Maharashtra State Board 12th Maths Solutions Chapter 4 Pair of Straight Lines Ex 4.1

Question 1.
Find the combined equation of the following pairs of lines:
(i) $2x + y = 0$ and $3x - y = 0$
Solution:
The combined equation of the lines $2x + y = 0$ and $3x - y = 0$ is
$(2x + y)(3x - y) = 0$
$\therefore 6x^2 - 2xy + 3xy - y^2 = 0$
$\therefore 6x^2 - xy - y^2 = 0$.

(ii) $x + 2y - 1 = 0$ and $x - 3y + 2 = 0$
Solution:
The combined equation of the lines $x + 2y - 1 = 0$ and $x - 3y + 2 = 0$ is
$(x + 2y - 1)(x - 3y + 2) = 0$
$\therefore x^2 - 3xy + 2x + 2xy - 6y^2 + 4y - x + 3y - 2 = 0$
$\therefore x^2 - xy - 6y^2 + x + 7y - 2 = 0$.

(iii) Passing through (2, 3) and parallel to the co-ordinate axes.
Solution:
Equations of the coordinate axes are $x = 0$ and $y = 0$.
$\therefore$ the equations of the lines passing through (2, 3) and parallel to the coordinate axes are $x = 2$ and
i.e. $x - 2 = 0$ and $y - 3 = 0$.
$\therefore$ their combined equation is
$(x - 2)(y - 3) = 0$.
$\therefore xy - 3x - 2y + 6 = 0$.

(iv) Passing through (2, 3) and perpendicular to lines $3x + 2y - 1 = 0$ and $x - 3y + 2 = 0$
Solution:
Let L_1 and L_2 be the lines passing through the point (2, 3) and perpendicular to the lines $3x + 2y - 1 = 0$ and $x - 3y + 2 = 0$ respectively.
Slopes of the lines $3x + 2y - 1 = 0$ and $x - 3y + 2 = 0$ are $\frac{-3}{2}$ and $\frac{-1}{-3}=\frac{1}{3}$ respectively.
$\therefore$ slopes of the lines L_1 and L_2 are $\frac{2}{3}$ and -3 respectively.
Since the lines L_1 and L_2 pass through the point (2, 3), their equations are
$y - 3 = \frac{2}{3}(x - 2)$ and $y - 3 = -3(x - 2)$
$\therefore 3y - 9 = 2x - 4$ and $y - 3 = -3x + 6$
$\therefore 2x - 3y + 5 = 0$ and $3x - y - 9 = 0$
$\therefore$ their combined equation is
$(2x - 3y + 5)(3x + y - 9) = 0$
$\therefore 6x^2 + 2xy - 18x - 9xy - 3y^2 + 27y + 15x + 5y - 45 = 0$
$\therefore 6x^2 - 7xy - 3y^2 - 3x + 32y - 45 = 0$.

(v) Passsing through (-1, 2),one is parallel to $x + 3y - 1 = 0$ and the other is perpendicular to $2x - 3y - 1 = 0$.
Solution:
Let L_1 be the line passing through (-1, 2) and parallel to the line $x + 3y - 1 = 0$ whose slope is $-\frac{1}{3}$.
$\therefore$ slope of the line L_1 is $-\frac{1}{3}$
$\therefore$ equation of the line L_1 is
$y - 2 = -\frac{1}{3}(x + 1)$
$\therefore 3y - 6 = -x - 1$
$\therefore x + 3y - 5 = 0$
Let L_2 be the line passing through (-1, 2) and perpendicular to the line $2x - 3y - 1 = 0$
whose slope is $\frac{-2}{-3}=\frac{2}{3}$.
$\therefore$ slope of the line L_2 is $-\frac{3}{2}$
$\therefore$ equation of the line L_2 is
$y - 2 = -\frac{3}{2}(x + 1)$
$\therefore 2y - 4 = -3x - 3$
$\therefore 3x + 2y - 1 = 0$
Hence, the equations of the required lines are
$x + 3y - 5 = 0$ and $3x + 2y - 1 = 0$
$\therefore$ their combined equation is
$(x + 3y - 5)(3x + 2y - 1) = 0$
$\therefore 3x^2 + 2xy - x + 9xy + 6y^2 - 3y - 15x - 10y + 5 = 0$
$\therefore 3x^2 + 11xy + 6y^2 - 16x - 13y + 5 = 0$

Question 2.
Find the separate equations of the lines represented by following equations:

(i) $3y^2 + 7xy = 0$
Solution:
$3y^2 + 7xy = 0$
$\therefore y(3y + 7x) = 0$
$\therefore$ the separate equations of the lines are $y = 0$ and $7x + 3y = 0$.

(ii) $5x^2 – 9y^2 = 0$
Solution:
$5x^2 – 9y^2 = 0$
$\therefore (\sqrt{5}x)^2 – (3y)^2 = 0$
$\therefore (\sqrt{5}x + 3y)(\sqrt{5}x – 3y) = 0$
$\therefore$ the separate equations of the lines are
$\sqrt{5}x + 3y = 0$ and $\sqrt{5}x – 3y = 0$.

(iii) $x^2 – 4xy = 0$
Solution:
$x^2 – 4xy = 0$
$\therefore x(x – 4y) = 0$
$\therefore$ the separate equations of the lines are $x = 0$ and $x – 4y = 0$

(iv) $3x^2 – 10xy – 8y^2 = 0$
Solution:
$3x^2 – 10xy – 8y^2 = 0$
$\therefore 3x^2 – 12xy + 2xy – 8y^2 = 0$
$\therefore 3x(x – 4y) + 2y(x – 4y) = 0$
$\therefore (x – 4y)(3x +2y) = 0$
$\therefore$ the separate equations of the lines are $x – 4y = 0$ and $3x + 2y = 0$.

(v) $3x^2 – 2\sqrt{3}xy – 3y^2 = 0$
Solution:
$3x^2 – 2\sqrt{3}xy – 3y^2 = 0$
$\therefore 3x^2 – 3\sqrt{3}xy + \sqrt{3}xy – 3y^2 = 0$
$\therefore 3x(x – \sqrt{3}y) + \sqrt{3}y(x – \sqrt{3}y) = 0$
$\therefore (x – \sqrt{3}y)(3x + \sqrt{3}y) = 0$
$\therefore$ the separate equations of the lines are
$\therefore x – \sqrt{3}y = 0$ and $3x + \sqrt{3}y = 0$.

(vi) $x^2 + 2(\text{cosec} \propto)xy + y^2 = 0$
Solution:
$x^2 + 2(\text{cosec} \propto)xy – y^2 = 0$
i.e. $y^2 + 2(\text{cosec}\propto)xy + x^2 = 0$
Dividing by x^2, we get,

$$\left(\frac{y}{x}\right)^2 + 2\operatorname{cosec}\alpha \cdot \left(\frac{y}{x}\right) + 1 = 0$$

$$\therefore \frac{y}{x} = \frac{-2\operatorname{cosec}\alpha \pm \sqrt{4\operatorname{cosec}^2\alpha - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{-2\operatorname{cosec}\alpha \pm 2\sqrt{\operatorname{cosec}^2\alpha - 1}}{2}$$

$$= -\operatorname{cosec}\alpha \pm \cot\alpha$$

$$\therefore \frac{y}{x} = (\cot\alpha - \operatorname{cosec}\alpha) \quad \text{and}$$

$$\frac{y}{x} = -(\operatorname{cosec}\alpha + \cot\alpha)$$

$\therefore$ the separate equations of the lines are
$(\text{cosec} \propto – \cot \propto)x + y = 0$ and $(\text{cosec} \propto + \cot \propto)x + y = 0$.

(vii) $x^2 + 2xy \tan \propto – y^2 = 0$
Solution:

$x^2 + 2xy \tan \propto - y^2 = 0$
Dividind by y^2

$$\left(\frac{x}{y}\right)^2 + 2\left(\frac{x}{y}\right)\tan\alpha - 1 = 0$$

$$\therefore \frac{x}{y} = \frac{-2\tan\alpha \pm \sqrt{4\tan^2\alpha - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{-2\tan\alpha \pm 2\sqrt{\tan^2\alpha - 1}}{2}$$

$$= -\tan\alpha \pm \sec\alpha$$

$$\left(\frac{x}{y}\right) = (\sec\alpha - \tan\alpha) \text{ and}$$

$$\left(\frac{x}{y}\right) = -(\tan\alpha + \sec\alpha)$$

The separate equations of the lines are
$(\sec\propto - \tan\propto)x + y = 0$ and $(\sec\propto + \tan\propto)x - y = 0$

Question 3.
Find the combined equation of a pair of lines passing through the origin and perpendicular to the lines represented by following equations :
(i) $5x^2 - 8xy + 3y^2 = 0$
Solution:
Comparing the equation $5x^2 - 8xy + 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
a = 5, 2h = -8, b = 3
Let m_1 and m_2 be the slopes of the lines represented by $5x^2 - 8xy + 3y^2 = 0$.
$\therefore m_1 + m_2 = -2hb=83$

amd $m_1m_2 = ab=53$ ...(1)
Now required lines are perpendicular to these lines
$\therefore$ their slopes are $-1/m_1$ and $-1/m_2$ Since these lines are passing through the origin, their separate equations are
$y = -1m_1x$ and $y = -1m_2x$
i.e. $m_1y = -x$ and $m_2y = -x$
i.e. $x + m_1y = 0$ and $x + m_2y = 0$
$\therefore$ their combined equation is
$(x + m_1y)(x + m_2y) = 0$
$\therefore x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$
$\therefore x^2 + 83xy + 53y^2 = 0$ ... [By (1)]
$\therefore x^2 + 8xy + 5y83 = 0$

(ii) $5x^2 + 2xy - 3y^2 = 0$
Solution:
Comparing the equation $5x^2 + 2xy - 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
a = 5, 2h = 2, b = -3
Let m_1 and m_2 be the slopes of the lines represented by $5x^2 + 2xy - 3y^2 = 0$
$\therefore m_1 + m_2 = -2hb=-2-3=23$ and $m_1m_2 = ab=5-3$ ..(1)
Now required lines are perpendicular to these lines
$\therefore$ their slopes are $-1m_1$ and $-1m_2$
Since these lines are passing through the origin, their separate equations are
$y = -1\ m_1x$ and $y = -1\ m_2x$
i.e. $m_1y = -x$ amd $m_2y = -x$
i.e. $x + m_1y = 0$ and $x + m_2y = 0$
$\therefore$ their combined equation is
$\therefore (x + m_1y)(x + m_2y) = 0$
$x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$
$\therefore x^2 + 23xy - 53y = 0$ ...[By (1)]
$\therefore 3x^2 + 2xy - 5y^2 = 0$

(iii) $xy + y^2 = 0$
Solution:
Comparing the equation $xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

a = 0, 2h = 1, b = 1
Let m_1 and m_2 be the slopes of the lines represented by $xy + y^2 = 0$

$$\left.\begin{aligned}\therefore\ m_1 + m_2 &= \frac{-2h}{b} = \frac{-1}{1} = -1\\ \text{and } m_1 m_2 &= \frac{a}{b} = \frac{0}{1} = 0\end{aligned}\right\}\quad \dots (1)$$

Now required lines are perpendicular to these lines
∴ their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$.
Since these lines are passing through the origin, their separate equations are
$y = -\frac{1}{m_1}x$ and $y = -\frac{1}{m_2}x$
i.e. $m_1 y = -x$ and $m_2 y = -x$
i.e. $x + m_1 y = 0$ and $x + m_2 y = 0$
∴ their combined equation is
$(x + m_1 y)(x + m_2 y) = 0$
∴ $x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$
∴ $x^2 - xy = 0.y^2 = 0$ … [By (1)]
∴ $x^2 - xy = 0$.
Alternative Method :
Consider $xy + y^2 = 0$
∴ $y(x + y) = 0$
∴ separate equations of the lines are y = 0 and
$3x^2 + 8xy + 5y^2 = 0$.
x + y = 0.
Let m_1 and m_2 be the slopes of these lines.
Then $m_1 = 0$ and $m_2 = -1$
Now, required lines are perpendicular to these lines.
∴ their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$

Since, $m_1 = 0$, $-\frac{1}{m_1}$ does not exist.

Also, $m_2 = -1$, $-\frac{1}{m_2} = 1$

Since these lines are passing through the origin, their separate equations are x = 0 and y = x,
i.e. x – y = 0
∴ their combined equation is
x(x – y) = 0
$x^2 - xy = 0$.

(iv) $3x^2 - 4xy = 0$
Solution:
Consider $3x^2 - 4xy = 0$
∴ x(3x – 4y) = 0
∴ separate equations of the lines are x = 0 and 3x – 4y = 0.
Let m_1 and m_2 be the slopes of these lines.
Then m_1 does not exist and and $m_1 = \frac{3}{4}$.
Now, required lines are perpendicular to these lines.
∴ their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$.

Since m_1 does not exist, $-\frac{1}{m_1} = 0$

Also $m_2 = \frac{3}{4}, -\frac{1}{m_2} = -\frac{4}{3}$

Since these lines are passing through the origin, their separate equations are y = 0 and $y = -\frac{4}{3}x$,
i.e. 4x + 3y = 0
∴ their combined equation is
y(4x + 3y) = 0
∴ $4xy + 3y^2 = 0$.

Question 4.
Find k if,
(i) the sum of the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$ is twice their product.
Solution:
Comparing the equation $x^2 + kxy - 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get, a = 1, 2h = k, b = -3.
Let m_1 and m_2 be the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$.
∴ $m_1 + m_2 = -\frac{2h}{b} = -\frac{k}{(-3)} = \frac{k}{3}$

and $m_1 m_2 = \frac{a}{b} = \frac{1}{(-3)} = -\frac{1}{3}$

Now, $m_1 + m_2 = 2(m_1 m_2)$ ..(Given)
∴ $\frac{k}{3} = 2\left(-\frac{1}{3}\right)$ ∴ k = -2

(ii) slopes of lines represent by $3x^2 + kxy - y^2 = 0$ differ by 4.
Solution:
(ii) Comparing the equation $3x^2 + kxy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get, a = 3, 2h = k, b = -1.
Let m_1 and m_2 be the slopes of the lines represented by $3x^2 + kxy - y^2 = 0$.

