= a[-\cos(3 - 2x)].\ \frac{d}{dx}(3 - 2x)$

= a[– cos(3 – 2x)] x (a x 1 + 0)

= a^2 . cos[π + (3 – 2x)]

$= a^2.\cos\left[\frac{2\pi}{2} + (3 - 2x)\right]$

$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[-a^2\cos(ax + b)\right]$

$= -a^2\frac{d}{dx}[\cos(3 - 2x)]$

$= -a^2.\cos(3 - 2x).\ \frac{d}{dx}(3 - 2x)$

= – a^2. cos(3 – 2x) x (a x 1 + 0)

$= a^3.\cos\left[\frac{3\pi}{2} + (3 - 2x)\right]$

In general, the n^{th} order derivative is given by

$\frac{d^ny}{dx^n} = (-2)^n\cos\left[\frac{n\pi}{2} + (3 - 2x)\right].$

(ix) log(2x + 3)
Solution:

Let y = log (2x + 3)

Then $\frac{\text{dy}}{\text{dx}} = \frac{\text{d}}{\text{dx}}[\log(2x+3)]$

= $\frac{1}{2x+3} \cdot \frac{\text{d}}{\text{dx}}(2x+3)$

= $\frac{1}{2x+3} \times (a \times 1 + 0)$

= $\frac{a}{2x + 3}$

$\frac{d^2y}{dx^2} = \frac{\text{d}}{\text{dx}}\left(\frac{a}{2x+3}\right)$

= $a\frac{\text{d}}{\text{dx}}(2x+3)^{-1}$

= $a(-1)(2x+3)^{-2}. \frac{\text{d}}{\text{dx}}(2x+3)$

= $\frac{(-1)a}{(2x+3)^2} \times (a \times 1 + 0)$

= $\frac{(-1)a}{(2x+3)^2}$

$\frac{d^3y}{dx^3} = \frac{\text{d}}{\text{dx}}\left[\frac{(-1)^1 a^2}{(2x+3)^2}\right]$

= $(-1)^1 a^2. \frac{\text{d}}{\text{dx}}(2x+3)^{-2}$

= $(-1)^1 a^2. (-2)(2x+3)^{-3}. \frac{\text{d}}{\text{dx}}(2x+3)$

= $\frac{(-1)^2. 1.2. a^2}{(2x+3)^3} \times (a \times 1 + 0)$

= $\frac{(-1).2.2!a^3}{(2x+3)^3}$

In general, the n$^{\text{th}}$ order derivative is given by

$\frac{d^ny}{dx^2} = \frac{(-1)^{n-1}. (n-1)!2^n}{(2x+3)^n}$.

(x) 13x-5

Solution:

Let y = 13x-5

$$\text{Then } \frac{dy}{dx} = \frac{d}{dx}(3x-5)$$

$$= -1(3x-5)^{-2} \cdot \frac{d}{dx}(3x-5)$$

$$= \frac{-1}{(3x-5)^2} \times (3 \times 1 - 0)$$

$$= \frac{(-1)^1 \cdot 3}{(3x-5)^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{(-1)^1 \cdot 3}{(3x-5)^2}\right]$$

$$= (-1)^1 \cdot 3 \frac{d}{dx}(3x-5)^{-2}$$

$$= (-1)^1 \cdot 3 \cdot (-2)(3x-5)^{-3} \cdot \frac{d}{dx}(3x-5)$$

$$= \frac{(-1)^2 \cdot 3 \cdot 2}{(3x-5)^3} \times (3 \times 1 - 0)$$

$$= \frac{(-1)^2 \cdot 2! \cdot 3^2}{(3x-5)^3}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{(-1)^2 \cdot 2! \cdot 3^2}{(3x-5)^3}\right]$$

$$= (-1)^2 \cdot 2! \cdot 3^2 \cdot \frac{d}{dx}(3x-5)^{-3}$$

$$= (-1)^2 \cdot 2! \cdot 3^2 \cdot (-3)(3x-5)^{-4} \cdot \frac{d}{dx}(3x-5)$$

$$= \frac{(-1)^3 \times 3 \cdot 2! \times 3^2}{(3x-5)^4} \times (3 \times 1 - 0)$$

$$= \frac{(-1)^3 \cdot 3! \cdot 3^3}{(3x-5)^4}$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n} = \frac{(-1)^n \cdot n! \cdot 3^n}{(3x-5)^{n+1}}.$$

