Maharashtra State Board 12th Maths Solutions Chapter 3 Indefinite Integration Ex 3.1

I. Integrate the following functions w.r.t. x:

(i) x3 + x2 – x + 1
Solution:

$$\int (x^3+x^2-x+1)\,dx = \int x^3\,dx + \int x^2\,dx - \int x\,dx + \int 1\,dx$$

$$= \frac{x^4}{4}+\frac{x^3}{3}-\frac{x^2}{2}+x+c.$$

(ii) x2(1−2x)2
Solution:

$$\int x^2\left(1-\frac{2}{x}\right)^2 dx = \int x^2\left(1-\frac{4}{x}+\frac{4}{x^2}\right)dx$$

$$= \int (x^2-4x+4)\,dx$$

$$= \int x^2\,dx - 4\int x\,dx + 4\int 1\,dx$$

$$= \frac{x^3}{3}-4\left(\frac{x^2}{2}\right)+4x+c$$

$$= \frac{1}{3}x^3-2x^2+4x+c.$$

(iii) 3sec2x−4x+1xx√−7
Solution:

$$\int \left(3\sec^2 x-\frac{4}{x}+\frac{1}{x\sqrt{x}}-7\right)dx$$

$$= 3\int \sec^2 x\,dx - 4\int \frac{1}{x}dx + \int x^{-\frac{3}{2}}dx - 7\int 1\,dx$$

$$= 3\tan x - 4\log|x| + \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} - 7x + c$$

$$= 3\tan x - 4\log|x| - \frac{2}{\sqrt{x}} - 7x + c$$

(iv) 2x3−5x+3x+4x5
Solution:

$$\int \left(2x^3-5x+\frac{3}{x}+\frac{4}{x^5}\right)dx$$

$$= 2\int x^3\,dx - 5\int x\,dx + 3\int \frac{1}{x}dx + 4\int x^{-5}\,dx$$

$$= 2\left(\frac{x^4}{4}\right)-5\left(\frac{x^2}{2}\right)+3\log|x|+4\left(\frac{x^{-4}}{-4}\right)+c$$

$$= \frac{x^4}{2}-\frac{5}{2}x^2+3\log|x|-\frac{1}{x^4}+c$$

(v) 3x3−2x+5xx√
Solution:

$$\int \frac{3x^3 - 2x + 5}{x\sqrt{x}} dx$$

$$= \int x^{-\frac{3}{2}}(3x^3 - 2x + 5)\, dx$$

$$= \int \left(3x^{\frac{3}{2}} - 2x^{-\frac{1}{2}} + 5x^{-\frac{3}{2}}\right) dx$$

$$= 3\int x^{\frac{3}{2}} dx - 2\int x^{-\frac{1}{2}} dx + 5\int x^{-\frac{3}{2}} dx$$

$$= 3\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right) - 2\left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + 5\left(\frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right) + c$$

$$= \frac{6}{5}x^2\sqrt{x} - 4\sqrt{x} - \frac{10}{\sqrt{x}} + c.$$

II. Evaluate:

(i) ∫tan2 x . dx
Solution:

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$$
$$= \int \sec^2 x\, dx - \int 1\, dx$$
$$= \tan x - x + c.$$

(ii) $\int \frac{\sin 2x}{\cos x} \cdot dx$
Solution:

$$\int \frac{\sin 2x}{\cos x} dx = \int \frac{2\sin x \cos x}{\cos x} dx$$
$$= 2\int \sin x\, dx$$
$$= -2\cos x + c.$$

(iii) $\int \frac{\sin x}{\cos^2 x} \cdot dx$
Solution:

$$\int \frac{\sin x}{\cos^2 x} dx = \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) dx$$
$$= \int \sec x \tan x\, dx$$
$$= \sec x + c.$$

(iv) $\int \frac{\cos 2x}{\sin^2 x} \cdot dx$
Solution:

$$\int \frac{\cos 2x}{\sin^2 x} dx = \int \frac{(1 - 2\sin^2 x)}{\sin^2 x} dx$$

$$= \int \left(\frac{1}{\sin^2 x} - \frac{2\sin^2 x}{\sin^2 x}\right) dx$$
$$= \int \operatorname{cosec}^2 x\, dx - 2\int dx$$
$$= -\cot x - 2x + c.$$

(v) $\int \frac{\cos 2x}{\sin^2 x \cdot \cos^2 x} \cdot dx$
Solution:

$$\int \frac{\cos 2x}{\sin^2x \cdot \cos^2x} dx$$

$$= \int \frac{\cos^2x - \sin^2x}{\sin^2x \cdot \cos^2x} dx$$

$$= \int \left(\frac{1}{\sin^2x} - \frac{1}{\cos^2x}\right) dx$$

$$= \int \operatorname{cosec}^2x\, dx - \int \sec^2x\, dx$$

= -cot x – tan x + c

(vi) ∫sinx1+sinx⋅ dx

Solution:

$$\int \frac{\sin x}{1+\sin x} dx$$

$$= \int \frac{\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx$$

$$= \int \frac{\sin x - \sin^2x}{1-\sin^2x} dx$$

$$= \int \frac{\sin x - \sin^2x}{\cos^2x} dx$$

$$= \int \left(\frac{\sin x}{\cos^2x} - \frac{\sin^2x}{\cos^2x}\right) dx$$

$$= \int \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) dx - \int \tan^2x\, dx$$

$$= \int \sec x \tan x\, dx - \int (\sec^2x - 1)\, dx$$

$$= \int \sec x \tan x\, dx - \int \sec^2x\, dx + \int 1\, dx$$

$$= \sec x - \tan x + x + c.$$

(vii) ∫tanxsecx+tanx⋅ dx

Solution:

$$\int \frac{\tan x}{\sec x + \tan x} dx$$

$$= \int \frac{\tan x}{\sec x + \tan x} \times \frac{\sec x - \tan x}{\sec x - \tan x} dx$$

$$= \int \frac{\sec x \tan x - \tan^2x}{\sec^2x - \tan^2x} dx$$

$$= \int \frac{\sec x \tan x - (\sec^2x - 1)}{1} dx$$

$$= \int \sec x \tan x\, dx - \int \sec^2x\, dx + \int 1\, dx$$

$$= \sec x - \tan x + x + c.$$

(viii) ∫1+sin2x--------√⋅ dx

Solution:

$$\int \sqrt{1+\sin 2x}\, dx$$

$$= \int \sqrt{\cos^2x + \sin^2x + 2\sin x \cos x}\, dx$$

$$= \int \sqrt{(\cos x + \sin x)^2}\, dx$$

$$= \int (\cos x + \sin x)\, dx$$

$$= \int \cos x\, dx + \int \sin x\, dx$$

$$= \sin x - \cos x + c.$$

(ix) ∫1−cos2x--------√ dx

Solution:

$$\int \sqrt{1-\cos 2x}\,dx$$
$$= \int \sqrt{2\sin^2 x}\,dx = \sqrt{2}\int \sin x\,dx$$
$$= -\sqrt{2}\cos x + c.$$

(x) ∫sin 4x cos 3x dx

Solution:

$$\int \sin 4x \cos 3x\,dx$$
$$= \frac{1}{2}\int \sin 4x \cos 3x\,dx$$
$$= \frac{1}{2}\int [\sin(4x+3x) + \sin(4x-3x)]\,dx$$
$$= \frac{1}{2}\int \sin 7x\,dx + \frac{1}{2}\int \sin x\,dx$$
$$= \frac{1}{2}\left(\frac{-\cos 7x}{7}\right) - \frac{1}{2}\cos x + c$$
$$= -\frac{1}{14}\cos 7x - \frac{1}{2}\cos x + c.$$

III. Evaluate:

(i) ∫xx+2 dx

Solution:

$$\int \frac{x}{x+2}\,dx$$
$$= \int \frac{(x+2)-2}{x+2}\,dx$$
$$= \int \left(\frac{x+2}{x+2} - \frac{2}{x+2}\right)dx$$
$$= \int 1\,dx - 2\int \frac{1}{x+2}\,dx$$
$$= x - 2\log|x+2| + c.$$

(ii) ∫4x+32x+1 dx

Solution:

$$\int \frac{4x+3}{2x+1}\,dx$$
$$= \int \frac{2(2x+1)+1}{2x+1}\,dx$$
$$= \int \left(\frac{2(2x+1)}{2x+1} + \frac{1}{2x+1}\right)dx$$
$$= 2\int 1\,dx + \int \frac{1}{2x+1}\,dx$$
$$= 2x + \frac{1}{2}\log|2x+1| + c.$$

(iii) ∫5x+23x−4⋅ dx

Solution:

$$\int \frac{5x+2}{3x-4}dx$$

$$= \int \frac{\frac{5}{3}(3x-4)+\frac{20}{3}+2}{3x-4}\,dx$$

$$= \int \frac{\frac{5}{3}(3x-4)+\frac{26}{3}}{3x-4}dx$$

$$= \int \left[\frac{5}{3}+\frac{\left(\frac{26}{3}\right)}{3x-4}\right]dx$$

$$= \frac{5}{3}\int 1\,dx + \frac{26}{3}\int \frac{1}{3x-4}dx$$

$$= \frac{5x}{3}+\frac{26}{3}\cdot\frac{1}{3}\log|3x-4|+c$$

$$= \frac{5x}{3}+\frac{26}{9}\log|3x-4|+c.$$

(iv) ∫x−2x+5√⋅ dx

Solution:

$$\int \frac{x-2}{\sqrt{x+5}}dx = \int \frac{(x+5)-7}{\sqrt{x+5}}dx$$

$$= \int \left(\frac{x+5}{\sqrt{x+5}}-\frac{7}{\sqrt{x+5}}\right)dx$$

$$= \int (x+5)^{\frac{1}{2}}dx - 7\int (x+5)^{-\frac{1}{2}}dx$$

$$= \frac{(x+5)^{\frac{3}{2}}}{(3/2)} - \frac{7(x+5)^{\frac{1}{2}}}{(1/2)}+c$$

$$= \frac{2}{3}(x+5)^{\frac{3}{2}} - 14\sqrt{x+5}+c.$$

(v) ∫2x−74x−1√⋅ dx

Solution:

$$\int \frac{2x-7}{\sqrt{4x-1}} dx$$

$$= \frac{1}{2}\int \frac{2(2x-7)}{\sqrt{4x-1}} dx$$

$$= \frac{1}{2}\int \frac{(4x-1)-13}{\sqrt{4x-1}} dx$$

$$= \frac{1}{2}\int\left(\frac{4x-1}{\sqrt{4x-1}} - \frac{13}{\sqrt{4x-1}}\right)dx$$

$$= \frac{1}{2}\int (4x-1)^{\frac{1}{2}} dx - \frac{13}{2}\int (4x-1)^{-\frac{1}{2}} dx$$

$$= \frac{1}{2}\cdot\frac{(4x-1)^{\frac{3}{2}}}{(4)\left(\frac{3}{2}\right)} - \frac{13}{2}\cdot\frac{(4x-1)^{\frac{1}{2}}}{(4)\left(\frac{1}{2}\right)} + c$$

$$= \frac{1}{12}(4x-1)^{\frac{3}{2}} - \frac{13}{4}\sqrt{4x-1} + c.$$

(vi) ∫sin4xcos2x · dx

Solution:

$$\int \frac{\sin 4x}{\cos 2x} dx$$

$$= \int \frac{2\sin 2x \cos 2x}{\cos 2x} dx$$

$$= 2\int \sin 2x\, dx$$

$$= 2\left(-\frac{\cos 2x}{2}\right) + c$$

$$= -\cos 2x + c.$$

(vii) ∫1+sin5x--------√ · dx

Solution:

$$\int \sqrt{1+\sin 5x}.\, dx$$

$$= \int \sqrt{\sin^2 x + \cos^2 x + 5\sin x\cos x}.\, dx$$

$$= \int \sqrt{(\sin x + \cos x)^2}.\, dx$$

$$= \int (\sin x + \cos x).\, dx$$

$$= \int \sin x\, dx + \int \cos x.\, dx$$

$$= \left(\frac{2}{5}\sin\ \frac{5x}{2} - \cos\ \frac{5x}{2}\right) + c.$$

(viii) ∫cos2 x . dx
Solution:

Recall the identity cos 2x = 2 $\cos^2$x – 1, which gives

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

Therefore, $\int \cos^2 x . dx$

$$= \frac{1}{2}\int (1+\cos 2x) . dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x . dx$$

$$= \frac{x}{2} + \frac{1}{4}\sin 2x + C.$$

(ix) ∫2x√−x+3√· dx
Solution:

$$\int \frac{2}{\sqrt{x}-\sqrt{x+3}}dx$$

$$= \int \frac{2}{\sqrt{x}-\sqrt{x+3}} \times \frac{\sqrt{x}+\sqrt{x+3}}{\sqrt{x}+\sqrt{x+3}}dx$$

$$= \int \frac{2(\sqrt{x}+\sqrt{x+3})}{x-(x+3)}dx$$

$$= -\frac{2}{3}\int (\sqrt{x}+\sqrt{x+3})\,dx$$

$$= -\frac{2}{3}\int x^{\frac{1}{2}}dx - \frac{2}{3}\int (x+3)^{\frac{1}{2}}dx$$

$$= -\frac{2}{3}\cdot\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} - \frac{2}{3}\cdot\frac{(x+3)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + c$$

$$= -\frac{4}{9}[x^{\frac{3}{2}} + (x+3)^{\frac{3}{2}}] + c.$$

(x) ∫37x−2√−7x−5√· dx
Solution:

$$\int \frac{3}{\sqrt{7x-2}-\sqrt{7x-5}}dx$$

$$= \int \frac{3}{\sqrt{7x-2}-\sqrt{7x-5}} \times \frac{\sqrt{7x-2}+\sqrt{7x-5}}{\sqrt{7x-2}-\sqrt{7x-5}}dx$$

$$= \int \frac{3(\sqrt{7x-2}+\sqrt{7x-5})}{(7x-2)-(7x-5)}\,dx$$

$$= \int (\sqrt{7x-2}+\sqrt{7x-5})\,dx$$

$$= \int (7x-2)^{\frac{1}{2}}dx + \int (7x-5)^{\frac{1}{2}}dx$$

$$= \frac{(7x-2)^{\frac{3}{2}}}{3/2} \times \frac{1}{7} + \frac{(7x-5)^{\frac{3}{2}}}{3/2} \times \frac{1}{7} + c$$

$$= \frac{2}{21}(7x-2)^{\frac{3}{2}} + \frac{2}{21}(7x-5)^{\frac{3}{2}} + c.$$

IV.

Question 1.
If f'(x) = $x - \frac{3}{x^3}$, f(1) = $\frac{11}{2}$, find f(x).
Solution:
By the definition of integral,

$$f(x) = \int f'(x)\,dx$$
$$= \int \left(x - \frac{3}{x^3}\right) dx$$
$$= \int x\,dx - 3\int x^{-3}\,dx$$
$$= \frac{x^2}{2} - \frac{3x^{(-2)}}{(-2)} + c$$
$$= \frac{x^2}{2} + \frac{3}{2x^2} + c \quad \text{.........(1)}$$
$$f(1) = \frac{11}{2} \quad \text{.......(Given)}$$
$$\therefore \frac{1}{2} + \frac{3}{2} + c = \frac{11}{2}$$
$$\therefore c = \frac{7}{2}$$
$$\therefore f(x) = \frac{x^2}{2} + \frac{3}{2x^2} + \frac{7}{2}. \quad \text{...... [By (1)]}$$

Maharashtra State Board 12th Maths Solutions Chapter 3 Indefinite Integration Ex 3.2(A)

I. Integrate the following functions w.r.t. x:

Question 1.
$\frac{(\log x)^n}{x}$
Solution:

$$\text{Let } I = \int \frac{(\log x)^n}{x}\,dx$$
$$\text{Put } \log x = t. \qquad \therefore \frac{1}{x}\,dx = dt$$
$$\therefore I = \int t^n\,dt = \frac{t^{n+1}}{n+1} + c$$
$$= \frac{1}{n+1} \cdot (\log x)^{n+1} + c.$$

Question 2.
$\frac{(\sin^{-1}x)^{\frac{3}{2}}}{\sqrt{1-x^2}}$
Solution:

Let I = ∫(sin-1x)321−x2√dx

$$\text{Put } \sin^{-1}x = t. \qquad \therefore \frac{1}{\sqrt{1-x^2}}dx = dt$$

$$\therefore I = \int t^{\frac{3}{2}}dt = \frac{t^{\frac{5}{2}}}{5/2} + c$$

$$= \frac{2}{5}(\sin^{-1}x)^{\frac{5}{2}} + c.$$

Question 3.
1+xx⋅sin(x+logx)
Solution:

$$\text{Let } I = \int \frac{1+x}{x \cdot \sin(x+\log x)}dx$$

$$= \int \frac{1}{\sin(x+\log x)} \cdot \left(\frac{1+x}{x}\right)dx$$

$$= \int \frac{1}{\sin(x+\log x)} \cdot \left(\frac{1}{x}+1\right)dx$$

$$\text{Put } x + \log x = t \qquad \therefore \left(1+\frac{1}{x}\right)dx = dt$$

$$\therefore I = \int \frac{1}{\sin t}dt = \int \operatorname{cosec} t\, dt$$

$$= \log|\operatorname{cosec} t - \cot t| + c$$

$$= \log|\operatorname{cosec}(x+\log x) - \cot(x+\log x)| + c.$$

Question 4.
x⋅sec2(x2)tan3(x2)√
Solution:

$$\text{Let } I = \int \frac{x \cdot \sec^2(x^2)}{\sqrt{\tan^3(x^2)}}dx$$

$$\text{Put } \tan(x^2) = t \qquad \therefore \sec^2(x^2) \times 2x\,dx = dt$$

$$\therefore x \cdot \sec^2(x^2)dx = \frac{dt}{2}$$

$$\therefore I = \int \frac{1}{\sqrt{t^3}} \cdot \frac{dt}{2} = \frac{1}{2}\int t^{-\frac{3}{2}}dt$$

$$= \frac{1}{2} \cdot \frac{t^{-\frac{1}{2}}}{-1/2} + c = \frac{-1}{\sqrt{t}} + c = \frac{-1}{\sqrt{\tan(x^2)}} + c.$$

Question 5.
e3xe3x+1
Solution:

$$\text{Let } I = \int \frac{e^{3x}}{e^{3x}+1}dx$$

$$\text{Put } e^{3x} + 1 = t.$$

$$\therefore 3e^{3x}dx = dt$$

$$\therefore e^{3x}dx = \frac{dt}{3}$$

$$\therefore I = \int \frac{1}{t} \cdot \frac{dt}{3} = \frac{1}{3}\int \frac{1}{t}dt$$

$$= \frac{1}{3}\log|t| + c = \frac{1}{3}\log|e^{3x}+1| + c.$$

Question 6.

(x2+2)(x2+1) · ax+tan-1x

Solution:

$$\text{Let } I=\int \frac{x^2+2}{(x^2+1)}\cdot a^{x+\tan^{-1}x}\,dx$$

$$=\int a^{x+\tan^{-1}x}\cdot\left(\frac{x^2+2}{x^2+1}\right)dx$$

Put $x+\tan^{-1}x=t$

$$\therefore\ \left(1+\frac{1}{1+x^2}\right)dx=dt$$

$$\therefore\ \left(\frac{1+x^2+1}{1+x^2}\right)dx=dt$$

$$\therefore\ \left(\frac{x^2+2}{x^2+1}\right)dx=dt$$

$$\therefore\ I=\int a^t\,dt=\frac{a^t}{\log a}+c$$

$$=\frac{a^{x+\tan^{-1}x}}{\log a}+c.$$

Question 7.

ex· log(sinex)tan(ex)

Solution:

$$\text{Let } I=\int\frac{e^x\cdot\log(\sin e^x)}{\tan(e^x)}\,dx$$

$$=\int\log(\sin e^x)\cdot e^x\cot(e^x)\,dx$$

Put $\log(\sin e^x)=t \qquad \therefore\ \frac{1}{\sin(e^x)}\cdot\cos(e^x)\cdot e^x\,dx=dt$

$$\therefore\ e^x\cdot\cot(e^x)\,dx=dt$$

$$\therefore\ I=\int t\,dt=\frac{t^2}{2}+c$$

$$=\tfrac{1}{2}[\log(\sin e^x)]^2+c.$$

Question 8.

e2x+1e2x−1

Solution:

$$\text{Let } I=\int\frac{e^{2x}+1}{e^{2x}-1}\,dx=\int\frac{\left(\frac{e^{2x}+1}{e^x}\right)}{\left(\frac{e^{2x}-1}{e^x}\right)}\,dx$$

$$=\int\left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)dx$$

$$=\int\frac{\frac{d}{dx}(e^x-e^{-x})}{e^x-e^{-x}}\,dx$$

$$=\log|e^x-e^{-x}|+c \qquad \ldots\left[\because\ \int\frac{f'(x)}{f(x)}dx=\log|f(x)|+c\right]$$

Question 9.

sin4x . cos3x

Solution:

$$\text{Let } I = \int \sin^4 x \cdot \cos^3 x\, dx$$

$$= \int \sin^4 x \cdot \cos^2 x \cdot \cos x\, dx$$

$$= \int \sin^4 x (1 - \sin^2 x) \cos x\, dx$$

Put $\sin x = t \quad \therefore \cos x\, dx = dt$

$$\therefore I = \int t^4 (1 - t^2)\, dt = \int (t^4 - t^6)\, dt$$

$$= \int t^4 dt - \int t^6 dt$$

$$= \frac{t^5}{5} - \frac{t^7}{7} + c$$

$$= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c.$$

Question 10.

14x+5x-11

Solution:

$$\text{Let } I = \int \frac{1}{4x + 5x^{-11}}\, dx$$

$$= \int \frac{x^{11}}{x^{11}(4x + 5x^{-11})}\, dx$$

$$= \int \frac{x^{11}}{4x^{12} + 5}\, dx$$

$$= \frac{1}{48}\int \frac{48x^{11}}{4x^{12} + 5}\, dx$$

$$= \frac{1}{48}\int \frac{\frac{d}{dx}(4x^{12} + 5)}{4x^{12} + 5}\, dx$$

$$= \frac{1}{48}\log|4x^{12} + 5| + c \quad \ldots \left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$$

Question 11.

x9 . sec2(x10)

Solution:

$$\text{Let } I = \int x^9 \cdot \sec^2(x^{10})\, dx$$

Put $x^{10} = t \quad \therefore 10x^9 dx = dt \quad \therefore x^9 dx = \frac{1}{10}dt$

$$\therefore I = \int \sec^2 t \cdot \frac{dt}{10}$$

$$= \frac{1}{10}\int \sec^2 t\, dt$$

$$= \frac{1}{10}\tan t + c$$

$$= \frac{1}{10}\tan(x^{10}) + c.$$

Question 12.

e3logx · (x4+1)-1

Solution:

Let $I = e^{3\log x}(x^4+1)^{-1}\,dx$

$$= \int \frac{e^{\log x^3}}{x^4+1}\,dx$$

$$= \int \frac{x^3}{x^4+1}\,dx \qquad \ldots [\because\ e^{\log N} = N]$$

$$= \frac{1}{4}\int \frac{4x^3}{x^4+1}\,dx$$

$$= \frac{1}{4}\int \frac{\frac{d}{dx}(x^4+1)}{x^4+1}\,dx$$

$$= \frac{1}{4}\log|x^4+1| + c. \ \ldots \left[\because\ \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

Question 13.
tanx√sinx· cosx
Solution:
Let I = ∫tanx√sinx· cosxdx
Dividing numerator and denominator by cos2x, we get

$$I = \int \frac{\left(\frac{\sqrt{\tan x}}{\cos^2 x}\right)}{\left(\frac{\sin x}{\cos x}\right)}\,dx$$

$$= \int \frac{\sqrt{\tan x}\cdot \sec^2 x}{\tan x}\,dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x}}\,dx$$

Put $\tan x = t \qquad \therefore\ \sec^2 x\,dx = dt$

$$\therefore\ I = \int \frac{1}{\sqrt{t}}\,dt = \int t^{-\frac{1}{2}}\,dt$$

$$= \frac{t^{\frac{1}{2}}}{1/2} + c = 2\sqrt{t} + c$$

$$= 2\sqrt{\tan x} + c.$$

Question 14.
(x−1)2(x2+1)2
Solution:

$$\text{Let } I = \int \frac{(x-1)^2}{(x^2+1)^2} dx$$

$$= \int \frac{x^2 - 2x + 1}{(x^2+1)^2} dx$$

$$= \int \frac{(x^2+1) - 2x}{(x^2+1)^2} dx$$

$$= \int \left[\frac{x^2+1}{(x^2+1)^2} - \frac{2x}{(x^2+1)^2} \right] dx$$

$$= \int \frac{1}{x^2+1} dx - \int \frac{2x}{(x^2+1)^2} dx$$

$$= I_1 - I_2 \quad \dots \text{ (Let)}$$

In I_2, Put $x^2 + 1 = t$ $\quad \therefore\ 2x\,dx = dt$

$$\therefore\ I = \int \frac{1}{x^2+1} dx - \int t^{-2} dt$$

$$= \tan^{-1} x - \frac{t^{-1}}{(-1)} + c$$

$$= \tan^{-1} x + \frac{1}{x^2+1} + c.$$

Question 15.
2sinxcosx3cos2x+4sin2x
Solution:

$$\text{Let } I = \int \frac{2 \sin x \cos x}{3\cos^2 x + 4\sin^2 x} dx$$

Put $3\cos^2 x + 4\sin^2 x = t$

$$\therefore\ \left[3(2\cos x)\frac{d}{dx}(\cos x) + 4(2\sin x)\frac{d}{dx}(\sin x) \right] dx = dt$$

$$\therefore\ [-6\cos x \sin x + 8 \sin x \cos x]\, dx = dt$$

$$\therefore\ 2\sin x \cos x\, dx = dt$$

$$I = \int \frac{dt}{t} = \log|t| + c$$

$$= \log|3\cos^2 x + 4\sin^2 x| + c.$$

Question 16.
1x√+x3√
Solution:

$$\text{Let } I = \int \frac{1}{\sqrt{x} + \sqrt{x^3}}\, dx$$

$$= \int \frac{1}{x^{\frac{1}{2}} + x^{\frac{3}{2}}}\, dx$$

Put $x = t^2 \qquad \therefore\ dx = 2t\, dt$

Also $x^{\frac{1}{2}} = (t^2)^{\frac{1}{2}} = t$ and $x^{\frac{3}{2}} = (t^2)^{\frac{3}{2}} = t^3$

$$\therefore\ I = \int \frac{2t\, dt}{t + t^3}$$

$$= 2\int \frac{t\, dt}{t(1 + t^2)}$$

$$= 2\int \frac{1}{1 + t^2}\, dt$$

$$= 2\tan^{-1} t + c$$

$$= 2\tan^{-1}(\sqrt{x}) + c.$$

Question 17.
10x9+10x· log1010x+x10
Solution:

$$\text{Let } I = \int \frac{10x^9 + 10^x \cdot \log 10}{10^x + x^{10}}\, dx$$

Put $10^x + x^{10} = t$

$\therefore\ (10^x \cdot \log 10 + 10x^9)\, dx = dt$

$$\therefore\ I = \int \frac{1}{t}\, dt = \log|t| + c$$

$$= \log|10^x + x^{10}| + c.$$

Question 18.
xn−11+4xn√
Solution:

$$\text{Let } I = \int \frac{x^{n-1}}{\sqrt{1 + 4x^n}}\, dx$$

Put $x^n = t \qquad \therefore\ nx^{n-1} dx = dt$

$$\therefore\ x^{n-1} dx = \frac{dt}{n}$$

$$\therefore\ I = \int \frac{1}{\sqrt{1 + 4t}} \cdot \frac{dt}{n} = \frac{1}{n} \int (1 + 4t)^{-\frac{1}{2}} dt$$

$$= \frac{1}{n} \cdot \frac{(1 + 4t)^{\frac{1}{2}}}{1/2} \times \frac{1}{4} + c$$

$$= \frac{1}{2n} \cdot \sqrt{1 + 4x^n} + c.$$

Question 19.
(2x + 1) x+2−−−−−√

Solution:

Let $I=\int (2x+1)\sqrt{x+2}\,dx$

Put $x+2=t \qquad \therefore\ dx=dt$

Also, $x=t-2$

$\therefore\ 2x+1=2(t-2)+1=2t-3$

$$\therefore\ I=\int (2t-3)\sqrt{t}\,dt$$
$$=\int \left(2t^{\frac{3}{2}}-3t^{\frac{1}{2}}\right)dt$$
$$=2\int t^{\frac{3}{2}}dt-3\int t^{\frac{1}{2}}dt$$
$$=2\cdot\frac{t^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}-3\cdot\frac{t^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+c$$
$$=\frac{4}{5}(x+2)^{\frac{5}{2}}-2(x+2)^{\frac{3}{2}}+c.$$

Question 20.

x5a2+x2−−−−−−√

Solution:

Let $I=\int x^5\sqrt{a^2+x^2}\,dx$

Put, $a^2+x^2=t$

$\therefore\ 2x\,dx=dt \qquad \therefore\ x\,dx=\frac{1}{2}dt$

Also, $x^2=t-a^2$

$$I=\int x^2\cdot x^2\sqrt{a^2+x^2}\,x\,dx$$
$$=\frac{1}{2}\int (t-a^2)^2\sqrt{t}\,dt$$
$$=\frac{1}{2}\int (t^2-2a^2t+a^4)\sqrt{t}\,dt$$
$$=\frac{1}{2}\int\left(t^{\frac{5}{2}}-2a^2t^{\frac{3}{2}}+a^4t^{\frac{1}{2}}\right)dt$$
$$=\frac{1}{2}\int t^{\frac{5}{2}}dt-a^2\int t^{\frac{3}{2}}dt+\frac{a^4}{2}\int t^{\frac{1}{2}}dt$$
$$=\frac{1}{2}\cdot\frac{t^{\frac{7}{2}}}{\left(\frac{7}{2}\right)}-a^2\cdot\frac{t^{\frac{5}{2}}}{\left(\frac{5}{2}\right)}+\frac{a^4}{2}\cdot\frac{t^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+c$$
$$=\frac{1}{7}(a^2+x^2)^{\frac{7}{2}}-\frac{2a^2}{5}(a^2+x^2)^{\frac{5}{2}}+\frac{a^4}{3}(a^2+x^2)^{\frac{3}{2}}+c.$$

Question 21.

(5−3x)(2−3x)−12

Solution:

$$\text{Let } I = \int (5-3x)(2-3x)^{-\frac{1}{2}}\,dx$$

Put $2-3x=t$

$\therefore\ -3dx = dt$

$\therefore\ dx = \frac{-dt}{3}$

Also, $x = \frac{2-t}{3}$

$$\therefore\ I = \int \left[5-3\left(\frac{2-t}{3}\right)\right] t^{-\frac{1}{2}} \cdot \left(\frac{-dt}{3}\right)$$

$$= -\frac{1}{3}\int (5-2+t)t^{-\frac{1}{2}}\,dt$$

$$= -\frac{1}{3}\int (3+t)t^{-\frac{1}{2}}\,dt$$

$$= -\frac{1}{3}\int \left(3t^{-\frac{1}{2}} + t^{\frac{1}{2}}\right)dt$$

$$= -\frac{3}{3}\int t^{-\frac{1}{2}}\,dt - \frac{1}{3}\int t^{\frac{1}{2}}\,dt$$

$$= -\frac{t^{\frac{1}{2}}}{(1/2)} - \frac{1}{3}\cdot\frac{t^{\frac{3}{2}}}{(3/2)} + c$$

$$= -2\sqrt{2-3x} - \frac{2}{9}(2-3x)^{\frac{3}{2}} + c.$$

Question 22.