$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{-k}{-1} = k$

and $m_{12} = \frac{a}{b} = \frac{3}{-1} = -3$

$\therefore (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$
$= k^2 - 4(-3)$
$= k^2 + 12$ ... (1)
But $|m_1 - m_2| = 4$
$\therefore (m_1 - m_2)^2 = 16$ ... (2)
$\therefore$ from (1) and (2), $k^2 + 12 = 16$
$\therefore k^2 = 4 \therefore k = \pm 2$.

(iii) slope of one of the lines given by $kx^2 + 4xy - y^2 = 0$ exceeds the slope of the other by 8.
Solution:
Comparing the equation $kx^2 + 4xy - y^2 = 0$ with ${}^2 + 2hxy + by^2 = 0$, we get, a = k, 2h = 4, b = -1. Let m_1 and m_2 be the slopes of the lines represented by $kx^2 + 4xy - y^2 = 0$.

$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{-4}{-1} = 4$

and $m_1m_2 = \frac{a}{b} = \frac{k}{-1} = -k$

We are given that $m_2 = m_1 + 8$
$m_1 + m_1 + 8 = 4$
$\therefore 2m_1 = -4 \therefore m_1 = -2$ ... (1)
Also, $m_1(m_1 + 8) = -k$
$(-2)(-2 + 8) = -k$ ... [By(1)]
$\therefore (-2)(6) = -k$
$\therefore -12 = -k \therefore k = 12$.

Question 5.
Find the condition that :
(i) the line $4x + 5y = 0$ coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$.
Solution:
The auxiliary equation of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$.
Given that $4x + 5y = 0$ is one of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

The slope of the line $4x + 5y = 0$ is $-\frac{4}{5}$.

$\therefore m = -\frac{4}{5}$ is a root of the auxiliary equation $bm^2 + 2hm + a = 0$.

$\therefore b\left(-\frac{4}{5}\right)^2 + 2h\left(-\frac{4}{5}\right) + a = 0$

$\therefore \frac{16b}{25} - \frac{8h}{5} + a = 0$

$\therefore 16b - 40h + 25a = 0$
$\therefore 25a + 16b = 40k$.
This is the required condition.

(ii) the line $3x + y = 0$ may be perpendicular to one of the lines given by $ax^2 + 2hxy + by^2 = 0$.
Solution:
The auxiliary equation of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$.
Since one line is perpendicular to the line $3x + y = 0$

whose slope is $-\frac{3}{1} = -3$

$\therefore$ slope of that line = $m = \frac{1}{3}$
$\therefore m = \frac{1}{3}$is the root of the auxiliary equation $bm^2 + 2hm + a = 0$.

$\therefore b\left(\frac{1}{3}\right)^2 + 2h\left(\frac{1}{3}\right) + a = 0$

$\therefore \frac{b}{9} + \frac{2h}{3} + a = 0$

$\therefore b + 6h + 9a = 0$
$\therefore 9a + b + 6h = 0$
This is the required condition.

Question 6.
If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$ then show that $ap^2 + 2hpq + bq^2 = 0$.
Solution:
To prove $ap^2 + 2hpq + bq^2 = 0$.
Let the slope of the pair of straight lines $ax^2 + 2hxy + by^2 = 0$ be m_1 and m_2
Then, $m_1 + m_2 = \frac{-2h}{b}$ and $m_1m_2 = \frac{a}{b}$
Slope of the line $px + qy = 0$ is $\frac{-p}{q}$
But one of the lines of $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$

$$\Rightarrow m_1 = \frac{q}{p}$$

$$\text{Now, } m_1 + m_2 = \frac{-2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

$$\Rightarrow \frac{q}{p} + m_2 = \frac{-2h}{b} \text{ and } \left(\frac{q}{p}\right)m_2 = \frac{a}{b}$$

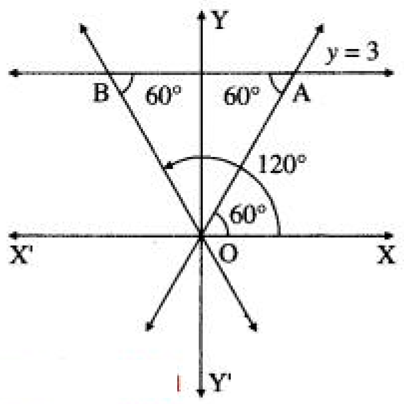
$$\Rightarrow \frac{q}{p} + m_2 = \frac{-2h}{b} \text{ and } m_2 = \frac{ap}{bq}$$

$$\Rightarrow \frac{q}{p} + \frac{ap}{bq} = \frac{-2h}{b}$$

$$\Rightarrow \frac{bq^2 + ap^2}{pq} = -2h$$

$\Rightarrow bq^2 + ap^2 = -2hpq$
$\Rightarrow ap^2 + 2hpq + bq^2 = 0$

Question 7.
Find the combined equation of the pair of lines passing through the origin and making an equilateral triangle with the line y = 3.
Solution:
Let OA and OB be the lines through the origin making.an angle of 60° with the line y = 3.
∴ OA and OB make an angle of 60° and 120° with the positive direction of X-axis.
∴ slope of OA = tan60° = $\sqrt{3}$
∴ equation of the line OA is
y = $\sqrt{3}$ x, i.e. $\sqrt{3}$ x – y = 0



Slope of OB = tan 120° = tan (180° – 60°)
= -tan 60°= $-\sqrt{3}$
∴ equation of the line OB is
y = $-\sqrt{3}$ x, i.e. $\sqrt{3}$ x + y = 0
∴ required joint equation of the lines is
($\sqrt{3}$ x – y)($\sqrt{3}$ x + y) = 0
i.e. $3x^2 – y^2 = 0$.

Question 8.
If slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is four times the other then show that $16h^2 = 25ab$.
Solution:
Let m_1 and m_2 be the slopes of the lines given by $ax^2 + 2hxy + by^2 = 0$.
∴ $m_1 + m_2 = \frac{-2h}{b}$
and $m_1m_2 = \frac{a}{b}$
We are given that $m_2 = 4m_1$

$\therefore\ m_1 + 4m_1 = -\frac{2h}{b}$

$\therefore\ 5m_1 = \frac{-2h}{b}$

$\therefore\ m_1 = -\frac{2h}{5b}$ ... (1)

Also, $m_1(4m_1) = \frac{a}{b}$

$\therefore\ 4m_1{}^2 = \frac{a}{b}$

$\therefore\ m_1{}^2 = \frac{a}{4b}$

$\therefore\ \left(\frac{-2h}{5b}\right)^2 = \frac{a}{4b}$ ... [By (1)]

$\therefore\ \frac{4h^2}{25b^2} = \frac{a}{4b}$

$\therefore\ \frac{4h^2}{25b} = \frac{a}{4}$, as $b \neq 0$

∴ $16h^2 = 25ab$
This is the required condition.

Question 9.
If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ bisects an angle between co-ordinate axes then show that $(a + b)^2 = 4h^2$.
Solution:
The auxiliary equation of the lines given by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$.
Since one of the line bisects an angle between the coordinate axes, that line makes an angle of 45° or 135° with the positive direction of X-axis.
∴ slope of that line = tan45° or tan 135°
∴ m = tan45° = 1
or m = tan 135° = tan (180° – 45°)
= -tan 45°= -1
∴ m = ±1 are the roots of the auxiliary equation $bm^2 + 2hm + a = 0$.
∴ $b(\pm1)^2 + 2h(\pm1) + a = 0$
∴ b ± 2h + a = 0
∴ a + b = ±2h
∴ $(a + b)^2 = 4h^2$
This is the required condition.

Maharashtra State Board 12th Maths Solutions Chapter 4 Pair of Straight Lines Ex 4.2

Question 1.
Show that lines represented by $3x^2 – 4xy – 3y^2 = 0$ are perpendicular to each other.
Solution:
Comparing the equation $3x^2 – 4xy – 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get, a = 3, 2h = -4, b = -3 Since a + b = 3 + (-3) = 0, the lines represented by $3x^2 – 4xy – 3y^2 = 0$ are perpendicular to each other.

Question 2.
Show that lines represented by $x^2 + 6xy + gy^2= 0$ are coincident.
The question is modified.
Show that lines represented by $x^2 + 6xy + 9y^2= 0$ are coincident.

Solution:
Comparing the equation $x^2 + 6xy + 9y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
a = 1, 2h = 6, i.e. h = 3 and b = 9
Since $h^2 – ab = (3)^2 – 1(9)$
= 9 – 9 = 0, .
the lines represented by $x^2 + 6xy + 9y^2 = 0$ are coincident.

Question 3.
Find the value of k if lines represented by $kx^2 + 4xy – 4y^2 = 0$ are perpendicular to each other.
Solution:
Comparing the equation $kx^2 + 4xy – 4y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
a = k, 2h = 4, b = -4
Since lines represented by $kx^2 + 4xy – 4y^2 = 0$ are perpendicular to each other,
a + b = 0
∴ k – 4 = 0 ∴ k = 4.

Question 4.
Find the measure of the acute angle between the lines represented by:
(i) $3x^2 – 4\sqrt{3}xy + 3y^2 = 0$
Solution:
Comparing the equation $3x^2 – 4\sqrt{3}xy + 3y^2 = 0$ with
$ax^2 + 2hxy + by^2 = 0$, we get,
a = 3, 2h = $-4\sqrt{3}$, i.e. h = $-2\sqrt{3}$ and b = 3
Let θ be the acute angle between the lines.

$$\therefore\ \tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{(-2\sqrt{3})^2-3(3)}}{3+3}\right|$$

$$= \left|\frac{2\sqrt{12-9}}{6}\right| = \left|\frac{2\sqrt{3}}{6}\right|$$

$$\therefore\ \tan\theta = \frac{1}{\sqrt{3}} = \tan 30^\circ$$

∴ θ = 30°.

(ii) $4x^2 + 5xy + y^2 = 0$
Solution:
Comparing the equation $4x^2 + 5xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
a = 4, 2h = 5, i.e. h = $\frac{5}{2}$ and b = 1.
Let θ be the acute angle between the lines.

$$\therefore\ \tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{\left(\frac{5}{2}\right)^2-4(1)}}{4+1}\right|$$

$$= \left|\frac{2\sqrt{\frac{25}{4}-4}}{5}\right| = \left|\frac{2\times\frac{3}{2}}{5}\right|$$

$$\therefore\ \tan\theta = \frac{3}{5}$$

$$\therefore\ \theta = \tan^{-1}\left(\frac{3}{5}\right).$$

(iii) $2x^2 + 7xy + 3y^2 = 0$
Solution:

Comparing the equation
$2x^2 + 7xy + 3y^2 = 0$ with
$ax^2 + 2hxy + by^2 = 0$, we get,
$a = 2$, $2h = 7$ i.e. $h = \frac{7}{2}$ and $b = 3$
Let θ be the acute angle between the lines.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$$= \left|\frac{2\sqrt{\left(\frac{7}{2}\right)^2 - 2(3)}}{2+3}\right|$$

$$= \left|\frac{2\sqrt{\left(\frac{49}{4}\right) - 6}}{5}\right|$$

$$= \left|\frac{2\sqrt{\left(\frac{49-24}{4}\right)}}{5}\right|$$

$$= \left|\frac{2\sqrt{\left(\frac{25}{4}\right)}}{5}\right|$$

$$= \frac{2 \times \left(\frac{5}{2}\right)}{5}$$

$$= \frac{5}{5}$$

$\tan\theta = 1$
$\therefore \theta = \tan 1 = 45°$
$\therefore \theta = 45°$

(iv) $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$
Solution:
Comparing the equation
$(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$, with
$Ax^2 + 2Hxy + By^2 = 0$, we have,
$A = a^2 - 3b^2$, $H = 4ab$, $B = b^2 - 3a^2$.
$\therefore H^2 - AB = 16a^2b^2 - (a^2 - 3b^2)(b^2 - 3a^2)$
$= 16a^2b^2 + (a^2 - 3b^2)(3a^2 - b^2)$
$= 16a^2b^2 + 3a^4 - 10a^2b^2 + 3b^4$
$= 3a^4 + 6a^2b^2 + 3b^4$
$= 3(a^4 + 2a^2b^2 + b^4)$
$= 3(a^2 + b^2)^2$
$\therefore \sqrt{H^2 - AB} = \sqrt{3}(a^2 + b^2)$
Also, $A + B = (a^2 - 3b^2) + (b^2 - 3a^2)$
$= -2(a^2 + b^2)$
If θ is the acute angle between the lines, then
$\tan\theta = \left|\frac{2\sqrt{H^2 - AB}}{A + B}\right| = \left|\frac{2\sqrt{3}(a^2+b^2)}{-2(a^2+b^2)}\right|$
$= \sqrt{3} = \tan 60°$
$\therefore \theta = 60°$

Question 5.
Find the combined equation of lines passing through the origin each of which making an angle of 30° with the line $3x + 2y - 11 = 0$
Solution:
The slope of the line $3x + 2y - 11 = 0$ is $m_1 = -\frac{3}{2}$.
Let m be the slope of one of the lines making an angle of 30° with the line $3x + 2y - 11 = 0$.
The angle between the lines having slopes m and m1 is 30°.

$$\therefore \tan 30° = \left|\frac{m - m_1}{1 + m \cdot m_1}\right|, \text{ where } \tan 30° = \frac{1}{\sqrt{3}}$$

$$\therefore \frac{1}{\sqrt{3}} = \left|\frac{m - \left(-\frac{3}{2}\right)}{1 + m\left(-\frac{3}{2}\right)}\right|$$

$$\therefore \frac{1}{\sqrt{3}} = \left|\frac{2m + 3}{2 - 3m}\right|$$

On squaring both sides, we get,

$\frac{1}{3} = \frac{(2m+3)^2}{(2-3m)^2}$

$\therefore (2 - 3m)^2 = 3(2m + 3)^2$

$\therefore 4 - 12m + 9m^2 = 3(4m^2 + 12m + 9)$

$\therefore 4 - 12m + 9m^2 = 12m^2 + 36m + 27$

$3m^2 + 48m + 23 = 0$

This is the auxiliary equation of the two lines and their joint equation is obtained by putting $m = \frac{y}{x}$.