(xi) $y = e^{ax} . \cos(bx + c)$
Solution:
$y = e^{ax} . \cos(bx + c)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}[e^{ax} \cdot \cos(bx+c)]$$

$$= e^{ax} \cdot \frac{d}{dx}[\cos(bx+c)] + \cos(bx+c) \cdot \frac{d}{dx}(e^{ax})$$

$$= e^{ax} \cdot [-\sin(bx+c)] \cdot \frac{d}{dx}(bx+c) + \cos(bx+c) \cdot e^{ax} \cdot \frac{d}{dx}(ax)$$

$$= -e^{ax}\sin(bx+c)\times(b\times 1+0)+ e^{ax}\cos(bx+c)\times a\times 1$$

$$= e^{ax}[a\cos(bx+c)-b\sin(bx+c)]$$

$$= e^{ax}\cdot\sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos(bx+c)-\frac{b}{\sqrt{a^2+b^2}}\sin(bx+c)\right]$$

Let $\frac{a}{\sqrt{a^2+b^2}}=\cos\alpha$ and $\frac{b}{\sqrt{a^2+b^2}}=\sin\alpha$

Then $\tan\alpha=\frac{b}{a}$ $\quad\therefore\ \alpha=\tan^{-1}\left(\frac{b}{a}\right)$

$$\therefore\ \frac{dy}{dx}=e^{ax}\cdot\sqrt{a^2+b^2}[\cos\alpha\cdot\cos(bx+c)-\sin\alpha\cdot\sin(bx+c)]$$

$$= e^{ax}\cdot(a^2+b^2)^{\frac{1}{2}}\cdot\cos(bx+c+\alpha)$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}[e^{ax}\cdot(a^2+b^2)^{\frac{1}{2}}\cdot\cos(bx+c+\alpha)]$$

$$=(a^2+b^2)^{\frac{1}{2}}\cdot\frac{d}{dx}[e^{ax}\cdot\cos(bx+c+\alpha)]$$

$$=(a^2+b^2)^{\frac{1}{2}}\left[e^{ax}\cdot\frac{d}{dx}\{\cos(bx+c+\alpha)\}+\cos(bx+c+\alpha)\cdot\frac{d}{dx}(e^{ax})\right]$$

$$=(a^2+b^2)^{\frac{1}{2}}\left[e^{ax}\cdot\{-\sin(bx+c+\alpha)\}\cdot\frac{d}{dx}(bx+c+\alpha)+\cos(bx+c+\alpha)\cdot e^{ax}\cdot\frac{d}{dx}(ax)\right]$$

$$= (a^2+b^2)^{\frac{1}{2}}[-e^{ax}\sin(bx+c+\alpha)\times(b\times 1+0+0)+ \cos(bx+c+\alpha)\cdot e^{ax}\times a\times 1]$$

$$= e^{ax}\cdot(a^2+b^2)^{\frac{1}{2}}[a\cos(bx+c+\alpha)-b\sin(bx+c+\alpha)]$$

$$= e^{ax}\cdot(a^2+b^2)^{\frac{1}{2}}\cdot\sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos(bx+c+\alpha)- \frac{b}{\sqrt{a^2+b^2}}\sin(bx+c+\alpha)\right]$$

$$= e^{ax}\cdot(a^2+b^2)^{\frac{2}{2}}[\cos\alpha\cdot\cos(bx+c+\alpha)- \sin\alpha\cdot\sin(bx+c+\alpha)]$$

$$= e^{ax}\cdot(a^2+b^2)^{\frac{2}{2}}\cdot\cos(bx+c+\alpha+\alpha)$$

$$= e^{ax}\cdot(a^2+b^2)^{\frac{2}{2}}\cdot\cos(bx+c+2\alpha)$$

Similarly,

$$\frac{d^3y}{dx^3} = e^{ax}\cdot(a^2+b^2)^{\frac{3}{2}}\cdot\cos(bx+c+3\alpha)$$

In general, the n^{th} order derivative is given by

$$\frac{d^ny}{dx^n} = e^{ax}\cdot(a^2+b^2)^{\frac{n}{2}}\cdot\cos(bx+c+n\alpha),$$

$$\text{where } \alpha = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore\ \frac{d^ny}{dx^n} = e^{ax}\cdot(a^2+b^2)^{\frac{n}{2}}\cdot\cos\left[bx+c+n\tan^{-1}\left(\frac{b}{a}\right)\right]$$