$\frac{7+4x+5x^2}{(2x+3)^{\frac{3}{2}}}$

Solution:

$$\text{Let } I=\int \frac{7+4x+5x^2}{(2x+3)^{\frac{3}{2}}}dx=\int \frac{5x^2+4x+7}{(2x+3)^{\frac{3}{2}}}dx$$

Put $2x+3=t \qquad \therefore\ 2dx=dt$

$$\therefore\ dx=\frac{dt}{2}$$

Also, $x=\dfrac{t-3}{2}$

$$\therefore\ I=\int \frac{5\left(\frac{t-3}{2}\right)^2+4\left(\frac{t-3}{2}\right)+7}{t^{\frac{3}{2}}}\cdot\frac{dt}{2}$$

$$=\frac{1}{2}\int \frac{5\left(\frac{t^2-6t+9}{4}\right)+2(t-3)+7}{t^{\frac{3}{2}}}dt$$

$$=\frac{1}{2}\int \frac{5t^2-30t+45+8t-24+28}{4t^{\frac{3}{2}}}dt$$

$$=\frac{1}{8}\int \frac{5t^2-22t+49}{t^{\frac{3}{2}}}dt$$

$$=\frac{1}{8}\int (5t^{\frac{1}{2}}-22t^{-\frac{1}{2}}+49t^{-\frac{3}{2}})\,dt$$

$$=\frac{5}{8}\int t^{\frac{1}{2}}dt-\frac{22}{8}\int t^{-\frac{1}{2}}dt+\frac{49}{8}\int t^{-\frac{3}{2}}dt$$

$$=\frac{5}{8}\cdot\frac{t^{\frac{3}{2}}}{(3/2)}-\frac{11}{4}\cdot\frac{t^{\frac{1}{2}}}{(1/2)}+\frac{49}{8}\cdot\frac{t^{-\frac{1}{2}}}{(-1/2)}+c$$

$$=\frac{5}{12}(2x+3)^{\frac{3}{2}}-\frac{11}{2}\sqrt{2x+3}-\frac{49}{4}\cdot\frac{1}{\sqrt{2x+3}}+c.$$

Question 23.
x29−x6√
Solution:

$$\text{Let } I=\int \frac{x^2}{\sqrt{9-x^6}}\,dx$$

Put $x^3=t \qquad \therefore\ 3x^2\,dx=dt \qquad \therefore\ x^2dx=\dfrac{1}{3}dt$

$$\therefore\ I=\int \frac{1}{\sqrt{9-t^2}}\cdot\frac{dt}{3}$$

$$=\frac{1}{3}\int \frac{dt}{\sqrt{3^2-t^2}}$$

$$=\frac{1}{3}\sin^{-1}\left(\frac{t}{3}\right)+c$$

$$=\frac{1}{3}\sin^{-1}\left(\frac{x^3}{3}\right)+c.$$

Question 24.
1x(x3−1)
Solution:

Let $I = \int \frac{1}{x(x^3-1)} dx$

$= \int \frac{x^{-4}}{x^{-4}x(x^3-1)} dx$

$= \int \frac{x^{-4}}{1-x^{-3}} dx$

$= \frac{1}{3}\int \frac{3x^{-4}}{1-x^{-3}} dx$

$= \frac{1}{3}\int \frac{\frac{d}{dx}(1-x^{-3})}{1-x^{-3}} dx$

$= \frac{1}{3}\log|1-x^{-3}| + c$

$\dots \left[\because \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c\right]$

$= \frac{1}{3}\log\left|1-\frac{1}{x^3}\right| + c$

$= \frac{1}{3}\log\left|\frac{x^3-1}{x^3}\right| + c.$

Alternative Method :

Let $I = \int \frac{1}{x(x^3-1)} dx$

$= \int \frac{x^2}{x^3(x^3-1)} dx$

Put $x^3 = t$ $\quad \therefore\ 3x^2 dx = dt$

$\therefore\ x^2 dx = \frac{dt}{3}$

$\therefore\ I = \int \frac{1}{t(t-1)} \cdot \frac{dt}{3} = \frac{1}{3}\int \frac{1}{t(t-1)} dt$

$= \frac{1}{3}\int \frac{t-(t-1)}{t(t-1)} dt = \frac{1}{3}\int \left(\frac{1}{t-1} - \frac{1}{t}\right) dt$

$= \frac{1}{3}\left[\int \frac{1}{t-1} dt - \int \frac{1}{t} dt\right]$

$= \frac{1}{3}[\log|t-1| - \log|t|] + c$

$= \frac{1}{3}\log\left|\frac{t-1}{t}\right| + c$

$= \frac{1}{3}\log\left|\frac{x^3-1}{x^3}\right| + c.$

Question 25.

1x· logx· log(logx)

Solution:

$$\text{Let } I=\int \frac{1}{x\cdot\log x\cdot\log(\log x)}dx$$

$$=\int \frac{1}{\log(\log x)}\cdot\frac{1}{x\cdot\log x}dx$$

$$\text{Put } \log(\log x)=t \qquad \therefore\ \frac{1}{\log x}\cdot\frac{1}{x}dx=dt$$

$$\therefore\ \frac{1}{x\cdot\log x}dx=dt$$

$$\therefore\ I=\int\frac{1}{t}dt=\log|t|+c$$

$$=\log|\log(\log x)|+c.$$

II. Integrate the following functions w.r.t x:

Question 1.
cos3x−cos4xsin3x+sin4x
Solution:

$$\text{Let } I=\int\frac{\cos 3x-\cos 4x}{\sin 3x+\sin 4x}dx$$

$$=\int\frac{-2\sin\left(\frac{3x+4x}{2}\right)\sin\left(\frac{3x-4x}{2}\right)}{2\sin\left(\frac{3x+4x}{2}\right)\cos\left(\frac{3x-4x}{2}\right)}dx$$

$$=\int -\frac{\sin\left(-\frac{x}{2}\right)}{\cos\left(-\frac{x}{2}\right)}dx=\int\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}dx$$

$$=\int\tan\left(\frac{x}{2}\right)dx$$

$$=\frac{\log\left|\sec\left(\frac{x}{2}\right)\right|}{\left(\frac{1}{2}\right)}+c$$

$$=2\log\left|\sec\left(\frac{x}{2}\right)\right|+c.$$

Question 2.
cosxsin(x−a)
Solution:

$$\text{Let } I=\int\frac{\cos x}{\sin(x-a)}dx$$

$$=\int\frac{\cos[(x-a)+a]}{\sin(x-a)}dx$$

$$=\int\frac{\cos(x-a)\cos a-\sin(x-a)\sin a}{\sin(x-a)}dx$$

$$=\int\left[\frac{\cos(x-a)\cos a}{\sin(x-a)}-\frac{\sin(x-a)\sin a}{\sin(x-a)}\right]dx$$

$$=\cos a\int\cot(x-a)\,dx-\sin a\int 1\,dx$$

$$=\cos a\log|\sin(x-a)|-x\sin a+c.$$

Question 3.
$\frac{\sin(x-a)}{\cos(x+b)}$
Solution:

$$\text{Let } I=\int \frac{\sin(x-a)}{\cos(x+b)}\,dx$$
$$=\int \frac{\sin[(x+b)-(a+b)]}{\cos(x+b)}\,dx$$
$$=\int \frac{\sin(x+b)\cos(a+b)-\cos(x+b)\sin(a+b)}{\cos(x+b)}\,dx$$
$$=\int\left[\frac{\sin(x+b)\cos(a+b)}{\cos(x+b)}-\frac{\cos(x+b)\sin(a+b)}{\cos(x+b)}\right]dx$$
$$=\cos(a+b)\int \tan(x+b)\,dx-\sin(a+b)\int 1\,dx$$
$$=\cos(a+b)\log|\sec(x+b)|-x\sin(a+b)+c.$$

Question 4.
$\frac{1}{\sin x \cdot \cos x+2\cos^2 x}$
Solution:
Let I = $\int \frac{1}{\sin x \cdot \cos x+2\cos^2 x}dx$
Dividing numerator and denominator of $\cos^2 x$, we get

$$I=\int \frac{\left(\frac{1}{\cos^2 x}\right)}{\frac{\sin x}{\cos x}+2}\,dx$$
$$=\int \frac{\sec^2 x}{\tan x+2}\,dx$$
$$\text{Put } \tan x=t \qquad \therefore\ \sec^2 x\,dx=dt$$
$$\therefore\ I=\int \frac{1}{t+2}\,dt=\log|t+2|+c$$
$$=\log|\tan x+2|+c.$$

Question 5.
$\frac{\sin x+2\cos x}{3\sin x+4\cos x}$
Solution:
Let I = $\int \frac{\sin x+2\cos x}{3\sin x+4\cos x}dx$
Put, Numerator = A (Denominator) + B [$\frac{d}{dx}$ (Denominator)]
∴ sin x+ 2 cos x = A(3 sin x + 4 cos x) + B [$\frac{d}{dx}$ (3 sin x + 4 cos x)]
= A(3 sin x + 4 cos x) + B (3 cos x – 4 sin x)
∴ sin x + 2 cos x = (3A – 4B) sin x + (4A + 3B) cos x
Equating the coefficients of sin x and cos x on both the sides, we get
3A – 4B = 1 …… (1)
and 4A + 3B = 2 …… (2)
Multiplying equation (1) by 3 and equation (2) by 4, we get
9A – 12B = 3
16A + 12B = 8

On adding, we get

$$25A = 11 \qquad \therefore\ A = \frac{11}{25}$$

$$\therefore\ \text{from (2)},\ 4\left(\frac{11}{25}\right) + 3B = 2$$

$$\therefore\ 3B = 2 - \frac{44}{25} = \frac{6}{25} \qquad \therefore\ B = \frac{2}{25}$$

$$\therefore\ \sin x + 2\cos x = \frac{11}{25}(3\sin x + 4\cos x) + \frac{2}{25}(3\cos x - 4\sin x)$$

$$\therefore\ I = \int\left[\frac{\frac{11}{25}(3\sin x + 4\cos x) + \frac{2}{25}(3\cos x - 4\sin x)}{3\sin x + 4\cos x}\right]dx$$

$$= \int\left[\frac{11}{25} + \frac{\frac{2}{25}(3\cos x - 4\sin x)}{(3\sin x + 4\cos x)}\right]dx$$

$$= \frac{11}{25}\int 1\,dx + \frac{2}{25}\int\frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}dx$$

$$= \frac{11}{25}x + \frac{2}{25}\log|3\sin x + 4\cos x| + c$$

$$\ldots\left[\because\ \int\frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$$

Question 6.
12+3tanx
Solution:
Let I = ∫12+3tanxdx

$$= \int\frac{1}{2 + 3\left(\frac{\sin x}{\cos x}\right)}dx$$

$$= \int\frac{\cos x}{2\cos x + 3\sin x}dx$$

Numerator = A (Denominator) + B [ddx (Denominator)]
∴ cos x = A(2 cos x + 3 sin x) + B [ddx (2 cos x + 3 sin x)]
= A (2 cos x + 3 sin x) + B (-2 sin x + 3 cos x)
∴ cos x = (2A + 3B) cos x + (3A – 2B) sin x
Equating the coefficients of cosx and sinx on both the sides, we get
2A + 3B = 1 …… (1)
and 3A – 2B = 0 ……. (2)
Multiplying equation (1) by 2 and equation (2) by 3, we get
4A + 6B = 2
9A – 6B = 0

On adding, we get

$$13A = 2 \qquad \therefore\ A = \frac{2}{13}$$

$$\therefore\ \text{from (2), } 2B = 3A = 3\left(\frac{2}{13}\right) = \frac{6}{13}$$

$$\therefore\ B = \frac{3}{13}$$

$$\therefore\ \cos x = \frac{2}{13}(2\cos x + 3\sin x) + \frac{3}{13}(-2\sin x + 3\cos x)$$

$$\therefore\ I = \int\left[\frac{\frac{2}{13}(2\cos x + 3\sin x) + \frac{3}{13}(-2\sin x + 3\cos x)}{2\cos x + 3\sin x}\right]dx$$

$$= \int\left[\frac{2}{13} + \frac{\frac{3}{13}(-2\sin x + 3\cos x)}{2\cos x + 3\sin x}\right]dx$$

$$= \frac{2}{13}\int 1\,dx + \frac{3}{13}\int\frac{-2\sin x + 3\cos x}{2\cos x + 3\sin x}\,dx$$

$$= \frac{2}{13}x + \frac{3}{13}\log|2\cos x + 3\sin x| + c$$

$$\ldots\left[\because\ \int\frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$$

Question 7.
$\frac{4e^x-25}{2e^x-5}$
Solution:
Let I = $\int\frac{4e^x-25}{2e^x-5}dx$
Put, Numerator = A (Denominator) + B [$\frac{d}{dx}$ (Denominator)]
∴ 4ex – 25 = A(2ex – 5) + B[$\frac{d}{dx}$ (2ex – 5)]
= A(2ex – 5) + B(2ex – 0)
∴ 4ex – 25 = (2A + 2B) ex – 5A
Equating the coefficient of ex and constant on both sides, we get
2A + 2B = 4 …….(1)
and 5A = 25
∴ A = 5
from (1), 2(5) + 2B = 4
∴ 2B = -6
∴ B = -3

$$\therefore\ 4e^x - 25 = 5(2e^x - 5) - 3(2e^x)$$

$$\therefore\ I = \int\left[\frac{5(2e^x - 5) - 3(2e^x)}{2e^x - 5}\right]dx$$

$$= \int\left[5 - \frac{3(2e^x)}{2e^x - 5}\right]dx$$

$$= 5\int 1dx - 3\int\frac{2e^x}{2e^x - 5}dx$$

$$= 5x - 3\log|2e^x - 5| + c$$

$$\ldots\left[\because\ \int\frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$$

Question 8.
$\frac{20+12e^x}{3e^x+4}$
Solution:

Let I = $\int \frac{20+12e^x}{3e^x+4}.dx$

Put,

Numerator = A (Denominator) + B $\left[\frac{d}{dx}\text{(Denominator)}\right]$

$\therefore$ 20 +12e^x = A$(3e^x+4)$ + B$\left[\frac{d}{dx}(3e^x+4)\right]$

= A($3e^x$ + 4) + B($3e^x$ + 0)

$\therefore$ 20 +12e^x = (2A + 2B)e^x – 5A

Equating the coefficient of e^x and constant on both sides, we get

2A + 2B = 4 ...(1)

and

5A = 25

$\therefore$ A = 5

$\therefore$ from (1),2(5) + 2B = 4

$\therefore$ 2B = – 6

$\therefore$ B = – 3

$\therefore$ 20 +12e^x = 5($3e^x$ + 4) – 3 ($3e^x$)

$\therefore$ I = $\int \left[\frac{5(3e^x+5)-3(3e^x)}{3e^x+4}\right].dx$

= $\int \left[5-\frac{3(3e^x)}{3^x+4}\right].dx$

= $5\int 1dx - 3\int \frac{3e^x}{3e^x+4}.dx$

= 5x – log|$3e^x$ + 4| + c .

Question 9.

$\frac{3e^{2x}+5}{4e^{2x}-5}$

Solution:

Let I = $\int \frac{3e^{2x}+5}{4e^{2x}-5}dx$

Put, Numerator = A (Denominator) + B [$\frac{d}{dx}$ (Denominator)]

$\therefore$ $3e^{2x}$ + 5 = A($4e^{2x}$ – 5) + B [$\frac{d}{dx}$ ($4e^{2x}$ – 5)]

= A($4e^{2x}$ – 5) + B(4 . e^{2x} × 2 – 0)

$\therefore$ $3e^{2x}$ + 5 = (4A + 8B) e^{2x} – 5A

Equating the coefficient of e^{2x} and constant on both sides, we get

4A + 8B = 3 .... (1)

and -5A = 5

$\therefore$ A = -1

$\therefore$ from (1), 4(-1) + 8B = 3

$\therefore$ 8B = 7

∴ B = 78

$$\therefore\ 3e^{2x}+5=-(4e^{2x}-5)+\frac{7}{8}(8e^{2x})$$

$$\therefore\ I=\int\left[\frac{-(4e^{2x}-5)+\frac{7}{8}(8e^{2x})}{4e^{2x}-5}\right]dx$$

$$=\int\left[-1+\frac{\frac{7}{8}(8e^{2x})}{4e^{2x}-5}\right]dx$$

$$=-\int 1\,dx+\frac{7}{8}\int\frac{8e^{2x}}{4e^{2x}-5}\,dx$$

$$=-x+\frac{7}{8}\log|4e^{2x}-5|+c$$

$$\ldots\left[\because\ \int\frac{f'(x)}{f(x)}dx=\log|f(x)|+c\right]$$

Question 10.
$\cos^8 x \cdot \cot x$
Solution:

Let $I=\int\cos^8 x\cot x\,dx$

$$=\int\cos^8 x\cdot\frac{\cos x}{\sin x}dx$$

Put $\sin x=t$ $\quad\therefore\ \cos x\,dx=dt$

$$\cos^8 x=(\cos^2 x)^4=(1-\sin^2 x)^4$$

$$=(1-t^2)^4=1-4t^2+6t^4-4t^6+t^8$$

$$I=\int\frac{1-4t^2+6t^4-4t^6+t^8}{t}dt$$

$$=\int\left[\frac{1}{t}-4t+6t^3-4t^5+t^7\right]dt$$

$$=\int\frac{1}{t}dx-4\int t\,dt+6\int t^3\,dt-4\int t^5\,dt+\int t^7\,dt$$

$$=\log|t|-4\left(\frac{t^2}{2}\right)+6\left(\frac{t^4}{4}\right)-4\left(\frac{t^6}{6}\right)+\frac{t^8}{8}+c$$

$$=\log|\sin x|-2\sin^2 x+\frac{3}{2}\sin^4 x-\frac{2}{3}\sin^6 x+\frac{\sin^8 x}{8}+c.$$

Question 11.
$\tan^5 x$
Solution:

Let I = $\int \tan^5 x\, dx$

$$= \int \tan^3 x \tan^2 x\, dx$$
$$= \int \tan^3 x (\sec^2 x - 1)\, dx$$
$$= \int (\tan^3 x \sec^2 x - \tan^3 x)\, dx$$
$$= \int (\tan^3 x \sec^2 x - \tan x \cdot \tan^2 x)\, dx$$
$$= \int [\tan^3 x \sec^2 x - \tan x (\sec^2 x - 1)]\, dx$$
$$= \int (\tan^3 x \sec^2 x - \tan x \sec^2 x + \tan x)\, dx$$
$$= \int [(\tan^3 x - \tan x) \sec^2 x + \tan x]\, dx$$
$$= \int (\tan^3 x - \tan x) \sec^2 x\, dx + \int \tan x\, dx$$
$$= I_1 + I_2 \qquad \text{... (Let)}$$

In I_1, put $\tan x = t \qquad \therefore\ \sec^2 x\, dx = dt$

$$\therefore\ I = \int (t^3 - t)\, dt + \int \tan x\, dx$$
$$= \frac{t^4}{4} - \frac{t^2}{2} + \log|\sec x| + c$$
$$= \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log|\sec x| + c.$$

Question 12.
$\cos^7 x$
Solution:

Let $I = \int \cos^7 x\, dx$

$$= \int \cos^6 x \cdot \cos x\, dx$$
$$= \int (1 - \sin^2 x)^3 \cos x\, dx$$

Put, $\sin x = t \qquad \therefore\ \cos x\, dx = dt$

$$I = \int (1 - t^2)^3\, dt$$
$$= \int (1 - 3t^2 + 3t^4 - t^6)\, dt$$
$$= \int 1\, dt - 3\int t^2 dt + 3\int t^4\, dt - \int t^6\, dt$$
$$= t - 3\left(\frac{t^3}{3}\right) + 3\left(\frac{t^5}{3}\right) - \frac{t^7}{7} + c$$
$$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c.$$

Question 13.
tan 3x tan 2x tan x
Solution:
Let I = ∫ tan 3x tan 2x tan x dx
Consider tan 3x = tan (2x + x) = $\frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$
tan 3x (1 – tan 2x tan x) = tan 2x + tan x
tan 3x – tan 3x tan 2x tan x = tan 2x + tan x
tan 3x – tan 2x – tan x = tan 3x tan 2x tan x
I = ∫(tan 3x – tan 2x – tan x) dx
= ∫tan3x dx – ∫tan 2x dx – ∫tan x dx
= $\frac{1}{3}$ log | sec 3x| – $\frac{1}{2}$ log |sec 2x| – log |sec x| + c.

Question 14.
$\sin^5 x \cos^8 x$

Solution:

$$\text{Let } I = \int \sin^5 x \cos^8 x \, dx$$
$$= \int \sin^4 x \cos^8 x \sin x \, dx$$
$$= \int (1-\cos^2 x)^2 \cos^8 x \sin x \, dx$$

$$\text{Put } \cos x = t \qquad \therefore \; -\sin x \, dx = dt$$

$$\therefore \; \sin x \, dx = -dt$$

$$I = -\int (1-t^2)^2 t^8 \, dt$$
$$= -\int (1-2t^2+t^4) t^8 \, dt$$
$$= -\int (t^8 - 2t^{10} + t^{12}) \, dt$$
$$= -\int t^8 dt + 2\int t^{10} dt - \int t^{12} dt$$
$$= -\frac{t^9}{9} + 2\left(\frac{t^{11}}{11}\right) - \frac{t^{13}}{13} + c$$
$$= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + c.$$

Question 15.

3cos2x· sin2x

Solution:

$$\text{Let } I = \int 3^{\cos^2 x} \sin 2x \, dx$$

$$\text{Put } \cos^2 x = t$$

$$\therefore \; \left[2\cos x \frac{d}{dx}(\cos x)\right] dx = dt$$

$$\therefore \; -2\sin x \cos x \, dx = dt$$

$$\therefore \; \sin 2x \, dx = -dt$$

$$I = -\int 3^t \, dt = -\frac{1}{\log 3} \cdot 3^t + c$$
$$= -\frac{1}{\log 3} \cdot 3^{\cos^2 x} + c.$$

Question 16.

sin6xsin10xsin4x

Solution:

$$\text{Let } I = \int \frac{\sin 6x}{\sin 10x \sin 4x} dx = \int \frac{\sin(10x-4x)}{\sin 10x \sin 4x} dx$$
$$= \int \frac{\sin 10x \cos 4x - \cos 10x \sin 4x}{\sin 10x \sin 4x} dx$$
$$= \int \left[\frac{\sin 10x \cos 4x}{\sin 10x \sin 4x} - \frac{\cos 10x \sin 4x}{\sin 10x \sin 4x}\right] dx$$
$$= \int \cot 4x \, dx - \int \cot 10x \, dx$$
$$= \frac{1}{4}\log|\sin 4x| - \frac{1}{10}\log|\sin 10x| + c.$$

Question 17.

sinxcos3x1+cos2x

Solution:

Let $I = \int \frac{\sin x \cos^3 x}{1+\cos^2 x} dx$

Put $\cos x = t$

$\therefore \quad -\sin x\, dx = dt \quad \therefore \ \sin x\, dx = -dt$

$$I = -\int \frac{t^3}{t^2+1} dt$$

$$= -\int \frac{t(t^2+1)-t}{t^2+1} dt$$

$$= -\int \left[\frac{t(t^2+1)}{t^2+1} - \frac{t}{t^2+1}\right] dt$$

$$= -\int t\, dt + \int \frac{t}{t^2+1} dt$$

$$= -\int t\, dt + \frac{1}{2}\int \frac{2t}{t^2+1} dt$$

$$= -\frac{t^2}{2} + \frac{1}{2}\log|t^2+1| + c$$

$$\ldots \left[\because \frac{d}{dt}(t^2+1) = 2t \text{ and } \int \frac{f'(x)}{f(x)} dx = \log[f(x)] + c\right]$$

$$= -\frac{1}{2}\cos^2 x + \frac{1}{2}\log|\cos^2 x + 1| + c$$

$$= \frac{1}{2}[\log|\cos^2 x + 1| - \cos^2 x] + c.$$

Maharashtra State Board 12th Maths Solutions Chapter 3 Indefinite Integration Ex 3.2(B)

I. Evaluate the following:

Question 1.
$\int \frac{1}{4x^2-3} \cdot dx$
Solution:

$$I=\int \frac{1}{4x^2-3}\,dx=\frac{1}{4}\int \frac{1}{x^2-\frac{3}{4}}\,dx$$

$$=\frac{1}{4}\int \frac{1}{x^2-\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$=\frac{1}{4}\,\frac{1}{2\left(\frac{\sqrt{3}}{2}\right)}\log\left|\frac{x-\frac{\sqrt{3}}{2}}{x+\frac{\sqrt{3}}{2}}\right|+c$$

$$=\frac{1}{4\sqrt{3}}\log\left|\frac{2x-\sqrt{3}}{2x+\sqrt{3}}\right|+c.$$

Question 2.
$\int \frac{1}{25-9x^2}\,dx$
Solution:

$$I=\int \frac{1}{25-9x^2}\,dx=\int \frac{1}{5^2-(3x)^2}\,dx$$

$$=\frac{1}{2(5)}\log\left|\frac{5+3x}{5-3x}\right|\cdot\frac{1}{3}+c$$

$$=\frac{1}{30}\log\left|\frac{5+3x}{5-3x}\right|+c.$$

Alternative Method :

$$\int \frac{1}{25-9x^2}\,dx=\frac{1}{9}\int \frac{1}{\frac{25}{9}x^2}\,dx$$

$$=\frac{1}{9}\int \frac{1}{\left(\frac{5}{3}\right)^2-x^2}\,dx=\frac{1}{9}\times\frac{1}{2\times\frac{5}{3}}\log\left|\frac{\frac{5}{3}+x}{\frac{5}{3}-x}\right|+c$$

$$=\frac{1}{30}\log\left|\frac{5+3x}{5-3x}\right|+c.$$

Question 3.
$\int \frac{1}{7+2x^2}\,dx$
Solution:

$$I=\int \frac{1}{7+2x^2}\,dx=\frac{1}{2}\int \frac{1}{\frac{7}{2}+x^2}\,dx$$

$$=\frac{1}{2}\int \frac{1}{\left(\sqrt{\frac{7}{2}}\right)^2+x^2}\,dx$$

$$=\frac{1}{2}\cdot\frac{1}{\left(\sqrt{\frac{7}{2}}\right)}\tan^{-1}\left|\frac{x}{\sqrt{\frac{7}{2}}}\right|+c$$

$$=\frac{1}{\sqrt{14}}\tan^{-1}\left|\frac{\sqrt{2}x}{\sqrt{7}}\right|+c.$$

Question 4.

∫13x2+8√· dx

Solution:

$$\int \frac{1}{\sqrt{3x^2+8}}\,dx = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2+\frac{8}{3}}}\,dx$$

$$= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{x^2+\left(\sqrt{\frac{8}{3}}\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{3}}\log\left|x+\sqrt{x^2+\left(\sqrt{\frac{8}{3}}\right)^2}\right|+c_1$$

$$= \frac{1}{\sqrt{3}}\log\left|x+\sqrt{x^2+\frac{8}{3}}\right|+c_1$$

$$= \frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{3}x+\sqrt{3x^2+8}}{\sqrt{3}}\right|+c_1$$

$$= \frac{1}{\sqrt{3}}\log|\sqrt{3}x+\sqrt{3x^2+8}|-\log\sqrt{3}+c_1$$

$$= \frac{1}{\sqrt{3}}\log|\sqrt{3}x+\sqrt{3x^2+8}|+c, \text{ where } c=c_1-\log\sqrt{3}$$

Alternative Method :

$$\int \frac{1}{\sqrt{3x^2+8}}\,dx = \int \frac{1}{\sqrt{(\sqrt{3}x)^2+(\sqrt{8})^2}}\,dx$$

$$= \frac{\log|\sqrt{3}x+\sqrt{(\sqrt{3}x)^2+(\sqrt{8})^2}|+c}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}\log|\sqrt{3}x+\sqrt{3x^2+8}|+c.$$

Question 5.

∫111−4x2√· dx

Solution:

$$\int \frac{1}{\sqrt{11-4x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{(\sqrt{11})^2-(2x)^2}}\,dx$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{11}}\right)+c.$$

Question 6.

∫12x2−5√· dx

Solution:

$$\int \frac{1}{\sqrt{2x^2-5}}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{x^2-\frac{5}{2}}}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{x^2-\left(\sqrt{\frac{5}{2}}\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{2}}\log\left|x+\sqrt{x^2-\frac{5}{2}}\right|+c.$$

Question 7.
∫9+x9−x−−−√· dx
Solution:

$$\text{Let } I = \int \sqrt{\frac{9+x}{9-x}}\, dx$$
$$= \int \sqrt{\frac{9+x}{9-x} \times \frac{9+x}{9+x}}\, dx$$
$$= \int \frac{9+x}{\sqrt{81-x^2}}\, dx$$
$$= \int \frac{9}{\sqrt{81-x^2}}\, dx + \int \frac{x}{\sqrt{81-x^2}}\, dx$$
$$= 9 \int \frac{1}{\sqrt{9^2-x^2}}\, dx + \frac{1}{2} \int \frac{2x}{\sqrt{81-x^2}}\, dx$$
$$= I_1 + I_2 \quad \text{... (Let)}$$
$$I_1 = 9 \int \frac{1}{\sqrt{9^2-x^2}}\, dx = 9 \sin^{-1}\left(\frac{x}{9}\right) + c_1$$

In I_2, put $81 - x^2 = t$

$\therefore \ -2x\,dx = dt \qquad \therefore \ 2x\,dx = -dt$

$$I_2 = -\frac{1}{2} \int t^{-\frac{1}{2}}\, dt = -\frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c_2$$
$$= -\sqrt{81-x^2} + c_2$$
$$I = 9 \sin^{-1}\left(\frac{x}{9}\right) - \sqrt{81-x^2} + c, \text{ where } c = c_1 + c_2$$

Question 8.
∫2+x2−x−−−√· dx
Solution:

Let I = $\int \sqrt{\frac{2+x}{2-x}}.dx$

= $\int \sqrt{\frac{2+x}{2-x} \times \frac{2+x}{2+x}}.dx$

= $\int \frac{2+x}{\sqrt{4-x^2}}.dx$

= $\int \frac{2}{\sqrt{4-x^2}}.dx + \int \frac{x}{\sqrt{4-x^2}}.dx$

= $2\int \frac{1}{\sqrt{2^2-x^2}}.dx + \frac{1}{2}\int \frac{2x}{\sqrt{4-x^2}}.dx$

= $I_1 + I_2$ ...(Let)

$I_1 = 2\int \frac{1}{\sqrt{2^2-x^2}}.dx$

= $2\sin^{-1}\left(\frac{x}{2}\right) + c_1$

In I_2, put $4 - x^2 = t$

$\therefore -2x\,dx = dt$

$\therefore 2x\,dx = -dt$

$I_2 = -\frac{1}{2}\int t^{-\frac{1}{2}}dt$

= $-\frac{1}{2}.\frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c_2$

= $-\sqrt{4-x^2} + c_2$

I = $2\sin^{-1}\left(\frac{x}{2}\right) - \sqrt{4-x^2} + c.$

Question 9.