$\therefore$ the combined equation of the two lines is

$3\left(\frac{y}{x}\right)^2 + 48\left(\frac{y}{x}\right) + 23 = 0$

$\therefore \frac{3y^2}{x^2} + \frac{48y}{x} + 23 = 0$

$\therefore 3y^2 + 48xy + 23x^2 = 0$

$\therefore 23x^2 + 48xy + 3y^2 = 0.$

Question 6.
If the angle between lines represented by $ax^2 + 2hxy + by^2 = 0$ is equal to the angle between lines represented by $2x^2 - 5xy + 3y^2 = 0$ then show that $100(h^2 - ab) = (a + b)^2$.
Solution:
The acute angle θ between the lines $ax^2 + 2hxy + by^2 = 0$ is given by

$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$ ..(1)

Comparing the equation $2x^2 - 5xy + 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 2$, $2h = -5$, i.e. $h = -\frac{5}{2}$ and $b = 3$

Let $\propto$ be the acute angle between the lines $2x^2 - 5xy + 3y^2 = 0$.

$$\therefore \tan\alpha = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$$= \left|\frac{2\sqrt{\left(\frac{5}{2}\right)^2 - 2(3)}}{2 + 3}\right|$$

$$= \left|\frac{2\sqrt{\frac{25}{4} - 6}}{5}\right| = \left|\frac{2 \times \frac{1}{2}}{5}\right|$$

$$\therefore \tan\alpha = \frac{1}{5} \qquad \ldots (2)$$

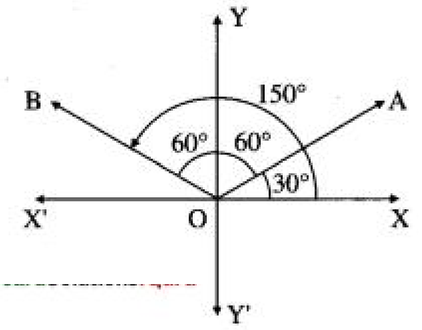
If $\theta = \alpha$, then $\tan\theta = \tan\alpha$

$$\therefore \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right| = \frac{1}{5} \qquad \ldots \text{[By (1) and (2)]}$$

$$\therefore \frac{4(h^2 - ab)}{(a + b)^2} = \frac{1}{25} \quad \therefore 100(h^2 - ab) = (a + b)^2$$

This is the required condition.

Question 7.
Find the combined equation of lines passing through the origin and each of which making angle 60° with the Y- axis.
Solution:



Let OA and OB be the lines through the origin making an angle of 60° with the Y-axis.
Then OA and OB make an angle of 30° and 150° with the positive direction of X-axis.
∴ slope of OA = tan 30° = $\frac{1}{\sqrt{3}}$
∴ equation of the line OA is
y = $\frac{1}{\sqrt{3}}$ = x, i.e. x – $\sqrt{3}$y = 0
Slope of OB = tan 150° = tan (180° – 30°)
= tan 30° = $-\frac{1}{\sqrt{3}}$
∴ equation of the line OB is
y = $-\frac{1}{\sqrt{3}}$x, i.e. x + $\sqrt{3}$ y = 0
∴ required combined equation is
(x – $\sqrt{3}$y)(x + $\sqrt{3}$y) = 0
i.e. $x^2 - 3y^2 = 0$.

Maharashtra State Board 12th Maths Solutions Chapter 4 Pair of Straight Lines Ex 4.3

Question 1.
Find the joint equation of the pair of lines:
(i) Through the point (2, -1) and parallel to lines represented by $2x^2 + 3xy - 9y^2 = 0$
Solution:
The combined equation of the given lines is
$2x^2 + 3xy - 9y^2 = 0$
i.e. $2x^2 + 6xy - 3xy - 9y^2 = 0$
i.e. 2x(x + 3y) – 3y(x + 3y) = 0
i.e. (x + 3y)(2x – 3y) = 0
∴ their separate equations are
x + 3y = 0 and 2x – 3y = 0
∴ their slopes are $m_1 = -\frac{1}{3}$ and $m_2 = \frac{-2}{-3} = \frac{2}{3}$.
The slopes of the lines parallel to these lines are m_1 and m_2, i.e. $-\frac{1}{3}$ and $\frac{2}{3}$.
∴ the equations of the lines with these slopes and through the point (2, -1) are
y + 1 = $-\frac{1}{3}$ (x – 2) and y + 1 = $\frac{2}{3}$(x – 2)
i.e. 3y + 3= -x + 2 and 3y + 3 = 2x – 4
i.e. x + 3y + 1 = 0 and 2x – 3y – 7 = 0
∴ the joint equation of these lines is
(x + 3y + 1)(2x – 3y – 7) = 0
∴ $2x^2 - 3xy - 7x + 6xy - 9y^2 - 21y + 2x - 3y - 7 = 0$
∴ $2x^2 + 3xy - 9y^2 - 5x - 24y - 7 = 0$.

(ii) Through the point (2, -3) and parallel to lines represented by $x^2 + xy - y^2 = 0$
Solution:
Comparing the equation
$x^2 + xy - y^2 = 0$ … (1)
with $ax^2 + 2hxy + by^2 = 0$, we get,
a = 1, 2h = 1, b = -1
Let m_1 and m_2 be the slopes of the lines represented by (1).

$$\left.\begin{aligned}\text{Then } m_1 + m_2 &= -\frac{2h}{b} = \frac{-1}{-1} = 1\\ \text{and } m_1m_2 &= \frac{a}{b} = \frac{1}{-1} = -1\end{aligned}\right\} \quad \ldots (2)$$

The slopes of the lines parallel to these lines are m_1 and m_2.
∴ the equations of the lines with these slopes and through the point (2, -3) are
$y + 3 = m_1(x - 2)$ and $y + 3 = m_2(x - 2)$
i.e. $m_1(x - 2) - (y + 3) = 0$ and $m_2(x - 2) - (y + 3) = 0$
∴ the joint equation of these lines is
$[m_1(x - 2) - (y + 3)][m_2(x - 2) - (y + 3)] = 0$
∴ $m_1m_2(x - 2)^2 - m_1(x - 2)(y + 3) - m_2(x - 2)(y + 3) + (y + 3)^2 = o$
∴ $m_1m_2(x - 2)^2 - (m_1 + m_2)(x - 2)(y + 3) + (y + 3)^3 = 0$
∴ $-(x - 2)^2 - (x - 2)(y + 3) + (y + 3)^2 = 0$ …… [By (2)]
∴ $(x - 2)^2 + (x - 2)(y + 3) - (y + 3)^2 = 0$
∴ $(x^2 - 4x + 4) + (xy + 3x - 2y - 6) - (y^2 + 6y + 9) = 0$
∴ $x^2 - 4x + 4 + xy + 3x - 2y - 6 - y^2 - 6y - 9 = 0$
∴ $x^2 + xy - y^2 - x - 8y - 11 = 0$.

Question 2.
Show that equation $x^2 + 2xy+ 2y^2 + 2x + 2y + 1 = 0$ does not represent a pair of lines.
Solution:
Comparing the equation
$x^2 + 2xy + 2y^2 + 2x + 2y + 1 = 0$ with
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,
a = 1, h = 1, b = 2, g = 1, f = 1, c = 1.
The given equation represents a pair of lines, if
D = ∣∣∣∣ ahghbfgfc ∣∣∣∣ = 0 and $h^2 - ab \geq 0$
Now, D = ∣∣∣∣ ahghbfgfc ∣∣∣∣ = ∣∣∣∣ 111121111 ∣∣∣∣
= 1 (2 – 1) – 1(1 – 1) + 1 (1 – 2)
= 1 – 0 – 1 = 0
and $h^2 - ab = (1)^2 - 1(2) = -1 < 0$
∴ given equation does not represent a pair of lines.

Question 3.
Show that equation $2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$ represents a pair of lines.
Solution:
Comparing the equation
$2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$
with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,
a = 2, h = −12, b = -3, g = -3, f = 192, c = -20.
∴ D = ∣∣∣∣ ahghbfgfc ∣∣∣∣ = ∣∣∣∣∣ 2−12−3−12−3192−3192−20 ∣∣∣∣∣
Taking 12 common from each row, we get,
D = 18 ∣∣∣∣ 4−1−6−1−619−619−40 ∣∣∣∣
= 18[4(240 – 361) + 1(40 + 114) – 6(-19 – 36)]
= 18[4(-121) + 154 – 6(-55)]
= 118[4(-11) + 14 – 6(-5)]
= 18(-44 + 14 + 30) = 0
Also $h^2 - ab$ = (−12)2 – 2(-3) = 14 + 6 = 254 > 0
∴ the given equation represents a pair of lines.

Question 4.
Show the equation $2x^2 + xy - y^2 + x + 4y - 3 = 0$ represents a pair of lines. Also find the acute angle between them.
Solution:
Comparing the equation
$2x^2 + xy — y^2 + x + 4y — 3 = 0$ with
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c — 0$, we get,
a = 2, h = 12, b = -1, g = 12, f = 2, c = – 3.
∴ D = ∣∣∣∣ ahghbfgfc ∣∣∣∣ = ∣∣∣∣∣ 2121212−12122−3 ∣∣∣∣∣
Taking 12 common from each row, we get,
D = 18 ∣∣∣∣ 4111−2414−6 ∣∣∣∣
= 18[4(12 —16) — 1(—6 — 4) + 1(4 + 2)]
= 18[4(– 4) – 1(-10) + 1(6)]

= 18(—16 + 10 + 6) = 0

Also, $h^2 - ab = (12)^2 - 2(-1) = 14 + 2 = 94 > 0$

∴ the given equation represents a pair of lines. Let θ be the acute angle between the lines

∴ tan θ = $\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|$

$$= \left|\frac{2\sqrt{\left(\frac{1}{2}\right)^2 - 2(-1)}}{2-1}\right|$$

$$= \left|\frac{2\sqrt{\frac{1}{4}+2}}{1}\right| = 2 \times \frac{3}{2} = 3$$

$$\therefore\ \theta = \tan^{-1}(3).$$

Question 5.
Find the separate equation of the lines represented by the following equations :
(i) $(x - 2)^2 - 3(x - 2)(y + 1) + 2(y + 1)^2 = 0$
Solution:
$(x - 2)^2 - 3(x - 2)(y + 1) + 2(y + 1)^2 = 0$
∴ $(x - 2)^2 - 2(x - 2)(y + 1) - (x - 2)(y + 1) + 2(y + 1)^2 = 0$
∴ $(x - 2)[(x - 2) - 2(y + 1)] - (y + 1)[(x - 2) - 2(y + 1)] = 0$
∴ $(x - 2)(x - 2 - 2y - 2) - (y + 1)(x - 2 - 2y - 2) = 0$
∴ $(x - 2)(x - 2y - 4) - (y + 1)(x - 2y - 4) = 0$
∴ $(x - 2y - 4)(x - 2 - y - 1) = 0$
∴ $(x - 2y - 4)(x - y - 3) = 0$
∴ the separate equations of the lines are
$x - 2y - 4 = 0$ and $x - y - 3 = 0$.
Alternative Method :
$(x - 2)^2 - 3(x - 2)(y + 1) + 2(y + 1)^2 = 0$ ... (1)
Put $x - 2 = X$ and $y + 1 = Y$
∴ (1) becomes,
$X^2 - 3XY + 2Y^2 = 0$
∴ $X^2 - 2XY - XY + 2Y^2 = 0$
∴ $X(X - 2Y) - Y(X - 2Y) = 0$
∴ $(X - 2Y)(X - Y) = 0$
∴ the separate equations of the lines are
∴ $X - 2Y = 0$ and $X - Y = 0$
∴ $(x - 2) - 2(y + 1) = 0$ and $(x - 2) - (y + 1) = 0$
∴ $x - 2y - 4 = 0$ and $x - y - 3 = 0$.

(ii) $10(x + 1)^2 + (x + 1)(y - 2) - 3(y - 2)^2 = 0$
Solution:
$10(x + 1)^2 + (x + 1)(y - 2) - 3(y - 2)^2 = 0$ ...(1)
Put $x + 1 = X$ and $y - 2 = Y$
∴ (1) becomes
$10x^2 + xy - 3y^2 = 0$
$10x^2 + 6xy - 5xy - 3y^2 = 0$
$2x(5x + 3y) - y(5x + 3y) = 0$
$(2x - y)(5x + 3y) = 0$
$5x + 3y = 0$ and $2x - y = 0$
$5x + 3y = 0$
$5(x + 1) + 3(y - 2) = 0$
$5x + 5 + 3y - 6 = 0$
∴ $5x + 3y - 1 = 0$
$2x - y = 0$
$2(x + 1) - (y - 2) = 0$
$2x + 2 - y + 2 = 0$
∴ $2x - y + 4 = 0$

Question 6.
Find the value of k if the following equations represent a pair of lines :
(i) $3x^2 + 10xy + 3y^2 + 16y + k = 0$
Solution:
Comparing the given equation with
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,
we get, a = 3, h = 5, b = 3, g = 0, f= 8, c = k.
Now, given equation represents a pair of lines.
∴ $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
∴ $(3)(3)(k) + 2(8)(0)(5) - 3(8)^2 - 3(0)^2 - k(5)^2 = 0$

$\therefore$ 9k + 0 – 192 – 0 – 25k = 0
$\therefore$ -16k – 192 = 0
$\therefore$ – 16k = 192
$\therefore$ k= -12.

(ii) kxy + 10x + 6y + 4 = 0
Solution:
Comparing the given equation with
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,
we get, a = 0, h = $\frac{k}{2}$, b = 0, g = 5, f = 3, c = 4
Now, given equation represents a pair of lines.
$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
$\therefore (0)(0)(4) + 2(3)(5)\left(\frac{k}{2}\right) - 0(3)^2 - 0(5)^2 - 4\left(\frac{k}{2}\right)^2 = 0$
$\therefore 0 + 15k - 0 - 0 - k^2 = 0$
$\therefore 15k - k^2 = 0$
$\therefore$ -k(k – 15) = 0
$\therefore$ k = 0 or k = 15.
If k = 0, then the given equation becomes
10x + 6y + 4 = 0 which does not represent a pair of lines.
$\therefore k \neq o$
Hence, k = 15.

