Maharashtra State Board 11th Maths Solutions Chapter 4 Methods of Induction and Binomial Theorem Ex 4.1

Prove by the method of induction, for all $n \in N$.

Question 1.
$2 + 4 + 6 + \ldots\ldots + 2n = n(n + 1)$
Solution:
Let $P(n) = 2 + 4 + 6 + \ldots\ldots + 2n = n(n + 1)$, for all $n \in N$.
Step I:
Put $n = 1$
L.H.S. = 2
R.H.S. = $1(1 + 1) = 2$
∴ L.H.S. = R.H.S.
∴ P(n) is true for $n = 1$.

Step II:
Let us assume that P(n) is true for $n = k$.
∴ $2 + 4 + 6 + \ldots\ldots + 2k = k(k + 1)$ ......(i)

Step III:
We have to prove that P(n) is true for $n = k + 1$,
i.e., to prove that
$2 + 4 + 6 + \ldots\ldots + 2(k + 1) = (k + 1)(k + 2)$
L.H.S. = $2 + 4 + 6 + \ldots + 2(k + 1)$
$= 2 + 4 + 6 + \ldots\ldots + 2k + 2(k + 1)$
$= k(k + 1) + 2(k + 1)$ .....[From (i)]
$= (k + 1).(k + 2)$
= R.H.S.
∴ P(n) is true for $n = k + 1$.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all $n \in N$.
∴ $2 + 4 + 6 + \ldots\ldots + 2n = n(n + 1)$ for all $n \in N$.

Question 2.
$3 + 7 + 11 + \ldots\ldots$ to n terms = $n(2n + 1)$
Solution:
Let $P(n) = 3 + 7 + 11 + \ldots\ldots$ to n terms = $n(2n + 1)$, for all $n \in N$.
But 3, 7, 11, ... are in A.P.
∴ $a = 3$ and $d = 4$
Let t_n be the nth term.
∴ $t_n = a + (n - 1)d = 3 + (n - 1)4 = 4n - 1$
∴ $P(n) = 3 + 7 + 11 + \ldots\ldots + (4n - 1) = n(2n + 1)$

Step I:
Put $n = 1$
L.H.S. = 3
R.H.S. = $1[2(1) + 1] = 3$
∴ L.H.S. = R.H.S.
∴ P(n) is true for $n = 1$.

Step II:
Let us assume that P(n) is true for $n = k$.
∴ $3 + 7 + 11 + \ldots\ldots + (4k - 1) = k(2k + 1)$ .....(i)

Sept III:
We have to prove that P(n) is true for $n = k + 1$,
i.e., to prove that
$3 + 7 + 11 + \ldots + [4(k + 1) - 1] = (k + 1)(2k + 3)$
L.H.S. = $3 + 7 + 11 + \ldots\ldots + [4(k + 1) - 1]$
$= 3 + 7 + 11 + \ldots\ldots + (4k - 1) + [4(k + 1) - 1]$
$= k(2k + 1) + (4k + 4 - 1)$ .....[From (i)]
$= 2k^2 + k + 4k + 3$
$= 2k^2 + 2k + 3k + 3$
$= 2k(k + 1) + 3(k + 1)$
$= (k + 1)(2k + 3)$

= R.H.S.

∴ P(n) is true for n = k + 1.

Step IV:

From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.

∴ 3 + 7 + 11 + ... to n terms = n(2n + 1) for all n ∈ N.

Question 3.

$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution:

Let $P(n) = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all n ∈ N.

Step I:

Put n = 1

L.H.S. = $1^2 = 1$

RHS = $\frac{1(1+1)[2(1)+1]}{6} = \frac{6}{6} = 1$

∴ L.H.S. = R.H.S.

∴ P(n) is true for n = 1.

Step II:

Let us assume that P(n) is true for n = k.

∴ $1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$ .....(i)

Step III:

We have to prove that P(n) is true for n = k + 1,

i.e., to prove that

$$1^2 + 2^2 + 3^2 + \ldots + (k+1)^2$$
$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\begin{aligned}
\text{L.H.S.} &= 1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 \\
&= 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \ldots\text{[From (i)]} \\
&= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\
&= (k+1)\left(\frac{2k^2 + k + 6k + 6}{6}\right) \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \text{R.H.S.}
\end{aligned}$$

∴ P(n) is true for n = k + 1.