∫10+x10−x−−−−√⋅ dx

Solution:

Let $I = \int \sqrt{\frac{10+x}{10-x}}.dx$

$= \int \sqrt{\frac{10+x}{10-x} \times \frac{10+x}{10+x}}.dx$

$= \int \frac{10+x}{\sqrt{100-x^2}}.dx$

$= \int \frac{10}{\sqrt{100-x^2}}.dx + \int \frac{x}{\sqrt{100-x^2}}.dx$

$= 10\int \frac{1}{\sqrt{10^2-x^2}}.dx + \frac{1}{2}\int \frac{2x}{\sqrt{100-x^2}}.dx$

$= I_1 + I_2$ ...(Let)

$I_1 = 10\int \frac{1}{\sqrt{10^2-x^2}}.dx$

$= 10\sin^{-1}\left(\frac{x}{10}\right) + c_1$

In I_2, put $100 - x^2 = t$

$\therefore -2x\,dx = dt$

$\therefore 2x\,dx = -dt$

$I_2 = -\frac{1}{2}\int t^{-\frac{1}{2}}dt$

$= -\frac{1}{2}.\frac{t^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + c_2$

$= -\sqrt{100-x^2} + c_2$

$I = 10\sin^{-1}\left(\frac{x}{10}\right) - \sqrt{100-x^2} + c.$

Question 10.
$\int \frac{1}{x^2+8x+12}\,dx$
Solution:

$\int \frac{1}{x^2+8x+12}dx$

$= \int \frac{1}{(x^2+8x+16)-16+12}dx$

$= \int \frac{1}{(x+4)^2-2^2}dx$

$= \frac{1}{2(2)}\log\left|\frac{(x+4)-2}{(x+4)+2}\right| + c$

$= \frac{1}{4}\log\left|\frac{x+2}{x+6}\right| + c.$

Question 11.
$\int \frac{1}{1+x-x^2}\,dx$

Solution:

$$\text{Let } I=\int \frac{1}{1+x-x^2}dx$$

$$1+x-x^2=1-(x^2-x)$$

$$=1-\left(x^2-x+\frac{1}{4}\right)+\frac{1}{4}$$

$$=\frac{5}{4}-\left(x^2-x+\frac{1}{4}\right)$$

$$=\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2$$

$$\therefore\ I=\int \frac{1}{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}dx$$

$$=\frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\log\left|\frac{\frac{\sqrt{5}}{2}+\left(x-\frac{1}{2}\right)}{\frac{\sqrt{5}}{2}-\left(x-\frac{1}{2}\right)}\right|+c$$

$$=\frac{1}{\sqrt{5}}\log\left|\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x}\right|+c.$$

Question 12.

∫14x2−20x+17⋅ dx

Solution:

$$\int \frac{1}{4x^2-20x+17}dx$$

$$=\frac{1}{4}\int \frac{1}{x^2-5x+\frac{17}{4}}dx$$

$$=\frac{1}{4}\int \frac{1}{\left(x^2-5x+\frac{25}{4}\right)-\frac{25}{4}+\frac{17}{4}}dx$$

$$=\frac{1}{4}\int \frac{1}{\left(x-\frac{5}{2}\right)^2-(\sqrt{2})^2}dx$$

$$=\frac{1}{4}\times\frac{1}{2\sqrt{2}}\log\left|\frac{x-\frac{5}{2}-\sqrt{2}}{x-\frac{5}{2}+\sqrt{2}}\right|+c$$

$$=\frac{1}{8\sqrt{2}}\log\left|\frac{2x-5-2\sqrt{2}}{2x-5+2\sqrt{2}}\right|+c.$$

Question 13.

∫15−4x−3x2⋅ dx

Solution:

$$\text{Let } I=\int \frac{1}{5-4x-3x^2}\,dx$$

$$5-4x-3x^2=3\left[\frac{5}{3}-\left(x^2+\frac{4}{3}x\right)\right]$$

$$=3\left[\frac{5}{3}-\left(x^2+\frac{4}{3}x+\frac{4}{9}\right)+\frac{4}{9}\right]$$

$$=3\left[\frac{19}{9}-\left(x^2+\frac{4x}{3}+\frac{4}{9}\right)\right]$$

$$=3\left[\left(\frac{\sqrt{19}}{3}\right)^2-\left(x+\frac{2}{3}\right)^2\right]$$

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$$I=\int \frac{1}{3\left[\left(\frac{\sqrt{19}}{3}\right)^2-\left(x+\frac{2}{3}\right)^2\right]}\,dx$$

$$=\frac{1}{3}\,\frac{1}{2\left(\frac{\sqrt{19}}{3}\right)}\log\left|\frac{\frac{\sqrt{19}}{3}+\left(x+\frac{2}{3}\right)}{\frac{\sqrt{19}}{3}-\left(x+\frac{2}{3}\right)}\right|+c$$

$$=\frac{1}{2\sqrt{19}}\log\left|\frac{\sqrt{19}+2+3x}{\sqrt{19}-2-3x}\right|+c$$

$$=\frac{1}{2\sqrt{19}}\log\left|\frac{3x+2+\sqrt{19}}{-(3x+2-\sqrt{19})}\right|+c$$

$$=\frac{1}{2\sqrt{19}}\log\left|\frac{3x+2+\sqrt{19}}{3x+2-\sqrt{19}}\right|+c \qquad \ldots [\because |-x|=x]$$

Question 14.
∫13x2+5x+7√⋅ dx
Solution:

$$\text{Let } I=\int \frac{1}{\sqrt{3x^2+5x+7}}\,dx$$

$$3x^2+5x+7=3\left[x^2+\frac{5}{3}x+\frac{7}{3}\right]$$

$$=3\left[\left(x^2+\frac{5x}{3}+\frac{25}{36}\right)+\left(\frac{7}{3}-\frac{25}{36}\right)\right]$$

$$=3\left[\left(x+\frac{5}{6}\right)^2+\left(\frac{\sqrt{59}}{6}\right)^2\right]$$

$$\therefore\ \sqrt{3x^2+5x+7}=\sqrt{3}\sqrt{\left(x+\frac{5}{6}\right)^2+\left(\frac{\sqrt{59}}{6}\right)^2}$$

$$\therefore\ I=\frac{1}{\sqrt{3}}\int \frac{1}{\left(x+\frac{5}{6}\right)^2+\left(\frac{\sqrt{59}}{6}\right)^2}\,dx$$

$$=\frac{1}{\sqrt{3}}\log\left|x+\frac{5}{6}+\sqrt{\left(x+\frac{5}{6}\right)^2+\left(\frac{\sqrt{59}}{6}\right)^2}\right|+c$$

$$=\frac{1}{\sqrt{3}}\log\left|x+\frac{5}{6}+\sqrt{x^2+\frac{5x}{3}+\frac{7}{3}}\right|+c.$$

Question 15.

∫1x2+8x−20√⋅ dx

Solution:

$$\int \frac{1}{\sqrt{x^2+8x-20}}\, dx$$

$$= \int \frac{1}{\sqrt{(x^2+8x+16)-16-20}}\, dx$$

$$= \int \frac{1}{\sqrt{(x+4)^2-(6)^2}}\, dx$$

$$= \log|(x+4)+\sqrt{(x-4)^2-(6)^2}|.+c$$

$$= \log|(x+4)+\sqrt{x^2-8x-20}|+c.$$

Question 16.

∫18−3x+2x2√⋅ dx

Solution:

$$\text{Let I} = \int \frac{1}{\sqrt{8-3x+2x^2}}.\, dx$$

$$8 - 3x + 2x^2 = 8\left[x^2+\frac{3}{2}x+\frac{2}{2}\right]$$

$$= 8\left[\left(x^2+\frac{3x}{2}+\frac{6}{4}\right)+\left(\frac{2}{2}-\frac{6}{4}\right)\right]$$

$$= 8\left[\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{1}}{4}\right)^2\right]$$

$$\therefore \sqrt{3x^2+5x+7} = \sqrt{3}\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{1}}{4}\right)^2}$$

$$\therefore \text{I} = \frac{1}{\sqrt{3}}\int \frac{1}{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{1}}{4}\right)^2}.\, dx$$

$$= \frac{1}{\sqrt{2}}\log\left|x-\frac{3}{4}+\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{1}}{4}\right)^2}\right|+c$$

$$= \frac{1}{\sqrt{2}}\log\left|x-\frac{3}{4}+\sqrt{x^2-\frac{3x}{2}+4}\right|+c.$$

Question 17.

∫1(x−3)(x+2)√⋅ dx

Solution:

$$\text{Let } I = \int \frac{1}{\sqrt{(x-3)(x+2)}}\, dx$$

$$= \int \frac{1}{\sqrt{x^2-x-6}}\, dx$$

$$= \int \frac{1}{\sqrt{\left(x^2-x+\frac{1}{4}\right)-\frac{1}{4}-6}}\, dx$$

$$= \int \frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{5}{2}\right)^2}}\, dx$$

$$= \log\left|\left(x-\frac{1}{2}\right)+\sqrt{\left(x-\frac{1}{2}\right)^2-\left(\frac{5}{2}\right)^2}\right| + c$$

$$= \log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2-x-6}\right| + c.$$

Question 18.
∫14+3cos2x⋅ dx
Solution:

$$\text{Let } I = \int \frac{1}{4+3\cos^2 x}\, dx$$

Dividing both numerator and denominator by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x}{4\sec^2 x + 3}\, dx$$

$$= \int \frac{\sec^2 x}{4(1+\tan^2 x)+3}\, dx$$

$$= \int \frac{\sec^2 x}{4\tan^2 x + 7}\, dx$$

Put $\tan x = t \qquad \therefore \ \sec^2 x\, dx = dt$

$$I = \int \frac{dt}{4t^2+7}$$

$$= \int \frac{dt}{(2t)^2+(\sqrt{7})^2}$$

$$= \frac{1}{\sqrt{7}} \tan^{-1}\left(\frac{2t}{\sqrt{7}}\right) \cdot \frac{1}{2} + c$$

$$= \frac{1}{2\sqrt{7}} \tan^{-1}\left(\frac{2\tan x}{\sqrt{7}}\right) + c.$$

Question 19.
∫1cos2x+3sin2x⋅ dx
Solution:

$$\text{Let } I = \int \frac{1}{\cos 2x + 3\sin^2 x} dx$$

$$= \int \frac{1}{1 - 2\sin^2 x + 3\sin^2 x} dx$$

$$= \int \frac{1}{1 + \sin^2 x} dx$$

Dividing both numerator and denominator by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x\, dx}{\sec^2 x + \tan^2 x} = \int \frac{\sec^2 x\, dx}{1 + \tan^2 x + \tan^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{2\tan^2 x + 1}$$

Put $\tan x = t \quad \therefore \sec^2 x\, dx = dt$

$$\therefore I = \int \frac{1}{2t^2 + 1} dt = \frac{1}{2} \int \frac{1}{t^2 + \left(\frac{1}{\sqrt{2}}\right)^2} dt$$

$$= \frac{1}{2} \times \frac{1}{\left(\frac{1}{\sqrt{2}}\right)} \tan^{-1}\left(\frac{t}{\frac{1}{\sqrt{2}}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}\tan x) + c.$$

Question 20.

$\int \frac{\sin x}{\sin 3x} dx$

Solution:

$$\text{Let } I = \int \frac{\sin x}{\sin 3x} dx$$

$$= \int \frac{\sin x}{3\sin x - 4\sin^3 x} dx$$

$$= \int \frac{1}{3 - 4\sin^2 x} dx$$

Dividing both numerator and denominator by $\cos^2 x$, we get

$$I = \int \frac{\sec^2 x}{3\sec^2 x - 4\tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{3(1 + \tan^2 x) - 4\tan^2 x} dx$$

$$= \int \frac{\sec^2 x}{3 - \tan^2 x} dx$$

Put $\tan x = t \qquad \therefore \sec^2 x\, dx = dt$

$$I = \int \frac{dt}{(\sqrt{3})^2 - t^2}$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{\sqrt{3} + t}{\sqrt{3} - t}\right| + c$$

$$= \frac{1}{2\sqrt{3}} \log\left|\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right| + c.$$

II. Integrate the following functions w. r. t. x:

Question 1.

$\int 13+2\sin x \cdot dx$

Solution:

$$\text{Let } I=\int \frac{1}{3+2\sin x}\,dx$$

$$\text{Put } \tan(x/2)=t \qquad \therefore\ x=2\tan^{-1}t$$

$$\therefore\ dx=\frac{2dt}{1+t^2} \text{ and } \sin x=\frac{2t}{1+t^2}$$

$$\therefore\ I=\int \frac{1}{3+2\left(\frac{2t}{1+t^2}\right)}\cdot\frac{2dt}{1+t^2}$$

$$=\int \frac{1+t^2}{3+3t^2+4t}\cdot\frac{2dt}{1+t^2}$$

$$=2\int \frac{1}{3t^2+4t+3}\,dt$$

$$=\frac{2}{3}\int \frac{1}{t^2+\frac{4}{3}t+1}\,dt$$

$$=\frac{2}{3}\int \frac{1}{\left(t^2+\frac{4}{3}t+\frac{4}{9}\right)-\frac{4}{9}+1}\,dt$$

$$=\frac{2}{3}\int \frac{1}{\left(t+\frac{2}{3}\right)^2+\left(\frac{\sqrt{5}}{3}\right)^2}\,dt$$

$$=\frac{2}{3}\times\frac{1}{\left(\frac{\sqrt{5}}{3}\right)}\tan^{-1}\left[\frac{t+\frac{2}{3}}{\frac{\sqrt{5}}{3}}\right]+c$$

$$=\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{3t+2}{\sqrt{5}}\right)+c$$

$$=\frac{2}{\sqrt{5}}\tan^{-1}\left[\frac{3\tan\left(\frac{x}{2}\right)+2}{\sqrt{5}}\right]+c.$$

Question 2.

$\int 14-5\cos x \cdot dx$

Solution:

$$\text{Let } I=\int \frac{1}{4-5\cos x}\,dx$$

Put $\tan\left(\frac{x}{2}\right)=t \quad \therefore\ x=2\tan^{-1}t$

$\therefore\ dx=\frac{2dt}{1+t^2}$ and $\cos x=\frac{1-t^2}{1+t^2}$

$$\therefore\ I=\int \frac{1}{4-5\left(\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2dt}{1+t^2}$$

$$=\int \frac{1+t^2}{4(1+t^2)-5(1-t^2)}\cdot\frac{2dt}{1+t^2}$$

$$=\int \frac{2dt}{9t^2-1}=\frac{2}{9}\int \frac{1}{t^2-\frac{1}{9}}\,dt$$

$$=\frac{2}{9}\int \frac{1}{t^2-\left(\frac{1}{3}\right)^2}\,dt$$

$$=\frac{2}{9}\times\frac{1}{2\times\frac{1}{3}}\log\left|\frac{t-\frac{1}{3}}{t+\frac{1}{3}}\right|+c$$

$$=\frac{1}{3}\log\left|\frac{3\tan\left(\frac{x}{2}\right)-1}{3\tan\left(\frac{x}{2}\right)+1}\right|+c.$$

Question 3.
$\int \frac{1}{2+\cos x-\sin x}\cdot dx$
Solution:

$$\text{Let } I=\int \frac{1}{2+\cos x-\sin x}\,dx$$

Put $\tan(x/2)=t \quad \therefore\ x=2\tan^{-1} t$

$$\therefore\ dx=\frac{2dt}{1+t^2} \text{ and } \sin x=\frac{2t}{1+t^2},\ \cos x=\frac{1-t^2}{1+t^2}$$

$$\therefore\ I=\int \frac{1}{2+\left(\frac{1-t^2}{1+t^2}\right)-\left(\frac{2t}{1+t^2}\right)}\cdot\frac{2dt}{1+t^2}$$

$$=\int \frac{1+t^2}{2+2t^2+1-t^2-2t}\cdot\frac{2dt}{1+t^2}$$

$$=2\int \frac{1}{t^2-2t+3}\,dt=2\int \frac{1}{(t^2-2t+1)+2}\,dt$$

$$=2\int \frac{1}{(t-1)^2+(\sqrt{2})^2}\,dt$$

$$=2\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-1}{\sqrt{2}}\right)+c$$

$$=\sqrt{2}\tan^{-1}\left[\frac{\tan\left(\frac{x}{2}\right)-1}{\sqrt{2}}\right]+c.$$

Question 4.
$\int \frac{1}{3+2\sin x-\cos x}\,dx$
Solution:

$$\text{Let } I=\int \frac{1}{3+2\sin x-\cos x}\,dx$$

Put $\tan\left(\frac{x}{2}\right)=t \qquad \therefore\ x=2\tan^{-1} t$

$$\therefore\ dx=\frac{2}{1+t^2}\,dt \text{ and}$$

$$\sin x=\frac{2t}{1+t^2},\ \cos x=\frac{1-t^2}{1+t^2}$$

$$\therefore\ I=\int \frac{1}{3+2\left(\frac{2t}{1+t^2}\right)-\left(\frac{1-t^2}{1+t^2}\right)}\cdot\frac{2dt}{1+t^2}$$

$$=\int \frac{1+t^2}{3(1+t^2)+4t-(1-t^2)}\cdot\frac{2dt}{1+t^2}$$

$$=2\int \frac{dt}{4t^2+4t+2}$$

$$=2\int \frac{dt}{4t^2+4t+1+1}$$

$$=2\int \frac{dt}{(2t+1)^2+1^2}$$

$$=\frac{2}{2}\tan^{-1}\left(\frac{2t+1}{1}\right)+c$$

$$=\tan^{-1}\left[2\tan^{-1}\left(\frac{x}{2}\right)+1\right]+c.$$

Question 5.
$\int \frac{1}{3-2\cos 2x}\,dx$

Solution:

$$=\frac{1}{5}\int\frac{1}{\left(\frac{1}{\sqrt{5}}\right)^2+t^2}\,dt$$

$$=\frac{1}{5}\times\frac{1}{(1/\sqrt{5})}\tan^{-1}\left(\frac{t}{1/\sqrt{5}}\right)+c$$

$$=\frac{1}{\sqrt{5}}\tan^{-1}(\sqrt{5}\tan x)+c.$$

Question 6.
∫12sin2x-3⋅ dx
Solution:

$$\text{Let } I=\int \frac{1}{2\sin 2x-3}\,dx$$

Put $\tan x=t \qquad \therefore\ x=\tan^{-1}t$

$$\therefore\ dx=\frac{dt}{1+t^2} \text{ and } \sin 2x=\frac{2t}{1+t^2}$$

$$\therefore\ I=\int \frac{1}{2\left(\frac{2t}{1+t^2}\right)-3}\cdot\frac{dt}{1+t^2}$$

$$=\int \frac{1+t^2}{4t-3-3t^2}\cdot\frac{dt}{1+t^2}$$

$$=\int \frac{1}{-3t^2+4t-3}\,dt$$

$$=-\frac{1}{3}\int \frac{1}{t^2-\frac{4}{3}t+1}\,dt$$

$$=-\frac{1}{3}\int \frac{1}{\left(t^2-\frac{4}{3}t+\frac{4}{9}\right)-\frac{4}{9}+1}\,dt$$

$$=-\frac{1}{3}\int \frac{1}{\left(t-\frac{2}{3}\right)^2+\left(\frac{\sqrt{5}}{3}\right)^2}\,dt$$

$$=-\frac{1}{3}\times\frac{1}{(\sqrt{5}/3)}\tan^{-1}\left(\frac{t-\frac{2}{3}}{\sqrt{5}/3}\right)+c$$

$$=-\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{3t-2}{\sqrt{5}}\right)+c$$

$$=-\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{3\tan x-2}{\sqrt{5}}\right)+c.$$

Question 7.
$\int 13+2sin2x+4cos2x \cdot\ dx$
Solution:

$$\text{Let } I=\int \frac{1}{3+2\sin 2x+4\cos 2x}dx$$

Put $\tan x = t \quad \therefore \; x=\tan^{-1} t$

$$\therefore \; dx=\frac{dt}{1+t^2} \text{ and } \sin 2x=\frac{2t}{1+t^2},\; \cos 2x=\frac{1-t^2}{1+t^2}$$

$$\therefore \; I=\int \frac{1}{3+2\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)}\cdot\frac{dt}{1+t^2}$$

$$=\int \frac{1+t^2}{3(1+t^2)+4t+4(1-t^2)}\cdot\frac{dt}{1+t^2}$$

$$=\int \frac{1}{7+4t-t^2}\,dt=\int \frac{1}{7-(t^2-4t+4)+4}\,dt$$

$$=\int \frac{1}{(\sqrt{11})^2-(t-2)^2}\,dt$$

$$=\frac{1}{2\sqrt{11}}\log\left|\frac{\sqrt{11}+t-2}{\sqrt{11}-t+2}\right|+c$$

$$=\frac{1}{2\sqrt{11}}\log\left|\frac{\sqrt{11}+\tan x-2}{\sqrt{11}-\tan x+2}\right|+c.$$

Question 8.

$\int \frac{1}{\cos x-\sin x}\cdot dx$

Solution:

$$\text{Let } I=\int \frac{1}{\cos x-\sin x}\,dx$$

Dividing each term by $\sqrt{1^2+(-1)^2}=\sqrt{2}$, we get

$$I=\frac{1}{\sqrt{2}}\int \frac{1}{\cos x\cdot\frac{1}{\sqrt{2}}-\sin x\cdot\frac{1}{\sqrt{2}}}\,dx$$

$$=\frac{1}{\sqrt{2}}\int \frac{1}{\cos x\cdot\cos\frac{\pi}{4}-\sin x\cdot\sin\frac{\pi}{4}}\,dx$$

$$=\frac{1}{\sqrt{2}}\int \frac{1}{\cos\left(x+\frac{\pi}{4}\right)}\,dx$$

$$=\frac{1}{\sqrt{2}}\int \sec\left(x+\frac{\pi}{4}\right)dx$$

$$=\frac{1}{\sqrt{2}}\log\left|\sec\left(x+\frac{\pi}{4}\right)+\tan\left(x+\frac{\pi}{4}\right)\right|+c.$$

Question 9.

$\int \frac{1}{\cos x-\sqrt{3}\sin x}\cdot dx$

Solution:

Let I = $\int \frac{1}{\cos x-\sqrt{3} \sin x} . d x$

Dividing each term by $\sqrt{1^{2}+(-1)^{2}}=\sqrt{3}$, we get

$$I = \frac{1}{2} \int \frac{1}{\cos x . \frac{1}{\sqrt{3}}-\sin x . \frac{1}{\sqrt{3}}} . d x$$

$$=\frac{1}{2} \int \frac{1}{\cos x . \cos \frac{\pi}{3}-\sin x . \sin \frac{\pi}{3}} . d x$$

$$=\frac{1}{2} \int \frac{1}{\cos \left(x+\frac{\pi}{3}\right)} . d x$$

$$=\frac{1}{2} \int \sec \left(x+\frac{\pi}{3}\right) . d x$$

$$=\frac{1}{2} \log \left|\sec \left(x+\frac{\pi}{3}\right)+\tan \left(x+\frac{\pi}{3}\right)\right|+c .$$

Maharashtra State Board 12th Maths Solutions Chapter 3 Indefinite Integration Ex 3.2(C)

I. Evaluate:

Question 1.
∫3x+4x2+6x+5dx
Solution:
Let I = ∫3x+4x2+6x+5dx
Let 3x + 4 = A[ddx(x2 + 6x + 5)] + B
= A(2x + B) + B
∴ 3x + 4 = 2Ax + (6A + B)
Comparing the coefficient of x and constant on both sides, we get
2A = 3 and 6A + B = 4
∴ A = 32 and 6(32) + B = 4
∴ B = -5

$3x + 4 = \frac{3}{2}(2x + 6) - 5$

$$\therefore\ I = \int \frac{\frac{3}{2}(2x+6)-5}{x^2+6x+5}\,dx$$

$$= \frac{3}{2}\int \frac{2x+6}{x^2+6x+5}\,dx - 5\int \frac{1}{x^2+6x+5}\,dx$$

$$= \frac{3}{2}I_1 - 5I_2$$

I_1 is of the type $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$

$$\therefore\ I_1 = \log|x^2+6x+5| + c_1$$

$$I_2 = \int \frac{1}{x^2+6x+5}\,dx = \int \frac{1}{(x^2+6x+9)-4}\,dx$$

$$= \int \frac{1}{(x+3)^2 - 2^2}\,dx$$

$$= \frac{1}{2\times 2}\log\left|\frac{x+3-2}{x+3+2}\right| + c_2 = \frac{1}{4}\log\left|\frac{x+1}{x+5}\right| + c_2$$

$$\therefore\ I = \frac{3}{2}\log|x^2+6x+5| - \frac{5}{4}\log\left|\frac{x+1}{x+5}\right| + c,$$

where $c = c_1 + c_2$.

Question 2.
$\int \frac{2x+1}{x^2+4x-5}\,dx$
Solution:
Let I = $\int \frac{2x+1}{x^2+4x-5}\,dx$
Let $2x + 1 = A\left[\frac{d}{dx}(x^2 + 4x - 5)\right] + B$
$2x + 1 = A(2x + 1) + B$
$\therefore 2x + 1 = 2Ax + (4A + B)$
Comparing the coefficient of x and constant on both sides, we get
$4A = 2$ and $4A + B = 4$
$\therefore A = \frac{3}{2}$ and $6(\frac{3}{2}) + B = 4$
$\therefore B = -5$
$\therefore 2x + 1 = \frac{3}{2}(2x + 1) - 5$

$$\therefore \mathrm{I} = \int \frac{\frac{3}{2}(2x+1) - 5}{x^2 + 6x + 5} . dx$$

$$= \frac{3}{2} \int \frac{2x+1}{x^2+4x-5} . dx - 5 \int \frac{1}{x^2+4x+5} . dx$$

$$= \frac{3}{2} \mathrm{I}_1 - 5\mathrm{I}_2$$

I_1 is of the type $\int \frac{f\prime(x)}{f(x)} . dx = \log|f(x)| + c$

$$\therefore \mathrm{I}_1 = \log\left|x^2 + 4x - 5\right| + c_1$$

$$\mathrm{I}_2 = \int \frac{1}{x^2+4x-5} . dx$$

$$= \int \frac{1}{(x^2+4x-9)-4} . dx$$

$$= \int \frac{1}{(x+3)^2 - 2^2} . dx$$

$$= \log\left|\frac{x+3-2}{x+3+2}\right| + c_2$$

$$= \log\left|\frac{x-1}{x+5}\right| + c_2$$

$$\therefore \mathrm{I} = \log\left|x^2+4x-5\right| - \frac{1}{2}\log\left|\frac{x-1}{x+5}\right| + c,$$

Question 3.
∫2x+32x2+3x−1dx
Solution:
Let I = ∫2x+32x2+3x−1dx
Let 2x+ 3 = A[ddx(2x2 + 3x – 1)] + B
2x + 1 = A(4x + 3) + B
∴ 2x + 1 = 4Ax + (3A + B)
Comparing the coefficient of x and constant on both sides, we get
4A = 2 and 3A + B = 3
∴ A = 12 and 3(12) + B = 3
∴ B = 32
∴ 2x + 3 = 12(4x + 3) + 32

$$\therefore\ I=\int \frac{\frac{1}{2}(4x+3)+\frac{3}{2}}{2x^2+3x-1}\,dx$$

$$=\frac{1}{2}\int \frac{4x+3}{2x^2+3x-1}\,dx+\frac{3}{2}\int \frac{1}{2x^2+3x-1}\,dx$$

$$=\frac{1}{2}I_1+\frac{3}{2}I_2$$

I_1 is of the type $\int \frac{f'(x)}{f(x)}\,dx=\log|f(x)|+c$

$$\therefore\ I_1=\log|2x^2+3x-1|+c_1$$

$$I_2=\int \frac{1}{2x^2+3x-1}\,dx=\frac{1}{2}\int \frac{1}{x^2+\frac{3}{2}x-\frac{1}{2}}\,dx$$

$$=\frac{1}{2}\int \frac{1}{\left(x^2+\frac{3}{2}x+\frac{9}{16}\right)-\frac{9}{16}-\frac{1}{2}}\,dx$$

$$=\frac{1}{2}\int \frac{1}{\left(x+\frac{3}{4}\right)^2-\left(\frac{\sqrt{17}}{4}\right)^2}\,dx$$

$$=\frac{1}{2}\times\frac{1}{2\times\frac{\sqrt{17}}{4}}\log\left|\frac{x+\frac{3}{4}-\frac{\sqrt{17}}{4}}{x+\frac{3}{4}+\frac{\sqrt{17}}{4}}\right|+c_2$$

$$=\frac{1}{\sqrt{17}}\log\left|\frac{4x+3-\sqrt{17}}{4x+3+\sqrt{17}}\right|+c_2$$

$$\therefore\ I=\frac{1}{2}\log|2x^2+3x-1|+\frac{3}{2\sqrt{17}}\log\left|\frac{4x+3-\sqrt{17}}{4x+3+\sqrt{17}}\right|+c,$$

where $c=c_1+c_2$.

Question 4.
$\int \frac{3x+4}{\sqrt{2x^2+2x+1}}\,dx$
Solution:
Let I = $\int \frac{3x+4}{\sqrt{2x^2+2x+1}}\,dx$
Let 3x + 4 = A[$\frac{d}{dx}$($2x^2$ + 2x + 1)] + B
∴ 3x + 4 = A (4x + 2) + B
∴ 3x + 4 = 4Ax + (2A + B)
Comparing the coefficient of x and the constant on both the sides, we get
4A = 3 and 2A + B = 4
∴ A = $\frac{3}{4}$ and 2($\frac{3}{4}$) + B = 4
∴ B = $\frac{5}{2}$
∴ 3x + 4 = $\frac{3}{4}$ (4x + 2) + $\frac{5}{2}$

$$\therefore\ I=\int \frac{\frac{3}{4}(4x+2)+\frac{5}{2}}{\sqrt{2x^2+2x+1}}\,dx$$

$$=\frac{3}{4}\int \frac{4x+2}{\sqrt{2x^2+2x+1}}\,dx+\frac{5}{2}\int \frac{1}{\sqrt{2x^2+2x+1}}\,dx$$

$$=\frac{3}{4}I_1+\frac{5}{2}I_2$$

In I_1, put $2x^2+2x+1=t$ $\quad\therefore\ (4x+2)dx=dt$

$$\therefore\ I_1=\int \frac{1}{\sqrt{t}}dt=\int t^{-\frac{1}{2}}dt$$

$$=\frac{t^{\frac{1}{2}}}{1/2}+c_1=2\sqrt{2x^2+2x+1}+c$$

$$I_2=\frac{5}{\sqrt{2}}\int \frac{1}{\sqrt{x^2+x+\frac{1}{2}}}\,dx$$

$$=\frac{5}{\sqrt{2}}\int \frac{1}{\sqrt{\left(x^2+x+\frac{1}{4}\right)+\frac{1}{4}}}\,dx$$

$$=\frac{5}{\sqrt{2}}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}\,dx$$

$$=\frac{5}{\sqrt{2}}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right|+c_2$$

$$=\frac{5}{\sqrt{2}}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+\frac{1}{2}}\right|+c_2$$

$$\therefore\ I=\frac{3}{2}\sqrt{2x^2+2x+1}+\frac{5}{2\sqrt{2}}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+\frac{1}{2}}\right|+c,$$

where $c=c_1+c_2$.