(iii) $x^2 + 3xy + 2y^2 + x - y + k = 0$
Solution:
Comparing the given equation with
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$,
we get, a = 1, h = $\frac{3}{2}$, b = 2, g = $\frac{1}{2}$, f= $-\frac{1}{2}$, c = k.
Now, given equation represents a pair of lines.

$$\therefore \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\text{i.e.} \begin{vmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ \frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & k \end{vmatrix} = 0$$

Taking out $\frac{1}{2}$ common from each row, we get,

$$\frac{1}{8}\begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & -1 \\ 1 & -1 & 2k \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & -1 \\ 1 & -1 & 2k \end{vmatrix} = 0$$

$\therefore$ 2(8k – 1) – 3(6k + 1) + 1(-3 – 4) = 0
$\therefore$ 16k – 2 – 18k – 3 – 7 = 0
$\therefore$ -2k – 12 = 0
$\therefore$ -2k = 12 $\therefore$ k = -6.

Question 7.
Find p and q if the equation $px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$ represents a pair of perpendicular lines.
Solution:
The given equation represents a pair of lines perpendicular to each other
$\therefore$ (coefficient of x^2) + (coefficient of y^2) = 0
$\therefore$ p + 3 = 0 p = -3
With this value of p, the given equation is
$-3x^2 - 8xy + 3y^2 + 14x + 2y + q = 0$.
Comparing this equation with
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we have,
a = -3, h = -4, b = 3, g = 7, f = 1 and c = q.

D = | | | | ahghbfgfc | | | | = | | | | -3-47-43171q | | | |
= -3(3q – 1) + 4(-4q – 7) + 7(-4 – 21)
= -9q + 3 – 16q – 28 – 175
= -25q – 200= -25(q + 8)
Since the given equation represents a pair of lines, D = 0
∴ -25(q + 8) = 0 ∴ q= -8.
Hence, p = -3 and q = -8.

Question 8.
Find p and q if the equation $2x^2 + 8xy + py^2 + qx + 2y – 15 = 0$ represents a pair of parallel lines.
Solution:
The given equation is
$2x^2 + 8xy + py^2 + qx + 2y – 15 = 0$
Comparing it with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,
a = 2, h = 4, b = p, g = $\frac{q}{2}$, f = 1, c = – 15
Since the lines are parallel, $h^2 = ab$
∴ $(4)^2 = 2p$ ∴ P = 8
Since the given equation represents a pair of lines

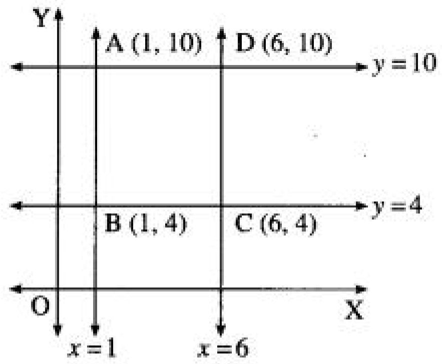
$$D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0, \text{ where } b=p=8$$

$$\text{i.e.} \begin{vmatrix} 2 & 4 & q/2 \\ 4 & 8 & 1 \\ q/2 & 1 & -15 \end{vmatrix}=0$$

$$\text{i.e, } 2(-120-1)-4\left(-60-\frac{q}{2}\right)+\frac{q}{2}(4-4q)=0$$

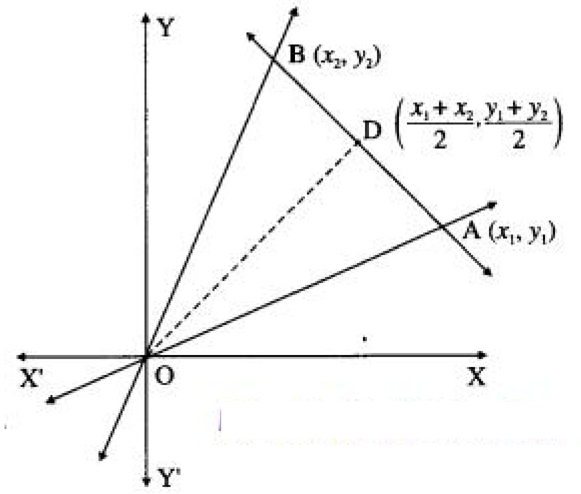
i.e. $– 242 + 240 + 2q + 2q – 2q^2 = 0$
i.e. $-2q^2 + 4q – 2 = 0$
i.e. $q^2 – 2q + 1 = 0$
i.e. $(q – 1)^2 = 0$ ∴ q – 1 = 0 ∴ q = 1.
Hence, p = 8 and q = 1.

Question 9.
Equations of pairs of opposite sides of a parallelogram are $x^2 – 7x+ 6 = 0$ and $y^2 – 14y + 40 = 0$. Find the joint equation of its diagonals.
Solution:
Let ABCD be the parallelogram such that the combined equation of sides AB and CD is $x^2 – 7x + 6 = 0$ and the combined equation of sides BC and AD is $y^2 – 14y + 40 = 0$.
The separate equations of the lines represented by $x^2 – 7x + 6 = 0$, i.e. (x – 1)(x – 6) = 0 are x – 1 = 0 and x – 6 = 0.
Let equation of the side AB be x – 1 = 0 and equation of side CD be x – 6 = 0.
The separate equations of the lines represented by $y^2 – 14y + 40 = 0$, i.e. (y – 4)(y – 10) = 0 are y – 4 = 0 and y – 10 = 0.
Let equation of the side BC be y – 4 = 0 and equation of side AD be y – 10 = 0.



Coordinates of the vertices of the parallelogram are A(1, 10), B(1, 4), C(6, 4) and D(6, 10).
∴ equation of the diagonal AC is
y-10x-1 = 10-41-6 = 6-5
∴ -5y + 50 = 6x – 6
∴ 6x + 5y – 56 = 0
and equation of the diagonal BD is
y-4x-1 = 4-101-6 = -6-5 = 65
∴ 5y – 20 = 6x – 6
∴ 6x – 5y + 14 = 0
Hence, the equations of the diagonals are 6x + 5y – 56 = 0 and 6x – 5y + 14 = 0.
∴ the joint equation of the diagonals is (6x + 5y – 56)(6x – 5y + 14) = 0
∴ $36x^2 – 30xy + 84x + 30xy – 25y^2 + 70y – 336x + 280y – 784 = 0$
∴ $36x^2 – 25y^2 – 252x + 350y – 784 = 0$.

Question 10.
ΔOAB is formed by lines $x^2 – 4xy + y^2 = 0$ and the line 2x + 3y – 1 = 0. Find the equation of the median of the triangle drawn from O.
Solution:



Let D be the midpoint of seg AB where A is (x_1, y_1) and B is (x_2, y_2).
Then D has coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
The joint (combined) equation of the lines OA and OB is $x^2 - 4xy + y^2 = 0$ and the equation of the line AB is $2x + 3y - 1 = 0$.
$\therefore$ points A and B satisfy the equations $2x + 3y - 1 = 0$
and $x^2 - 4xy + y^2 = 0$ simultaneously.
We eliminate x from the above equations, i.e.,
put $x = \frac{1-3y}{2}$ in the equation $x^2 - 4xy + y^2 = 0$, we get,
$\therefore \left(\frac{1-3y}{2}\right)^2 - 4\left(\frac{1-3y}{2}\right)y + y^2 = 0$
$\therefore (1 - 3y)^2 - 8(1 - 3y)y + 4y^2 = 0$
$\therefore 1 - 6y + 9y^2 - 8y + 24y^2 + 4y^2 = 0$
$\therefore 37y^2 - 14y + 1 = 0$
The roots y_1 and y_2 of the above quadratic equation are the y-coordinates of the points A and B.
$\therefore y_1 + y_2 = -\frac{b}{a} = \frac{14}{37}$
$\therefore$ y-coordinate of D $= \frac{y_1+y_2}{2} = \frac{7}{37}$.
Since D lies on the line AB, we can find the x-coordinate of D as
$2x + 3\left(\frac{7}{37}\right) - 1 = 0$
$\therefore 2x = 1 - \frac{21}{37} = \frac{16}{37}$
$\therefore x = \frac{8}{37}$
$\therefore$ D is (8/37, 7/37)
$\therefore$ equation of the median OD is $\frac{x}{8/37} = \frac{y}{7/37}$,
i.e., $7x - 8y = 0$.

Question 11.
Find the co-ordinates of the points of intersection of the lines represented by $x^2 - y^2 - 2x + 1 = 0$.
Solution:
Consider, $x^2 - y^2 - 2x + 1 = 0$
$\therefore (x^2 - 2x + 1) - y^2 = 0$
$\therefore (x - 1)^2 - y^2 = 0$
$\therefore (x - 1 + y)(x - 1 - y) = 0$
$\therefore (x + y - 1)(x - y - 1) = 0$
$\therefore$ separate equations of the lines are
$x + y - 1 = 0$ and $x - y + 1 = 0$.
To find the point of intersection of the lines, we have to solve
$x + y - 1 = 0$ ... (1)
and $x - y + 1 = 0$ ... (2)
Adding (1) and (2), we get,
$2x = 0 \therefore x = 0$
Substituting x = 0 in (1), we get,
$0 + y - 1 = 0 \therefore y = 1$
$\therefore$ coordinates of the point of intersection of the lines are (0, 1).

Maharashtra State Board 12th Maths Solutions Chapter 4 Pair of Straight Lines Miscellaneous Exercise 4

I : Choose correct alternatives.
Question 1.
If the equation $4x^2 + hxy + y^2 = 0$ represents two coincident lines, then h = ________.
(A) ± 2
(B) ± 3
(C) ± 4
(D) ± 5
Solution:
(C) ± 4

Question 2.
If the lines represented by $kx^2 – 3xy + 6y^2 = 0$ are perpendicular to each other then ________.
(A) k = 6
(B) k = -6
(C) k = 3
(D) k = -3
Solution:
(B) k = -6

Question 3.
Auxiliary equation of $2x^2 + 3xy – 9y^2 = 0$ is ________.
(A) $2m^2 + 3m – 9 = 0$
(B) $9m^2 – 3m – 2 = 0$
(C) $2m^2 – 3m + 9 = 0$
(D) $-9m^2 – 3m + 2 = 0$
Solution:
(B) $9m^2 – 3m – 2 = 0$

Question 4.
The difference between the slopes of the lines represented by $3x^2 – 4xy + y^2 = 0$ is ________.
(A) 2
(B) 1
(C) 3
(D) 4
Solution:
(A) 2

Question 5.
If the two lines $ax^2 + 2hxy + by^2 = 0$ make angles α and β with X-axis, then tan (α + β) = _____.
(A) *ha+b*
(B) *ha−b*
(C) *2ha+b*
(D) *2ha−b*
Solution:
(D) *2ha−b*

$$[\textbf{Hint : } m_1 = \tan\alpha,\ m_2 = \tan\beta$$
$$\therefore\ \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\cdot\tan\beta}$$
$$= \frac{m_1 + m_2}{1 - m_1 m_2} = \frac{(-2h/b)}{1-(a/b)} = \frac{2h}{a-b}\,]$$

Question 6.
If the slope of one of the two lines *x2a+2xyh+y2b* = 0 is twice that of the other, then ab:h^2 = ___.
(A) 1 : 2
(B) 2 : 1
(C) 8 : 9
(D) 9 : 8
Solution:

(D) 9 : 8

[**Hint :** $m_1 + m_2 = \frac{-2b}{h}$ and $m_1 m_2 = \frac{b}{a}$

where $m_1 = 2m_2$

$\therefore\ 2m_2 + m_2 = -\frac{2b}{h}$ and $2m_2 \times m_2 = \frac{b}{a}$

$\therefore\ m_2 = \frac{-2b}{3h}$ and $m_2{}^2 = \frac{b}{2a}$

$\therefore\ \left(\frac{-2b}{3h}\right)^2 = \frac{b}{2a}$ $\quad\therefore\ \frac{4b^2}{9h^2} = \frac{b}{2a}$

$\therefore\ \frac{ab}{h^2} = \frac{9}{8}$]

Question 7.
The joint equation of the lines through the origin and perpendicular to the pair of lines $3x^2 + 4xy – 5y^2 = 0$ is ________.
(A) $5x^2 + 4xy – 3y^2 = 0$
(B) $3x^2 + 4xy – 5y^2 = 0$
(C) $3x^2 – 4xy + 5y^2 = 0$
(D) $5x^2 + 4xy + 3y^2 = 0$
Solution:
(A) $5x^2 + 4xy – 3y^2 = 0$

Question 8.
If acute angle between lines $ax^2 + 2hxy + by^2 = 0$ is, $\frac{\pi}{4}$ then $4h^2$ = ________.
(A) $a^2 + 4ab + b^2$
(B) $a^2 + 6ab + b^2$
(C) (a + 2b)(a + 3b)
(D) (a – 2b)(2a + b)
Solution:
(B) $a^2 + 6ab + b^2$

Question 9.
If the equation $3x^2 – 8xy + qy^2 + 2x + 14y + p = 1$ represents a pair of perpendicular lines then the values of p and q are respectively ________.
(A) -3 and -7
(B) -7 and -3
(C) 3 and 7
(D) -7 and 3
Solution:
(B) -7 and -3

Question 10.
The area of triangle formed by the lines $x^2 + 4xy + y^2 = 0$ and x – y – 4 = 0 is ________.
(A) $\frac{4}{\sqrt{3}}$ Sq. units
(B) $\frac{8}{\sqrt{3}}$ Sq. units
(C) $\frac{16}{\sqrt{3}}$ Sq. units
(D) $\frac{15}{\sqrt{3}}$ Sq. units
Solution:
(B) $\frac{8}{\sqrt{3}}$ Sq. units
[Hint : Area = $\frac{p^2}{\sqrt{3}}$, where p is the length of perpendicular from the origin to x – y – 4 = 0]

Question 11.
The combined equation of the co-ordinate axes is ________.
(A) x + y = 0
(B) x y = k
(C) xy = 0
(D) x – y = k
Solution:
(C) xy = 0

Question 12.
If $h^2 = ab$, then slope of lines $ax^2 + 2hxy + by^2 = 0$ are in the ratio ________.
(A) 1 : 2
(B) 2 : 1
(C) 2 : 3

(D) 1 : 1
Solution:
(D) 1 : 1
[Hint: If $h^2 = ab$, then lines are coincident. Therefore slopes of the lines are equal.]