(xii) y = e^{8x} . cos (6x + 7)

Solution:

y = e^{8x} . cos (6x + 7)

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}[e^{ax}.\cos(6x+7)]$$

$$= e^{ax}.\frac{d}{dx}[\cos(6x+7)] + \cos(6x+7).\frac{d}{dx}(e^{ax})$$

$$= e^{ax}.[-\sin(6x+7)].\frac{d}{dx}(6x+7) + \cos(6x+7).e^{ax}.\frac{d}{dx}(ax)$$

= – e^{ax} sin (6x + 7) x (b x 1 + 0) + e^{ax}cos(6x + 7) x a x 1

= e^{ax} [a cos (6x + 7) – b sin (6x + 7)]

$$= e^{ax}.\sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\cos(6x+7) - \frac{b}{\sqrt{a^2+b^2}}\sin(6x+7)\right]$$

Let $\frac{a}{\sqrt{a^2+b^2}} = \cos x$ and $\frac{b}{\sqrt{a^2+b^2}} = \sin x$

Then tan ∞ = $\dfrac{b}{a}$

∴ ∞ = $\tan^{-1}\left(\dfrac{b}{a}\right)$

∴ $\dfrac{\text{dy}}{\text{dx}} = e^{ax}.\sqrt{a^2+b^2}[\cos\infty.\cos(bx+c) - \sin\infty.\sin(bx+c)]$

= $e^{ax}.\left(a^2+b^2\right)^{\frac{1}{2}}.\cos(6x+7+x)$

$\dfrac{d^2y}{dx^2} = \dfrac{\text{d}}{\text{dx}}\left[e^{ax}.\left(a^2+b^2\right)^{\frac{1}{2}}.\cos(6x+7+\infty)\right]$

= $\left(a^2+b^2\right)^{\frac{1}{2}}.\dfrac{\text{d}}{\text{dx}}[e^{ax}.\cos(6x+7+\infty)]$

= $\left(a^2+b^2\right)^{\frac{1}{2}}\left[e^{ax}.\dfrac{\text{d}}{\text{dx}}\{\cos(6x+7+\infty)\} + \cos(6x+7+\infty).\dfrac{\text{d}}{\text{dx}}(e^{ax})\right]$

= $\left(a^2+b^2\right)^{\frac{1}{2}}\left[e^{ax}.\{-\sin(6x+7+\infty)\}.\dfrac{\text{d}}{\text{dx}}(6x+7+\infty) + \cos(6x+7+\infty).e^{ax}.\dfrac{\text{d}}{\text{dx}}(ax)\right]$

= $\left(a^2+b^2\right)^{\frac{1}{2}}[-e^{ax}\sin(6x+7+\infty)\times(b\times1+0+0) + \cos(6x+7+\infty).e^{ax}\times a\times 1]$

= $e^{ax}.\left(a^2+b^2\right)^{\frac{1}{2}}[a\cos(6x+7+\infty) - b\sin(6x+7+\infty)]$

= $e^{ax}.\left(a^2+b^2\right)^{\frac{1}{2}}.\sqrt{a^2+b^2}\left[\dfrac{a}{\sqrt{a^2+b^2}}\cos(6x+7+\infty) = \dfrac{b}{\sqrt{a^2+b^2}}\sin(6x+7+\infty)\right]$

= $e^{ax}.\left(a^2+b^2\right)^{\frac{2}{2}}[\cos\infty.\cos(6x+7+\infty) - \sin\infty.\sin(6x+7+\infty)$

= $e^{ax}.\left(a^2+b^2\right)^{\frac{2}{2}}.\cos(6x+7+\infty+\infty)$

= $e^{ax}.\left(a^2+b^2\right)^{\frac{2}{2}}.\cos(6x+7+2\infty)$

Similarly.

$\dfrac{d^3y}{dx^3} = e^{ax}.\left(a^2+b^2\right)^{\frac{3}{2}}.\cos(6x+7+3\infty)$

In general, the n$^{\text{th}}$ order derivative is given by

$\dfrac{d^ny}{dx^n} = e^{ax}.\left(a^2+b^2\right)^{\frac{n}{2}}.\cos(6x+7+n\infty),$

Where ∞ = $\tan^{-1}\left(\dfrac{b}{a}\right)$

∴ $\dfrac{d^ny}{dx^n} = e^{8x}.(10)^n.\cos\left[6x+7+n\tan^{-1}\left(\dfrac{3}{4}\right)\right]$