Question 5.
$\int \frac{7x+3}{\sqrt{3+2x-x^2}}dx$
Solution:
Let I = $\int \frac{7x+3}{\sqrt{3+2x-x^2}}dx$
Let 7x + 3 = A[$\frac{d}{dx}$(3 + 2x – x^2)] + B
7x + 3 = A(2 – 2x) + B
∴ 7x + 3 = -2Ax + (2A + B)
Comparing the coefficient of x and constant on both the sides, we get
-2A = 7 and 2A + B = 3

$$\therefore\ A=\frac{-7}{2} \text{ and } 2\left(-\frac{7}{2}\right)+B=3 \quad \therefore\ B=10$$

$$\therefore\ 7x+3=\frac{-7}{2}(2-2x)+10$$

$$\therefore\ I=\int \frac{\frac{-7}{2}(2-2x)+10}{\sqrt{3+2x-x^2}}\,dx$$

$$=\frac{-7}{2}\int \frac{(2-2x)}{\sqrt{3+2x-x^2}}\,dx+10\int \frac{1}{\sqrt{3+2x-x^2}}\,dx$$

$$=\frac{-7}{2}I_1+10I_2$$

In I_1, put $3+2x-x^2=t \quad \therefore\ (2-2x)\,dx=dt$

$$\therefore\ I_1=\int \frac{1}{\sqrt{t}}\,dt=\int t^{-1/2}\,dt$$

$$=\frac{t^{\frac{1}{2}}}{1/2}+c_1=2\sqrt{3+2x-x^2}+c_1$$

$$I_2=\int \frac{1}{\sqrt{3-(x^2-2x+1)+1}}\,dx$$

$$=\int \frac{1}{\sqrt{(2)^2-(x-1)^2}}\,dx$$

$$=\sin^{-1}\left(\frac{x-1}{2}\right)+c_2$$

$$\therefore\ I=-7\sqrt{3+2x-x^2}+10\sin^{-1}\left(\frac{x-1}{2}\right)+c,$$

where $c=c_1+c_2$.

Question 6.
∫x−7x−9−−−√dx
Solution:

$$\text{Let } I=\sqrt{\frac{x-7}{x-9}}\,dx=\int \sqrt{\frac{x-7}{x-9}\times\frac{x-7}{x-7}}\,dx$$

$$\int \sqrt{\frac{(x-7)^2}{x^2-16x+63}}\,dx$$

$$\int \frac{x-7}{\sqrt{x^2-16x+63}}\,dx$$

$$\text{Let } x-7=A\left[\frac{d}{dx}(x^2-16x+63)\right]+B$$

$$=A(2x-16)+B$$

$$=2Ax+(B-16A)$$

Comparing the coefficients of x and constant term on both sides, we get

$$2A = 1 \qquad \therefore\ A = \frac{1}{2} \text{ and}$$

$$B - 16A = -7 \qquad \therefore\ B - 16\left(\frac{1}{2}\right) = -7$$

$$\therefore\ B = 1$$

$$\therefore\ x - 7 = \frac{1}{2}(2x - 16) + 1$$

$$\therefore\ I = \int \frac{\frac{1}{2}(2x-16)+1}{\sqrt{x^2 - 16x + 63}}\,dx$$

$$= \frac{1}{2}\int \frac{2x-16}{\sqrt{x^2-16x+63}}\,dx + \int \frac{1}{\sqrt{x^2-16x+63}}\,dx$$

$$= \frac{1}{2}I_1 + I_2$$

In I_1, put $x^2 - 16x + 63 = t$

$$\therefore\ (2x-16)\,dx = dt$$

$$\therefore\ I_1 = \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt = \frac{1}{2}\int t^{-\frac{1}{2}}\,dt$$

$$= \frac{1}{2}\frac{t^{\frac{1}{2}}}{(1/2)} + c_1 = \sqrt{x^2-16x+63} + c_1$$

$$I_2 = \int \frac{1}{\sqrt{x^2-16x+63}}\,dx$$

$$= \int \frac{1}{\sqrt{(x^2-16x+64)-1}}\,dx$$

$$= \int \frac{1}{\sqrt{(x-8)^2 - 1^2}}\,dx$$

$$= \log|x - 8 + \sqrt{(x-8)^2 - 1^2}| + c_2$$

$$= \log|x - 8 + \sqrt{x^2-16x+63}| + c_2$$

$$\therefore\ I = \sqrt{x^2-16x+63} + \log|x - 8 + \sqrt{x^2-16x+63}| + c,$$

where $c = c_1 + c_2$.

Question 7.
∫9−xx−−−√dx
Solution:

$$\text{Let } I = \int \sqrt{\frac{9-x}{x}}\,dx = \int \sqrt{\frac{9-x}{x}\cdot\frac{9-x}{9-x}}\,dx$$

$$= \int \frac{9-x}{\sqrt{9x-x^2}}\,dx$$

Let $9-x=A\left[\frac{d}{dx}(9x-x^2)\right]+B$

$$=A(9-2x)+B$$

$\therefore\ 9-x=(9A+B)-2Ax$

Comparing the coefficient of x and constant on both the sides, we get

$-2A=-1$ and $9A+B=9$

$\therefore\ A=\frac{1}{2}$ and $9\left(\frac{1}{2}\right)+B=9 \quad \therefore\ B=\frac{9}{2}$

$\therefore\ 9-x=\frac{1}{2}(9-2x)+\frac{9}{2}$

$$\therefore\ I=\int \frac{\frac{1}{2}(9-2x)+\frac{9}{2}}{\sqrt{9x-x^2}}\,dx$$

$$=\frac{1}{2}\int \frac{9-2x}{\sqrt{9x-x^2}}\,dx+\frac{9}{2}\int \frac{1}{\sqrt{9x-x^2}}\,dx$$

$$=\frac{1}{2}I_1+\frac{9}{2}I_2$$

In I_1, put $9x-x^2=t \qquad \therefore\ (9-2x)\,dx=dt$

$$\therefore\ I_1=\int \frac{1}{\sqrt{t}}\,dt=\int t^{-1/2}\,dt$$

$$=\frac{t^{\frac{1}{2}}}{1/2}+c_1=2\sqrt{9x-x^2}+c_1$$

$$I_2=\int \frac{1}{\sqrt{\frac{81}{4}-\left(x^2-9x+\frac{81}{4}\right)}}\,dx$$

$$=\int \frac{1}{\sqrt{\left(\frac{9}{2}\right)^2-\left(x-\frac{9}{2}\right)^2}}\,dx$$

$$=\sin^{-1}\left(\frac{x-\frac{9}{2}}{9/2}\right)+c_2=\sin^{-1}\left(\frac{2x-9}{9}\right)+c_2$$

$$\therefore\ I=\sqrt{9x-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{2x-9}{9}\right)+c,$$

where $c=c_1+c_2$.

Question 8.
$\int 3\cos x 4\sin 2x+4\sin x-1\,dx$
Solution:

$$\text{Let } I = \int \frac{3\cos x}{4\sin^2 x + 4\sin x - 1}\,dx$$

Put $\sin x = t \quad \therefore \cos x\,dx = dt$

$$\therefore I = 3\int \frac{dt}{4t^2 + 4t - 1}$$

$$= 3\int \frac{dt}{(4t^2 + 4t + 1) - 2}$$

$$= 3\int \frac{dt}{(2t+1)^2 - (\sqrt{2})^2}$$

$$= \frac{3}{2(2\sqrt{2})}\log\left|\frac{2t+1-\sqrt{2}}{2t+1+\sqrt{2}}\right| + c$$

$$= \frac{3}{4\sqrt{2}}\log\left|\frac{2\sin x+1-\sqrt{2}}{2\sin x+1+\sqrt{2}}\right| + c.$$

Question 9.

∫e3x−e2xex+1−−−−−√dx

Solution:

$$\text{Let } I = \int \sqrt{\frac{e^{3x} - e^{2x}}{e^x + 1}}\,dx$$

$$= \int \sqrt{\frac{e^{2x}(e^x - 1)}{e^x + 1}}\,dx$$

$$= \int e^x \sqrt{\frac{e^x - 1}{e^x + 1}}\,dx$$

Put $e^x = t \quad \therefore e^x\,dx = dt$

$$\therefore I = \int \sqrt{\frac{t-1}{t+1}}\,dt$$

$$= \int \sqrt{\frac{t-1}{t+1} \times \frac{t-1}{t+1}}\,dt$$

$$= \int \sqrt{\frac{(t-1)^2}{t^2-1}}\,dt$$

$$= \int \frac{t-1}{\sqrt{t^2-1}}\,dt$$

$$= \frac{1}{2}\int \frac{2t}{\sqrt{t^2-1}}\,dt - \int \frac{1}{\sqrt{t^2-1}}\,dt$$

$$= I_1 - I_2$$

In I_1, put $t^2 - 1 = \theta$

$\therefore\ 2t\,dt = d\theta$

$$\therefore\ I_1 = \frac{1}{2}\int \frac{d\theta}{\sqrt{\theta}} = \frac{1}{2}\int \theta^{-\frac{1}{2}}\,d\theta$$

$$= \frac{1}{2}\frac{\theta^{\frac{1}{2}}}{(1/2)} + c_1 = \sqrt{\theta} + c_1$$

$$= \sqrt{t^2-1} + c_1 = \sqrt{e^{2x}-1} + c_1$$

$$I_2 = \int \frac{1}{\sqrt{t^2-1}}\,dt$$

$$= \log|t + \sqrt{t^2-1}| + c_2$$

$$= \log|e^x + \sqrt{e^{2x}-1}| + c_2$$

$\therefore\ I = \sqrt{e^{2x}-1} - \log|e^x + \sqrt{e^{2x}-1}| + c$, where $c = c_1 + c_2$.

Maharashtra State Board 12th Maths Solutions Chapter 3 Indefinite Integration Ex 3.3

I. Evaluate the following:

Question 1.
$\int x^2 \log x\,dx$
Solution:

$$\text{Let } I = \int x^2 \log x\,dx = \int \log x \cdot x^2\,dx$$

$$= (\log x)\int x^2\,dx - \int\left[\left\{\frac{d}{dx}(\log x)\int x^2\,dx\right\}\right]dx$$

$$= (\log x)\cdot\frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx$$

$$= \frac{x^3\log x}{3} - \frac{1}{3}\int x^2\,dx$$

$$= \frac{x^3\log x}{3} - \frac{1}{3}\left(\frac{x^3}{3}\right) + c$$

$$= \frac{x^3}{9}(3\log x - 1) + c.$$

Question 2.
$\int x^2 \sin 3x\, dx$
Solution:

$$\text{Let } I = \int x^2 \sin 3x\, dx$$

$$= x^2 \int \sin 3x\, dx - \int \left[\left\{ \frac{d}{dx}(x^2) \int \sin 3x\, dx \right\} \right] dx$$

$$= x^2 \left(-\frac{\cos 3x}{3} \right) - \int 2x \left(\frac{-\cos 3x}{3} \right) dx$$

$$= \frac{x^2 \cos 3x}{3} + \frac{2}{3} \int x \cos 3x\, dx$$

$$= -\frac{x^2 \cos 3x}{3} +$$

$$\frac{2}{3}\left[x \int \cos 3x\, dx - \int \left\{ \frac{d}{dx}(x) \int \cos 3x\, dx \right\} dx \right]$$

$$= -\frac{x^2 \cos 3x}{3} + \frac{2}{3}\left[\frac{x \sin 3x}{3} - \int 1 \cdot \frac{\sin 3x}{3} dx \right]$$

$$= -\frac{x^2 \cos 3x}{3} + \frac{2}{9} x \sin 3x - \frac{2}{9} \int \sin 3x\, dx$$

$$= -\frac{x^2 \cos 3x}{3} + \frac{2}{9} x \sin 3x - \frac{2}{9} \int \left(\frac{-\cos 3x}{3} \right) + c$$

$$= -\frac{x^2 \cos 3x}{3} + \frac{2}{9} x \sin 3x + \frac{2}{27} \cos 3x + c.$$

Question 3.
$\int x \tan^{-1} x\, dx$
Solution:

$$\text{Let } I = \int x \tan^{-1} x\, dx = \int (\tan^{-1} x) \cdot x\, dx$$

$$= (\tan^{-1} x) \int x\, dx - \int \left[\left\{ \frac{d}{dx}(\tan^{-1} x) \int x\, dx \right\} \right] dx$$

$$= (\tan^{-1} x)\left(\frac{x^2}{2} \right) - \int \left(\frac{1}{1+x^2} \right)\left(\frac{x^2}{2} \right) dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{x^2+1} dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{(x^2+1)-1}{x^2+1}\, dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2}\left[\int \left(1 - \frac{1}{x^2+1} \right) dx \right]$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2}\left[\int 1\, dx - \int \frac{1}{x^2+1} dx \right]$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} (x - \tan^{-1} x) + c.$$

Question 4.
$\int x^2 \tan^{-1} x\, dx$

Solution:

$$\text{Let } I = \int x^2 \tan^{-1} x\,dx = \int (\tan^{-1} x)\cdot x^2\,dx$$

$$= (\tan^{-1} x)\int x^2\,dx - \int\left[\left\{\frac{d}{dx}(\tan^{-1} x)\int x^2\,dx\right\}\right]dx$$

$$= (\tan^{-1} x)\left(\frac{x^3}{3}\right) - \int\left(\frac{1}{1+x^2}\right)\left(\frac{x^3}{3}\right)dx$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\int \frac{x(x^2+1)-x}{x^2+1}\,dx$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\left[\int\left\{x - \frac{x}{x^2+1}\right\}dx\right]$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\left[\int x\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx\right]$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\left[\frac{x^2}{2} - \frac{1}{2}\log|x^2+1|\right] + c$$

$$\ldots\left[\because \frac{d}{dx}(x^2+1) = 2x \text{ and } \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c\right]$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{x^2}{6} + \frac{1}{6}\log|x^2+1| + c.$$

Question 5.
$\int x^3 \tan^{-1} x\, dx$
Solution:
Let I = $\int x^3 \tan^{-1} x\, dx$
= $\int (\tan^{-1} x) . x^3\, dx$

$$= (\tan^{-1} x)\int x^3\,dx - \int\left[\left\{\frac{d}{dx}(\tan^{-1} x)\int x^3\,dx\right\}\right]dx$$

$$= (\tan^{-1} x)\left(\frac{x^4}{4}\right) - \int\left(\frac{1}{1+x^2}\right)\frac{x^4}{4}\,dx$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4}\int \frac{(x^4-1)+1}{x^2+1}\,dx$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4}\int \frac{(x^2-1)(x^2+1)+1}{x^2+1}\,dx$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4}\int\left[x^2 - 1 + \frac{1}{x^2+1}\right]dx$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4}\left[\int x^2\,dx - \int 1\,dx + \int \frac{1}{x^2+1}\,dx\right]$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{1}{4}\left[\frac{x^3}{3} - x + \tan^{-1} x\right] + c$$

$$= \frac{x^4 \tan^{-1} x}{4} - \frac{\tan^{-1} x}{4} - \frac{x^3}{12} - \frac{x}{4} + c$$

$$= \frac{1}{4}(\tan^{-1} x)(x^4 - 1) - \frac{x}{12}(x^2 - 3) + c.$$

Question 6.
$\int (\log x)^2\, dx$
Solution:
Let I = $\int (\log x)^2\, dx$
Put log x = t

$\therefore\ x = e^t \qquad \therefore\ dx = e^t\, dt$

$\therefore\ I = \int t^2 e^t\, dt$

$= t^2 \int e^t\, dt - \int \left[\frac{d}{dt}(t^2) \int e^t - dt\right] dt$

$= t^2 e^t - \int 2t\, e^t\, dt$

$= t^2 e^t - 2\left[t \int e^t\, dt - \int \left\{\frac{d}{dt}(t) \int e^t\, dt\right\} dt\right]$

$= t^2 e^t - 2\,[t e^t - \int 1 \cdot e^t\, dt]$

$= t^2 e^t - 2\,t e^t + 2\,e^t + c$

$= e^t[t^2 - 2t + 2] + c$

$= x\,[(\log x)^2 - 2\,(\log x) + 2] + c.$

Alternative Method :

Let $I = \int (\log x)^2\, dx = \int (\log x)^2 \cdot 1\, dx$

$= (\log x)^2 \int 1\, dx - \int \left[\frac{d}{dx}(\log x)^2 \cdot \int 1\, dx\right] dx$

$= (\log x)^2 \cdot x - \int 2 \log x \cdot \frac{d}{dx}(\log x) \cdot x\, dx$

$= x\,(\log x)^2 - \int 2 \log x \times \frac{1}{x} \times x\, dx$

$= x\,(\log x)^2 - 2 \int (\log x) \cdot 1\, dx$

$= x(\log x)^2 - 2[(\log x) \int 1\, dx - \int \left\{\frac{d}{dx}(\log x) \int 1\, dx\right\} dx]$

$= x\,(\log x)^2 - 2\,[(\log x)x - \int \frac{1}{x} \times x\, dx]$

$= x\,(\log x)^2 - 2x \log x + 2 \int 1\, dx$

$= x\,(\log x)^2 - 2x \log x + 2x + c$

$= x\,[(\log x)^2 - 2 \log x + 2] + c.$

Question 7.
$\int \sec^3 x\, dx$
Solution:
Let I = $\int \sec^3 x\, dx$
= $\int \sec x \sec^2 x\, dx$
= $\sec x \int \sec^2 x\, dx - \int [\frac{d}{dx}(\sec x) \int \sec^2 x\, dx]\, dx$
= $\sec x \tan x - \int (\sec x \tan x)(\tan x)\, dx$
= $\sec x \tan x - \int \sec x \tan^2 x\, dx$
= $\sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$
= $\sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$
$\therefore$ I = $\sec x \tan x - I + \log|\sec x + \tan x|$
$\therefore$ 2I = $\sec x \tan x + \log|\sec x + \tan x|$
$\therefore$ I = $\frac{1}{2}$ [$\sec x \tan x + \log|\sec x + \tan x|$] + c.

Question 8.
$\int x \,.\, \sin^2 x\, dx$

Solution:

$$\int x \cdot \sin^2 x \, dx = \int x\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int (x - x\cos 2x)\, dx$$

$$= \frac{1}{2}\int x\, dx - \frac{1}{2}\int x\cos 2x\, dx$$

$$= \frac{1}{2}\cdot\frac{x^2}{2} - \frac{1}{2}\left[x\int \cos 2x\, dx - \int\left\{\frac{d}{dx}(x)\int \cos 2x\, dx\right\} dx\right]$$

$$= \frac{x^2}{4} - \frac{1}{2}\left[x\cdot\frac{\sin 2x}{2} - \int 1.\frac{\sin 2x}{2} dx\right]$$

$$= \frac{x^2}{4} - \frac{1}{4}x\cdot\sin 2x + \frac{1}{4}\int \sin 2x\, dx$$

$$= \frac{x^2}{4} - \frac{1}{4}x\cdot\sin 2x + \frac{1}{4}\cdot\frac{(-\cos 2x)}{2} + c$$

$$= \frac{x^2}{4} - \frac{1}{4}x\cdot\sin 2x - \frac{1}{8}\cos 2x + c$$

$$= \frac{1}{4}[x^2 - x\sin 2x - \frac{1}{2}\cos 2x] + c.$$

Question 9.
$\int x^3 \log x \, dx$
Solution:

$$\text{Let I} = \int x^3.\log x.\, dx$$

$$= \int \log x.\, x^3.\, dx$$

$$= (\log x)\int x^3.\, dx - \int\left[\left\{\frac{d}{dx}(\log x)\int x^3.\, dx\right\}\right].\, dx$$

$$= (\log x).\,\frac{x^4}{4} - \int \frac{1}{x}.\,\frac{x^4}{4}.\, dx$$

$$= \frac{x^4}{4}\log x - \frac{1}{4}\int x^3.\, dx$$

$$= \frac{x^4}{4}\log x - \frac{1}{4}\left(\frac{x^4}{4}\right) + c$$

$$= \frac{x^4}{4}\log x - \frac{x^4}{16} + c.$$

Question 10.
$\int e^{2x} \cos 3x \, dx$
Solution:

Let $I = \int e^{2x} \cos 3x \, dx$

$$= e^{2x} \int \cos 3x \, dx - \int \left[\frac{d}{dx}(e^{2x}) \int \cos 3x \, dx \right] dx$$

$$= e^{2x} \cdot \frac{\sin 3x}{3} - \int e^{2x} \times 2 \times \frac{\sin 3x}{3} dx$$

$$= \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} \int e^{2x} \sin 3x \, dx$$

$$= \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3}\left[e^{2x} \int \sin 3x \, dx - \int \left\{ \frac{d}{dx}(e^{2x}) \int \sin 3x \, dx \right\} dx \right]$$

$$= \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3}\left[e^{2x} \cdot \left(\frac{-\cos 3x}{3} \right) - \int e^{2x} \times 2 \times \left(\frac{-\cos 3x}{3} \right) dx \right]$$

$$= \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} \int e^{2x} \cos 3x \, dx$$

$$\therefore \; I = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} I$$

$$\therefore \; \left(1 + \frac{4}{9}\right) I = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x$$

$$\therefore \; \frac{13}{9} I = \frac{e^{2x}}{9}(3 \sin 3x + 2 \cos 3x)$$

$$\therefore \; I = \frac{e^{2x}}{13}(2 \cos 3x + 3 \sin 3x) + c.$$

Question 11.
$\int x \sin^{-1} x \, dx$
Solution:

Let $I = \int x \sin^{-1} x \, dx = \int (\sin^{-1} x) \cdot x \, dx$

$$= (\sin^{-1} x) \int x \, dx - \int \left[\left\{ \frac{d}{dx}(\sin^{-1} x) \int x \, dx \right\} \right] dx$$

$$= (\sin^{-1} x)\left(\frac{x^2}{2}\right) - \int \left(\frac{1}{\sqrt{1-x^2}} \right)\left(\frac{x^2}{2}\right) dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{-x^2}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \frac{(1-x^2)-1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \left[\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}} \right] dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2} \int \sqrt{1-x^2} \, dx - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x\right] - \frac{1}{2} \sin^{-1} x + c$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{4} x \sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + c.$$

Question 12.
$\int x^2 \cos^{-1} x \, dx$

Solution:

$$\text{Let } I = \int x^2 \cos^{-1} x\, dx = \int (\cos^{-1} x) \cdot x^2\, dx$$

$$= (\cos^{-1} x) \int x^2\, dx - \int \left[\frac{d}{dx} (\cos^{-1} x) \int x^2\, dx \right] dx$$

$$= (\cos^{-1} x) \left(\frac{x^3}{3} \right) - \int \left(\frac{-1}{\sqrt{1-x^2}} \right) \left(\frac{x^3}{3} \right) dx$$

$$= \frac{x^3 \cos^{-1} x}{3} + \frac{1}{3} \int \frac{x^2 \cdot x}{\sqrt{1-x^2}} dx$$

In $\int \frac{x^3}{\sqrt{1-x^2}} dx$, put $1 - x^2 = t$

$\therefore\ -2x\, dx = dt \qquad \therefore\ x\, dx = -\frac{1}{2} dt$

Also, $x^2 = 1 - t$

$$\therefore\ I = \frac{x^3 \cos^{-1} x}{3} + \frac{1}{3} \int \frac{(1-t)}{\sqrt{t}} \left(-\frac{1}{2} \right) dt$$

$$= \frac{x^3 \cos^{-1} x}{3} - \frac{1}{6} \int \left(\frac{1}{\sqrt{t}} - \sqrt{t} \right) dt$$

$$= \frac{x^3 \cos^{-1} x}{3} - \frac{1}{6} \int t^{-\frac{1}{2}} dt + \frac{1}{6} \int t^{\frac{1}{2}} dt$$

$$= \frac{x^3 \cos^{-1} x}{3} - \frac{1}{6} \left(\frac{t^{\frac{1}{2}}}{1/2} \right) + \frac{1}{6} \frac{t^{\frac{3}{2}}}{(3/2)} + c$$

$$= \frac{x^3 \cos^{-1} x}{3} - \frac{1}{3} \sqrt{1-x^2} + \frac{1}{9} (1-x^2)^{\frac{3}{2}} + c.$$

Question 13.

$\int \frac{\log(\log x)}{x} dx$

Solution:

$$\text{Let } I = \int \frac{\log(\log x)}{x} dx$$

$$= \int \log(\log x) \cdot \frac{1}{x} dx$$

Put $\log x = t \qquad \therefore\ \frac{1}{x} dx = dt$

$$\therefore\ I = \int \log t\, dt = \int (\log t) \cdot 1\, dt$$

$$= (\log t) \int 1\, dt - \int \left[\frac{d}{dt} (\log t) \int 1\, dt \right] dt$$

$$= (\log t)\, t - \int \frac{1}{t} \times t\, dt$$

$$= t \log t - \int 1\, dt$$

$$= t \log t - t + c$$

= t(log t – 1) + c

= (log x) . [log(log x) – 1] + c.

Question 14.

$\int t \cdot \frac{\sin^{-1} t}{\sqrt{1-t^2}} dt$

Solution:

$$\text{Let } I=\int \frac{t\cdot \sin^{-1} t}{\sqrt{1-t^2}}dt=\int t\cdot \sin^{-1} t\cdot \frac{1}{\sqrt{1-t^2}}dt$$

Put $\sin^{-1} t=\theta \qquad \therefore \ \frac{1}{\sqrt{1-t^2}}dt=d\theta$

and $t=\sin\theta$

$$\therefore \ I=\int (\sin\theta)\cdot\theta\, d\theta=\int \theta \sin\theta\, d\theta$$

$$=\theta\int \sin\theta\, d\theta-\int\left[\frac{d}{d\theta}(\theta)\int \sin\theta\, d\theta\right]d\theta$$

$$=\theta(-\cos\theta)-\int 1\cdot(-\cos\theta)\,d\theta$$

$$=-\theta\cos\theta+\int \cos\theta\, d\theta$$

$$=-\theta\cos\theta+\sin\theta+c$$

$$=-\theta\cdot\sqrt{1-\sin^2\theta}+\sin\theta+c$$

$$=-\sin^{-1} t\cdot\sqrt{1-t^2}+t+c$$

$$=-\sqrt{1-t^2}\cdot\sin^{-1} t+t+c.$$

Question 15.
∫cos√x dx
Solution:
Let I = ∫cos√x dx
Put √x = t
∴ x = t^2
∴ dx = 2t dt
∴ I = ∫(cos t) 2t dt
= ∫2t cos t dt
= 2t ∫cos t dt – ∫[$\frac{d}{dt}$(2t) ∫cos t dt]dt
= 2t sin t – ∫2 sin t dt
= 2t sin t + 2 cos t + c
= 2[√x sin√x + cos√x] + c.

Question 16.
∫sin θ . log(cos θ) dθ
Solution:
Let I = ∫sin θ . log (cos θ) dθ
= ∫log(cos θ) . sin θ dθ
Put cos θ = t
∴ -sin θ dθ = dt
∴ sin θ dθ = -dt

$$\therefore \ I=\int \log t\cdot(-dt)$$

$$=-\int (\log t)\cdot 1\, dt$$

$$=-\left[(\log t)\int 1\, dt-\int\left\{\frac{d}{dt}(\log t)\int 1\, dt\right\}dt\right]$$

$$=-\left[(\log t)t-\int\frac{1}{t}\cdot t\, dt\right]$$

$$=-t\log t+\int 1\, dt$$

= -t log t + t + c
= -cos θ . log(cos θ) + cos θ + c
= -cos θ [log(cos θ) – 1] + c.