Question 13.
If slope of one of the lines $ax^2 + 2hxy + by^2 = 0$ is 5 times the slope of the other, then $5h^2$ = ________.
(A) ab
(B) 2 ab
(C) 7 ab
(D) 9 ab
Solution:
(D) 9 ab

Question 14.
If distance between lines $(x - 2y)^2 + k(x - 2y) = 0$ is 3 units, then k =
(A) ± 3
(B) $\pm 5\sqrt{5}$
(C) 0
(D) $\pm 3\sqrt{5}$
Solution:
(D) $\pm 3\sqrt{5}$
[Hint: $(x - 2y)^2 + k(x - 2y) = 0$
∴ $(x - 2y)(x - 2y + k) = 0$
∴ equations of the lines are $x - 2y = 0$ and $x - 2y + k = 0$ which are parallel to each other.
∴ $\left|\frac{k-0}{\sqrt{1+4}}\right| = 3$
∴ $k = \pm 3\sqrt{5}$

II. Solve the following.
Question 1.
Find the joint equation of lines:
(i) $x - y = 0$ and $x + y = 0$
Solution:
The joint equation of the lines $x - y = 0$ and
$x + y = 0$ is
$(x - y)(x + y) = 0$
∴ $x^2 - y^2 = 0$.

(ii) $x + y - 3 = 0$ and $2x + y - 1 = 0$
Solution:
The joint equation of the lines $x + y - 3 = 0$ and $2x + y - 1 = 0$ is
$(x + y - 3)(2x + y - 1) = 0$
∴ $2x^2 + xy - x + 2xy + y^2 - y - 6x - 3y + 3 = 0$
∴ $2x^2 + 3xy + y^2 - 7x - 4y + 3 = 0$.

(iii) Passing through the origin and having slopes 2 and 3.
Solution:
We know that the equation of the line passing through the origin and having slope m is y = mx. Equations of the lines passing through the origin and having slopes 2 and 3 are y = 2x and y = 3x respectively.
i.e. their equations are
$2x - y = 0$ and $3x - y = 0$ respectively.
∴ their joint equation is $(2x - y)(3x - y) = 0$
∴ $6x^2 - 2xy - 3xy + y^2 = 0$
∴ $6x^2 - 5xy + y^2 = 0$.

(iv) Passing through the origin and having inclinations 60° and 120°.
Solution:
Slope of the line having inclination θ is tan θ .
Inclinations of the given lines are 60° and 120°
∴ their slopes are $m_1 = \tan 60° = \sqrt{3}$ and
$m_2 = \tan 120° = \tan (180° - 60°)$
$= -\tan 60° = -\sqrt{3}$
Since the lines pass through the origin, their equa-tions are
$y = \sqrt{3}x$ and $y = -\sqrt{3}x$
i.e., $\sqrt{3}x - y = 0$ and $\sqrt{3}x + y = 0$
∴ the joint equation of these lines is

$(\sqrt{3}x - y)(\sqrt{3}x + y) = 0$
∴ $3x^2 - y^2 = 0$.

(v) Passing through (1, 2) amd parallel to the co-ordinate axes.
Solution:
Equations of the coordinate axes are x = 0 and y = 0
∴ the equations of the lines passing through (1, 2) and parallel to the coordinate axes are x = 1 and y =1
i.e. x – 1 = 0 and y – 2 0
∴ their combined equation is
(x – 1)(y – 2) = 0
∴ x(y – 2) – 1(y – 2) = 0
∴ xy – 2x – y + 2 = 0

(vi) Passing through (3, 2) and parallel to the line x = 2 and y = 3.
Solution:
Equations of the lines passing through (3, 2) and parallel to the lines x = 2 and y = 3 are x = 3 and y = 2.
i.e. x – 3 = 0 and y – 2 = 0
∴ their joint equation is
(x – 3)(y – 2) = 0
∴ xy – 2x – 3y + 6 = 0.

(vii) Passing through (-1, 2) and perpendicular to the lines x + 2y + 3 = 0 and 3x – 4y – 5 = 0.
Solution:
Let L_1 and L_2 be the lines passing through the origin and perpendicular to the lines x + 2y + 3 = 0 and 3x – 4y – 5 = 0 respectively.
Slopes of the lines x + 2y + 3 = 0 and 3x – 4y – 5 = 0 are $-\frac{1}{2}$ and $-\frac{3}{-4} = \frac{3}{4}$ respectively.
∴ slopes of the lines $L_1$and L_2 are 2 and $-\frac{4}{3}$ respectively.
Since the lines L_1 and L_2 pass through the point (-1, 2), their equations are
∴ $(y - y_1) = m(x - x_1)$
∴ (y – 2) = 2(x + 1)
⇒ y – 1 = 2x + 2
⇒ 2x – y + 4 = 0 and
∴ (y – 2) = $\left(-\frac{4}{3}\right)$(x + 1)
⇒ 3y – 6 = (-4)(x + 1)
⇒ 3y – 6 = -4x + 4
⇒ 4x + 3y – 6 + 4 = 0
⇒ 4x + 3y – 2 = 0
their combined equation is
∴ (2x – y + 4)(4x + 3y – 2) = 0
∴ $8x^2 + 6xy - 4x - 4xy - 3y^2 + 2y + 16x + 12y - 8 = 0$
∴ $8x^2 + 2xy + 12x - 3y^2 + 14y - 8 = 0$

(viii) Passing through the origin and having slopes 1 + $\sqrt{3}$ and 1 – $\sqrt{3}$
Solution:
Let l_1 and l_2 be the two lines. Slopes of l_1 is 1 + $\sqrt{3}$ and that of l_2 is 1 – $\sqrt{3}$
Therefore the equation of a line (l_1) passing through the origin and having slope is
y = (1 + $\sqrt{3}$)x
∴ (1 + $\sqrt{3}$)x – y = 0 ..(1)
Similarly, the equation of the line (l_2) passing through the origin and having slope is
y = (1 – $\sqrt{3}$)x
∴ (1 – $\sqrt{3}$)x – y = 0 ...(2)
From (1) and (2) the required combined equation is

$$\left[\left(1+\sqrt{3}\right)x - y\right]\left[\left(1-\sqrt{3}\right)x - y\right] = 0$$

$$\therefore \left(1+\sqrt{3}\right)x\left[\left(1-\sqrt{3}\right)x - y\right] - y\left[\left(1-\sqrt{3}\right)x - y\right] = 0$$

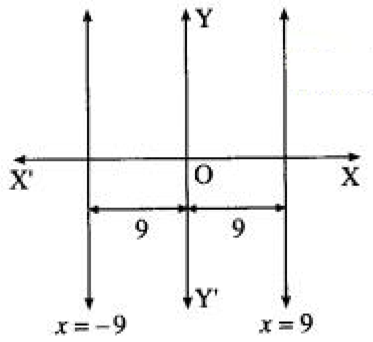
$$\therefore \left(1-\sqrt{3}\right)\left(1+\sqrt{3}\right)x^2 - \left(1+\sqrt{3}\right)xy - \left(1-\sqrt{3}\right)xy + y^2 = 0$$

$$\therefore \left((1)^2 - \left(\sqrt{3}\right)^2\right)x^2 - \left[\left(1+\sqrt{3}\right) + \left(1-\sqrt{3}\right)\right]xy + y^2 = 0$$

∴ $(1 - 3)x^2 - 2xy + y^2 = 0$
∴ $-2x^2 - 2xy + y^2 = 0$

∴ $2x^2 + 2xy - y^2 = 0$
This is the required combined equation.

(ix) Which are at a distance of 9 units from the Y – axis.
Solution:
Equations of the lines, which are parallel to the Y-axis and at a distance of 9 units from it, are x = 9 and x = -9
i.e. x – 9 = 0 and x + 9 = 0



∴ their combined equation is
(x – 9)(x + 9) = 0
∴ $x^2 - 81 = 0$.

(x) Passing through the point (3, 2), one of which is parallel to the line x – 2y = 2 and other is perpendicular to the line y = 3.
Solution:
Let L_1 be the line passes through (3, 2) and parallel to the line x – 2y = 2 whose slope is $\frac{-1}{-2} = \frac{1}{2}$
∴ slope of the line L_1 is $\frac{1}{2}$.
∴ equation of the line L_1 is
$y - 2 = \frac{1}{2}(x - 3)$
∴ 2y – 4 = x – 3 ∴ x – 2y + 1 = 0
Let L_2 be the line passes through (3, 2) and perpendicular to the line y = 3.
∴ equation of the line L_2 is of the form x = a.
Since L_2 passes through (3, 2), 3 = a
∴ equation of the line L_2 is x = 3, i.e. x – 3 = 0
Hence, the equations of the required lines are
x – 2y + 1 = 0 and x – 3 = 0
∴ their joint equation is
(x – 2y + 1)(x – 3) = 0
∴ $x^2 - 2xy + x - 3x + 6y - 3 = 0$
∴ $x^2 - 2xy - 2x + 6y - 3 = 0$.

(xi) Passing through the origin and perpendicular to the lines x + 2y = 19 and 3x + y = 18.
Solution:
Let L_1 and L_2 be the lines passing through the origin and perpendicular to the lines x + 2y = 19 and 3x + y = 18 respectively.
Slopes of the lines x + 2y = 19 and 3x + y = 18 are $-\frac{1}{2}$ and $-\frac{3}{1}$ = -3 respectively.
Since the lines L_1 and L_2 pass through the origin, their equations are
y = 2x and $y = \frac{1}{3}x$
i.e. 2x – y = 0 and x – 3y = 0
∴ their combined equation is
(2x – y)(x – 3y) = 0
∴ $2x^2 - 6xy - xy + 3y^2 = 0$
∴ $2x^2 - 7xy + 3y^2 = 0$.

Question 2.
Show that each of the following equation represents a pair of lines.
(i) $x^2 + 2xy - y^2 = 0$
Solution:
Comparing the equation $x^2 + 2xy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
a = 1, 2h = 2, i.e. h = 1 and b = -1
∴ $h^2 - ab = (1)^2 - 1(-1) = 1 + 1 = 2 > 0$
Since the equation $x^2 + 2xy - y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab > 0$, the given equation represents a pair of lines which are real and distinct.

(ii) $4x^2 + 4xy + y^2 = 0$
Solution:
Comparing the equation $4x^2 + 4xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
a = 4, 2h = 4, i.e. h = 2 and b = 1
∴ $h^2 - ab = (2)^2 - 4(1) = 4 - 4 = 0$
Since the equation $4x^2 + 4xy + y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab = 0$, the given equation represents a pair of lines which are real and coincident.

(iii) $x^2 - y^2 = 0$
Solution:

Comparing the equation $x^2 - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
$a = 1$, $2h = 0$, i.e. $h = 0$ and $b = -1$
$\therefore h^2 - ab = (0)^2 - 1(-1) = 0 + 1 = 1 > 0$
Since the equation $x^2 - y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab > 0$, the given equation represents a pair of lines which are real and distinct.

(iv) $x^2 + 7xy - 2y^2 = 0$
Solution:
Comparing the equation $x^2 + 7xy - 2y^2 = 0$
$a = 1$, $2h = 7$ i.e., $h = \frac{7}{2}$ and $b = -2$
$\therefore h^2 - ab = \left(\frac{7}{2}\right)^2 - 1(-2)$
$= \frac{49}{4} + 2$
$= \frac{57}{4}$ i.e. $14.25 = 14 > 0$
Since the equation $x^2 + 7xy - 2y^2 = 0$ is a homogeneous equation of second degree and $h^2 - ab > 0$, the given equation represents a pair of lines which are real and distinct.

(v) $x^2 - 2\sqrt{3} xy - y^2 = 0$
Solution:
Comparing the equation $x^2 - 2\sqrt{3} xy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
$a = 1$, $2h = -2\sqrt{3}$, i.e. $h = -\sqrt{3}$ and $b = 1$
$\therefore h^2 - ab = (-\sqrt{3})^2 - 1(1) = 3 - 1 = 2 > 0$
Since the equation $x^2 - 2\sqrt{3}xy - y^2 = 0$ is a homo¬geneous equation of second degree and $h^2 - ab > 0$, the given equation represents a pair of lines which are real and distinct.

Question 3.
Find the separate equations of lines represented by the following equations:
(i) $6x^2 - 5xy - 6y^2 = 0$
Solution:
$6x^2 - 5xy - 6y^2 = 0$
$\therefore 6x^2 - 9xy + 4xy - 6y^2 = 0$
$\therefore 3x(2x - 3y) + 2y(2x - 3y) = 0$
$\therefore (2x - 3y)(3x + 2y) = 0$
$\therefore$ the separate equations of the lines are
$2x - 3y = 0$ and $3x + 2y = 0$.

(ii) $x^2 - 4y^2 = 0$
Solution:
$x^2 - 4y^2 = 0$
$\therefore x^2 - (2y)^2 = 0$
$\therefore (x - 2y)(x + 2y) = 0$
$\therefore$ the separate equations of the lines are
$x - 2y = 0$ and $x + 2y = 0$.

(iii) $3x^2 - y^2 = 0$
Solution:
$3x^2 - y^2 = 0$
$\therefore (\sqrt{3} x)^2 - y^2 = 0$
$\therefore (\sqrt{3}x - y)(\sqrt{3}x + y) = 0$
$\therefore$ the separate equations of the lines are
$\sqrt{3}x - y = 0$ and $\sqrt{3}x + y = 0$.

(iv) $2x^2 + 2xy - y^2 = 0$
Solution:
$2x^2 + 2xy - y^2 = 0$
$\therefore$ The auxiliary equation is $-m^2 + 2m + 2 = 0$
$\therefore m^2 - 2m - 2 = 0$

$$\therefore\ m = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2 \times 1} = \frac{2 \pm \sqrt{4+8}}{2}$$

$$= \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

$m_1 = 1 + \sqrt{3}$ and $m_2 = 1 - \sqrt{3}$ are the slopes of the lines.

∴ their separate equations are

$y = m_1x$ and $y = m_2x$

i.e. $y = (1 + \sqrt{3})x$ and $y = (1 - \sqrt{3})x$

i.e. $(\sqrt{3} + 1)x - y = 0$ and $(\sqrt{3} - 1)x + y = 0$.