Step IV:

From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.

∴ $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ for all n ∈ N.

Question 4.

$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n}{3}(2n-1)(2n+1)$

Solution:

Let $P(n) = 1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n}{3}(2n-1)(2n+1)$, for all n ∈ N.

Step I:

Put n = 1

L.H.S. = $1^2 = 1$

R.H.S. = $\frac{1}{3}[2(1) - 1][2(1) + 1] = 1$

∴ L.H.S. = R.H.S.

∴ P(n) is true for n = 1.

Step II:

Let us assume that P(n) is true for n = k.

∴ $1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 = \frac{k}{3}(2k-1)(2k+1)$ ....(i)

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., to prove that

$$1^2 + 3^2 + 5^2 + \ldots + [2(k+1) - 1]^2$$
$$= \frac{(k+1)}{3}[2(k+1) - 1][2(k+1) + 1]$$
$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$\text{L.H.S.} = 1^2 + 3^2 + 5^2 + \ldots + [2(k+1) - 1]^2$$
$$= 1^2 + 3^2 + 5^2 + \ldots + (2k-1)^2 + (2k+1)^2$$
$$= \frac{k}{3}(2k-1)(2k+1) + (2k+1)^2 \quad \ldots[\text{From (i)}]$$
$$= (2k+1)\left[\frac{k(2k-1)}{3} + (2k+1)\right]$$
$$= (2k+1)\left[\frac{2k^2 - k + 6k + 3}{3}\right]$$
$$= \frac{(2k+1)}{3}(2k^2 + 2k + 3k + 3)$$
$$= \frac{(2k+1)}{3}[2k(k+1) + 3(k+1)]$$
$$= \frac{(2k+1)(k+1)(2k+3)}{3}$$
$$= \text{R.H.S.}$$

∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.
∴ $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n}{3}(2n-1)(2n+1)$ for all n ∈ N.

Question 5.
$1^3 + 3^3 + 5^3 + \ldots$ to n terms $= n^2(2n^2 - 1)$
Solution:
Let P(n) = $1^3 + 3^3 + 5^3 + \ldots$ to n terms $= n^2(2n^2 - 1)$, for all n ∈ N.
But 1, 3, 5, are in A.P.
∴ a = 1, d = 2
Let tn be the nth term.
$t_n = a + (n-1)d = 1 + (n-1)2 = 2n - 1$
∴ P(n) = $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = n^2(2n^2 - 1)$

Step I:
Put n = 1
L.H.S. = $1^3 = 1$
R.H.S. = $1^2[2(1)^2 - 1] = 1$
∴ L.H.S. = R.H.S.
∴ P(n) is true for n = 1.

Step II:
Let us assume that P(n) is true for n = k.
∴ $1^3 + 3^3 + 5^3 + \ldots + (2k-1)^3 = k^2(2k^2 - 1)$ .....(i)

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., to prove that

$$
\begin{aligned}
&1^3 + 3^3 + 5^3 + \ldots + [2(k+1)-1]^3 \\
&= (k+1)^2[2(k+1)^2 - 1] \\
&= (k^2 + 2k + 1)(2k^2 + 4k + 1) \\
\text{L.H.S.} &= 1^3 + 3^3 + 5^3 + \ldots + [2(k+1)-1]^3 \\
&= 1^3 + 3^3 + 5^3 + \ldots + (2k-1)^3 + (2k+1)^3 \\
&= k^2(2k^2 - 1) + (2k+1)^3 \quad \ldots\text{[From (i)]} \\
&= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1 \\
&= 2k^4 + 8k^3 + 11k^2 + 6k + 1 \\
&= 2k^2(k^2 + 2k + 1) + 4k^3 + 9k^2 + 6k + 1 \\
&= 2k^2(k^2 + 2k + 1) + 4k(k^2 + 2k + 1) \\
&\qquad + (k^2 + 2k + 1) \\
&= (k^2 + 2k + 1)(2k^2 + 4k + 1) \\
&= \text{R.H.S.}
\end{aligned}
$$

∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.
∴ $1^3 + 3^3 + 5^3 + \ldots$ to n terms = $n^2(2n^2 - 1)$ for all n ∈ N.