Question 17.
∫x $\cos^3$ x dx
Solution:
cos 3x = 4 $\cos^3$ x – 3 cos x
∴ $\cos^3$ x + 3 cos x = 4cos3x

$\therefore \cos^3 x = \frac{1}{4} \cos 3x + \frac{3}{4} \cos x$

$$\therefore \int \cos^3 x\,dx = \frac{1}{4}\int \cos 3x\,dx + \frac{3}{4}\int \cos x\,dx$$

$$= \frac{1}{4}\left(\frac{\sin 3x}{3}\right) + \frac{3}{4}\sin x$$

$$= \frac{\sin 3x}{12} + \frac{3\sin x}{4} \qquad \ldots (1)$$

Let $I = \int x\cos^3 x\,dx$

$$= x\int \cos^3 x\,dx - \int\left[\left\{\frac{d}{dx}(x)\int \cos^3 x\,dx\right\}\right]dx$$

$$= x\left[\frac{\sin 3x}{12} + \frac{3\sin x}{4}\right] - \int 1\cdot\left(\frac{\sin 3x}{12} + \frac{3\sin x}{4}\right)dx$$

$$\ldots \text{[By (1)]}$$

$$= \frac{x\sin 3x}{12} + \frac{3x\sin x}{4} - \frac{1}{12}\int \sin 3x\,dx - \frac{3}{4}\int \sin x\,dx$$

$$= \frac{x\sin 3x}{12} + \frac{3x\sin x}{4} - \frac{1}{12}\left(\frac{-\cos 3x}{3}\right) - \frac{3}{4}(-\cos x) + c$$

$$= \frac{1}{4}\left[\frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x + 3x\sin x + 3\cos x\right] + c.$$

Question 18.
$\int \frac{\sin(\log x)^2}{x} \cdot \log x\,dx$
Solution:

$$\text{Let } I = \int \frac{\sin(\log x)^2}{x}\cdot \log x\,dx$$

Put $(\log x)^2 = t$

$$\therefore\ 2\log x\cdot\frac{1}{x}dx = dt$$

$$\therefore\ \frac{1}{x}\log x\,dx = \frac{1}{2}dt$$

$$\therefore\ I = \frac{1}{2}\int \sin t\,dt$$

$$= -\frac{1}{2}\cos t + c$$

$$= -\frac{1}{2}\cos[(\log x)^2] + c.$$

Question 19.
$\int \frac{\log x}{x} dx$
Solution:
Let I = $\int \frac{\log x}{x} dx$
Put log x = t
$\frac{1}{x}$ dx = dt
∴ I = ∫t dt
= $\frac{1}{2}$ t^2 + c
= $\frac{1}{2}$ $(\log x)^2$ + c

Question 20.
∫x sin 2x cos 5x dx.
Solution:
Let I = ∫x sin 2x cos 5x dx
sin 2x cos 5x = $\frac{1}{2}$[2 sin 2x cos 5x]
= $\frac{1}{2}$ [sin(2x + 5x) + sin(2x – 5x)]

= 12 [sin 7x – sin 3x]
∴ ∫sin 2x cos 5x dx = 12 [∫sin 7x dx – ∫sin 3x dx]

$$=\frac{1}{2}\left(\frac{-\cos 7x}{7}\right)-\frac{1}{2}\left(\frac{-\cos 3x}{3}\right)$$

$$=-\frac{1}{14}\cos 7x+\frac{1}{6}\cos 3x \quad \ldots (1)$$

$$I=\int x\sin 2x\cos 5x\,dx$$

$$=x\int \sin 2x\cos 5x\,dx-\int\left[\frac{d}{dx}(x)\int \sin 2x\cos 5x\,dx\right]dx$$

$$=x\left[-\frac{1}{14}\cos 7x+\frac{1}{6}\cos 3x\right]-\int 1\cdot\left(-\frac{1}{14}\cos 7x+\frac{1}{6}\cos 3x\right)dx \quad \ldots \text{[By (1)]}$$

$$=-\frac{x}{14}\cos 7x+\frac{x}{6}\cos 3x+\frac{1}{14}\int \cos 7x\,dx-\frac{1}{6}\int \cos 3x\,dx$$

$$=-\frac{x}{14}\cos 7x+\frac{x}{6}\cos 3x+\frac{1}{14}\left(\frac{\sin 7x}{7}\right)-\frac{1}{6}\left(\frac{\sin 3x}{3}\right)+c$$

$$=-\frac{x}{14}\cos 7x+\frac{x}{6}\cos 3x+\frac{\sin 7x}{98}-\frac{\sin 3x}{18}+c.$$

Question 21.
∫cos(x−−√3)dx
Solution:
Let I = ∫cos(x−−√3)dx

$$\text{Put } \sqrt[3]{x}=t \qquad \therefore\ x=t^3 \qquad \therefore\ dx=3t^2\,dt$$

$$\therefore\ I=\int 3t^2\cos t\,dt$$

$$=3t^2\int \cos t\,dt-\int\left[\frac{d}{dt}(3t^2)\int \cos t\,dt\right]dt$$

$$=3t^2\sin t-\int 6t\sin t\,dt$$

$$=3t^2\sin t-\left[6t\int \sin t\,dt-\int\left\{\frac{d}{dt}(6t)\int \sin t\,dt\right\}dt\right]$$

$$=3t^2\sin t-[6t(-\cos t)-\int 6(-\cos t)\,dt]$$

$$=3t^2\sin t+6t\cos t-6\sin t+c$$

$$=3(t^2-2)\sin t+6t\cos t+c$$

$$=3(x^{\frac{2}{3}}-2)\sin(\sqrt[3]{x})+6\sqrt[3]{x}\cos(\sqrt[3]{x})+c.$$

II. Integrate the following functions w.r.t. x:

Question 1.
e^{2x} sin 3x
Solution:

Let I = $\int e^{2x}.\sin 3x$\`

= $e^{2x}.\sin 3x - \int\left[\frac{d}{dx}\left(e^{2x}\right)\int \sin 3x.\,dx\right].\,dx$

= $e^{2x}.\frac{\sin 3x}{3} - \int e^{2x} \times 2 \times \frac{\sin 3x}{3}.\,dx$

= $\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x.\,dx$

= $\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\left[e^{2x}\int \sin 3x.\,dx\right]$

= $\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\left[e^{2x}.\left(\frac{-\cos 3x}{3}\right) - \int e^{2x} \times 2 \times \left(\frac{-\sin 3x}{3}\right).\,dx\right]$

= $\frac{1}{3}e^{2x}\sin 3x - \frac{2}{13}e^{2x}\cos 3x - \frac{4}{13}\int e^{2x}\cos 3x.\,dx$

$\therefore$ I = $\frac{1}{3}e^{2x}\sin 3x - \frac{2}{13}e^{2x}\cos 3x - \frac{4}{13}\mathrm{I}$

$\therefore \left(1 + \frac{4}{13}\right)\mathrm{I} = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{13}e^{2x}\cos 3x$

$\therefore \frac{13}{13}\mathrm{I} = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x)$

$\therefore$ I = $\frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + c.$

Question 2.
$e^{-x}\cos 2x$
Solution:

Let I = $\int e^{-x}\cos 2x.\,dx$

$= e^{-x}\int \cos 2x.\,dx - \int\left[\frac{d}{dx}(e^{2x})\int \sin 2x.\,dx\right].dx$

$= e^{-x}.\,\frac{\cos 2x}{3} - \int e^{-x}\times 2\times\frac{\sin 2x}{3}.\,dx$

$= \frac{1}{3}e^{-x}\cos 2x - \frac{2}{3}\int e^{-x}\sin 2x.\,dx$

$= \frac{1}{3}e^{-x}\cos 2x - \frac{2}{3}\left[e^{-x}\int \sin 2x.\,dx\right]$

$= \frac{1}{3}e^{-x}\cos 2x - \frac{2}{3}\left[e^{-x}.\left(\frac{-\cos 2x}{3}\right) - \int e^{-x}\times 2\times\left(\frac{-\cos 2x}{3}\right).\,dx\right]$

$= \frac{1}{3}e^{-x}\cos 2x + \frac{2}{5}e^{-x}\cos 2x - \frac{2}{5}\int e^{-x}\sin 2x.\,dx$

$\therefore$ I = $\frac{1}{3}e^{-x}\cos 2x + \frac{2}{5}e^{-x}\cos 2x - \frac{3}{5}\mathrm{I}$

$\therefore \left(1+\frac{4}{5}\right)\mathrm{I} = \frac{1}{3}e^{-x}\cos 2x + \frac{2}{5}e^{-x}\sin 2x$

$\therefore \frac{e^{-x}}{5}\mathrm{I} = \frac{e^{-x}}{5}(2\cos 2x + 2\sin 2x)$

$\therefore$ I = $\frac{e^{-x}}{5}(2\cos 2x + 2\sin 2x) + c.$

Question 3.
sin(log x)
Solution:

$$\text{Let } I = \int \sin(\log x)\,dx$$

Put $\log x = t \quad \therefore\ x = e^t \quad \therefore\ dx = e^t\,dt$

$\therefore\ I = \int \sin t \times e^t\,dt = \int e^t \sin t\,dt$

$= e^t\int \sin t\,dt - \int\left[\frac{d}{dt}(e^t)\int \sin t\,dt\right]dt$

$= e^t(-\cos t) - \int e^t(-\cos t)\,dt$

$= -e^t\cos t + \int e^t\cos t\,dt$

$= -e^t\cos t + e^t\int \cos t\,dt - \int\left[\frac{d}{dt}(e^t)\int \cos t\,dt\right]dt$

$= -e^t\cos t + e^t\sin t - \int e^t\sin t\,dt$

$\therefore\ I = -e^t\cos t + e^t\sin t - I$

$\therefore\ 2I = e^t(\sin t - \cos t)$

$\therefore\ I = \frac{e^t}{2}(\sin t - \cos t) + c$

$= \frac{x}{2}[\sin(\log x) - \cos(\log x)] + c.$

Question 4.
5x2+3------√
Solution:

Let I = 5x2+3−−−−−−√ dx

$$= \sqrt{5} \int \sqrt{x^2 + \frac{3}{5}}\, dx$$

$$= \sqrt{5}\left[\frac{x}{2}\sqrt{x^2+\frac{3}{5}} + \frac{\left(\frac{3}{5}\right)}{2}\log\left|x+\sqrt{x^2+\frac{3}{5}}\right|\right] + c$$

$$= \frac{\sqrt{5}}{2}\left[x\sqrt{x^2+\frac{3}{5}} + \frac{3}{5}\log\left|x+\sqrt{x^2+\frac{3}{5}}\right|\right] + c.$$

Question 5.
x2a2−x6−−−−−−√
Solution:

$$\text{Let } I = \int x^2\sqrt{a^2 - x^6}\, dx$$

$$\text{Put } x^3 = t \qquad \therefore\ 3x^2\,dx = dt \qquad \therefore\ x^2\,dx = \frac{1}{3}dt$$

$$\therefore\ I = \int \sqrt{a^2 - t^2}\cdot\frac{dt}{3} = \frac{1}{3}\int \sqrt{a^2 - t^2}\,dt$$

$$= \frac{1}{3}\left[\frac{t}{2}\sqrt{a^2-t^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{t}{a}\right)\right] + c$$

$$= \frac{1}{6}\left[x^3\sqrt{a^2-x^6} + a^2\sin^{-1}\left(\frac{x^3}{a}\right)\right] + c.$$

Question 6.
(x−3)(7−x)−−−−−−−−−−−√
Solution:

$$\text{Let } I = \int \sqrt{(x-3)(7-x)}\, dx$$

$$= \int \sqrt{-x^2 + 10x - 21}\, dx$$

$$= \int \sqrt{-(x^2 - 10x + 21)}\, dx$$

$$= \int \sqrt{4 - (x^2 - 10x + 25)}\, dx$$

$$= \int \sqrt{2^2 - (x-5)^2}$$

$$= \left(\frac{x-5}{2}\right)\sqrt{2^2-(x-5)^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x-5}{2}\right) + c$$

$$= \left(\frac{x-5}{2}\right)\sqrt{(x-3)(7-x)} + 2\sin^{-1}\left(\frac{x-5}{2}\right) + c.$$

Question 7.
4x(4x+4)−−−−−−−−−√
Solution:

$$\text{Let } I = \int \sqrt{4^x(4^x+4)}\, dx$$

$$= \int 2^x\sqrt{(2^x)^2 + 2^2}\, dx$$

$$\text{Put } 2^x = t \qquad \therefore\ 2^x\log 2\,dx = dt \qquad \therefore\ 2^x\,dx = \frac{1}{\log 2}dt$$

$$\therefore\ I = \int \sqrt{t^2+2^2}\cdot\frac{dt}{\log 2} = \frac{1}{\log 2}\int \sqrt{t^2+2^2}\,dt$$

$$= \frac{1}{\log 2}\left[\frac{t}{2}\sqrt{t^2+2^2} + \frac{2^2}{2}\log|t+\sqrt{t^2+2^2}|\right] + c$$

$$= \frac{1}{\log 2}\left[\frac{2^x}{2}\sqrt{4^x+4} + 2\log|2^x+\sqrt{4^x+4}|\right] + c$$

Question 8.

$(x + 1)\sqrt{2x^2+3}$

Solution:

Let $I = \int (x + 1)\sqrt{2x^2+3}\, dx$

Let $x + 1 = A\left[\frac{d}{dx}(2x^2 + 3)\right] + B$

$= A(4x) + B$

$= 4Ax + B$

Comparing the coefficients of x and constant term on both the sides, we get

$4A = 1, B = 1$

$\therefore A = \frac{1}{4}, B = 1$

$$\therefore\ x+1=\frac{1}{4}(4x)+1$$

$$\therefore\ I=\int\left[\frac{1}{4}(4x)+1\right]\sqrt{2x^2+3}\,dx$$

$$=\frac{1}{4}\int 4x\sqrt{2x^2+3}\,dx+\int\sqrt{2x^2+3}\,dx.$$

$$=I_1+I_2$$

In I_1, put $2x^2+3=t$ $\quad\therefore\ 4x\,dx=dt$

$$\therefore\ I_1=\frac{1}{4}\int t^{\frac{1}{2}}\,dt=\frac{1}{4}\left(\frac{t^{\frac{3}{2}}}{3/2}\right)+c_1=\frac{1}{6}(2x^2+3)^{\frac{3}{2}}+c_1$$

$$I_2=\int\sqrt{2x^2+3}\,dx$$

$$=\sqrt{2}\int\sqrt{x^2+\frac{3}{2}}\,dx$$

$$=\sqrt{2}\left[\frac{x}{2}\sqrt{x^2+\frac{3}{2}}+\frac{\left(\frac{3}{2}\right)}{2}\log\left|x+\sqrt{x^2+\frac{3}{2}}\right|\right]+c_2$$

$$=\sqrt{2}\left[\frac{x}{2}\sqrt{x^2+\frac{3}{2}}+\frac{3}{4}\log\left|x+\sqrt{x^2+\frac{3}{2}}\right|\right]+c_2$$

$$\therefore\ I=\frac{1}{6}(2x^2+3)^{\frac{3}{2}}+$$

$$\sqrt{2}\left[\frac{x}{2}\sqrt{x^2+\frac{3}{2}}+\frac{3}{4}\log\left|x+\sqrt{x^2+\frac{3}{2}}\right|\right]+c,$$

where $c=c_1+c_2$.

Question 9.

$x\sqrt{5-4x-x^2}$

Solution:

Let $I = \int x\sqrt{5-4x-x^2}\, dx$

Let $x = A\left[\frac{d}{dx}(5 - 4x - x^2)\right] + B$

$= A[-4 - 2x] + B$

$= -2Ax + (B - 4A)$

Comparing the coefficients of x and the constant term on both sides, we get

$-2A = 1, B - 4A = 0$

$$\therefore\ A = -\frac{1}{2},\ B = 4A = 4\left(-\frac{1}{2}\right) = -2$$

$$\therefore\ x = -\frac{1}{2}(-4-2x) - 2$$

$$\therefore\ I = \int\left[-\frac{1}{2}(-4-2x) - 2\right]\sqrt{5-4x-x^2}\,dx$$

$$= -\frac{1}{2}\int(-4-2x)\sqrt{5-4x-x^2}\,dx - 2\int\sqrt{5-4x-x^2}\,dx$$

$$= I_1 - I_2$$

In I_1, put $5-4x-x^2 = t$

$$\therefore\ (-4-2x)\,dx = dt$$

$$\therefore\ I_1 = -\frac{1}{2}\int t^{\frac{1}{2}}\,dt = -\frac{1}{2}\left(\frac{t^{\frac{3}{2}}}{3/2}\right) + c_1$$

$$= -\frac{1}{3}(5-4x-x^2)^{\frac{3}{2}} + c_1$$

$$I_2 = 2\int\sqrt{5-4x-x^2}\,dx$$

$$= 2\int\sqrt{5-(x^2+4x)}\,dx$$

$$= 2\int\sqrt{9-(x^2+4x+4)}\,dx$$

$$= 2\int\sqrt{3^2-(x+2)^2}\,dx$$

$$= 2\left[\left(\frac{x+2}{2}\right)\sqrt{3^2-(x+2)^2} + \frac{3^2}{2}\sin^{-1}\left(\frac{x+2}{3}\right)\right] + c_2$$

$$= (x+2)\sqrt{5-4x-x^2} + 9\sin^{-1}\left(\frac{x+2}{3}\right) + c_2$$

$$\therefore\ I = -\frac{1}{3}(5-4x-x^2)^{\frac{3}{2}} - (x+2)\sqrt{5-4x-x^2} -$$

$$9\sin^{-1}\left(\frac{x+2}{3}\right) + c, \text{ where } c = c_1 + c_2$$

Question 10.

sec2xtan2x+tanx−7−−−−−−−−−−−−−−√

Solution:

$$\text{Let } I = \int \sec^2 x \sqrt{\tan^2 x + \tan x - 7}\, dx$$

Put $\tan x = t \quad \therefore \sec^2 x\, dx = dt$

$\therefore I = \int \sqrt{t^2 + t - 7}\, dt$

$$= \int \sqrt{t^2 + t + \frac{1}{4} - \frac{29}{4}}\, dt$$

$$= \int \sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{29}}{2}\right)^2}\, dt$$

$$= \left(\frac{t + \frac{1}{2}}{2}\right)\sqrt{\left(t + \frac{1}{2}\right)^2 - \frac{29}{4}}$$

$$- \frac{\left(\frac{29}{4}\right)}{2} \log\left|\left(t + \frac{1}{2}\right) + \sqrt{\left(t + \frac{1}{2}\right)^2 - \frac{29}{4}}\right| + c$$

$$= \frac{(2t+1)}{4}\sqrt{t^2 + t - 7} -$$

$$\frac{29}{8}\log\left|\left(t + \frac{1}{2}\right) + \sqrt{t^2 + t - 7}\right| + c$$

$$= \left(\frac{2\tan x + 1}{4}\right)\sqrt{\tan^2 x + \tan x - 7} -$$

$$\frac{29}{8}\log\left|\left(\tan x + \frac{1}{2}\right) + \sqrt{\tan^2 x + \tan x - 7}\right| + c.$$

Question 11.
$\sqrt{x^2+2x+5}$
Solution:

$$\text{Let } I = \int \sqrt{x^2 + 2x + 5}\, dx$$

$$= \int \sqrt{x^2 + 2x + 1 + 4}\, dx$$

$$= \int \sqrt{(x+1)^2 + 2^2}\, dx$$

$$= \left(\frac{x+1}{2}\right)\int \sqrt{(x+1)^2 + 2^2} +$$

$$\frac{2^2}{2}\log\left|(x+1) + \sqrt{(x+1)^2 + 2^2}\right| + c$$

$$= \left(\frac{x+1}{2}\right)\sqrt{x^2 + 2x + 5} +$$

$$2\log\left|(x+1) + \sqrt{x^2 + 2x + 5}\right| + c.$$

Question 12.
$\sqrt{2x^2+3x+4}$
Solution:

$$\text{Let } I = \int \sqrt{2x^2 + 3x + 4}\,dx$$

$$= \sqrt{2}\int \sqrt{x^2 + \frac{3}{2}x + 2}\,dx$$

$$= \sqrt{2}\int \sqrt{\left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) - \frac{9}{16} + 2}\,dx$$

$$= \sqrt{2}\int \sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2}\,dx$$

$$= \sqrt{2}\left[\frac{\left(x + \frac{3}{4}\right)}{2}\sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2}\right.$$

$$\left. + \frac{\left(\frac{23}{16}\right)}{2}\log\left|\left(x + \frac{3}{4}\right) + \sqrt{\left(x + \frac{3}{4}\right)^2 + \left(\frac{\sqrt{23}}{4}\right)^2}\right|\right] + c$$

$$= \sqrt{2}\left[\left(\frac{4x+3}{8}\right)\sqrt{x^2 + \frac{3}{2}x + 2}\right.$$

$$\left. + \frac{23}{32}\log\left|\left(x + \frac{3}{4}\right) + \sqrt{x^2 + \frac{3}{2}x + 2}\right|\right] + c.$$

III. Integrate the following functions w.r.t. x:

Question 1.
$[2 + \cot x - \operatorname{cosec}^2 x]\, e^x$
Solution:
Let $I = \int e^x [2 + \cot x - \operatorname{cosec}^2 x]\, dx$
Put $f(x) = 2 + \cot x$
$\therefore f'(x) = \frac{d}{dx}(2 + \cot x)$
$= \frac{d}{dx}(2) + \frac{d}{dx}(\cot x)$
$= 0 - \operatorname{cosec}^2 x$
$= -\operatorname{cosec}^2 x$
$\therefore I = \int e^x [f(x) + f'(x)]\, dx$
$= e^x f(x) + c$
$= e^x (2 + \cot x) + c.$

Question 2.
$\left(\frac{1+\sin x}{1+\cos x}\right)e^x$

Solution:

$$\text{Let } I=\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$$

$$=\int e^x\left[\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right]dx$$

$$=\int e^x\left[\frac{1}{2\cos^2\frac{x}{2}}+\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right]dx$$

$$=\int e^x\left[\frac{1}{2}\sec^2\frac{x}{2}+\tan\left(\frac{x}{2}\right)\right]dx$$

$$\text{Put } f(x)=\tan\left(\frac{x}{2}\right)$$

$$\therefore\ f'(x)=\frac{d}{dx}\left[\tan\frac{x}{2}\right]=\sec^2\frac{x}{2}\cdot\frac{1}{2}=\frac{1}{2}\sec^2\frac{x}{2}$$

$$\therefore\ I=\int e^x[f(x)+f'(x)]\,dx$$

$$=e^x f(x)+c$$

$$=e^x\cdot\tan\left(\frac{x}{2}\right)+c.$$

Question 3.
$e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$
Solution:
Let I = $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)$
Let f(x) = $\frac{1}{x}$, f'(x) = $-\frac{1}{x^2}$
∴ I = $\int e^x$ [f(x) + f'(x)] dx
= e^x f(x) + c
= e^x . $\frac{1}{x}$ + c

Question 4.
$\left[\frac{x}{(x+1)^2}\right]e^x$
Solution:

$$\text{Let } I=\int e^x\left[\frac{x}{(x+1)^2}\right]dx$$

$$=\int e^x\left[\frac{(x+1)-1}{(x+1)^2}\right]dx$$

$$=\int e^x\left[\frac{1}{x+1}-\frac{1}{(x+1)^2}\right]dx$$

$$\text{Let } f(x)=\frac{1}{x+1}=(x+1)^{-1}$$

$$\therefore\ f'(x)=\frac{d}{dx}(x+1)^{-1}]=-(x+1)^{-2}\frac{d}{dx}(x+1)$$

$$=\frac{-1}{(x+1)^2}\times 1=\frac{-1}{(x+1)^2}$$

$$\therefore\ I=\int e^x[f(x)+f'(x)]\,dx$$

$$=e^x\cdot f(x)+c$$

$$=\frac{e^x}{x+1}+c.$$

Question 5.
e^x . [x(log x)^2 + 2 log x]

Solution:

$$\text{Let } I=\int \frac{e^x}{x}[x(\log x)^2+2\log x]\,dx$$

$$=\int e^x\left[(\log x)^2+\frac{2\log x}{x}\right]dx$$

Put $f(x)=(\log x)^2$

$$\therefore\ f'(x)=\frac{d}{dx}(\log x)^2=2(\log x)\cdot\frac{d}{dx}(\log x)$$

$$=\frac{2\log x}{x}$$

$$\therefore\ I=\int e^x[f(x)+f'(x)]\,dx$$

$$=e^x\cdot f(x)+c=e^x\cdot(\log x)^2+c.$$

Question 6.

$e^{5x}\left[5x\log x+\frac{1}{x}\right]$

Solution:

Let I = $\int e^{5x}\left[5x\log x+\frac{1}{x}\right]$

$$=\int e^{5x}\left[5\log x+\frac{1}{x}\right]dx$$

Put $5x=t \qquad \therefore\ 5\,dx=dt \qquad \therefore\ dx=\frac{1}{5}dt$

Also, $x=\frac{t}{5}$

$$\therefore\ I=\frac{1}{5}\int e^t\left[5\log\left(\frac{t}{5}\right)+\frac{5}{t}\right]dt$$

Let $f(t)=5\log\left(\frac{t}{5}\right)=5\log t-5\log 5$

$$\therefore\ f'(t)=\frac{d}{dt}[5\log t-5\log 5]=\frac{5}{t}-0=\frac{5}{t}$$

$$\therefore\ I=\frac{1}{5}\int e^t[f(t)+f'(t)]\,dt$$

$$=\frac{1}{5}e^t f(t)+c$$

$$=\frac{1}{5}e^t\cdot 5\log\left(\frac{t}{5}\right)+c$$

$$=e^{5x}\log x+c.$$

Question 7.

$e^{\sin^{-1}x}\left[\frac{x+\sqrt{1-x^2}}{\sqrt{1-x^2}}\right]$

Solution:

$$\text{Let } I = \int e^{\sin^{-1}x}\left[\frac{x+\sqrt{1-x^2}}{\sqrt{1-x^2}}\right]dx$$

$$= \int e^{\sin^{-1}x}[x+\sqrt{1-x^2}]\cdot\frac{1}{\sqrt{1-x^2}}dx$$

Put $\sin^{-1}x = t \qquad \therefore \frac{1}{\sqrt{1-x^2}}dx = dt$

and $x = \sin t$

$$\therefore\ I = \int e^t[\sin t + \sqrt{1-\sin^2 t}]\,dt$$

$$= \int e^t[\sin t + \sqrt{\cos^2 t}]\,dt$$

$$= \int e^t(\sin t + \cos t)\,dt$$

Let $f(t) = \sin t \qquad \therefore\ f'(t) = \cos t$

$\therefore\ I = \int e^t[f(t) + f'(t)]\,dt$

$= e^t\cdot f(t) + c$

$= e^t\cdot \sin t + c$

$= e^{\sin^{-1}x}\cdot x + c = x\cdot e^{\sin^{-1}x} + c.$

Question 8.
$\log(1 + x)^{(1+x)}$
Solution :
Let I = $\int \log(1 + x)^{(1+x)}$ dx

$= \int (1+x)\log(1+x)\,dx$

$= \int [\log(1+x)](1+x)\,dx$

$= [\log(1+x)]\int(1+x)\,dx -$

$$\int\left[\frac{d}{dt}\{\log(1+x)\}\int(1+x)\,dx\right]dx$$

$$= [\log(1+x)]\left[\frac{(1+x)^2}{2}\right] - \int\frac{1}{x+1}\cdot\frac{(x+1)^2}{2}dx$$

$$= \frac{(x+1)^2}{2}\cdot\log(1+x) - \frac{1}{2}\int(x+1)\,dx$$

$$= \frac{(x+1)^2}{2}\cdot\log(1+x) - \frac{1}{2}\cdot\frac{(x+1)^2}{2} + c$$

$$= \frac{(x+1)^2}{2}\left[\log(1+x) - \frac{1}{2}\right] + c.$$

Question 9.
cosec (log x)[1 – cot(log x)]
Solution:
Let I = $\int$cosec (log x)[1 – cot(log x)] dx
Put log x = t
x = e^t
dx = e^t dt
I = $\int$cosec t (1 – cot t). e^t dt
= $\int e^t$ [cosec t – cosec t cot t] dt
= $\int e^t$ [cosec t + $\frac{d}{dt}$ (cosec t)] dt
= e^t cosec t + c ..... [$\because$ e^t [f(t) +f'(t)] dt = e^t f(t) + c]
= x . cosec(log x) + c.

Maharashtra State Board 12th Maths Solutions Chapter 3 Indefinite Integration Ex 3.4

I. Integrate the following w. r. t. x:

Question 1.
x2+2(x−1)(x+2)(x+3)
Solution:

$$\text{Let } I=\int \frac{x^2+2}{(x-1)(x+2)(x+3)}\,dx$$

$$\text{Let } \frac{x^2+2}{(x-1)(x+2)(x+3)}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{x+3}$$

$\therefore\ x^2+2=A(x+2)(x+3)+B(x-1)(x+3)+C(x-1)(x+2)$

Put $x-1=0$, i.e. $x=1$, we get

$1+2=A(3)(4)+B(0)(4)+C(0)(3)$

$\therefore\ 3=12A \qquad \therefore\ A=\frac{1}{4}$

Put $x+2=0$, i.e. $x=-2$, we get

$4+2=A(0)(1)+B(-3)(1)+C(-3)(0)$

$\therefore\ 6=-3B \qquad \therefore\ B=-2$

Put $x+3=0$, i.e. $x=-3$, we get

$9+2=A(-1)(0)+B(-4)(0)+C(-4)(-1)$

$\therefore\ 11=4C$

$\therefore\ C=\frac{11}{4}$

$$\therefore\ \frac{x^2+2}{(x-1)(x+2)(x+3)}=\frac{(1/4)}{x-1}+\frac{(-2)}{x+2}+\frac{(11/4)}{x+3}$$

$$\therefore\ I=\int\left[\frac{(1/4)}{x-1}+\frac{(-2)}{x+2}+\frac{(11/4)}{x+3}\right]dx$$

$$=\frac{1}{4}\int\frac{1}{x-1}dx-2\int\frac{1}{x+2}dx+\frac{11}{4}\int\frac{1}{x+3}dx$$

$$=\frac{1}{4}\log|x-1|-2\log|x+2|+\frac{11}{4}\log|x+3|+c.$$

Question 2.
x2(x2+1)(x2−2)(x2+3)
Solution:

$$\text{Let } I=\int\frac{x^2}{(x^2+1)(x^2-2)(x^2+3)}\,dx$$

$$\text{Consider, } \frac{x^2}{(x^2+1)(x^2-2)(x^2+3)}$$

For finding partial fractions only, put $x^2=t$.

$$\therefore\ \frac{x^2}{(x^2+1)(x^2-2)(x^2+3)}=\frac{t}{(t+1)(t-2)(t+3)}$$

$$=\frac{A}{t+1}+\frac{B}{t-2}+\frac{C}{t+3} \qquad \ldots \text{(Say)}$$

$\therefore\ t=A(t-2)(t+3)+B(t+1)(t+3)+C(t+1)(t-2)$

Put $t+1=0$, i.e. $t=-1$, we get

$$-1 = A(-3)(2) + B(0)(2) + C(0)(-3)$$

$$\therefore\ -1 = -6A \qquad \therefore\ A = \frac{1}{6}$$

Put $t-2=0$, i.e. $t=2$, we get

$$2 = A(0)(5) + B(3)(5) + C(3)(0)$$

$$\therefore\ 2 = 15B \qquad \therefore\ B = \frac{2}{15}$$

Put $t+3=0$, i.e. $t=-3$, we get

$$-3 = A(-5)(0) + B(-2)(0) + C(-2)(-5)$$

$$-3 = 10C \qquad \therefore\ C = -\frac{3}{10}$$

$$\therefore\ \frac{t}{(t+1)(t-2)(t+3)} = \frac{(1/6)}{t+1} + \frac{(2/15)}{t-2} + \frac{(-3/10)}{t+3}$$

$$\therefore\ \frac{x^2}{(x^2+1)(x^2-2)(x^2+3)} = \frac{(1/6)}{x^2+1} + \frac{(2/15)}{x^2-2} + \frac{(-3/10)}{x^2+3}$$

$$\therefore\ I = \int\left[\frac{(1/6)}{x^2+1} + \frac{(2/15)}{x^2-2} + \frac{(-3/10)}{x^2+3}\right]dx$$

$$= \frac{1}{6}\int\frac{1}{1+x^2}dx + \frac{2}{15}\int\frac{1}{x^2-(\sqrt{2})^2}dx - \frac{3}{10}\int\frac{1}{x^2+(\sqrt{3})^2}dx$$

$$= \frac{1}{6}\tan^{-1}x + \frac{2}{15}\times\frac{1}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| - \frac{3}{10}\times\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$$

$$= \frac{1}{6}\tan^{-1}x + \frac{1}{15\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| - \frac{\sqrt{3}}{10}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c.$$

Question 3.
12x+36x2+13x−63
Solution:

$$\text{Let } I=\int \frac{12x+3}{6x^2+13x-63}\,dx$$

$$\text{Let } \frac{12x+3}{6x^2+13x-63}=\frac{12x+3}{(2x+9)(3x-7)}=\frac{A}{2x+9}+\frac{B}{3x-7}$$

$$\therefore\ 12x+3=A(3x-7)+B(2x+9)$$

Put $2x+9=0$, i.e. $x=\frac{-9}{2}$, we get

$$12\left(\frac{-9}{2}\right)+3=A\left(\frac{-27}{2}-7\right)+B(0)$$

$$\therefore\ -51=\frac{-41}{2}A \qquad \therefore\ A=\frac{102}{41}$$

Put $3x-7=0$, i.e. $x=\frac{7}{3}$, we get

$$12\left(\frac{7}{3}\right)+3=A(0)+B\left(\frac{14}{3}+9\right)$$

$$\therefore\ 31=\frac{41}{3}B \qquad \therefore\ B=\frac{93}{41}$$

$$\therefore\ \frac{12x+3}{6x^2+13x-63}=\frac{(102/41)}{2x+9}+\frac{(93/41)}{3x-7}$$

$$\therefore\ I=\int\left[\frac{(102/41)}{2x+9}+\frac{(93/41)}{3x-7}\right]dx$$

$$=\frac{102}{41}\int\frac{1}{2x+9}\,dx+\frac{93}{41}\int\frac{1}{3x-7}\,dx$$

$$=\frac{102}{41}\cdot\frac{\log|2x+9|}{2}+\frac{93}{41}\cdot\frac{\log|3x-7|}{3}+c$$

$$=\frac{51}{41}\log|2x+9|+\frac{31}{41}\log|3x-7|+c.$$

Question 4.