Question 4.
Find the joint equation of the pair of lines through the origin and perpendicular to the lines given by :

(i) $x^2 + 4xy - 5y^2 = 0$

Solution:

Comparing the equation $x^2 + 4xy - 5y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 1$, $2h = 4$, $b = -5$

Let m_1 and m_2 be the slopes of the lines represented by $x^2 + 4xy - 5y^2 = 0$.

$$\left.\begin{aligned} \therefore\ m_1 + m_2 &= \frac{-2h}{b} = \frac{4}{5} \\ \text{and } m_1m_2 &= \frac{a}{b} = \frac{-1}{5} \end{aligned}\right\} \quad \ldots (1)$$

Now, required lines are perpendicular to these lines

∴ their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$

Since these lines are passing through the origin, their separate equations are

$y = -\frac{1}{m_1}x$ and $y = -\frac{1}{m_2}x$

i.e. $m_1y = -x$ and $m_2y = -x$

i.e. $x + m_1y = 0$ and $x + m_2y = 0$

∴ their combined equation is

$(x + m_1x + m_2y) = 0$

∴ $x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$

∴ $x^2 + \frac{4}{5}xy - \frac{1}{5}y^2 = 0$ …[By (1)]

∴ $5x^2 + 4xy - y^2 = 0$

(ii) $2x^2 - 3xy - 9y^2 = 0$

Solution:

Comparing the equation $2x^2 - 3xy - 9y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 2$, $2h = -3$, $b = -9$

Let m_1 and m_2 be the slopes of the lines represented by $2x^2 - 3xy - 9y^2 = 0$

∴ $m_1 + m_2 = \frac{-2h}{b} = \frac{-3}{9}$ and $m_1m_2 = \frac{a}{b} = \frac{-2}{9}$ …(1)

Now, required lines are perpendicular to these lines

∴ their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$

Since these lines are passing through the origin, their separate equations are

$y = -\frac{1}{m_1}x$ and $y = -\frac{1}{m_2}x$

i.e. $m_1y = -x$ and $m_2y = -x$

i.e. $x + m_1y = 0$ and $x + m_2y = 0$

∴ their combined equation is

$(x + m_1y)(x + m_2y) = 0$

∴ $x^2 + (m_1 + m_2)xy + m_1m_2y^2 = 0$

∴ $x^2 + \left(\frac{-3}{9}\right)xy + \left(\frac{-2}{9}\right)y^2 = 0$ …[By (1)]

∴ $9x^2 - 3xy - 2y^2 = 0$

(iii) $x^2 + xy - y^2 = 0$

Solution:

Comparing the equation $x^2 + xy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 1$, $2h = 1$, $b = -1$

Let m_1 and m_2 be the slopes of the lines represented by $x^2 + xy - y^2 = 0$

∴ $m_1 + m_2 = \frac{-2h}{b} = \frac{-1}{-1}$ and $m_1m_2 = \frac{a}{b} = \frac{1}{-1} = -1$ ..(1)

Now, required lines are perpendicular to these lines

∴ their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$

Since these lines are passing through the origin, their separate equations are

$y = -\frac{1}{m_1}x$ and $y = -\frac{1}{m_2}x$

i.e. $m_1y = -x$ and $m_2y = -x$

i.e. $x + m_1y = 0$ and $x + m_2y = 0$

∴ their combined equation is

$(x + m_1y)(x + m_2y) = 0$

∴ $x^2 + (m_1 + m_2) + m_1m_2y^2 = 0$

∴ $x^2 + 1xy + (-1)y^2 = 0$ …[By (1)]

∴ $x^2 + xy - y^2 = 0$

Question 5.
Find k if
(i) The sum of the slopes of the lines given by $3x^2 + kxy - y^2 = 0$ is zero.
Solution:
Comparing the equation $3x^2 + kxy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
$a = 3, 2h = k, b = -1$
Let m_1 and m_2 be the slopes of the lines represented by $3x^2 + kxy - y^2 = 0$.
$\therefore m_1 + m_2 = \frac{-2h}{b} = \frac{-k}{-1} = k$
Now, $m_1 + m_2 = 0$ ... (Given)
$\therefore k = 0$.

(ii) The sum of slopes of the lines given by $2x^2 + kxy - 3y^2 = 0$ is equal to their product.
Question is modified.
The sum of slopes of the lines given by $x^2 + kxy - 3y^2 = 0$ is equal to their product.
Solution:
Comparing the equation $x^2 + kxy - 3y^2 = 0$, with $ax^2 + 2hxy + by^2 = 0$, we get,
$a = 1, 2h = k, b = -3$
Let m_1 and m_2 be the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$.
$\therefore m_1 + m_2 = -\frac{2h}{b} = \frac{-k}{-3} = \frac{k}{3}$
and $m_1m_2 = \frac{a}{b} = \frac{1}{-3} = -\frac{1}{3}$
Now, $m_1 + m_2 = m_1m_2$ ... (Given)
$\therefore \frac{k}{3} = -\frac{1}{3}$
$\therefore k = -1$.

(iii) The slope of one of the lines given by $3x^2 - 4xy + ky^2 = 0$ is 1.
Solution:
The auxiliary equation of the lines given by $3x^2 - 4xy + ky^2 = 0$ is $km^2 - 4m + 3 = 0$.
Given, slope of one of the lines is 1.
$\therefore m = 1$ is the root of the auxiliary equation $km^2 - 4m + 3 = 0$.
$\therefore k(1)^2 - 4(1) + 3 = 0$
$\therefore k - 4 + 3 = 0$
$\therefore k = 1$.

(iv) One of the lines given by $3x^2 - kxy + 5y^2 = 0$ is perpendicular to the $5x + 3y = 0$.
Solution:
The auxiliary equation of the lines represented by $3x^2 - kxy + 5y^2 = 0$ is $5m^2 - km + 3 = 0$.
Now, one line is perpendicular to the line $5x + 3y = 0$, whose slope is $-\frac{5}{3}$.
$\therefore$ slope of that line $= m = \frac{3}{5}$
$\therefore m = \frac{3}{5}$ is the root of the auxiliary equation 5
$5m^2 - km + 3 = 0$.
$\therefore 5\left(\frac{3}{5}\right)^2 - k\left(\frac{3}{5}\right) + 3 = 0$
$\therefore \frac{9}{5} - \frac{3k}{5} + 3 = 0$
$\therefore 9 - 3k + 15 = 0$
$\therefore 3k = 24$
$\therefore k = 8$.

(v) The slope of one of the lines given by $3x^2 + 4xy + ky^2 = 0$ is three times the other.
Solution:
$3x^2 + 4xy + ky^2 = 0$
$\therefore$ divide by x^2

$$\frac{3x^2}{x^2} + \frac{4xy}{x^2} + \frac{ky^2}{x^2} = 0$$

$$3 + \frac{4y}{x} + \frac{ky^2}{x^2} = 0 \quad \text{....(1)}$$

$\therefore y = mx$
$\therefore \frac{y}{x} = m$
put $\frac{y}{x} = m$ in equation (1)
Comparing the equation $km^2 + 4m + 3 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
$a = k, 2h = 4, b = 3$
$m_1 = 3m_2$ ..(given condition)
$m_1 + m_2 = \frac{-2h}{k} = -\frac{4}{k}$

m_1m_2 = ab=3k

$m_1 + m_2$ = −4k

$4m_2$ = −4k ...($m_1 = 3m_2$)

m_2 = −1k

m_1m_2 = 3k

3 m22=3k ...($m_1 = 3m_2$)

3(−1k)2=3k ...(m_2 = −1k)

1k2=1k

$k^2 = k$
k = 1 or k = 0

(vi) The slopes of lines given by $kx^2 + 5xy + y^2 = 0$ differ by 1.
Solution:
Comparing the equation $kx^2 + 5xy + y^2 = 0$ with $ax^2 + 2hxy + by^2$
a = k, 2h = 5 i.e. h = 52

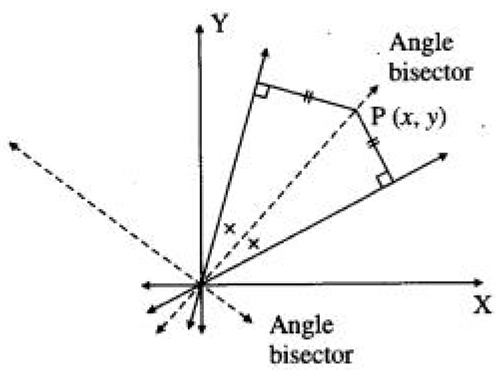
$m_1 + m_2$ = -2hb=−51 = -5

and m_1m_2 = ab=k1 = k

the slope of the line differ by $(m_1 – m_2) = 1$ ...(1)
$\therefore (m_1 – m_2)^2 = (m_1 + m_2)^2 – 4m_1m_2$
$(m_1 – m_2)^2 = (-5)^2 – 4(k)$
$(m_1 – m_2)^2 = 25 – 4k$
1 = 25 – 4k ..[By (1)]
4k = 24
k = 6

(vii) One of the lines given by $6x^2 + kxy + y^2 = 0$ is 2x + y = 0.
Solution:
The auxiliary equation of the lines represented by $6x^2 + kxy + y^2 = 0$ is
$m^2 + km + 6 = 0$.
Since one of the line is 2x + y = 0 whose slope is m = -2.
∴ m = -2 is the root of the auxiliary equation $m^2 + km + 6 = 0$.
∴ $(-2)^2 + k(-2) + 6 = 0$
∴ 4 – 2k + 6 = 0
∴ 2k = 10 ∴ k = 5

Question 6.
Find the joint equation of the pair of lines which bisect angle between the lines given by $x^2 + 3xy + 2y^2 = 0$
Solution:
$x^2 + 3xy + 2y^2 = 0$
∴ $x^2 + 2xy + xy + 2y^2 = 0$
∴ x(x + 2y) + y(x + 2y) = 0
∴ (x + 2y)(x + y) = 0
∴ separate equations of the lines represented by $x^2 + 3xy + 2y^2 = 0$ are x + 2y = 0 and x + y = 0.
Let P (x, y) be any point on one of the angle bisector. Since the points on the angle bisectors are equidistant from both the lines,



the distance of P (x, y) from the line x + 2y = 0
= the distance of P(x, y) from the line x + y = 0

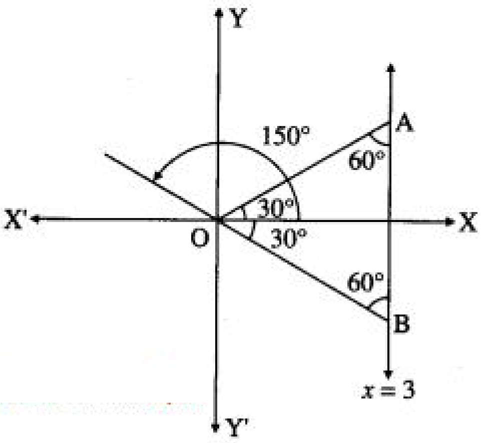
$$\therefore \left|\frac{x+2y}{\sqrt{1+4}}\right| = \left|\frac{x+y}{\sqrt{1+1}}\right|$$

$$\therefore \frac{(x+2y)^2}{5} = \frac{(x+y)^2}{2}$$

∴ $2(x + 2y)^2 = 5(x + y)^2$

$\therefore 2(x^2 + 4xy + 4y^2) = 5(x^2 + 2xy + y^2)$
$\therefore 2x^2 + 8xy + 8y^2 = 5x^2 + 10xy + 5y^2$
$\therefore 3x^2 + 2xy - 3y^2 = 0$.
This is the required joint equation of the lines which bisect the angles between the lines represented by $x^2 + 3xy + 2y^2 = 0$.

Question 7.
Find the joint equation of the pair of lies through the origin and making equilateral triangle with the line x = 3.
Solution:



Let OA and OB be the lines through the origin making an angle of 60° with the line x = 3.
$\therefore$ OA and OB make an angle of 30° and 150° with the positive direction of X-axis
$\therefore$ slope of OA = tan 30° = $1/\sqrt{3}$
$\therefore$ equation of the line OA is $y = \frac{1}{\sqrt{3}}x$
$\therefore \sqrt{3}y = x \therefore x - \sqrt{3}y = 0$
Slope of OB = tan 150° = tan (180° – 30°)
= – tan 30°= $-1/\sqrt{3}$
$\therefore$ equation of the line OB is $y = -\frac{1}{\sqrt{3}}x$
$\therefore \sqrt{3}y = -x \therefore x + \sqrt{3}y = 0$
$\therefore$ required combined equation of the lines is
$(x - \sqrt{3}y)(x + \sqrt{3}y) = 0$
i.e. $x^2 - 3y^2 = 0$.

Question 8.
Show that the lines $x^2 - 4xy + y^2 = 0$ and x + y = 10 contain the sides of an equilateral triangle. Find the area of the triangle.
Solution:
We find the joint equation of the pair of lines OA and OB through origin, each making an angle of 60° with x + y = 10 whose slope is -1.
Let OA (or OB) has slope m.
$\therefore$ its equation is y = mx … (1)
Also, $\tan 60° = \left|\frac{m-(-1)}{1+m(-1)}\right|$
$\therefore \sqrt{3} = \left|\frac{m+1}{1-m}\right|$
Squaring both sides, we get,
$3 = \frac{(m+1)^2}{(1-m)^2}$
$\therefore 3(1 - 2m + m^2) = m^2 + 2m + 1$
$\therefore 3 - 6m + 3m^2 = m^2 + 2m + 1$
$\therefore 2m^2 - 8m + 2 = 0$
$\therefore m^2 - 4m + 1 = 0$
$\therefore \left(\frac{y}{x}\right)^2 - 4\left(\frac{y}{x}\right) + 1 = 0$ …[By (1)]
$\therefore y^2 - 4xy + x^2 = 0$
$\therefore x^2 - 4xy + y\left(\frac{y}{x}\right) = 0$ is the joint equation of the two lines through the origin each making an angle of 60° with x + y = 10
$\therefore x^2 - 4xy + y^2 = 0$ and x + y = 10 form a triangle OAB which is equilateral.
Let seg OM $\perp$r line AB whose question is x + y = 10

$$\therefore \text{OM} = \left|\frac{-10}{\sqrt{1+1}}\right| = 5\sqrt{2}$$

$$\therefore \text{ area of equilateral } \triangle\text{OAB} = \frac{(\text{OM})^2}{\sqrt{3}} = \frac{(5\sqrt{2})^2}{\sqrt{3}}$$

$$= \frac{50}{\sqrt{3}} \text{ sq units.}$$

Question 9.
If the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is three times the other then prove that $3h^2 = 4ab$.
Solution:
Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

$\therefore m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$

We are given that $m_2 = 3m_1$

$\therefore m_1 + 3m_1 = -\frac{2h}{b}$ $4m_1 = -\frac{2h}{b}$

$\therefore m_1 = -\frac{h}{2b}$ ...(1)

Also, $m_1(3m_1) = \frac{a}{b}$ $\therefore 3m_1^2 = \frac{a}{b}$

$\therefore 3\left(-\frac{h}{2b}\right)^2 = \frac{a}{b}$ ....[By (1)]

$\therefore \frac{3h^2}{4b^2} = \frac{a}{b}$

$\therefore 3h^2 = 4ab$, as $b \neq 0$.