Question 6.
$1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \frac{n}{3}(n+1)(n+2)$
Solution:
Let P(n) = $1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$, for all n ∈ N.

Step I:
Put n = 1
L.H.S. = 1.2 = 2
R.H.S. = $\frac{1}{3}(1+1)(1+2) = 2$
∴ L.H.S. = R.H.S.
∴ P(n) is true for n = 1.

Step II:
Let us assume that P(n) is true for n = k.
∴ $1.2 + 2.3 + 3.4 + \ldots + k(k+1) = \frac{k}{3}(k+1)(k+2)$ ……(i)

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., to prove that

$$
\begin{aligned}
&1.2 + 2.3 + 3.4 + \ldots + (k+1)[(k+1)+1] \\
&= \frac{(k+1)}{3}[(k+1)+1][(k+1)+2] \\
&= \frac{(k+1)}{3}(k+2)(k+3) \\
\text{L.H.S.} &= 1.2 + 2.3 + 3.4 + \ldots + (k+1)[(k+1)+1] \\
&= 1.2 + 2.3 + 3.4 + \ldots + k(k+1) \\
&\qquad + (k+1)(k+2) \\
&= \frac{k}{3}(k+1)(k+2) + (k+1)(k+2) \\
&\qquad \text{.[From (i)]} \\
&= (k+1)(k+2)\left(\frac{k}{3} + 1\right) \\
&= \frac{(k+1)(k+2)(k+3)}{3} \\
&= \text{R.H.S.}
\end{aligned}
$$

∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.
∴ $1.2 + 2.3 + 3.4 + \ldots + n(n+1) = \frac{n}{3}(n+1)(n+2)$, for all n ∈ N.

Question 7.
$1.3 + 3.5 + 5.7 + \ldots$ to n terms = $\frac{n}{3}(4n^2 + 6n - 1)$
Solution:

Let P(n) = 1.3 + 3.5 + 5.7 +... to n terms = $\frac{n}{3}$ ($4n^2$ + 6n -1), for all n ∈ N.
But first factor in each term, i.e., 1, 3, 5,... are in A.P. with a = 1 and d = 2.
∴ nth term = a + (n – 1)d = 1 + (n – 1) 2 = (2n – 1)
Also, second factor in each term,
i.e., 3, 5, 7, ... are in A.P. with a = 3 and d = 2.
∴ nth term = a + (n – 1) d = 3 + (n – 1) 2 = (2n + 1)
∴ nth term, t_n = (2n – 1) (2n + 1)
∴ P(n) ≡ 1.3 + 3.5 + 5.7 + ... + (2n – 1) (2n + 1) = $\frac{n}{3}$ ($4n^2$ + 6n – 1)

Step I:
Put n = 1
L.H.S. = 1.3 = 3
R.H.S. = $\frac{1}{3}$ [$4(1)^2$ + 6(1) – 1] = 3
∴ L.H.S. = R.H.S.
∴ P(n) is trae for n = 1.

Step II:
Let us assume that P(n) is true for n = k.
∴ 1.3 + 3.5 + 5.7 +....+ (2k – 1)(2k + 1) = $\frac{k}{3}$ ($4k^2$ + 6k – 1) ......(i)

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., to prove that

$$1.3 + 3.5 + 5.7 + \ldots + [2(k+1)-1][2(k+1)+1]$$
$$= \frac{(k+1)}{3}[4(k+1)^2 + 6(k+1) - 1]$$
$$= \frac{(k+1)}{3}(4k^2 + 8k + 4 + 6k + 6 - 1)$$
$$= \frac{(k+1)}{3}(4k^2 + 14k + 9)$$

$$\text{L.H.S.} = 1.3 + 3.5 + 5.7 + \ldots + [2(k+1)-1][2(k+1)+1]$$
$$= 1.3 + 3.5 + 5.7 + \ldots + (2k-1)(2k+1) + (2k+1)(2k+3)$$
$$= \frac{k}{3}(4k^2 + 6k - 1) + (2k+1)(2k+3) \quad \ldots\text{[From (i)]}$$
$$= \frac{1}{3}[4k^3 + 6k^2 - k + 3(2k+1)(2k+3)]$$
$$= \frac{1}{3}(4k^3 + 6k^2 - k + 12k^2 + 24k + 9)$$
$$= \frac{1}{3}(4k^3 + 18k^2 + 23k + 9)$$
$$= \frac{1}{3}(k+1)(4k^2 + 14k + 9)$$
$$= \text{R.H.S.}$$

∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.
∴ 1.3 + 3.5 + 5.7 +... to n terms = $\frac{n}{3}$ ($4n^2$ + 6n – 1) for all n ∈ N.

Question 8.

$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

Solution:

Let P(n) ≡ $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$, for all n ∈ N.

Step I:
Put n = 1

L.H.S. = $\frac{1}{1.3}=\frac{1}{3}$

R.H.S. = $\frac{1}{2(1)+1}=\frac{1}{3}$

∴ L.H.S. = R.H.S.
∴ P(n) is true for n = 1.

Step II:
Let us assume that P(n) is true for n = k.

∴ 11.3+13.5+15.7+…+1(2k−1)(2k+1)=k2k+1 .....(i)

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., to prove that

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{[2(k+1)-1][2(k+1)+1]}$$

$$=\frac{k+1}{2(k+1)+1}=\frac{k+1}{2k+3}$$

$$\text{L.H.S.}=\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{[2(k+1)-1][2(k+1)+1]}$$

$$=\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \quad \ldots\text{[From (i)]}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+2k+k+1}{(2k+1)(2k+3)}$$

$$=\frac{2k(k+1)+1(k+1)}{(2k+1)(2k+3)}$$

$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$=\frac{k+1}{2k+3}$$

$$=\text{R.H.S.}$$

∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.

∴ 11.3+13.5+15.7+…+1(2n−1)(2n+1)=n2n+1, for all n ∈ N.

Question 9.
13.5+15.7+17.9+… to n terms =n3(2n+3)
Solution:
Let P(n) ≡ 13.5+15.7+17.9+… to n terms =n3(2n+3), for all n ∈ N.
But first factor in each term of the denominator,
i.e., 3, 5, 7, ... are in A.P. with a = 3 and d = 2.
∴ nth term = a + (n – 1)d = 3 + (n – 1) 2 = (2n + 1)
Also, second factor in each term of the denominator,
i.e., 5, 7, 9, ... are in A.P. with a = 5 and d = 2.
∴ nth term = a + (n – 1) d = 5 + (n – 1) 2 = (2n + 3)
∴ nth term, t_n = 1(2n+1)(2n+3)

P(n) ≡ 13.5+15.7+17.9+…+1(2n+1)(2n+3) = n3(2n+3)

Step I:
Put n = 1

L.H.S. = 13.5=115

R.H.S. = 13[2(1)+3]=13(2+3)=115

∴ L.H.S. = R.H.S.
∴ P(n) is true for n = 1.

Step II:
Let us assume that P(n) is true for n = k.

$\therefore \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2k+1)(2k+3)} = \frac{k}{3(2k+3)}$ ….(i)

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., to prove that

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{[2(k+1)+1][2(k+1)+3]}$$
$$=\frac{k+1}{3[2(k+1)+3]}=\frac{k+1}{3(2k+5)}$$

$$\text{L.H.S.} = \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{[2(k+1)+1][2(k+1)+3]}$$
$$= \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2k+1)(2k+3)}+\frac{1}{(2k+3)(2k+5)}$$
$$= \frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)} \quad \ldots\text{[From (i)]}$$
$$= \frac{k(2k+5)+3}{3(2k+3)(2k+5)}$$
$$= \frac{2k^2+5k+3}{3(2k+3)(2k+5)}$$
$$= \frac{2k^2+2k+3k+3}{3(2k+3)(2k+5)}$$
$$= \frac{2k(k+1)+3(k+1)}{3(2k+3)(2k+5)}$$
$$= \frac{(2k+3)(k+1)}{3(2k+3)(2k+5)}$$
$$= \frac{k+1}{3(2k+5)}$$
$$= \text{R.H.S.}$$

∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.