2x4−3x−x2

Solution:

$$\text{Let } I=\int \frac{2x}{4-3x-x^2}\,dx$$

$$\text{Let } \frac{2x}{4-3x-x^2}=\frac{2x}{(4+x)(1-x)}=\frac{A}{4+x}+\frac{B}{1-x}$$

$\therefore\ 2x=A(1-x)+B(4+x)$

Put $4+x=0$, i.e. $x=-4$, we get

$-8=A(5)+B(0) \qquad \therefore\ A=-\frac{8}{5}$

Put $1-x=0$, i.e. $x=1$, we get

$2=A(0)+B(5) \qquad \therefore\ B=\frac{2}{5}$

$$\therefore\ \frac{2x}{4-3x-x^2}=\frac{(-8/5)}{4+x}+\frac{(2/5)}{1-x}$$

$$\therefore\ I=\int\left[\frac{(-8/5)}{4+x}+\frac{(2/5)}{1-x}\right]dx$$

$$=-\frac{8}{5}\int\frac{1}{4+x}\,dx+\frac{2}{5}\int\frac{1}{1-x}\,dx$$

$$=-\frac{8}{5}\log|4+x|+\frac{2}{5}\cdot\frac{\log|1-x|}{-1}+c$$

$$=-\frac{8}{5}\log|4+x|-\frac{2}{5}\log|1-x|+c.$$

Question 5.

x2+x−1x2+x−6

Solution:

$$\text{Let } I=\int\frac{x^2+x-1}{x^2+x-6}\,dx$$

$$=\int\frac{(x^2+x-6)+5}{x^2+x-6}\,dx$$

$$=\int\left[1+\frac{5}{x^2+x-6}\right]dx$$

$$=\int 1\,dx+5\int\frac{1}{x^2+x-6}\,dx$$

$$\text{Let } \frac{1}{x^2+x-6}=\frac{1}{(x+3)(x-2)}=\frac{A}{x+3}+\frac{B}{x-2}$$

$\therefore\ 1=A(x-2)+B(x+3)$

Put $x+3=0$, i.e. $x=-3$, we get

$1=A(-5)+B(0) \qquad \therefore\ A=\frac{-1}{5}$

Put $x-2=0$, i.e. $x=2$, we get

$1=A(0)+B(5) \qquad \therefore\ B=\frac{1}{5}$

$$\therefore\ \frac{1}{x^2+x-6}=\frac{(-1/5)}{x+3}+\frac{(1/5)}{x-2}$$

$$\therefore\ I=\int 1\,dx+5\int\left[\frac{(-1/5)}{x+3}+\frac{(1/5)}{x-2}\right]dx$$

$$=\int 1\,dx-\int\frac{1}{x+3}\,dx+\int\frac{1}{x-2}\,dx$$

$$=x-\log|x+3|+\log|x-2|+c$$

$$=x+\log\left|\frac{x-2}{x+3}\right|+c.$$

Question 6.

6x3+5x2−73x2−2x−1

Solution:

$$\text{Let } I=\int \frac{6x^3+5x^2-7}{3x^2-2x-1}\,dx$$

$$\begin{array}{r}
3x^2-2x-1\,\overline{)\,6x^3+5x^2-7\,(}\,2x+3 \\
6x^3-4x^2-2x \\
-\quad+\quad+ \\
\hline
9x^2+2x-7 \\
9x^2-6x-3 \\
-\quad+\quad+ \\
\hline
8x-4
\end{array}$$

$$\therefore\ I=\int\left[(2x+3)+\frac{8x-4}{3x^2-2x-1}\right]dx$$

$$=\int 2x+3+\int\frac{8x-4}{(x-1)(3x+1)}\,dx$$

$$\text{Let } \frac{8x-4}{(x-1)(3x+1)}=\frac{A}{x-1}+\frac{B}{3x+1}$$

$$\therefore\ 8x-4=A(3x+1)+B(x-1)$$

Put $x-1=0$, i.e. $x=1$, we get

$$8-4=A(4)+B(0)\qquad\therefore\ A=1$$

Put $3x+1=0$, i.e. $x=-\frac{1}{3}$, we get

$$8\left(-\frac{1}{3}\right)-4=A(0)+B\left(-\frac{4}{3}\right)$$

$$\therefore\ \frac{-8-12}{3}=-\frac{4B}{3}\qquad\therefore\ B=5$$

$$\therefore\ \frac{8x-4}{(x-1)(3x+1)}=\frac{1}{x-1}+\frac{5}{3x+1}$$

$$\therefore\ I=2\int x\,dx+3\int 1dx+\int\left[\frac{1}{x-1}+\frac{5}{3x+1}\right]dx$$

$$=2\left(\frac{x^2}{2}\right)+3x+\int\frac{1}{x-1}dx+5\int\frac{1}{3x+1}dx$$

$$=x^2+3x+\log|x-1|+\frac{5}{3}\log|3x+1|+c.$$

Question 7.

12x2−2x−9(4x2−1)(x+3)

Solution:

Let I = $\int \frac{12x^2 - 2x - 9}{(4x^2 - 1)(x + 3)}.dx$

Let $\frac{12x^2 - 2x - 9}{(4x^2 - 1)(x + 3)} = \frac{A}{4x^2 - 1} + \frac{B}{x + 3}$

$\therefore$ $12x^2 - 2x - 9 = A(x + 3) + B(4x^2 - 1)$

Put $4x^2 - 1 = 0$, i.e. $x^2 = \frac{1}{4}$, i.e $x = \frac{1}{2}$ we get

$$12 \times \left(\frac{1}{2}\right)^2 - 2 \times \left(\frac{1}{2}\right) - 9 = A\left(\frac{7}{2}\right) + B(0)$$

$\therefore -7 = \frac{7A}{2}$

$\therefore$ A = -2

Put x + 3 = 0, i.e. x = −3, we get

$$12(-3)^2 - 2(-3) - 9 = A(0) + B(4(3^2) - 1)$$

$\therefore$ 105 = 35B

$\therefore$ B = 3

$$\frac{12x^2 - 2x - 9}{(4x^2 - 1)(x + 3)} = \frac{-2}{4x^2 - 1} + \frac{3}{x + 3}$$

$\therefore$ I = $\int \left[\frac{-2}{4x^2 - 1} + \frac{3}{x + 3}\right].dx$

$= (-2)\int \frac{1}{(2x)^2 - 1}.dx + 3\int \frac{1}{x + 3}.dx$

$= \frac{1}{2}\log\left|\frac{2x + 1}{2x - 1}\right| + 3\log|x + 3| + c.$

Question 8.

1x(xs+1)

Solution:

$$\text{Let } I = \int \frac{1}{x(x^5 + 1)}\,dx = \int \frac{x^4}{x^5(x^5 + 1)}\,dx$$

Put $x^5 = t$. Then $5x^4\,dx = dt$

$\therefore\ x^4\,dx = \frac{dt}{5}$

$\therefore\ I = \int \frac{1}{t(t+1)} \cdot \frac{dt}{5}$

$= \frac{1}{5}\int \frac{(t+1) - t}{t(t+1)}\,dt$

$= \frac{1}{5}\int \left(\frac{1}{t} - \frac{1}{t+1}\right)dt$

$= \frac{1}{5}\left[\int \frac{1}{t}dt - \int \frac{1}{t+1}dt\right]$

$= \frac{1}{5}[\log|t| - \log|t+1|] + c$

$= \frac{1}{5}\log\left|\frac{t}{t+1}\right| + c.$

$= \frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right| + c.$

Question 9.

2x2−1x4+9x2+20

Solution:

Let I = $\int \frac{2x^2-1}{x^4+9x^2+20} . dx$

Consider, $\frac{2x^2-1}{x^4+9x^2+20}$

For finding partial fractions only, put x^2 = t.

$\therefore \frac{2x^2-1}{x^4+9x^2+20} = \frac{t}{(t-1)(t-2)(t+3)}$

$= \frac{A}{t+1} + \frac{B}{t-2} + \frac{C}{t+3}$ ...(Say)

∴ t = A(t − 2)(t + 3) + B(t + 1)(t + 3) + C(t + 1)(t −2)

Put t + 1 = 0, i.e. t = − 1, we get

−1 = A(− 3)(2) + B(0)(2) + C(0)(− 3)

∴ − 1 = − 6A

∴ A = $\frac{1}{6}$

Put t − 2 = 0, i.e. t = 2, we get

2 = A(0)(5) + B(3)(5) + C(3)(0)

∴ 2 = 15B

∴ B = $\frac{2}{15}$

Put t + 3 = 0, i.e. t = − 3, we get

− 3 = A(− 5)(0) + B(− 2)(0) + C(− 2)(− 5)

−3 = 10C

∴ C = $-\frac{3}{10}$

$\therefore \frac{t}{(t+1)(t-2)(t+3)} = \frac{\left(\frac{1}{6}\right)}{t+1} + \frac{\left(\frac{2}{15}\right)}{x^2-2} + \frac{\left(\frac{-3}{10}\right)}{x^2+3}$

$\therefore \frac{x^2}{(x^2+1)(x^2-2)(x^2+3)} = \frac{\left(\frac{1}{6}\right)}{x^2+1} + \frac{\left(\frac{2}{15}\right)}{x^2-2} + \frac{\left(\frac{-3}{10}\right)}{x^2+3}$

∴ I = $\int \left[\frac{\left(\frac{1}{6}\right)}{x^2+1} + \frac{\left(\frac{2}{15}\right)}{x^2-2} + \frac{\left(\frac{-3}{10}\right)}{x^2+3}\right] . dx$

$= \frac{1}{6} \int \frac{1}{1+x^2} . dx + \frac{2}{15} \int \frac{1}{x^2-\left(\sqrt{2}\right)^2} . dx - \frac{3}{10} \int \frac{1}{x^2+\left(\sqrt{3}\right)^2} . dx$

$= \frac{1}{6} \tan^{-1} x + \frac{2}{15} \times \frac{1}{2\sqrt{2}} \log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| - \frac{3}{10} \times \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$

$= \frac{11}{\sqrt{5}} \tan^{-1}\left(\frac{x}{\sqrt{5}}\right) - \frac{9}{2} \tan^{-1}\left(\frac{x}{2}\right) + c.$

Question 10.

x2+3(x2−1)(x2−2)

Solution:

Let I = $\int \frac{x^2+3}{(x^2-1)(x^2-2)}.dx$

Consider, $\frac{x^2+3}{(x^2-1)(x^2-2)}$

For finding partial fractions only, put x^2 = t.

$\therefore \frac{x^2+3}{(x^2-1)(x^2-2)} = \frac{t}{(t+1)(t-2)}$

= $\frac{A}{t+1} + \frac{B}{t-2}$ ...(Say)

∴ t = A(t – 2)(t + 3) + B(t + 1)(t + 3) + C(t + 1)(t –2)

Put t + 1 = 0, i.e. t = – 1, we get

–1 = A(– 3)(2) + B(0)(2) + C(0)(– 3)

∴ – 1 = – 6A

∴ A = $\frac{1}{6}$

Put t – 2 = 0, i.e. t = 2, we get

2 = A(0)(5) + B(3)(5) + C(3)(0)

∴ 2 = 15B

∴ B = $\frac{2}{15}$

Put t + 3 = 0, i.e. t = – 3, we get

– 3 = A(– 5)(0) + B(– 2)(0) + C(– 2)(– 5)

–3 = 10C

∴ C = $-\frac{3}{10}$

$\therefore \frac{t}{(t+1)(t-2)(t+3)} = \frac{\left(\frac{1}{6}\right)}{t+1} + \frac{\left(\frac{2}{15}\right)}{x^2-2} + \frac{\left(\frac{-3}{10}\right)}{x^2+3}$

$\therefore \frac{x^2}{(x^2+1)(x^2-2)(x^2+3)} = \frac{\left(\frac{1}{6}\right)}{x^2+1} + \frac{\left(\frac{2}{15}\right)}{x^2-2} + \frac{\left(\frac{-3}{10}\right)}{x^2+3}$

$\therefore$ I = $\int \left[\frac{\left(\frac{1}{6}\right)}{x^2+1} + \frac{\left(\frac{2}{15}\right)}{x^2-2} + \frac{\left(\frac{-3}{10}\right)}{x^2+3}\right].dx$

= $\frac{1}{6}\int \frac{1}{1+x^2}.dx + \frac{2}{15}\int \frac{1}{x^2-\left(\sqrt{2}\right)^2}.dx - \frac{3}{10}\int \frac{1}{x^2+\left(\sqrt{3}\right)^2}.dx$

= $\frac{1}{6}\tan^{-1}x + \frac{2}{15} \times \frac{1}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| - \frac{3}{10} \times \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$

= $2\log\left|\frac{x+1}{x-1}\right| + \frac{5}{2\sqrt{2}}\log\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + c.$

Question 11.

$2x(2+x_2)(3+x_2)$

Solution:

$$\text{Let } I = \int \frac{2x}{(2+x^2)(3+x^2)}\,dx$$

Put $x^2 = t \qquad \therefore\ 2x\,dx = dt$

$$\therefore\ I = \int \frac{1}{(2+t)(3+t)}\,dt$$

$$= \int \frac{(3+t)-(2+t)}{(2+t)(3+t)}\,dt$$

$$= \int \left[\frac{1}{2+t} - \frac{1}{3+t}\right] dt$$

$$= \int \frac{1}{2+t}\,dt - \int \frac{1}{3+t}\,dt$$

$$= \log|2+t| - \log|3+t| + c$$

$$= \log\left|\frac{2+t}{3+t}\right| + c$$

$$= \log\left|\frac{2+x^2}{3+x^2}\right| + c.$$

Question 12.

2x4x-3· 2x-4

Solution:

$$\text{Let } I = \int \frac{2^x}{4^x - 3\cdot 2^x - 4}\,dx$$

$$= \int \frac{2^x}{(2^x)^2 - 3\cdot 2^x - 4}$$

Put $2^x = t$

$\therefore\ 2^x \log 2\,dx = dt \qquad \therefore\ 2^x\,dx = \frac{1}{\log 2}dt$

$$\therefore\ I = \frac{1}{\log 2}\int \frac{dt}{t^2 - 3t - 4}$$

$$= \frac{1}{\log 2}\int \frac{1}{(t+1)(t-4)}\,dt$$

$$= \frac{1}{5\log 2}\int \frac{(t+1)-(t-4)}{(t+1)(t-4)}\,dt$$

$$= \frac{1}{5\log 2}\int \left[\frac{1}{t-4} - \frac{1}{t+1}\right] dt$$

$$= \frac{1}{5\log 2}\left[\int \frac{1}{t-4}dt - \int \frac{1}{t+1}dt\right]$$

$$= \frac{1}{5\log 2}[\log|t-4| - \log|t+1|] + c$$

$$= \frac{1}{5\log 2}\log\left|\frac{2^x-4}{2^x+1}\right| + c.$$

Question 13.

3x-2(x+1)2(x+3)

Solution:

$$\text{Let } I=\int \frac{3x-2}{(x+1)^2(x+3)}\,dx$$

Let $\frac{3x-2}{(x+1)^2(x+3)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+3}$

$\therefore\ 3x-2=A(x+1)(x+3)+B(x+3)+C(x+1)^2$

Put $x+1=0$, i.e. $x=-1$, we get

$-3-2=A(0)(2)+B(2)+C(0)$

$\therefore\ -5=2B \qquad \therefore\ B=-\frac{5}{2}$

Put $x+3=0$, i.e. $x=-3$, we get

$-9-2=A(-2)(0)+B(0)+C(-2)^2$

$\therefore\ -11=4C \qquad \therefore\ C=-\frac{11}{4}$

Put $x=0$, we get

$-2=A(1)(3)+B(3)+C(1)$

$\therefore\ -2=3A+3B+C$

$\therefore\ -2=3A-\frac{15}{2}-\frac{11}{4}$

$\therefore\ 3A=-2+\frac{15}{2}+\frac{11}{4}=\frac{-8+30+11}{4}=\frac{33}{4} \qquad \therefore\ A=\frac{11}{4}$

$$\therefore\ \frac{3x-2}{(x+1)^2(x+3)}=\frac{\left(\frac{11}{4}\right)}{x+1}+\frac{\left(-\frac{5}{2}\right)}{(x+1)^2}+\frac{\left(-\frac{11}{4}\right)}{x+3}$$

$$\therefore\ I=\int\left[\frac{(11/4)}{x+1}+\frac{(-5/2)}{(x+1)^2}+\frac{(-11/4)}{x+3}\right]dx$$

$$=\frac{11}{4}\int\frac{1}{x+1}dx-\frac{5}{2}\int(x+1)^{-2}dx-\frac{11}{4}\int\frac{1}{x+3}dx$$

$$=\frac{11}{4}\log|x+1|-\frac{5}{2}\cdot\frac{(x+1)^{-1}}{-1}\cdot\frac{1}{1}-\frac{11}{4}\log|x+3|+c$$

$$=\frac{11}{4}\log\left|\frac{x+1}{x+3}\right|+\frac{5}{2(x+1)}+c.$$

Question 14.

$5x^2+20x+6x^3+2x^2+x$

Solution:

Let I = $\int 5x^2+20x+6x^3+2x^2+x$ dx

$$= \int \frac{5x^2+20x+6}{x(x^2+2x+1)}\,dx$$

$$= \int \frac{5x^2+20x+6}{x(x+1)^2}\,dx$$

Let $\frac{5x^2+20x+6}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$\therefore\ 5x^2+20x+6 = A(x+1)^2 + Bx(x+1) + Cx$

Put $x = 0$, we get

$0+0+6 = A(1) + B(0)(1) + C(0) \quad \therefore\ A = 6$

Put $x+1 = 0$, i.e. $x = -1$, we get

$5(1) + 20(-1) + 6 = A(0) + B(-1)(0) + C(-1)$

$\therefore\ -9 = -C \quad \therefore\ C = 9$

Put $x = 1$, we get

$5(1) + 20(1) + 6 = A(4) + B(1)(2) + C(1)$

But $A = 6$ and $C = 9$

$\therefore\ 31 = 24 + 2B + 9 \quad \therefore\ B = -1$

$\therefore\ \frac{5x^2+20x+6}{x(x+1)^2} = \frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}$

$$\therefore\ I = \int \left[\frac{6}{x} - \frac{1}{x+1} + \frac{9}{(x+1)^2}\right] dx$$

$$= 6\int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx + 9\int (x+1)^{-2}\,dx$$

$$= 6\log|x| - \log|x+1| + 9\cdot\frac{(x+1)^{-1}}{-1} + c$$

$$= \log|x^6| - \log|x+1| - \frac{9}{(x+1)} + c$$

$$= \log\left|\frac{x^6}{x+1}\right| - \frac{9}{(x+1)} + c.$$

Question 15.
1x(1+4x3+3x6)
Solution:

$$\text{Let } I = \int \frac{1}{x(1+4x^3+3x^6)}\,dx$$

$$= \int \frac{x^2}{x^3(1+4x^3+3x^6)}\,dx$$

Put $x^3 = t \quad \therefore\ 3x^2dx = dt \quad \therefore\ x^2dx = \frac{1}{3}dt$

$$\therefore\ I = \frac{1}{3}\int \frac{1}{t(1+4t+3t^2)}\,dt$$

$$= \frac{1}{3}\int \frac{1}{t(t+1)(3t+1)}\,dt$$

Let $\frac{1}{t(t+1)(3t+1)} = \frac{A}{t} + \frac{B}{t+1} + \frac{C}{2t+1}$

$\therefore\ 1 = A(t+1)(3t+1) + Bt(3t+1) + Ct(t+1)$

Put $t=0$, we get

$1=A(1)+B(0)+C(0) \qquad \therefore \; A=1$

Put $t+1=0$, i.e. $t=-1$, we get

$1=A(0)+B(-1)(-2)+C(0) \qquad \therefore \; B=\frac{1}{2}$

Put $3t+1=0$, i.e. $t=-\frac{1}{3}$, we get

$1=A(0)+B(0)+C\left(-\frac{1}{3}\right)\left(\frac{2}{3}\right) \qquad \therefore \; C=-\frac{9}{2}$

$$\therefore \; \frac{1}{t(t+1)(3t+1)}=\frac{1}{t}+\frac{\left(\frac{1}{2}\right)}{t+1}+\frac{\left(-\frac{9}{2}\right)}{3t+1}$$

$$\therefore \; I=\frac{1}{3}\int\left[\frac{1}{t}+\frac{\left(\frac{1}{2}\right)}{t+1}+\frac{\left(-\frac{9}{2}\right)}{3t+1}\right]dt$$

$$=\frac{1}{3}\left[\int\frac{1}{t}dt+\frac{1}{2}\int\frac{1}{t+1}dt-\frac{9}{2}\int\frac{1}{3t+1}dt\right]$$

$$=\frac{1}{3}\left[\log|t|+\frac{1}{2}\log|t+1|-\frac{9}{2}\cdot\frac{1}{3}\log|3t+1|\right]+c$$

$$=\frac{1}{3}\log|x^3|+\frac{1}{2}\log|x^3+1|-\frac{3}{2}\log|3x^3+1|+c$$

$$=\log|x|+\frac{1}{2}\log|x^3+1|-\frac{3}{2}\log|3x^3+1|+c.$$

Question 16.
1x3−1
Solution:

$$\text{Let } I=\int\frac{1}{x^3-1}\,dx$$

$$=\int\frac{1}{(x-1)(x^2+x+1)}\,dx$$

Let $\frac{1}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$

$\therefore \; 1=A(x^2+x+1)+(Bx+C)(x-1)$

Put $x-1=0$ i.e. $x=1$, we get

$1=A(3)+(B+C)(0) \qquad \therefore \; A=\frac{1}{3}$

Put $x=0$, we get

$1=A(1)+C(-1) \qquad \therefore \; C=A-1=-\frac{2}{3}$

Comparing the coefficients of x^2 on both the sides, we get

$0=A+B \qquad \therefore \; B=-A=-\frac{1}{3}$

$$\therefore \frac{1}{(x-1)(x^2+x+1)} = \frac{\left(\frac{1}{3}\right)}{x-1} + \frac{\left(-\frac{1}{3}x - \frac{2}{3}\right)}{x^2+x+1}$$

$$= \frac{1}{3}\left[\frac{1}{x-1} - \frac{x+2}{x^2+x+1}\right]$$

Let $x+2 = p\left[\frac{d}{dx}(x^2+x+1)\right] + q$

$$= p(2x+1) + q = 2px + (p+q)$$

Comparing coefficients of x and the constant term on both the sides, we get

$2p = 1$ i.e. $p = \frac{1}{2}$ and $p + q = 2$

$$\therefore q = 2 - p = 2 - \frac{1}{2} = \frac{3}{2}$$

$$\therefore x+2 = \frac{1}{2}(2x+1) + \frac{3}{2}$$

$$\therefore \frac{1}{(x+1)(x^2+x+1)} = \frac{1}{3}\left[\frac{1}{x-1} - \frac{\frac{1}{2}(2x+1)+\frac{3}{2}}{(x^2+x+1)}\right]$$

$$= \frac{1}{3}\left[\frac{1}{x-1} - \frac{1}{2}\left(\frac{2x+1}{x^2+x+1}\right) - \frac{\left(\frac{3}{2}\right)}{(x^2+x+1)}\right]$$

$$\therefore I = \frac{1}{3}\int\left[\frac{1}{x-1} - \frac{1}{2}\left(\frac{2x+1}{x^2+x+1}\right) - \frac{\left(\frac{3}{2}\right)}{(x^2+x+1)}\right]dx$$

$$= \frac{1}{3}\int\frac{1}{x-1}dx - \frac{1}{6}\int\frac{2x+1}{x^2+x+1}dx - \frac{1}{2}\int\frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx$$

$$= \frac{1}{3}\log|x-1| -$$

$$\frac{1}{6}\int\frac{\frac{d}{dx}(x^2+x+1)}{x^2+x+1}dx - \frac{1}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= \frac{1}{3}\log|x-1| - \frac{1}{6}\log|x^2+x+1| -$$

$$\frac{1}{2}\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left[\frac{\left(x+\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}\right] + c$$

$$= \frac{1}{3}\log|x-1| - \frac{1}{6}\log|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c.$$

Question 17.
(3sinx−2)· cosx5−4sinx−cos2x
Solution:

$$\text{Let } I=\int \frac{(3\sin x-2)\cos x}{5-4\sin x-\cos^2 x}dx$$

$$=\int \frac{(3\sin x-2)\cos x}{5-(1-\sin^2 x)-4\sin x}dx$$

$$=\int \frac{(3\sin x-2)\cos x}{5-1+\sin^2 x-4\sin x}dx$$

$$=\int \frac{(3\sin x-2)\cos x}{\sin^2 x-4\sin x+4}dx$$

Put $\sin x=t$ $\quad \therefore \cos x\,dx=dt$

$$\therefore I=\int \frac{3t-2}{t^2-4t+4}dt$$

$$=\int \frac{3t-2}{(t-2)^2}dt$$

$$\text{Let } \frac{3t-2}{(t-2)^2}=\frac{A}{t-2}+\frac{B}{(t-2)^2}$$

$\therefore 3t-2=A(t-2)+B$

Put $t-2=0$, i.e. $t=2$, we get

$4=A(0)+B \quad \therefore B=4$

Put $t=0$, we get

$-2=A(-2)+B$

$\therefore -2=-2A+4$

$\therefore 2A=6 \quad \therefore A=3$

$$\therefore \frac{3t-2}{(t-2)^2}=\frac{3}{t-2}+\frac{4}{(t-2)^2}$$

$$\therefore I=\int\left[\frac{3}{t-2}+\frac{4}{(t-2)^2}\right]dt$$

$$=3\int\frac{1}{t-2}dt+4\int(t-2)^{-2}dt$$

$$=3\log|t-2|+4\cdot\frac{(t-2)^{-1}}{-1}\cdot\frac{1}{1}+c$$

$$=3\log|t-2|-\frac{4}{(t-2)}+c$$

$$=3\log|\sin x-2|-\frac{4}{(\sin x-2)}+c.$$

Question 18.

1sinx+sin2x

Solution:

$$\text{Let } I = \int \frac{1}{\sin x + \sin 2x} dx$$

$$= \int \frac{1}{\sin x + 2\sin x \cos x} dx$$

$$= \int \frac{dx}{\sin x(1 + 2\cos x)} = \int \frac{\sin x\, dx}{\sin^2 x(1 + 2\cos x)}$$

$$= \int \frac{\sin x\, dx}{(1 - \cos^2 x)(1 + 2\cos x)}$$

$$= \int \frac{\sin x\, dx}{(1 - \cos x)(1 + \cos x)(1 + 2\cos x)}$$

Put $\cos x = t \quad \therefore \; -\sin x\, dx = dt$

$\therefore \; \sin x\, dx = -dt$

$$\therefore I = \int \frac{-dt}{(1-t)(1+t)(1+2t)}$$

$$= -\int \frac{dt}{(1-t)(1+t)(1+2t)}$$

$$\text{Let } \frac{1}{(1-t)(1+t)(1+2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1+2t}$$

$\therefore \; 1 = A(1+t)(1+2t) + B(1-t)(1+2t) + C(1-t)(1+t)$

Putting $1 - t = 0$, i.e. $t = 1$, we get

$1 = A(2)(3) + B(0)(3) + C(0)(2) \quad \therefore \; A = 1/6$

Putting $1 + t = 0$, i.e. $t = -1$, we get

$1 = A(0)(-1) + B(2)(-1) + C(2)(0)$

$\therefore \; B = -1/2$

Putting $1 + 2t = 0$, i.e. $t = -1/2$, we get

$$1 = A(0) + B(0) + C\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \quad \therefore \; C = 4/3$$

$$\therefore \frac{1}{(1-t)(1+t)(1+2t)} = \frac{(1/6)}{1-t} + \frac{(-1/2)}{1+t} + \frac{(4/3)}{1+2t}$$

$$\therefore \; I = -\int \left[\frac{(1/6)}{1-t} + \frac{(-1/2)}{1+t} + \frac{(4/3)}{1+2t}\right] dt$$

$$= -\frac{1}{6}\int \frac{1}{1-t}\, dt + \frac{1}{2}\int \frac{1}{1+t}\, dt - \frac{4}{3}\int \frac{1}{1+2t}\, dt$$

$$= -\frac{1}{6} \cdot \frac{\log|1-t|}{-1} + \frac{1}{2}\log|1+t| - \frac{4}{3} \cdot \frac{\log|1+2t|}{2} + c$$

$$= \frac{1}{6}\log|1 - \cos x| + \frac{1}{2}\log|1 + \cos x| - \frac{2}{3}\log|1 + 2\cos x| + c$$

$$= \frac{1}{2}\log|\cos x + 1| + \frac{1}{6}\log|\cos x - 1| - \frac{2}{3}\log|2\cos x + 1| + c.$$

Question 19.
12sinx+sin2x
Solution:

$$\text{Let } I=\int \frac{1}{2\sin x+\sin 2x}\,dx$$

$$=\int \frac{1}{2\sin x+2\sin x\cos x}\,dx$$

$$=\int \frac{1}{2\sin x(1+\cos x)}\,dx$$

$$=\int \frac{\sin x}{2\sin^2 x(1+\cos x)}\,dx$$

$$=\int \frac{\sin x\,dx}{2(1-\cos^2 x)(1+\cos x)}$$

$$=\int \frac{\sin x\,dx}{2(1-\cos x)(1+\cos x)(1+\cos x)}$$

$$=\int \frac{\sin x\,dx}{2(1-\cos x)(1+\cos x)^2}$$

Put $\cos x=t$ $\quad\therefore\ -\sin x\,dx=dt$

$\therefore\ \sin x\,dx=-dt$

$$\therefore\ I=-\frac{1}{2}\int \frac{1}{(1-t)(1+t)^2}\,dt$$

$$=\frac{1}{2}\int \frac{1}{(t-1)(t+1)^2}\,dt$$

Let $\dfrac{1}{(t-1)(t+1)^2}=\dfrac{A}{t-1}+\dfrac{B}{t+1}+\dfrac{C}{(t+1)^2}$

$\therefore\ 1=A(t+1)^2+B(t-1)(t+1)+C(t-1)$

Put $t+1=0$, i.e., $t=1$, we get

$\therefore\ 1=A(0)+B(0)+C(-2)$

$\therefore\ C=-\dfrac{1}{2}$

Put $t-1=0$, i.e., $t=1$, we get

$\therefore\ 1=A(4)+B(0)+C(0) \qquad \therefore\ A=\dfrac{1}{4}$

Comparing coefficients of t^2 on both the sides, we get

$0=A+B$

$\therefore\ B=-A=-\dfrac{1}{4}$

$$\therefore\ \frac{1}{(t-1)(t+1)^2}=\frac{\left(\frac{1}{4}\right)}{t-1}+\frac{\left(-\frac{1}{4}\right)}{t+1}+\frac{\left(-\frac{1}{2}\right)}{(t+1)^2}$$

$$\therefore\ I=\frac{1}{2}\int\left[\frac{\left(\frac{1}{4}\right)}{t-1}+\frac{\left(-\frac{1}{4}\right)}{t+1}+\frac{\left(-\frac{1}{2}\right)}{(t+1)^2}\right]dt$$

$$=\frac{1}{8}\int\frac{1}{t-1}\,dt-\frac{1}{8}\int\frac{1}{t+1}\,dt-\frac{1}{4}\int\frac{1}{(t-1)^2}\,dt$$

$$=\frac{1}{8}\log|t-1|-\frac{1}{8}\log|t+1|-\frac{1}{4}\frac{(t+1)^{-1}}{(-1)}+c$$

$$=\frac{1}{8}\log\left|\frac{t-1}{t+1}\right|+\frac{1}{4}\cdot\frac{1}{t+1}+c$$

$$=\frac{1}{8}\log\left|\frac{\cos x-1}{\cos x+1}\right|+\frac{1}{4(\cos x+1)}+c.$$

Question 20.