Question 10.
Find the combined equation of the bisectors of the angles between the lines represented by $5x^2 + 6xy - y^2 = 0$.
Solution:
Comparing the equation $5x^2 + 6xy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
$a = 5$, $2h = 6$, $b = -1$
Let m_1 and m_2 be the slopes of the lines represented by $5x^2 + 6xy - y^2 = 0$.

$$\left.\begin{aligned} \therefore\ m_1 + m_2 &= \frac{-2h}{b} = \frac{-6}{-1} = 6 \\ \text{and } m_1m_2 &= \frac{a}{b} = \frac{5}{-1} = -5. \end{aligned}\right\} \quad \ldots (1)$$

The separate equations of the lines are
$y = m_1x$ and $y = m_2x$, where $m_1 \neq m_2$
i.e. $m_1x - y = 0$ and $m_1x - y = 0$.
Let P (x, y) be any point on one of the bisector of the angles between the lines.
$\therefore$ the distance of P from the line $m_1x - y = 0$ is equal to the distance of P from the line $m_2x - y = 0$.

$$\therefore \left|\frac{m_1x - y}{\sqrt{m_1^2 + 1}}\right| = \left|\frac{m_2x - y}{\sqrt{m_2^2 + 1}}\right|$$

Squaring both sides, we get,

$$\frac{(m_1x - y)^2}{m_1^2 + 1} = \frac{(m_2x - y)^2}{m_2^2 + 1}$$

$\therefore (m_2^2 + 1)(m_1x - y)^2 = (m_1^2 + 1)(m_2x - y)^2$
$\therefore (m_2^2 + 1)(m_1^2x^2 - 2m_1xy + y^2) = (m_1^2 + 1)(m_2^2x^2 - 2m_2xy + y^2)$
$\therefore m_1^2m_2^2x^2 - 2m_1m_1^2y^2xy + m_2^2y^2 + m_1^2x^2 - 2m_1^2xy + y^2$
$= m_1^2m_2^2x^2 - 2m_1^2m_2xy + m_1^2y^2 + m_2^2x^2 - 2m_2xy + y^2$
$\therefore (m_1^2 - m_2^2)x^2 + 2m_1m_2(m_1 - m_2)xy - 2(m_1 - m_2)xy - (m_1^2 - m_2^2)y^2 = 0$
Dividing throughout by $m_1 - m_2$ ($\neq 0$), we get,
$(m_1 + m_2)x^2 + 2m_1m_2xy - 2xy - (m_1 + m_2)y^2 = 0$
$\therefore 6x^2 - 10xy - 2xy - 6y^2 = 0$ ...[By (1)]
$\therefore 6x^2 - 12xy - 6y^2 = 0$
$\therefore x^2 - 2xy - y^2 = 0$
This is the joint equation of the bisectors of the angles between the lines represented by $5x^2 + 6xy - y^2 = 0$.

Question 11.
Find a, if the sum of the slopes of the lines represented by $ax^2 + 8xy + 5y^2 = 0$ is twice their product.
Solution :
Comparing the equation $ax^2 + 8xy + 5y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$,
we get, $a = a$, $2h = 8$, $b = 5$
Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 8xy + 5y^2 = 0$.

$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{8}{5}$

and $m_1m_2 = \frac{a}{b} = \frac{a}{5}$

Now, $(m_1 + m_2) = 2(m_1m_2)$

$-\frac{8}{5} = 2\left(\frac{a}{5}\right)$

$a = -4$

Question 12.
If the line $4x - 5y = 0$ coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$, then show that $25a + 40h + 16b = 0$.
Solution :
The auxiliary equation of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bm^2 + 2hm + a = 0$
Given that $4x - 5y = 0$ is one of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

The slope of the line $4x - 5y = 0$ is $\frac{-4}{-5} = \frac{4}{5}$

$\therefore m = \frac{4}{5}$ is a root of the auxiliary equation $bm^2 + 2hm + a = 0$.

$\therefore b\left(\frac{4}{5}\right)^2 + 2h\left(\frac{4}{5}\right) + a = 0$

∴ 16b25+8h5 + a = 0
∴ 16b + 40h + 25a = 0 i.e.
∴ 25a + 40h + 16b = 0

Question 13.
Show that the following equations represent a pair of lines. Find the acute angle between them :
(i) $9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$
Solution:
Comparing this equation with
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,
a = 9, h = -3, b = 1, g = 9, f = -3 and c = 8.
∴ D = | | | | ahghbfgfc | | | | = | | | | 9-39-31-39-38 | | | |
= 9(8 – 9) + 3(-24 + 27) + 9(9 – 9)
= 9(-1) + 3(3) + 9(0)
= -9 + 9 + 0 = 0
and $h^2 - ab = (-3)^2 - 9(1) = 9 - 9 = 0$
∴ the given equation represents a pair of lines.
Let θ be the acute angle between the lines.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{(-3)^2 - 9(1)}}{9+1}\right|$$

$$= \left|\frac{2\sqrt{9-9}}{10}\right| = 0$$

∴ tan θ = tan0°
∴ θ = 0°.

(ii) $2x^2 + xy - y^2 + x + 4y - 3 = 0$
Solution:
Comparing this equation with
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,
a = 2, h = 12, b = -1, g = 12, f = 2 and c = -3

$$\therefore D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 2 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & 2 \\ \frac{1}{2} & 2 & -3 \end{vmatrix}$$

$$= 2(3-4) - \frac{1}{2}\left(-\frac{3}{2} - 1\right) + \frac{1}{2}\left(1 + \frac{1}{2}\right)$$

$$= -2 + \frac{3}{4} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4}$$

= -2 + 1 + 1
= -2 + 2= 0
∴ the given equation represents a pair of lines.
Let θ be the acute angle between the lines.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$$= \left|\frac{2\sqrt{\left(\frac{1}{2}\right)^2 - (2)(-1)}}{2 - 1}\right|$$

$$= \left|\frac{2\sqrt{\frac{1}{4} + 2}}{1}\right|$$

$$= 2\sqrt{\frac{9}{4}} = 3$$

∴ tan θ = tan 3
∴ θ = $\tan^{-1}(3)$

(iii) $(x - 3)^2 + (x - 3)(y - 4) - 2(y - 4)^2 = 0$.
Solution :
Put x – 3 = X and y – 4 = Y in the given equation, we get,
$X^2 + XY - 2Y^2 = 0$
Comparing this equation with $ax^2 + 2hxy + by^2 = 0$, we get,
a = 1, h = 12, b = -2

This is the homogeneous equation of second degreeand $h^2 - ab$ = (12)2 – 1(-2)
= 14 + 2 = 94 > 0
Hence, it represents a pair of lines passing through the new origin (3, 4).
Let θ be the acute angle between the lines.

$$\therefore\ \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

Here $a = 1$, $2h = 1$, i.e. $h = \dfrac{1}{2}$ and $b = -2$

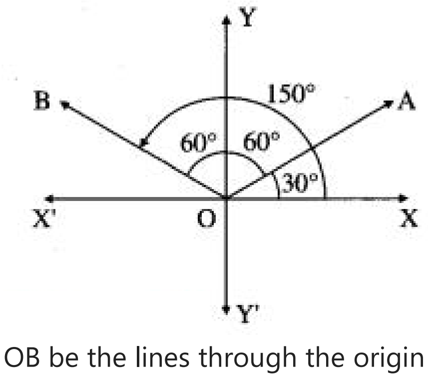
$$\therefore\ \tan\theta = \left|\frac{2\sqrt{\left(\frac{1}{2}\right)^2 - 1(-2)}}{1 - 2}\right|$$

$$= \left|\frac{2\sqrt{\frac{1}{4} + 2}}{-1}\right|$$

$$= \left|\frac{2 \times \frac{3}{2}}{-1}\right|$$

∴ tanθ = 3 ∴ θ = $\tan^{-1}(3)$

Question 14.
Find the combined equation of pair of lines through the origin each of which makes angle of 60° with the Y-axis.
Solution:



Let OA and OB be the lines through the origin making an angle of 60° with the Y-axis.
Then OA and OB make an angle of 30° and 150° with the positive direction of X-axis.
∴ slope of OA = tan 30° = 13√

∴ equation of the line OA is

$y = \frac{1}{\sqrt{3}} x$, i.e. $x - \sqrt{3}y = 0$

Slope of OB = tan 150° = tan (180° – 30°)

= tan 30° = $-\frac{1}{\sqrt{3}}$

∴ equation of the line OB is

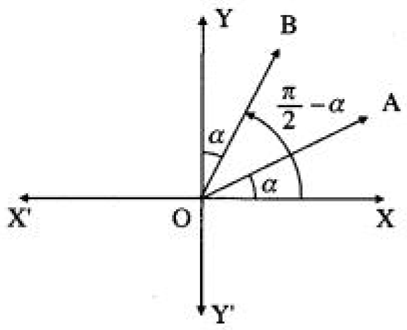
$y = -\frac{1}{\sqrt{3}}x$, i.e. $x + \sqrt{3} y = 0$

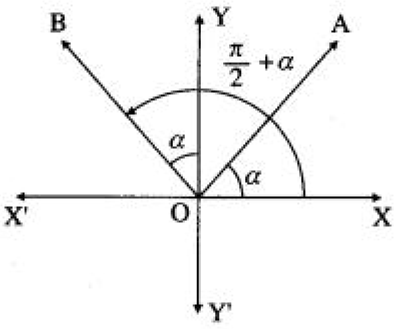
∴ required combined equation is

$(x - \sqrt{3}y)(x + \sqrt{3}y) = 0$

i.e. $x^2 - 3y^2 = 0$.

Question 15.
If lines representedby $ax^2 + 2hxy + by^2 = 0$ make angles of equal measures with the co-ordinate axes then show that $a = \pm b$.
OR
Show that, one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ will make an angle of the same measure with the X-axis as the other makes with the Y-axis, if $a = \pm b$.
Solution:





Let OA and OB be the two lines through the origin represented by $ax^2 + 2hxy + by^2 = 0$.
Since these lines make angles of equal measure with the coordinate axes, they make angles $\propto$ and $\frac{\pi}{2} - \propto$ with the positive direction of X-axis or $\propto$ and $\frac{\pi}{2} + \propto$ with thepositive direction of X-axis.
∴ slope of the line OA = $m_1 = \tan \propto$
and slope of the line OB = m_2
$= \tan(\frac{\pi}{2} - \propto)$ or $\tan(\frac{\pi}{2} + \propto)$
i.e. $m_2 = \cot \propto$ or $m_2 = -\cot \propto$
∴ $m_1m_2 - \tan \propto \times \cot \propto = 1$
OR $m_1m_2 = \tan \propto (-\cot \propto) = -1$
i.e. $m_1m_2 = \pm 1$
But $m_1m_2 = \frac{a}{b}$
∴ $\frac{a}{b} = \pm 1$ ∴ $a = \pm b$
This is the required condition.

Question 16.
Show that the combined equation of a pair of lines through the origin and each making an angle of $\propto$ with the line $x + y = 0$ is $x^2 + 2(\sec 2\propto) xy + y^2 = 0$.
Solution:
Let OA and OB be the required lines.
Let OA (or OB) has slope m.
∴ its equation is y = mx ... (1)
It makes an angle $\propto$ with x + y = 0 whose slope is -1. m +1
∴ $\tan \propto = \left| \frac{m+1}{1+m(-1)} \right|$
Squaring both sides, we get,
$\tan^2\propto = \frac{(m+1)^2}{(1-m)^2}$
∴ $\tan^2\propto(1 - 2m + m^2) = m^2 + 2m + 1$
∴ $\tan^2\propto - 2m \tan^2\propto + m^2\tan^2\propto = m^2 + 2m + 1$
∴ $(\tan^2\propto - 1)m^2 - 2(1 + \tan^2\propto)m + (\tan^2\propto - 1) = 0$

$$\therefore\ m^2 - 2\left(\frac{1+\tan^2\alpha}{\tan^2\alpha - 1}\right)m + 1 = 0$$

$$\therefore\ m^2 + 2\left(\frac{1+\tan^2\alpha}{1-\tan^2\alpha}\right)m + 1 = 0$$

$$\therefore\ m^2 + 2(\sec 2\alpha)m + 1 = 0\ \ldots\left[\because\ \cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}\right]$$

$$\therefore\ \frac{y^2}{x^2} + 2(\sec 2\alpha)\frac{y}{x} + 1 = 0 \qquad \ldots[\text{By } (1)]$$

$\therefore$ $y^2 + 2xy\sec 2\alpha + x^2 = 0$
$\therefore$ $x^2 + 2(\sec 2\alpha)xy + y^2 = 0$ is the required equation.