1sin2x+cosx

Solution:

Let I = $\int \frac{1}{\sin 2x + \cos x} \cdot dx$

= $\int \frac{1}{\sin x + \sin 2x \cos x} \cdot dx$

= $\int \frac{dx}{\sin x(1 + 2\cos x)}$

= $\int \frac{\sin x \cdot dx}{\sin^2 x(1 + 2\cos x)}$

= $\int \frac{\sin \cdot dx}{(1 - \cos^2 x)(1 + 2\cos x)}$

= $\int \frac{\sin \cdot dx}{(1 - \cos x)(1 + \sin 2\times)(1 + \cos x)}$

Put cos x = t

∴ – sinx . dx = dt

∴ sinx .dx = – dt

$\therefore$ I = $\int \frac{-dt}{(1-t)(1+t)(1+2t)}$

= $-\int \frac{dt}{(1-t)(1+t)(1+2t)}$

Let $\frac{1}{(1-t)(1+t)(1+2t)} = \frac{A}{1-\ t} + \frac{B}{1+t} + \frac{C}{1+2t}$

$\therefore$ 1 = A(1 + t)(1 + 2t) + B(1 – t)(1 + 2t) + C(1 – t)(1 + t)

Putting 1 – t = 0, i.e. t = 1, we get

1 = A(2)(3) + B(0)(3) + C(0)(2)

$\therefore$ A = $\frac{1}{6}$

Putting 1 – t = 0, i.e. t = – 1, we get

1 = A(0)(– 1) + B(2)(– 1) + C(2)(0)

$\therefore$ B = $-\frac{1}{2}$

Putting 1 + 2t = 0, i.e. t = $-\frac{1}{2}$, we get

1 = A$(0) +$ B$(0) +$ C$\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)$

$\therefore$ C = $\frac{4}{3}$

$\therefore \frac{1}{(1-t)(1+t)(1+2t)} = \frac{\left(\frac{1}{6}\right)}{1-t} + \frac{\left(\frac{-1}{2}\right)}{1+t} + \frac{\left(\frac{4}{3}\right)}{1+2t}$

$\therefore$ I = $\int \left[\frac{\left(\frac{1}{6}\right)}{1-t} + \frac{\left(\frac{-1}{2}\right)}{1+t} + \frac{\left(\frac{4}{3}\right)}{1+2t}\right] \cdot dt$

= $-\frac{1}{6}\int \frac{1}{1-t} \cdot dt + \frac{1}{2}\int \frac{1}{1+t} \cdot dt - \frac{4}{3}\int \frac{1}{1+2t} \cdot dt$

= $-\frac{1}{6} \cdot \frac{\log|1-t|}{-1} + \frac{1}{2}\log|1+t| - \frac{4}{3} \cdot \frac{\log|1+2t|}{2} + c$

= $-\frac{1}{6}\log|\sin x + 1| + \frac{1}{2}\log|\sin x - 1| - \frac{2}{3}\log|\sin x1 + 2| + c$

= $-\frac{1}{6}\log|1 - \sin x| - \frac{1}{2}\log|1 + \sin x| + \frac{2}{3}\log|1 + 2\sin x| + c.$

Question 21.

1sinx· (3+2cosx)

Solution:

$$\text{Let } I=\int \frac{1}{\sin x(3+2\cos x)}\,dx$$

$$=\int \frac{\sin x}{\sin^2 x(3+2\cos x)}\,dx$$

$$=\int \frac{\sin x}{(1-\cos^2 x)(3+2\cos x)}\,dx$$

$$=\int \frac{\sin x}{(1-\cos x)(1+\cos x)(3+2\cos x)}\,dx$$

Put $\cos x=t \qquad \therefore\ -\sin x\,dx=dt$

$\therefore\ \sin x\,dx=-dt$

$$\therefore\ I=\int \frac{1}{(1-t)(1+t)(3+2t)}(-dt)$$

$$=\int \frac{-1}{(1-t)(1+t)(3+2t)}\,dt$$

$$\text{Let } \frac{-1}{(1-t)(1+t)(3+2t)}=\frac{A}{1-t}+\frac{B}{1+t}+\frac{C}{3+2t}$$

$\therefore\ -1=A(1+t)(3+2t)+B(1-t)(3+2t)+C(1-t)(1+t)$

Put $1-t=0$, i.e. $t=1$, we get

$-1=A(2)(5)+B(0)(5)+C(0)(2)$

$\therefore\ -1=10A \quad \therefore\ A=\frac{-1}{10}$

Put $1+t=0$, i.e. $t=-1$, we get

$-1=A(0)(1)+B(2)(1)+C(2)(0)$

$\therefore\ -1=2B \quad \therefore\ B=-\frac{1}{2}$

Put $3+2t=0$, i.e. $t=-\frac{3}{2}$, we get

$$-1=A\left(-\frac{1}{2}\right)(0)+B\left(\frac{5}{2}\right)(0)+C\left(\frac{5}{2}\right)\left(-\frac{1}{2}\right)$$

$\therefore\ -1=-\frac{5}{4}C \quad \therefore\ C=\frac{4}{5}$

$$\therefore\ \frac{-1}{(1-t)(1+t)(3+2t)}=\frac{\left(\frac{-1}{10}\right)}{1-t}+\frac{\left(-\frac{1}{2}\right)}{1+t}+\frac{\left(\frac{4}{5}\right)}{3+2t}$$

$$\therefore\ I=\int\left[\frac{\left(\frac{-1}{10}\right)}{1-t}+\frac{\left(-\frac{1}{2}\right)}{1+t}+\frac{\left(\frac{4}{5}\right)}{3+2t}\right]dt$$

$$=-\frac{1}{10}\int\frac{1}{1-t}dt-\frac{1}{2}\int\frac{1}{1+t}dt+\frac{4}{5}\int\frac{1}{3+2t}dt$$

$$=-\frac{1}{10}\frac{\log|1-t|}{-1}-\frac{1}{2}\log|1+t|+\frac{4}{5}\frac{\log|3+2t|}{2}+c$$

$$=\frac{1}{10}\log|1-\cos x|-\frac{1}{2}\log|1+\cos x|+$$

$$\frac{2}{5}\log|3+2\cos x|+c.$$

Question 22.

5. $e^x(e^x+1)(e^{2x}+9)$

Solution:

$$\text{Let } I = \int \frac{5e^x}{(e^x+1)(e^{2x}+9)}\,dx$$

Put $e^x = t \qquad \therefore\ e^x dx = dt$

$$\therefore\ I = 5\int \frac{1}{(t+1)(t^2+9)}\,dt$$

$$\text{Let } \frac{1}{(t+1)(t^2+9)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+9}$$

$\therefore\ 1 = A(t^2+9) + (Bt+C)(t+1)$

Put $t+1=0$, i.e. $t=-1$, we get

$1 = A(1+9) + C(0) \qquad \therefore\ A = \frac{1}{10}$

Put $t=0$, we get

$1 = A(9) + C(1) \qquad \therefore\ C = 1-9A = 1-\frac{9}{10} = \frac{1}{10}$

Comparing coefficients of t^2 on both the sides, we get

$0 = A + B \qquad \therefore\ B = -A = -\frac{1}{10}$

$$\therefore\ \frac{1}{(t+1)(t^2+9)} = \frac{\left(\frac{1}{10}\right)}{t+1} + \frac{\left(-\frac{1}{10}t+\frac{1}{10}\right)}{t^2+9}$$

$$\therefore\ I = 5\int\left[\frac{\left(\frac{1}{10}\right)}{t+1} + \frac{\left(-\frac{1}{10}t+\frac{1}{10}\right)}{t^2+9}\right]dt$$

$$= \frac{1}{2}\int\frac{1}{t+1}\,dt - \frac{1}{2}\int\frac{t}{t^2+9}\,dt + \frac{1}{2}\int\frac{t}{t^2+9}\,dt$$

$$= \frac{1}{2}\log|t+1| - \frac{1}{4}\int\frac{2t}{t^2+9}\,dt + \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)$$

$$= \frac{1}{2}\log|t+1| - \frac{1}{4}\int\frac{\frac{d}{dt}(t^2+9)}{t^2+9}\,dt + \frac{1}{6}\tan^{-1}\left(\frac{t}{3}\right)$$

$$= \frac{1}{2}\log|t+1| - \frac{1}{4}\log|t^2+9| + \frac{1}{6}\tan^{-1}\left(\frac{t}{3}\right) + c$$

$$= \frac{1}{2}\log|e^x+1| - \frac{1}{4}\log|e^{2x}+9| + \frac{1}{6}\tan^{-1}\left(\frac{e^x}{3}\right) + c.$$

Question 23.

2logx+3x(3logx+2)[(logx)2+1]

Solution:

$$\text{Let } I = \int \frac{2\log x + 3}{x(3\log x + 2)[(\log x)^2 + 1]}\, dx$$

Put $\log x = t$ $\quad \therefore \frac{1}{x} dx = dt$

$$\therefore\ I = \int \frac{2t+3}{(3t+2)(t^2+1)}\, dt$$

$$\text{Let } \frac{2t+3}{(3t+2)(t^2+1)} = \frac{A}{3t+2} + \frac{Bt+C}{t^2+1}$$

$$\therefore\ 2t + 3 = A(t^2+1) + (Bt+C)(3t+2)$$

Put $3t + 2 = 0$ i.e. $t = -\frac{2}{3}$, we get

$$2\left(\frac{-2}{3}\right) + 3 = A\left(\frac{4}{9} + 1\right) + \left(\frac{-2}{3}B + C\right)(0)$$

$$\therefore\ \frac{5}{3} = A\left(\frac{13}{9}\right) \qquad \therefore\ A = \frac{15}{13}$$

Put $t = 0$, we get

$$3 = A(1) + C(2) = \frac{15}{13} + 2C$$

$$\therefore\ 2C = 3 - \frac{15}{13} = \frac{24}{13} \qquad \therefore\ C = \frac{12}{13}$$

Comparing coefficients of t^2 on both the sides, we get

$$0 = A + 3B \qquad \therefore\ B = -\frac{A}{3} = -\frac{5}{13}$$

$$\therefore\ \frac{2t+3}{(3t+2)(t^2+1)} = \frac{\left(\frac{15}{13}\right)}{3t+2} + \frac{\left(-\frac{5}{13}t + \frac{12}{13}\right)}{t^2+1}$$

$$\therefore\ I = \int \left[\frac{\left(\frac{15}{13}\right)}{3t+2} + \frac{\left(-\frac{5}{13}t + \frac{12}{13}\right)}{t^2+1}\right] dt$$

$$= \frac{15}{13}\int \frac{1}{3t+2}\, dt - \frac{5}{26}\int \frac{2t}{t^2+1}\, dt + \frac{12}{13}\int \frac{1}{t^2+1}\, dt$$

$$= \frac{15}{13}\cdot\frac{1}{3}\log|3t+2| - \frac{5}{26}\log|t^2+1| + \frac{12}{13}\tan^{-1}(t) + c$$

$$\ldots \left[\because\ \frac{d}{dt}(t^2+1) = 2t \text{ and } \int \frac{f'(t)}{f(t)}\, dt = \log|f(t)| + c\right]$$

$$= \frac{5}{13}\log|3\log x + 2| - \frac{5}{26}\log|(\log x)^2 + 1| + \frac{12}{13}\tan^{-1}(\log x) + c.$$

Maharashtra State Board 12th Maths Solutions Chapter 3 Indefinite Integration Miscellaneous Exercise 3

I. Choose the correct options from the given alternatives:

Question 1.
$\int \frac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}} dx=$

(a) $\frac{1}{2}\sqrt{x+1}+c$

(b) $\frac{2}{3}(x+1)^{\frac{3}{2}}+c$

(c) $\sqrt{x+1}+c$

(d) $2(x+1)^{\frac{3}{2}}+c$

Answer:

(b) $\frac{2}{3}(x+1)^{\frac{3}{2}}+c$

Question 2.
$\int \frac{1}{x+x^5} dx$ = f(x) + c, then $\int \frac{x^4}{x+x^5} dx=$

(a) log x – f(x) + c
(b) f(x) + log x + c
(c) f(x) – log x + c
(d) $\frac{1}{5}x^5$ f(x) + c
Answer:
(a) log x – f(x) + c

$$\text{Hint : } \int \frac{x^4}{x+x^5} dx = \int \frac{(x^4+1)-1}{x(x^4+1)} dx$$
$$= \int \left(\frac{1}{x} - \frac{1}{x+x^5}\right) dx = \log x - f(x) + c.$$

Question 3.
$\int \frac{\log(3x)}{x\log(9x)} dx=$

(a) log(3x) – log(9x) + c
(b) log(x) – (log 3) . log(log 9x) + c
(c) log 9 – (log x) . log(log 3x) + c
(d) log(x) + log(3) . log(log 9x) + c
Answer:
(b) log(x) – (log 3) . log(log 9x) + c

$$\text{Hint : } \int \frac{\log 3x}{x\log(9x)} dx = \int \frac{\log\left(\frac{9x}{3}\right)}{x\log(9x)} dx$$
$$= \int \frac{\log(9x) - \log 3}{x\log(9x)} dx$$
$$= \int \left[\frac{1}{x} - \frac{\log 3}{x\log(9x)}\right] dx$$
$$= \int \frac{1}{x} dx - (\log 3) \int \frac{\left(\frac{1}{x}\right)}{\log(9x)} dx$$
$$= \log(x) - (\log 3) \cdot \log(\log 9x) + c.$$

Question 4.
$\int \frac{\sin^m x}{\cos^{m+2} x} dx=$

(a) $\frac{\tan^{m+1} x}{m+1}+c$

(b) (m + 2) $\tan^{m+1}$ x + c

(c) $\frac{\tan^m x}{m}+c$

(d) (m + 1) $\tan^{m+1} x + c$

Answer:

(a) $\frac{\tan^{m+1} x}{m+1}+c$

Question 5.

$\int \tan(\sin^{-1} x) \cdot dx =$

(a) $(1-x^2)^{-\frac{1}{2}}+c$

(b) $(1-x^2)^{\frac{1}{2}}+c$

(c) $\frac{\tan^m x}{\sqrt{1-x^2}}+c$

(d) $-\sqrt{1-x^2}+c$

Answer:

(d) $-\sqrt{1-x^2}+c$

Hint: $\sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Question 6.

$\int \frac{x-\sin x}{1-\cos x} \cdot dx=$

(a) $x \cot\left(\frac{x}{2}\right) + c$

(b) $-x \cot\left(\frac{x}{2}\right) + c$

(c) $\cot\left(\frac{x}{2}\right) + c$

(d) $x \tan\left(\frac{x}{2}\right) + c$

Answer:

(b) $-x \cot\left(\frac{x}{2}\right) + c$

$$\textbf{Hint :} \int \frac{x-\sin x}{1-\cos x}dx = \int \frac{x-2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)}dx$$

$$=\frac{1}{2}\int x\,\mathrm{cosec}^2\left(\frac{x}{2}\right)dx - \int \cot\left(\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\left[x\int \mathrm{cosec}^2\left(\frac{x}{2}\right)dx - \int\left[\frac{d}{dx}(x)\int \mathrm{cosec}^2\left(\frac{x}{2}\right)dx\right]dx\right] - \int \cot\left(\frac{x}{2}\right)dx$$

$$=\frac{1}{2}\left[x\left\{\frac{-\cot\left(\frac{x}{2}\right)}{\left(\frac{1}{2}\right)}\right\} - \int 1\cdot\frac{-\cot\left(\frac{x}{2}\right)}{\left(\frac{1}{2}\right)}dx\right] - \int \cot\left(\frac{x}{2}\right)dx$$

$$=-x\cot\left(\frac{x}{2}\right)+\int \cot\left(\frac{x}{2}\right)dx - \int \cot\left(\frac{x}{2}\right)dx$$

$$=-x\cot\left(\frac{x}{2}\right)+c.]$$

Question 7.

If f(x) = $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$, g(x) = $e^{\sin^{-1}x}$, then $\int f(x) \cdot g(x) \cdot dx =$

(a) $e^{\sin^{-1}x} \cdot (\sin^{-1}x-1)+c$

(b) $e^{\sin^{-1}x} \cdot (1-\sin^{-1}x)+c$

(c) $e^{\sin^{-1}x} \cdot (\sin^{-1}x+1)+c$

(d) $e^{\sin^{-1}x} \cdot (\sin^{-1}X-1)+c$

Answer:

(a) $e^{\sin^{-1}x} \cdot (\sin^{-1}x-1)+c$

Question 8.

If $\int \tan^3 x \cdot \sec^3 x \cdot dx = \left(\frac{1}{m}\right) \sec^m x - \left(\frac{1}{n}\right) \sec^n x + c$, then (m, n) =

(a) (5, 3)
(b) (3, 5)
(c) (15,13)
(d) (4, 4)
Answer:
(a) (5, 3)

Hint: ∫$\tan^3 x . \sec^3 x$ dx
= ∫$\sec^2 x . \tan^2 x . \sec x \tan x$ dx
= ∫$\sec^2 x (\sec^2 x – 1) \sec x \tan x$ dx
Put sec x = t.

Question 9.
∫1cosx−cos2x⋅ dx=
(a) log(cosec x – cot x) + tan(x2) + c
(b) sin 2x – cos x + c
(c) log(sec x + tan x) – cot(x2) + c
(d) cos 2x – sin x + c
Answer:
(c) log(sec x + tan x) – cot(x2) + c

$$\textbf{Hint :} \int \frac{1}{\cos x-\cos^2 x} d x$$
$$=\int \frac{1}{\cos x(1-\cos x)} d x$$
$$=\int \frac{(1-\cos x)+\cos x}{\cos x(1-\cos x)} d x$$
$$=\int\left(\frac{1}{\cos x}+\frac{1}{1-\cos x}\right) d x$$
$$=\int\left[\sec x+\frac{1}{2} \operatorname{cosec}^2\left(\frac{x}{2}\right)\right] d x$$
$$=\log |\sec x+\tan x|+\frac{1}{2} \frac{\left(-\cot \frac{x}{2}\right)}{1 / 2}+c$$
$$=\log |\sec x+\tan x|-\cot \left(\frac{x}{2}\right)+c .$$

Question 10.
∫cotx√sinx⋅ cosx⋅ dx=
(a) 2cotx−−−−√+c
(b) −2cotx−−−−√+c
(c) 12cotx−−−−√+c
(d) cotX−−−−−√+c
Answer:
(b) −2cotx−−−−√+c

Question 11.
∫ex(x−1)x2⋅ dx=
(a) exx+C
(b) exx2+C
(c) (X−1x)ex+C
(d) x e-x + c
Answer:
(a) exx+C

Question 12.
∫sin(log x) . dx =
(a) $x2$ [sin(log x) – cos(log x)] + c
(b) $x2$ [sin(log x) + cos(log x)] + c
(c) $x2$ [cos(log x) – sin(log x)] + c
(d) $x4$ [cos(log x) – sin(log x)] + c
Answer:
(a) $x2$ [sin(log x) – cos(log x)] + c

Question 13.
∫x^x (1 + log x) . dx =
(a) 12 $(1 + \log x)^2$ + c
(b) x^{2x} + c
(c) x^x log x + c
(d) x^x + c
Answer:
(d) x^x + c

Hint: ddx(xx) = x^x (1 + log x)

Question 14.
$\int cos_{-37}x \cdot \; sin_{-117}x \cdot \; dx=$
(a) $log(sin_{-47}x)+c$
(b) $47tan_{47}x+c$
(c) $-74tan_{-47}x+c$
(d) $log(cos_{37}x)+c$
Answer:
(c) $-74tan_{-47}x+c$

Hint: $\int cos_{-37}xsin_{-117}xdx$
= $\int sin_{-117}xcos_{-117}x \cdot \; cos_2xdx$
= $\int tan_{-117}xsec2xdx$
Put tan x = t.

Question 15.
$2\int cos_2x-sin_2xcos_2x+sin_2x \cdot \; dx=$
(a) sin 2x + c
(b) cos 2x + c
(c) tan 2x + c
(d) 2 sin 2x + c
Answer:
(a) sin 2x + c

Question 16.
$\int dxcosxsin_2x-cos_2x\surd \cdot \; dx=$
(a) log(tan x – $tan2x-1--------\surd$) + c
(b) $\sin^{-1}$ (tan x) + c
(c) 1 + $\sin^{-1}$ (cot x) + c
(d) log(tan x + $tan2x-1--------\surd$) + c
Answer:
(d) log(tan x + $tan2x-1--------\surd$) + c

Hint : $\int \frac{dx}{\cos x\sqrt{\sin^2 x - \cos^2 x}}$

$= \int \frac{\sec^2 x\, dx}{\sqrt{\tan^2 x - 1}}$ ... [Dividing by $\cos^2 x$]

Put $\tan x = t$.

Question 17.
$\int \frac{\log x}{(\log ex)^2} \cdot dx =$

(a) $\frac{x}{1+\log x}+c$

(b) x(1 + log x) + c

(c) $\frac{1}{1+\log x}+c$

(d) $\frac{1}{1-\log x}+c$

Answer:

(a) $\frac{x}{1+\log x}+c$

Question 18.
∫[sin(log x) + cos(log x)] . dx =
(a) x cos(log x) + c
(b) sin(log x) + c
(c) cos(log x) + c
(d) x sin(log x) + c
Answer:
(d) x sin(log x) + c

Question 19.
$\int \frac{\cos 2x-1}{\cos 2x+1} \cdot dx =$
(a) tan x – x + c
(b) x + tan x + c
(c) x – tan x + c
(d) -x – cot x + c
Answer:
(c) x – tan x + c

Hint : $\int \frac{\cos 2x-1}{\cos 2x+1} dx$

$$= \int \frac{-(1-\cos 2x)}{1+\cos 2x} dx$$

$$= \int \frac{-2\sin^2 x}{2\cos^2 x} dx$$

$$= -\int (\sec^2 x - 1) dx$$

$$= -\tan x + x + c.$$

Question 20.
$\int \frac{e^{2x}+e^{-2x}}{e^x} \cdot dx =$

(a) $e^x-\frac{1}{3e^{3x}}+c$

(b) $e^x+\frac{1}{3e^{3x}}+c$

(c) $e^{-x}+\frac{1}{3e^{3x}}+c$

(d) $e^{-x}-\frac{1}{3e^{3x}}+c$

Answer:

(a) $e^x-\frac{1}{3e^{3x}}+c$

Hint.: $\int \frac{e^{2x}+e^{-2x}}{e^x} dx$

$$= \int e^x dx + \int e^{-3x} dx$$

$$= e^x + \frac{e^{-3x}}{(-3)} + c$$

$$= e^x - \frac{1}{3e^{3x}} + c.$$

II. Integrate the following with respect to the respective variable:

Question 1.
$(x-2)^2 \sqrt{x}$
Solution:

$$\text{Let } I = \int (x-2)^2\sqrt{x}\, dx$$

$$= \int (x^2 - 4x + 4)\sqrt{x}\, dx$$

$$= \int (x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + 4x^{\frac{1}{2}})\, dx$$

$$= \int x^{\frac{5}{2}}\, dx - 4\int x^{\frac{3}{2}}\, dx + 4\int x^{\frac{1}{2}}\, dx$$

$$= \frac{x^{\frac{7}{2}}}{(7/2)} - 4\frac{x^{\frac{5}{2}}}{(5/2)} + 4\frac{x^{\frac{3}{2}}}{(3/2)} + c$$

$$= \frac{2}{7}x^{\frac{7}{2}} - \frac{8}{5}x^{\frac{5}{2}} + \frac{8}{3}x^{\frac{3}{2}} + c.$$

Question 2.
x7x+1
Solution:

$$\text{Let } I = \int \frac{x^7}{x+1}\, dx$$

$$= \int \frac{(x^7+1)-1}{x+1}\, dx$$

$$= \int \frac{(x+1)(x^6 - x^5 + x^4 - x^3 + x^2 - x + 1) - 1}{x+1}\, dx$$

$$= \int \left[x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 - \frac{1}{x+1}\right] dx$$

$$= \int x^6 dx - \int x^5 dx + \int x^4 dx - \int x^3 dx + \int x^2 dx - \int x\, dx + \int 1\, dx - \int \frac{1}{x+1}\, dx$$

$$= \frac{x^7}{7} - \frac{x^6}{6} + \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + c.$$

Question 3.
(6x+5)32
Solution:

$$\int (6x+5)^{\frac{3}{2}}\, dx$$

$$= \frac{(6x+5)^{\frac{5}{2}}}{6 \times \frac{5}{2}} + c = \frac{1}{15}(6x+5)^{\frac{5}{2}} + c.$$

Question 4.
t3(t+1)2
Solution:

$$\text{Let } I = \int \frac{t^3}{(t+1)^2} dt$$

$$= \int \frac{(t^3+1)-1}{(t+1)^2} dt = \int \frac{(t+1)(t^2-t+1)-1}{(t+1)^2} dt$$

$$= \int \left[\frac{t^2-t+1}{t+1} - \frac{1}{(t+1)^2} \right] dt$$

$$= \int \left[\frac{(t+1)(t-2)+3}{t+1} - \frac{1}{(t+1)^2} \right] dt$$

$$= \int \left[t-2+\frac{3}{t+1} - \frac{1}{(t+1)^2} \right] dt$$

$$= \int t\,dt - 2\int 1\,dt + 3\int \frac{1}{t+1} dt - \int \frac{1}{(t+1)^2} dt$$

$$= \frac{t^2}{2} - 2t + 3\log|t+1| - \frac{(t+1)^{-1}}{(-1)} + c$$

$$= \frac{t^2}{2} - 2t + 3\log|t+1| + \frac{1}{t+1} + c.$$

Question 5.
3–2sinxcos2x
Solution:

$$\text{Let } I = \int \frac{3-2\sin x}{\cos^2 x} dx$$

$$= \int \left(\frac{3}{\cos^2 x} - \frac{2\sin x}{\cos^2 x} \right) dx$$

$$= 3\int \sec^2 x\,dx - 2\int \sec x \tan x\,dx$$

$$= 3\tan x - 2\sec x + c.$$

Question 6.
sin6θ+cos6θsin2θ · cos2θ
Solution:

$$\int \frac{\sin^6\theta + \cos^6\theta}{\sin^2\theta \cdot \cos^2\theta} d\theta$$

$$= \int \left[\frac{(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta \cdot \cos^2\theta(\sin^2\theta + \cos^2\theta)}{\sin^2\theta \cdot \cos^2\theta} \right] d\theta$$

$$\ldots \left[\because\ a^3 + b^3 = (a+b)^3 - 3ab(a+b) \right]$$

$$= \int \left[\frac{(1)^3 - 3\sin^2\theta \cdot \cos^2\theta(1)}{\sin^2\theta \cdot \cos^2\theta} \right] d\theta$$

$$= \int \left[\frac{1}{\sin^2\theta \cdot \cos^2\theta} - 3 \right] d\theta$$

$$= \int \left[\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta \cdot \cos^2\theta} - 3 \right] d\theta$$

$$= \int \left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} - 3 \right) d\theta$$

$$= \int (\sec^2\theta + \operatorname{cosec}^2\theta - 3)\,d\theta$$

$$= \int \sec^2\theta\,d\theta + \int \operatorname{cosec}^2\theta\,d\theta - 3\int 1\,d\theta$$

$$= \tan\theta - \cot\theta - 3\theta + c.$$

Question 7.
cos 3x cos 2x cos x
Solution:

$$\text{Let } I = \int \cos 3x \cos 2x \cos x \, dx$$

Consider $\cos 3x \cos 2x \cos x = \frac{1}{2} \cos 3x \, [2 \cos 2x \cos x]$

$$= \frac{1}{2} \cos 3x \, [\cos (2x + x) + \cos (2x - x)]$$

$$= \frac{1}{2} [\cos^2 3x + \cos 3x \cos x]$$

$$= \frac{1}{4} [2 \cos^2 3x + 2 \cos 3x \cos x]$$

$$= \frac{1}{4} [1 + \cos 6x + \cos (3x + x) + \cos (3x - x)]$$

$$= \frac{1}{4} [1 + \cos 6x + \cos 4x + \cos 2x]$$

$$\therefore \; I = \frac{1}{4} \int [1 + \cos 6x + \cos 4x + \cos 2x] \, dx$$

$$= \frac{1}{4} \int 1 \, dx + \frac{1}{4} \int \cos 6x \, dx + \frac{1}{4} \int \cos 4x \, dx + \frac{1}{4} \int \cos 2x \, dx$$

$$= \frac{x}{4} + \frac{1}{4} \left(\frac{\sin 6x}{6} \right) + \frac{1}{4} \left(\frac{\sin 4x}{4} \right) + \frac{1}{4} \left(\frac{\sin 2x}{2} \right) + c$$

$$= \frac{1}{48} [12x + 2 \sin 6x + 3 \sin 4x + 6 \sin 2x] + c.$$

Question 8.
$cos7x-cos8x1+2cos5x$
Solution:

$$\int \frac{\cos 7x - \cos 8x}{1 + 2 \cos 5x} \, dx$$

$$= \int \frac{\sin 5x \, (\cos 7x - \cos 8x)}{\sin 5x \, (1 + 2 \cos 5x)} \, dx$$

$$= \int \frac{\sin 5x \, (\cos 7x - \cos 8x)}{\sin 5x + 2 \sin 5x \cos 5x} \, dx$$

$$= \int \frac{\sin 5x \, (\cos 7x - \cos 8x)}{\sin 5x + \sin 10x} \, dx$$

$$= \int \frac{2 \sin \left(\frac{5x}{2} \right) \cdot \cos \left(\frac{5x}{2} \right) \times 2 \sin \left(\frac{7x + 8x}{2} \right) \cdot \sin \left(\frac{8x - 7x}{2} \right)}{2 \sin \left(\frac{10x + 5x}{2} \right) \cdot \cos \left(\frac{10x - 5x}{2} \right)} dx$$

$$= \int \frac{2 \sin \left(\frac{5x}{2} \right) \cdot \cos \left(\frac{5x}{2} \right) \times 2 \sin \left(\frac{15x}{2} \right) \cdot \sin \left(\frac{x}{2} \right)}{2 \sin \left(\frac{15x}{2} \right) \cdot \cos \left(\frac{5x}{2} \right)} dx$$

$$= \int 2 \sin \left(\frac{5x}{2} \right) \cdot \sin \left(\frac{x}{2} \right) \, dx$$

$$= \int \left[\cos \left(\frac{5x}{2} - \frac{x}{2} \right) - \cos \left(\frac{5x}{2} + \frac{x}{2} \right) \right] dx$$

$$= \int (\cos 2x - \cos 3x) \, dx$$

$$= \int \cos 2x \, dx - \int \cos 3x \, dx$$

$$= \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + c.$$

Question 9.
$\cot^{-1}\left(\frac{1+\sin x}{\cos x}\right)$
Solution:
Let I = $\int \cot^{-1}\left(\frac{1+\sin x}{\cos x}\right) dx$

$$\frac{1+\sin x}{\cos x}=\frac{1+\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)}$$

$$=\frac{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cdot\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}$$

$$=\cot\left(\frac{\pi}{6}-\frac{x}{2}\right)$$

$$\therefore\ I=\int \cot^{-1}\left[\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]dx$$

$$=\int\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$$

$$=\frac{\pi}{4}\int 1\,dx-\frac{1}{2}\int x\,dx$$

$$=\frac{\pi}{4}\cdot x-\frac{1}{2}\cdot\frac{x^2}{2}+c=\frac{\pi x}{4}-\frac{x^2}{4}+c.$$

III. Integrate the following w.r.t. x:

Question 1.
$\frac{(1+\log x)^3}{x}$
Solution:

$$\text{Let } I=\int\frac{(1+\log x)^3}{x}dx$$

$$\text{Put } 1+\log x=t \qquad \therefore\ \frac{1}{x}dx=dt$$

$$\therefore\ I=\int t^3\,dt=\frac{1}{4}t^4+c=\frac{1}{4}(1+\log x)^4+c.$$

Question 2.
$\cot^{-1}(1-x+x^2)$
Solution:

Let $I=\int \cot^{-1}(1-x+x^2)\,dx$

$=\int \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx$

$=\int \tan^{-1}\left[\frac{x+(1-x)}{1-x(1-x)}\right]$

$=\int [\tan^{-1}x+\tan^{-1}(1-x)]\,dx$

$=\int \tan^{-1}x\,dx+\int \tan^{-1}(1-x)\,dx$

$\therefore\ I=I_1+I_2$ ... (1)

$I_1=\int \tan^{-1}x\,dx=\int (\tan^{-1}x)\cdot 1\,dx$

$=(\tan^{-1}x)\cdot\int 1\,dx-\left[\frac{d}{dx}(\tan^{-1}x)\cdot\int 1\,dx\right]dx$

$=(\tan^{-1}x)x-\int\frac{1}{1+x^2}\cdot x\,dx$

$=x\tan^{-1}x-\frac{1}{2}\int\frac{2x}{1+x^2}dx$

$\therefore\ I_1=x\tan^{-1}x-\frac{1}{2}\log|1+x^2|+c_1$ ... (2)

... $\left[\because\ \frac{d}{dx}(1+x^2)=2x \text{ and } \int\frac{f'(x)}{f(x)}dx=\log|f(x)|+c\right]$

$I_2=\int \tan^{-1}(1-x)\,dx=\int[\tan^{-1}(1-x)]\cdot 1\,dx$

$=[\tan^{-1}(1-x)]\cdot\int 1\,dx-\int\left\{\frac{d}{dx}[\tan^{-1}(1-x)]\cdot\int 1\,dx\right\}dx$

$=[\tan^{-1}(1-x)]\cdot x-\int\frac{1}{1+(1-x)^2}\cdot(-1)\cdot x\,dx$

$=x\tan^{-1}(1-x)+\int\frac{x}{1+1-2x+x^2}dx$

$=x\tan^{-1}(1-x)+\int\frac{x}{2-2x+x^2}dx$

Let $x=A\left[\frac{d}{dx}(2-2x+x^2)\right]+B$

$\therefore\ x=A(-2+2x)+B=2Ax+(-2A+B)$

Comparing the coefficient of x and constant on both the sides, we get

$1=2A$ and $0=-2A+B$

$\therefore\ A=\frac{1}{2}$ and $0=-2\left(\frac{1}{2}\right)+B \qquad \therefore\ B=1$

$\therefore\ x=\frac{1}{2}(-2+2x)+1$

$\therefore\ I_2=x\tan^{-1}(1-x)+\int\frac{\frac{1}{2}(-2+2x)+1}{2-2x+x^2}dx$

$=x\tan^{-1}(1-x)+\frac{1}{2}\int\frac{-2+2x}{2-2x+x^2}dx+\int\frac{1}{2-2x+x^2}dx$

$=x\tan^{-1}(1-x)+\frac{1}{2}\log|2-2x+x^2|+\int\frac{1}{1+(1-2x+x^2)}dx$

$=x\tan^{-1}(1-x)+\frac{1}{2}\log|x^2-2x+2|+\int\frac{1}{1+(1-x)^2}dx$

$=x\tan^{-1}(1-x)+\frac{1}{2}\log|x^2-2x+2|+\frac{1}{1}\frac{\tan^{-1}(1-x)}{-1}+c_2$

$$= x\tan^{-1}(1-x) + \frac{1}{2}\log|x^2-2x+2| - \tan^{-1}(1-x) + c_2$$

$$= (x-1)\tan^{-1}(1-x) + \frac{1}{2}\log|x^2-2x+2| + c_2$$

$$\therefore\ I_2 = -(1-x)\tan^{-1}(1-x) + \frac{1}{2}\log|x^2-2x+2| + c_2 \quad \ldots (3)$$

From (1), (2) and (3), we get

$$I = x\tan^{-1}x - \frac{1}{2}\log|1+x^2| + c_1 - (1-x)\tan^{-1}(1-x) + \frac{1}{2}\log|x^2-2x+2| + c_2$$

$$= x\tan^{-1}x - \frac{1}{2}\log|1+x^2| - (1-x)\tan^{-1}(1-x) + \frac{1}{2}\log|x^2-2x+2| + c, \text{ where } c = c_1 + c_2.$$

Question 3.
1xsin2(logx)
Solution:

$$\text{Let } I = \int \frac{1}{x\sin^2(\log x)}dx$$

Put $\log x = t \qquad \therefore\ \frac{1}{x}dx = dt$

$$\therefore\ I = \int \frac{1}{\sin^2 t}dt = \int \operatorname{cosec}^2 t\, dt = -\cot t + c$$

$$= -\cot(\log x) + c.$$

Question 4.
x−−√sec(x32)tan(x32)
Solution:

$$\text{Let } I = \int \sqrt{x}\sec(x^{\frac{3}{2}})\tan(x^{\frac{3}{2}})dx$$

Put $x^{\frac{3}{2}} = t \qquad \therefore\ \frac{3}{2}\sqrt{x}\,dx = dt \qquad \therefore\ \sqrt{x}\,dx = \frac{2}{3}dt$

$$\therefore\ I = \frac{2}{3}\int \sec t\tan t\, dt = \frac{2}{3}\sec t + c = \frac{2}{3}\sec(x^{\frac{3}{2}}) + c.$$

Question 5.
log(1 + cos x) – x tan(x2)
Solution:

$$\text{Let } I=\int\left[\log(1+\cos x)-x\tan\left(\frac{x}{2}\right)\right]dx$$

$$=\int\left[\log(1+\cos x)\cdot 1\,dx-\int x\tan\left(\frac{x}{2}\right)dx\right.$$

$$=[\log(1+\cos x)]\cdot\int 1\,dx-\int\left\{\frac{d}{dx}[\log(1+\cos x)]\cdot\int 1\,dx\right\}dx-\int x\tan\left(\frac{x}{2}\right)dx$$

$$=[\log(1+\cos x)]\cdot(x)-\int\frac{1}{1+\cos x}\cdot(0-\sin x)\cdot x\,dx-\int x\tan\left(\frac{x}{2}\right)dx$$

$$=x\cdot\log(1+\cos x)+\int x\cdot\frac{\sin x}{1+\cos x}dx-\int x\tan\left(\frac{x}{2}\right)dx+c$$

$$=x\cdot\log(1+\cos x)+\int x\cdot\frac{2\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}dx-\int x\tan\left(\frac{x}{2}\right)dx+c$$

$$=x\log(1+\cos x)+\int x\cdot\tan\left(\frac{x}{2}\right)dx-\int x\tan\left(\frac{x}{2}\right)dx+c$$

$$=x\cdot\log(1+\cos x)+c.$$

Question 6.
x21−x6√
Solution:

$$\text{Let } I=\int\frac{x^2}{\sqrt{1-x^6}}dx$$

$$\text{Put } x^3=t \quad \therefore\ 3x^2\,dx=dt \quad \therefore\ x^2\,dx=\frac{1}{3}dt$$

$$\therefore\ I=\frac{1}{3}\int\frac{1}{\sqrt{1-t^2}}dt=\frac{1}{3}\sin^{-1}(t)+c$$

$$=\frac{1}{3}\sin^{-1}(x^3)+c.$$

Question 7.
1(1−cos4x)(3−cot2x)
Solution:

$$\text{Let } I=\int\frac{1}{(1-\cos 4x)(3-\cot 2x)}dx$$

$$=\int\frac{1}{2\sin^2 2x\,(3-\cot 2x)}dx=\frac{1}{2}\int\frac{\operatorname{cosec}^2 2x}{3-\cot 2x}dx$$

$$\text{Put } 3-\cot 2x=t \quad \therefore\ 2\operatorname{cosec}^2 2x\,dx=dt$$

$$\therefore\ \operatorname{cosec}^2 2x\,dx=\frac{1}{2}dt$$

$$\therefore\ I=\frac{1}{4}\int\frac{1}{t}dt=\frac{1}{4}\log|t|+c=\frac{1}{4}\log|3-\cot 2x|+c.$$

Question 8.
log(log x) + (log x)-2
Solution:

Let $I = \int [\log(\log x) + (\log x)^{-2}]\,dx$

$$= \int \left[\log(\log x) + \frac{1}{(\log x)^2} \right] dx$$

Put $\log x = t \quad \therefore\ x = e^t$

$\therefore\ dx = e^t\,dt$

$$\therefore\ I = \int \left(\log t + \frac{1}{t^2} \right) e^t\,dt$$

$$= \int e^t \left(\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2} \right) dt$$

$$= \int \left[e^t \left(\log t + \frac{1}{t} \right) + e^t \left(-\frac{1}{t} + \frac{1}{t^2} \right) \right] dt$$

$$= \int e^t \left(\log t + \frac{1}{t} \right) dt - \int e^t \left(\frac{1}{t} - \frac{1}{t^2} \right) dt$$

$= I_1 - I_2$

In I_1, Put $f(t) = \log t$. Then $f'(t) = (1/t)$

$\therefore\ I_1 = \int e^t [f(t) + f'(t)]\,dt$

$= e^t f(t) = e^t \log t$

In I_2, Put $g(t) = (1/t)$. Then $g'(t) = -(1/t^2)$

$\therefore\ I_2 = \int e^t [g(t) + g'(t)]\,dt$

$= e^t g(t) = e^t \cdot (1/t)$

$$\therefore\ I = e^t \log t - \frac{e^t}{t} + c$$

$$= x \log(\log x) - \frac{x}{\log x} + c.$$

Question 9.
12cosx+3sinx
Solution:

$$\text{Let } I = \int \frac{1}{2\cos x + 3\sin x}\,dx = \int \frac{1}{3\sin x + 2\cos x}\,dx$$

Dividing numerator and denominator by

$\sqrt{3^2 + 2^2} = \sqrt{13}$, we get

$$I = \int \frac{\left(\frac{1}{\sqrt{13}} \right)}{\frac{3}{\sqrt{13}}\sin x + \frac{2}{\sqrt{13}}\cos x}\,dx$$

Since, $\left(\frac{3}{\sqrt{13}} \right)^2 + \left(\frac{2}{\sqrt{13}} \right)^2 = \frac{9}{13} + \frac{4}{13} = 1,$

we take $\frac{3}{\sqrt{13}} = \cos\alpha,\ \frac{2}{\sqrt{13}} = \sin\alpha$

so that $\tan\alpha = \frac{2}{3}$ and $\alpha = \tan^{-1}\left(\frac{2}{3} \right)$

$$\therefore\ I=\frac{1}{\sqrt{13}}\int\frac{1}{\sin x\cos\alpha+\cos x\sin\alpha}dx$$

$$=\frac{1}{\sqrt{13}}\int\frac{1}{\sin(x+\alpha)}dx$$

$$=\frac{1}{\sqrt{13}}\int\operatorname{cosec}(x+\alpha)\,dx$$

$$=\frac{1}{\sqrt{13}}\log\left|\tan\left(\frac{x+\alpha}{2}\right)\right|+c$$

$$=\frac{1}{\sqrt{13}}\log\left|\tan\left(\frac{x+\tan^{-1}\frac{2}{3}}{2}\right)\right|+c.$$

Alternative Method :

Let $I=\int\frac{1}{2\cos x+3\sin x}dx$

Put $\tan\left(\frac{x}{2}\right)=t$ $\quad\therefore\ \frac{x}{2}=\tan^{-1}t$

$\therefore\ x=2\tan^{-1}t$ $\quad\therefore\ dx=\frac{2}{1+t^2}dt$

and $\sin x=\frac{2t}{1+t^2}$ and $\cos x=\frac{1-t^2}{1+t^2}$

$$\therefore\ I=\int\frac{1}{2\left(\frac{1-t^2}{1+t^2}\right)+3\left(\frac{2t}{1+t^2}\right)}\cdot\frac{2dt}{1+t^2}$$

$$=\int\frac{1+t^2}{2-2t^2+6t}\cdot\frac{2dt}{1+t^2}$$

$$=\int\frac{1}{1-t^2+3t}dt$$

$$=\int\frac{1}{1-\left(t^2-3t+\frac{9}{4}\right)+\frac{9}{4}}dt$$

$$=\int\frac{1}{\left(\frac{\sqrt{13}}{2}\right)^2-\left(t-\frac{3}{2}\right)^2}dt$$

$$=\frac{1}{2\times\frac{\sqrt{13}}{2}}\log\left|\frac{\frac{\sqrt{13}}{2}+t-\frac{3}{2}}{\frac{\sqrt{13}}{2}-t+\frac{3}{2}}\right|+c$$

$$=\frac{1}{\sqrt{13}}\log\left|\frac{\sqrt{13}+2t-3}{\sqrt{13}-2t+3}\right|+c$$

$$=\frac{1}{\sqrt{13}}\log\left|\frac{\sqrt{13}+2\tan\left(\frac{x}{2}\right)-3}{\sqrt{13}-2\tan\left(\frac{x}{2}\right)-3}\right|+c.$$

Question 10.
1x3x2−1√
Solution:

Let $I=\int \frac{1}{x^3\sqrt{x^2-1}}dx$

Put $x=\sec\theta$ $\quad\therefore\ dx=\sec\theta\tan\theta\, d\theta$

$\therefore\ I=\int \frac{\sec\theta\tan\theta\, d\theta}{\sec^3\theta\sqrt{\sec^2\theta-1}}=\int \frac{\sec\theta\tan\theta\, d\theta}{\sec^3\theta\sqrt{\tan^2\theta}}d\theta$

$=\int \cos^2\theta\, d\theta=\frac{1}{2}\int(1+\cos 2\theta)d\theta$

$=\frac{1}{2}\int d\theta+\frac{1}{2}\int\cos 2\theta\, d\theta=\frac{\theta}{2}+\frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)+c$ $\quad\ldots$ (1)

$\because\ x=\sec\theta$ $\quad\therefore\ \theta=\sec^{-1}x$

$\sin 2\theta=2\sin\theta\cos\theta=2\sqrt{1-\cos^2\theta}\cdot\cos\theta$

$=2\sqrt{1-\frac{1}{x^2}}\left(\frac{1}{x}\right)$ $\quad\ldots\left[\because\ \sec\theta=x\Rightarrow\cos\theta=\frac{1}{x}\right]$

$=\frac{2\sqrt{x^2-1}}{x^2}$

$\therefore$ from (1), we have

$I=\frac{1}{2}\sec^{-1}x+\frac{1}{2}\frac{\sqrt{x^2-1}}{x^2}+c.$

Question 11.

3x+1−2x2+x+3√

Solution:

Let I $=\int \frac{3x+1}{\sqrt{-2x^2+x+3}}.dx$

Let 3x + 1 = A$\left[\frac{d}{dx}(-2x^2+x+3)\right]$ + B

= A(2 − 2x) + B

∴ 3x + 1 = 2Ax + (2A + B)

Comparing the coefficient of x and constant on both the sides, we get

− 2A = 7 and 2A + B = 3

∴ A = $\frac{-7}{2}$ and $2\left(-\frac{7}{2}\right)$ + B = 3

∴ B = 10

∴ 7x + 3 = $\frac{-7}{2}(2-2x)+10$

∴ I = $\int \frac{\frac{-7}{2}(2-2x)+10}{\sqrt{3+2x-x^2}}.dx$

= $\frac{-7}{2}\int \frac{(2-2x)}{\sqrt{3+2x-x^2}}.dx+10\int\frac{1}{\sqrt{3+2x-x^2}}x$

$= \frac{-7}{2} I_1 + 10 I_2$

In I_1, put $3 + 2x - x^2 = t$

$\therefore (2 - 2x)dx = dt$

$\therefore I_1 = \int \frac{1}{\sqrt{t}} dt$

$= \int t^{-\frac{1}{2}} dt$

$= \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c_1$

$= 2\sqrt{3 + 2x - x^2} + c_1$

$I_2 = \int \frac{1}{\sqrt{3 - (x^2 - 2x + 1) + 1}} . dx$

$= \int \frac{1}{\sqrt{(2)^2 - (x - 1)^2}} . dx$

$= \sin^{-1}\left(\frac{x - 1}{2}\right) + c_2$

$-\frac{3}{2}\sqrt{-2x^2 + x + 3} + \frac{7}{4\sqrt{2}} \sin^{-1}\left(\frac{4x - 1}{5}\right) + c.$

Question 12.
$\log(x^2 + 1)$
Solution:

$$\text{Let } I = \int \log(x^2 + 1)\,dx = \int [\log(x^2 + 1)] \cdot 1\,dx$$

$$= [\log(x^2 + 1)] \int 1\,dx - \int \left[\frac{d}{dx}\{\log(x^2 + 1)\} \int 1\,dx \right] dx$$

$$= [\log(x^2 + 1)] \cdot x - \int \frac{1}{x^2 + 1} \cdot \frac{d}{dx}(x^2 + 1) \cdot x\,dx$$

$$= x \log(x^2 + 1) - \int \frac{1}{x^2 + 1} \times (2x + 0) \cdot x\,dx$$

$$= x \log(x^2 + 1) - \int \frac{2x^2}{x^2 + 1}\,dx$$

$$= x \log(x^2 + 1) - \int \frac{2x^2 + 2 - 2}{x^2 + 1}\,dx$$

$$= x \log(x^2 + 1) - \int \left[\frac{2(x^2 + 1)}{x^2 + 1} - \frac{2}{x^2 + 1} \right] dx$$

$$= x \log(x^2 + 1) - \left[2\int 1\,dx - 2\int \frac{1}{x^2 + 1}\,dx \right]$$

$$= x \log(x^2 + 1) - 2x + 2\tan^{-1} x + c.$$

Question 13.
$e^{2x} \sin x \cos x$
Solution:

Let $I = \int e^{2x} \cdot \sin x \cos x \, dx = \frac{1}{2} \int e^{2x} \cdot 2 \sin x \cos x \, dx$

$= \frac{1}{2} \int e^{2x} \cdot \sin 2x \, dx$ ... (1)

$= \frac{1}{2} [e^{2x} \int \sin 2x \, dx - \int \{ \frac{d}{dx}(e^{2x}) \int \sin 2x \, dx \} dx]$

$= \frac{1}{2} \left[e^{2x} \left(\frac{-\cos 2x}{2} \right) - \int e^{2x} \times 2 \times \left(\frac{-\cos 2x}{2} \right) dx \right]$

$= -\frac{1}{4} e^{2x} \cos 2x + \frac{1}{2} \int e^{2x} \cos 2x \, dx$

$= -\frac{1}{4} e^{2x} \cos 2x +$

$\frac{1}{2} \left[e^{2x} \int \cos 2x \, dx - \int \left\{ \frac{d}{dx}(e^{2x}) \int \cos 2x \, dx \right\} dx \right]$

$= -\frac{1}{4} e^{2x} \cos 2x + \frac{1}{2} \left[e^{2x} \cdot \frac{\sin 2x}{2} - \int e^{2x} \times 2 \times \frac{\sin 2x}{2} dx \right]$

$= -\frac{1}{4} e^{2x} \cos 2x + \frac{1}{4} e^{2x} \sin 2x - \frac{1}{2} \int e^{2x} \sin 2x \, dx$

$\therefore\ I = -\frac{1}{4} e^{2x} \cos 2x + \frac{1}{4} e^{2x} \sin 2x - I$ ... [By (1)]

$\therefore\ 2I = -\frac{1}{4} e^{2x} \cos 2x + \frac{1}{4} e^{2x} \sin 2x$

$\therefore\ I = \frac{e^{2x}}{8} (\sin 2x - \cos 2x) + c$

Question 14.
$x^2(x-1)(3x-1)(3x-2)$
Solution:

Let I = $\int \frac{x^2}{(x-1)(3x-1)(3x-2)} . dx$

Let $\frac{x^2}{(x-1)(3x-1)(3x-2)}$

= $\frac{A}{x-1} + \frac{B}{3x-1} + \frac{C}{3x-2}$

∴ x^2 = A(3x -1)(3x - 2) + B(x – 1)(3x - 2) + C(x – 1)(3x -1)

Put x – 1 = 0, i.e. x = 1, we get

∴ x^2 = A(2)(1) + B(0)(1) + C(0)(2)

∴ 2 = 4A

∴ A = $\frac{1}{2}$

Put x + 2 = 0, i.e. x = – 2, we get

2 + 2 = A(0)(1) + B(– 3)(1) + C(– 3)(0)

∴ 6 = – 3B

∴ B = – 2

Put x + 3 = 0, i.e. x = – 3we get

9 + 2 = A(– 1)(0) + B(– 4)(0) + C(– 4)(– 1)

∴ 11 = 4C

∴ C = $\frac{11}{4}$

∴ $\frac{x^2+2}{(3x-1)(x-1)(3x-2)} = \frac{\left(\frac{1}{4}\right)}{3x-1} + \frac{-2}{x-1} + \frac{\left(\frac{11}{4}\right)}{3x-2}$

∴ I = $\int \left[\frac{\left(\frac{1}{4}\right)}{3x-1} + \frac{-2}{x-1} + \frac{\left(\frac{11}{4}\right)}{3x-2}\right] . dx$

= $\frac{1}{18} \int \frac{1}{3x-1} . dx - 2 \int \frac{1}{x-1} . dx + \frac{4}{9} \int \frac{1}{3x-2} . dx$

= $\frac{1}{18} \log|3x-1| + \frac{1}{2} \log|x-1| - \frac{4}{9} \log|3x-2| + c.$

Question 15.

1sinx+sin2x

Solution:

Let I = $\int \frac{1}{\sin x + \sin 2x} \cdot dx$

= $\int \frac{1}{\sin x + 2\sin x \cos x} \cdot dx$

= $\int \frac{dx}{\sin x(1 + 2\cos x)}$

= $\int \frac{\sin x \cdot dx}{\sin^2 x(1 + 2\cos x)}$

= $\int \frac{\sin \cdot dx}{(1 - \cos^2 x)(1 + 2\cos x)}$

= $\int \frac{\sin \cdot dx}{(1 - \cos x)(1 + \cos x)(1 + 2\cos x)}$

Put cos x = t

∴ – sinx . dx = dt

∴ sinx .dx = – dt

∴ I = $\int \frac{-dt}{(1-t)(1+t)(1+2t)}$

= $-\int \frac{dt}{(1-t)(1+t)(1+2t)}$

Let $\frac{1}{(1-t)(1+t)(1+2t)} = \frac{\text{A}}{1-\ t} + \frac{\text{B}}{1+t} + \frac{\text{C}}{1+2t}$

∴ 1 = A(1 + t)(1 + 2t) + B(1 – t)(1 + 2t) + C(1 – t)(1 + t)

Putting 1 – t = 0, i.e. t = 1, we get

1 = A(2)(3) + B(0)(3) + C(0)(2)

∴ A = $\frac{1}{6}$

Putting 1 – t = 0, i.e. t = – 1, we get

1 = A(0)(– 1) + B(2)(– 1) + C(2)(0)

∴ B = $-\frac{1}{2}$

Putting 1 + 2t = 0, i.e. t = $-\frac{1}{2}$, we get

1 = A$(0) + \text{B}(0) + \text{C}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)$

∴ C = $\frac{4}{3}$

∴ $\frac{1}{(1-t)(1+t)(1+2t)} = \frac{\left(\frac{1}{6}\right)}{1-t} + \frac{\left(\frac{-1}{2}\right)}{1+t} + \frac{\left(\frac{4}{3}\right)}{1+2t}$

∴ I = $\int \left[\frac{\left(\frac{1}{6}\right)}{1-t} + \frac{\left(\frac{-1}{2}\right)}{1+t} + \frac{\left(\frac{4}{3}\right)}{1+2t}\right] \cdot dt$

= $\frac{1}{6}\int \frac{1}{1-t} \cdot dt + \frac{1}{2}\int \frac{1}{1+t} \cdot dt - \frac{4}{3}\int \frac{1}{1+2t} \cdot dt$

= $\frac{1}{6} \cdot \frac{\log|1-t|}{-1} + \frac{1}{2}\log|1+t| - \frac{4}{3} \cdot \frac{\log|1+2t|}{2} + c$

= $\frac{1}{6}\log|1-\cos x| + \frac{1}{2}\log|1+\cos x| - \frac{2}{3}\log|1+2\cos x| + c.$

Question 16.
sec2x7+2tanx−tan2x−−−−−−−−−−−−−−−−√
Solution:

$$\text{Let } I=\int \sec^2 x\sqrt{7+2\tan x-\tan^2 x}\,dx$$

Put $\tan x=t \quad \therefore \sec^2 x\,dx=dt$

$\therefore \ I=\int\sqrt{7+2t-t^2}\,dt$

$=\int\sqrt{7-(t^2-2t)}\,dt$

$=\int\sqrt{8-(t^2-2t+1)}\,dt$

$=\int\sqrt{(2\sqrt{2})^2-(t-1)^2}\,dt$

$=\left(\frac{t-1}{2}\right)\sqrt{(2\sqrt{2})^2-(t-1)^2}+\frac{(2\sqrt{2})^2}{2}\sin^{-1}\left(\frac{t-1}{2\sqrt{2}}\right)+c$

$=\left(\frac{t-1}{2}\right)\sqrt{7+2t-t^2}+4\sin^{-1}\left(\frac{t-1}{2\sqrt{2}}\right)+c$

$=\left(\frac{\tan x-1}{2}\right)\sqrt{7+2\tan x-\tan^2 x}+4\sin^{-1}\left(\frac{\tan x-1}{2\sqrt{2}}\right)+c.$

Question 17.
x+5x3+3x2−x−3
Solution:

Let I = $\int \frac{x+5}{x^3+3x^2-x-3} \cdot dx$

= $\int \frac{x+5}{x^2(x+3)-(x+3)} \cdot dx$

= $\int \frac{x+5}{(x+3)(x^2-1)}$

= $\int \frac{x+5}{(x+3)(x-1)(x+1)} \cdot dx$

∴ $x^2 + 2 = A(x + 2)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 2)$

Put x – 1 = 0, i.e. x = 1, we get

1 + 2 = A(3)(4) + B(0)(4) + C(0)(3)

∴ 3 = 12A

∴ A = $\frac{1}{4}$

Put x + 2 = 0, i.e. x = – 2, we get

4 + 2 = A(0)(1) + B(– 3)(1) + C(– 3)(0)

∴ 6 = – 3B

∴ B = – 2

Put x + 3 = 0, i.e. x = – 3we get

9 + 2 = A(– 1)(0) + B(– 4)(0) + C(– 4)(– 1)

∴ 11 = 4C

∴ C = $\frac{11}{4}$

∴ $\frac{x^2+2}{(x-1)(x+2)(x+3)} = \frac{\left(\frac{1}{4}\right)}{x-1} + \frac{-2}{x+1} + \frac{\left(\frac{11}{4}\right)}{x+3}$

∴ I = $\int \left[\frac{\left(\frac{1}{4}\right)}{x-1} + \frac{-2}{x+1} + \frac{\left(\frac{11}{4}\right)}{x+3}\right] . dx$

= $\frac{1}{4} \int \frac{1}{x-1} . dx - 2 \int \frac{1}{x+1} . dx + \frac{11}{4} \int \frac{1}{x+3} . dx$

$\frac{3}{4} \log|x-1| - \log|x+1| + \frac{1}{4} \log|x+3| + c.$

Question 18.

1x(x5+1)

Solution:

Let I = $\int \frac{1}{x(x^5+1)} \cdot dx$

= $\int \frac{x^4}{x^5(x^5+1)} \cdot dx$

Put x^5 = t.

Then $5x^4$ dx = dt

$\therefore$ x^4 dx= $\frac{dt}{5}$

$\therefore I = \int \frac{-}{t(t+1)} \cdot \frac{---}{5}$

= $\frac{1}{5} \int \frac{(t+1)-t}{t(t+1)} \cdot dt$

= $\frac{1}{5} \int \left(\frac{1}{t} - \frac{1}{t+1}\right) \cdot dt$

= $\frac{1}{5} \left[\int \frac{1}{t} dt - \int \frac{1}{t+1} dt\right]$

= $\frac{1}{5} [\log|t| - \log|t+1|] + c$

= $\frac{1}{5} \log\left|\frac{t}{t+1}\right| + c$

= $\frac{1}{5} \log\left|\frac{x^5}{x^5+1}\right| + c.$

Question 19.
tanx√sinx· cosx
Solution:

$$\text{Let } I = \int \frac{\sqrt{\tan x}}{\sin x \cdot \cos x} dx$$

Dividing numerator and denominator by $\cos^2 x$, we get

$$I = \int \frac{\left(\frac{\sqrt{\tan x}}{\cos^2 x}\right)}{\left(\frac{\sin x}{\cos x}\right)} dx$$

$$= \int \frac{\sqrt{\tan x} \cdot \sec^2 x}{\tan x} dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$$

Put $\tan x = t$ $\quad \therefore \sec^2 x \, dx = dt$

$$\therefore I = \int \frac{1}{\sqrt{t}} dt = \int t^{-\frac{1}{2}} dt$$

$$= \frac{t^{\frac{1}{2}}}{1/2} + c = 2\sqrt{t} + c$$

$$= 2\sqrt{\tan x} + c.$$

Question 20.
sec4 x cosec2 x

Solution:

$$\text{Let } I=\int \sec^4 x \operatorname{cosec}^2 x\, dx$$

$=\int \sec^2 x \operatorname{cosec}^2 x \cdot \sec^2 x\, dx$

Put $\tan x = t$ $\quad \therefore\ \sec^2 x\, dx = dt$

Also, $\sec^2 x \operatorname{cosec}^2 x = (1+\tan^2 x)(1+\cot^2 x)$

$=(1+t^2)\left(1+\dfrac{1}{t^2}\right)$

$=(1+t^2)\left(\dfrac{t^2+1}{t^2}\right)=\dfrac{t^4+2t^2+1}{t^2}$

$=t^2+2+\dfrac{1}{t^2}$

$\therefore\ I=\displaystyle\int\left(t^2+2+\frac{1}{t^2}\right)dt$

$=\displaystyle\int t^2\,dt+2\int dt+\int\frac{1}{t^2}\,dt$

$=\dfrac{t^3}{3}+2\text{t}+\dfrac{t^{-1}}{(-1)}+c$

$=\dfrac{1}{3}\tan^3 x+2\tan x-\dfrac{1}{\tan x}+c$

$=\dfrac{1}{3\cot^3 x}+\dfrac{2}{\cot x}-\cot x+c.$