Question 17.
Show that the line 3x + 4y+ 5 = 0 and the lines $(3x + 4y)^2 - 3(4x - 3y)^2 = 0$ form an equilateral triangle.
Solution:
The slope of the line 3x + 4y + 5 = 0 is $-\frac{3}{4}$
Let m be the slope of one of the line making an angle of 60° with the line 3x + 4y + 5 = 0. The angle between the lines having slope m and m_1 is 60°.

$$\therefore\ \tan 60° = \left|\frac{m-m_1}{1+m\cdot m_1}\right|, \text{ where } \tan 60° = \sqrt{3}$$

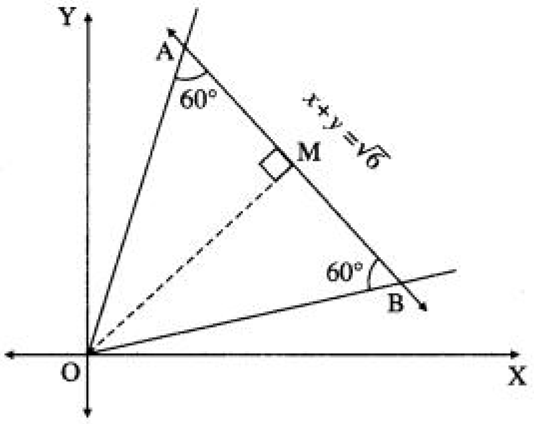
$$\therefore\ \sqrt{3} = \left|\frac{m-\left(\frac{-3}{4}\right)}{1+m\left(\frac{-3}{4}\right)}\right|$$

$$\therefore\ \sqrt{3} = \left|\frac{4m+3}{4-3m}\right|$$

On squaring both sides, we get,
$3 = \frac{(4m+3)^2}{(4-3m)^2}$
$\therefore$ $3(4 - 3m)^2 = (4m + 3)^2$
$\therefore$ $3(16 - 24m + 9m^2) = 16m^2 + 24m + 9$
$\therefore$ $48 - 72m + 27m^2 = 16m^2 + 24m + 9$
$\therefore$ $11m^2 - 96m + 39 = 0$
This is the auxiliary equation of the two lines and their joint equation is obtained by putting $m = \frac{y}{x}$.
$\therefore$ the combined equation of the two lines is
$11\left(\frac{y}{x}\right)^2 - 96\left(\frac{y}{x}\right) + 39 = 0$
$\therefore$ $\frac{11y^2}{x^2} - \frac{96y}{x} + 39 = 0$
$\therefore$ $11y^2 - 96xy + 39x^2 = 0$
$\therefore$ $39x^2 - 96xy + 11y^2 = 0$.
$\therefore$ $39x^2 - 96xy + 11y^2 = 0$ is the joint equation of the two lines through the origin each making an angle of 60° with the line 3x + 4y + 5 = 0.
The equation $39x^2 - 96xy + 11y^2 = 0$ can be written as :
$-39x^2 + 96xy - 11y^2 = 0$
i.e., $(9x^2 - 48x^2) + (24xy + 72xy) + (16y^2 - 27y^2) = 0$
i.e. $(9x^2 + 24xy + 16y^2) - (48x^2 - 72xy + 27y^2) = 0$
i.e. $(9x^2 + 24xy + 16y^2) - 3(16x^2 - 24xy + 9y^2) = 0$
i.e. $(3x + 4y)^2 - 3(4x - 3y)^2 = 0$
Hence, the line 3x + 4y + 5 = 0 and the lines
$(3x + 4y)^2 - 3(4x - 3y)^2$ form the sides of an equilateral triangle.

Question 18.
Show that lines $x^2 - 4xy + y^2 = 0$ and $x + y = \sqrt{6}$ form an equilateral triangle. Find its area and perimeter.
Solution:
$x^2 - 4xy + y^2 = 0$ and $x + y = \sqrt{6}$ form a triangle OAB which is equilateral.

Let OM be the perpendicular from the origin O to AB whose equation is $x + y = \sqrt{6}$



$$\therefore\ OM = \left|\frac{-\sqrt{6}}{\sqrt{1+1}}\right| = \sqrt{3}$$

$$\therefore\ \text{area of } \triangle OAB = \frac{(OM)^2}{\sqrt{3}}$$

$$= \frac{(\sqrt{3})^2}{\sqrt{3}} = \sqrt{3} \text{ sq units.}$$

In right angled triangle OAM,
$\sin 60° = \frac{OM}{OA}$ ∴ $\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{OA}$
∴ OA = 2
∴ length of the each side of the equilateral triangle OAB = 2 units.
∴ perimeter of Δ OAB = 3 × length of each side
= 3 × 2 = 6 units.

Question 19.
If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is square of the other then show that $a^2b + ab^2 + 8h^3 = 6abh$.
Solution:
Let m be the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$.
Then the other line has slope m^2

$$\therefore\ m + m^2 = \frac{-2h}{b} \qquad \dots (1) \text{ and}$$

$$(m)(m^2) = \frac{a}{b}$$

$$\text{i.e. } m^3 = \frac{a}{b} \qquad \dots (2)$$

$$\therefore\ (m+m^2)^3 = m^3 + (m^2)^3 + 3(m)(m^2)(m+m^2)$$

$$\dots [\because\ (p+q)^3 = p^3 + q^3 + 3pq(p+q)]$$

$$\therefore\ \left(\frac{-2h}{b}\right)^3 = \frac{a}{b} + \frac{a^2}{b^2} + 3\frac{a}{b}\left(\frac{-2h}{b}\right)$$

$$\therefore\ \frac{-8h^3}{b^3} = \frac{a}{b} + \frac{a^2}{b^2} - \frac{6ah}{b^2}$$

Multiplying by b^3, we get,
$-8h^3 = ab^2 + a^2b - 6abh$
∴ $a^2b + ab^2 + 8h^3 = 6abh$
This is the required condition.

Question 20.
Prove that the product of lengths of perpendiculars drawn from P (x_1, y_1) to the lines repersented by $ax^2 + 2hxy + by^2 = 0$ is $\left|\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{(a-b)^2+4h^2}}\right|$
Solution:
Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$.
∴ $m_1 + m_2 = -\frac{2h}{b}$ and $m_1m_2 = \frac{a}{b}$ ...(1)
The separate equations of the lines represented by
$ax^2 + 2hxy + by^2 = 0$ are
$y = m_1x$ and $y = m_2x$
i.e. $m_1x - y = 0$ and $m_2x - y = 0$
Length of perpendicular from P(x_1, 1) on

$m_1x - y = 0$ is $\left|\frac{m_1x_1 - y_1}{\sqrt{m_1^2 + 1}}\right|$.

Length of perpendicular form $P(x_1, y_1)$ on

$m_2x - y = 0$ is $\left|\frac{m_2x_1 - y_1}{\sqrt{m_2^2 + 1}}\right|$.

$\therefore$ product of lengths of perpendiculars

$$= \left|\frac{m_1x_1 - y_1}{\sqrt{m_1^2 + 1}}\right| \times \left|\frac{m_2x_1 - y_1}{\sqrt{m_2^2 + 1}}\right|$$

$$= \left|\frac{m_1m_2x_1^2 - (m_1 + m_2)x_1y_1 + y_1^2}{\sqrt{m_1^2m_2^2 + m_1^2 + m_2^2 + 1}}\right|$$

$$= \left|\frac{m_1m_2x_1^2 - (m_1 + m_2)x_1y_1 + y_1^2}{\sqrt{m_1^2m_2^2 + (m_1 + m_2)^2 - 2m_1m_2 + 1}}\right|$$

$$= \left|\frac{\frac{a}{b}.x_1^2 - \left(-\frac{2h}{b}\right)x_1y_1 + y_1^2}{\sqrt{\frac{a^2}{b^2} + \left(-\frac{2h}{b}\right)^2 - \frac{2a}{b} + 1}}\right| \quad \ldots [\text{By } (1)]$$

$$= \left|\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{a^2 + 4h^2 - 2ab + b^2}}\right|$$

$$= \left|\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a^2 - 2ab + b^2) + 4h^2}}\right|$$

$$= \left|\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}\right|.$$

Question 21.
Show that the difference between the slopes of lines given by $(\tan^2\theta + \cos^2\theta)x^2 - 2xy\tan\theta + (\sin^2\theta)y^2 = 0$ is two.
Solution:
Comparing the equation $(\tan^2\theta + \cos^2\theta)x^2 - 2xy \tan \theta + (\sin^2\theta) y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get,
$a = \tan^2\theta + \cos^2\theta$, $2h = -2 \tan \theta$ and $b = \sin^2\theta$
Let m_1 and m_2 be the slopes of the lines represented by the given equation.

$$\therefore\ m_1 + m_2 = \frac{-2h}{b} = -\left[\frac{-2\tan\theta}{\sin^2\theta}\right] = \frac{2\tan\theta}{\sin^2\theta} \quad \ldots (1)$$

$$\text{and } m_1 m_2 = \frac{a}{b} = \frac{\tan^2\theta + \cos^2\theta}{\sin^2\theta} \quad \ldots (2)$$

$$\therefore\ (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$$

$$= \left(\frac{2\tan\theta}{\sin^2\theta}\right)^2 - 4\left(\frac{\tan^2\theta + \cos^2\theta}{\sin^2\theta}\right)$$

$$= \frac{4\tan^2\theta}{\sin^4\theta} - 4\left(\frac{\tan^2\theta + \cos^2\theta}{\sin^2\theta}\right)$$

$$= \frac{4\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}{\sin^4\theta} - 4\left[\frac{\left(\frac{\sin^2\theta}{\cos^2\theta}\right) + \cos^2\theta}{\sin^2\theta}\right]$$

$$= \frac{4}{\sin^2\theta\ \cos^2\theta} - \frac{4(\sin^2\theta + \cos^4\theta)}{\sin^2\theta\ \cos^2\theta}$$

$$= 4\left[\frac{1 - \sin^2\theta - \cos^4\theta}{\sin^2\theta\ \cos^2\theta}\right]$$

$$= 4\left[\frac{\cos^2\theta - \cos^4\theta}{\sin^2\theta\ \cos^2\theta}\right]$$

$$= 4\left[\frac{\cos^2\theta(1 - \cos^2\theta)}{\sin^2\theta\ \cos^2\theta}\right] = 4$$

$$\therefore\ |m_1 - m_2| = 2$$

$\therefore$ the slopes differ by 2.

Question 22.
Find the condition that the equation $ay^2 + bxy + ex + dy = 0$ may represent a pair of lines.
Solution:
Comparing the equation
$ay^2 + bxy + ex + dy = 0$ with
$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$, we get,
$A = 0$, $H = \frac{b}{2}$, $B = a$, $G = \frac{e}{2}$, $F = \frac{d}{2}$, $C = 0$
The given equation represents a pair of lines,

$$\text{if} \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0$$

$$\text{i.e. if} \begin{vmatrix} 0 & \frac{b}{2} & \frac{e}{2} \\ \frac{b}{2} & a & \frac{d}{2} \\ \frac{e}{2} & \frac{d}{2} & 0 \end{vmatrix} = 0$$

$$\text{i.e. if } 0 - \frac{b}{2}\left(0 - \frac{ed}{4}\right) + \frac{e}{2}\left(\frac{bd}{4} - \frac{ae}{2}\right) = 0$$

$$\text{i.e. if } \frac{bed}{8} + \frac{bed}{8} - \frac{ae^2}{4} = 0$$

$$\text{i.e. if } \frac{bed}{4} - \frac{ae^2}{4} = 0$$

i.e. if $bed - ae^2 = 0$
i.e. if $e(bd - ae) = 0$
i.e. $e = 0$ or $bd - ae = 0$
i.e. $e = 0$ or $bd = ae$
This is the required condition.

Question 23.
If the lines given by $ax^2 + 2hxy + by^2 = 0$ form an equilateral triangle with the line $lx + my = 1$ then show that $(3a + b)(a + 3b) = 4h^2$.

Solution:
Since the lines $ax^2 + 2hxy + by^2 = 0$ form an equilateral triangle with the line $lx + my = 1$, the angle between the lines $ax^2 + 2hxy + by^2 = 0$ is 60°.

$$\therefore \tan 60^\circ = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$$\therefore \sqrt{3} = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

$\therefore 3(a + b)^2 = 4(h^2 - ab)$
$\therefore 3(a^2 + 2ab + b^2) = 4h^2 - 4ab$
$\therefore 3a^2 + 6ab + 3b^2 + 4ab = 4h^2$
$\therefore 3a^2 + 10ab + 3b^2 = 4h^2$
$\therefore 3a^2 + 9ab + ab + 3b^2 = 4h^2$
$\therefore 3a(a + 3b) + b(a + 3b) = 4h^2$
$\therefore (3a + b)(a + 3b) = 4h^2$
This is the required condition.

Question 24.
If line $x + 2 = 0$ coincides with one of the lines represented by the equation $x^2 + 2xy + 4y + k = 0$ then show that $k = -4$.
Solution:
One of the lines represented by
$x^2 + 2xy + 4y + k = 0$ … (1)
is $x + 2 = 0$.
Let the other line represented by (1) be $ax + by + c = 0$.
$\therefore$ their combined equation is $(x + 2)(ax + by + c) = 0$
$\therefore ax^2 + bxy + cx + 2ax + 2by + 2c = 0$
$\therefore ax^2 + bxy + (2a + c)x + 2by + 2c — 0$ … (2)
As the equations (1) and (2) are the combined equations of the same two lines, they are identical.
$\therefore$ by comparing their corresponding coefficients, we get,

$$\frac{a}{1} = \frac{b}{2} = \frac{2b}{4} = \frac{2c}{k} \text{ and } 2a + c = 0$$

$$\therefore a = \frac{2c}{k} \text{ and } c = -2a$$

$$\therefore a = \frac{2(-2a)}{k}$$

$\therefore 1 = \frac{-4}{k}$
$\therefore k = -4$.

Question 25.
Prove that the combined equation of the pair of lines passing through the origin and perpendicular to the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$
Solution:
Let m_1 and m_2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

$$\left. \begin{aligned} &\therefore m_1 + m_2 = \frac{-2h}{b} \\ &\text{and } m_1 m_2 = \frac{a}{b} \end{aligned} \right\} \quad \ldots (1)$$

Now, required lines are perpendicular to these lines.
$\therefore$ their slopes are and $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$
Since these lines are passing through the origin, their separate equations are
$y = -\frac{1}{m_1}x$ and $y = -\frac{1}{m_2}x$
i.e. $m_1 y = -x$ and $m_2 y = -x$
i.e. $x + m_1 y = 0$ and $x + m_2 y = 0$
$\therefore$ their combined equation is
$(x + m_1 y)(x + m_2 y) = 0$
$\therefore x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$
$\therefore x^2 - \frac{2h}{b}x + \frac{a}{b}y^2 = 0$
$\therefore bx^2 - 2hxy + ay^2 = 0$.

Question 26.
If equation $ax^2 - y^2 + 2y + c = 1$ represents a pair of perpendicular lines then find a and c.

Solution:
The given equation represents a pair of lines perpendicular to each other.
∴ coefficient of x^2 + coefficient of y^2 = 0
∴ $a - 1 = 0$ ∴ $a = 1$
With this value of a, the given equation is
$x^2 - y^2 + 2y + c - 1 = 0$
Comparing this equation with
$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$, we get,
$A = 1, H = 0, B = -1, G = 0, F = 1, C = c - 1$
Since the given equation represents a pair of lines,

$$D = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & c-1 \end{vmatrix} = 0$$

∴ $1(-c + 1 - 1) - 0 + 0 = 0$
∴ $-c = 0$
∴ $c = 0$.
Hence, $a = 1$, $c = 0$.