$\therefore \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots$ *to n terms* $=\frac{n}{3(2n+3)}$, for all n ∈ N.

Question 10.
$(2^{3n} – 1)$ is divisible by 7.
Solution:
$(2^{3n} – 1)$ is divisible by 7 if and only if $(2^{3n} – 1)$ is a multiple of 7.
Let P(n) ≡ $(2^{3n} – 1) = 7m$, where m ∈ N.

Step I:
Put n = 1
∴ $2^{3n} – 1 = 2^{3(1)} – 1 = 2^3 – 1 = 8 – 1 = 7$
∴ $(2^{3n} – 1)$ is a multiple of 7.
∴ P(n) is true for n = 1.

Step II:
Let us assume that P(n) is true for n = k.
i.e., $2^{3k} – 1$ is a multiple of 7.
∴ $2^{3k} – 1 = 7a$, where a ∈ N
∴ $2^{3k} = 7a + 1$ ……(i)

Step III:
We have to prove that P(n) is true for n = k + 1,

i.e., to prove that
$2^{3(k+1)} - 1 = 7b$, where $b \in N$.
$\therefore P(k + 1) = 2^{3(k+1)} - 1$
$= 2^{3k+3} - 1$
$= 2^{3k} . (2^3) - 1$
$= (7a + 1)8 - 1$ …..[From (i)]
$= 56a + 8 - 1$
$= 56a + 7$
$= 7(8a + 1)$
$7b$, where $b = (8a + 1) \in N$
$\therefore$ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all $n \in N$.
$\therefore (2^{4n} - 1)$ is divisible by 7, for all $n \in N$.

Question 11.
$(2^{4n} - 1)$ is divisible by 15.
Solution:
$(2^{4n} - 1)$ is divisible by 15 if and only if $(2^{4n} - 1)$ is a multiple of 15.
Let $P(n) \equiv (2^{4n} - 1) = 15m$, where $m \in N$.

Step I:
Put n = 1
$\therefore 2^{4(1)} - 1 = 16 - 1 = 15$
$\therefore (2^{4n} - 1)$ is a multiple of 15.
$\therefore$ P(n) is true for n = 1.

Step II:
Let us assume that P(n) is true for n = k.
$\therefore 2^{4k} - 1 = 15a$, where $a \in N$
$\therefore 2^{4k} = 15a + 1$ …..(i)

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., to prove that
$\therefore 2^{4(k+1)} - 1 = 15b$, where $b \in N$
$\therefore P(k + 1) = 2^{4(k+1)} - 1 = 2^{4k+4} - 1$
$= 2^{4k} . 2^4 - 1$
$= 16 . (2^{4k}) - 1$
$= 16(15a + 1) - 1$ …..[From (i)]
$= 240a + 16 - 1$
$= 240a + 15$
$= 15(16a + 1)$
$= 15b$, where $b = (16a + 1) \in N$
$\therefore$ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all $n \in N$.
$\therefore (2^{4n} - 1)$ is divisible by 15, for all $n \in N$.

Question 12.
$3^n - 2n - 1$ is divisible by 4.
Solution:
$(3^n - 2n - 1)$ is divisible by 4 if and only if $(3^n - 2n - 1)$ is a multiple of 4.
Let $P(n) \equiv (3^n - 2n - 1) = 4m$, where $m \in N$.

Step I:
Put n = 1
$\therefore (3^n - 2n - 1) = 3^{(1)} - 2(1) - 1 = 0 = 4(0)$
$\therefore (3^n - 2n - 1)$ is a multiple of 4.
$\therefore$ P(n) is tme for n = 1.

Step II:
Let us assume that P(n) is true for n = k.
$\therefore 3^k - 2k - 1 = 4a$, where $a \in N$
$\therefore 3^k = 4a + 2k + 1$ ….(i)

Step III:
We have to prove that P(n) is tme for n = k + 1,
i.e., to prove that
$3^{(k+1)} – 2(k + 1) – 1 = 4b$, where b ∈ N
$P(k + 1) = 3^{k+1} – 2(k + 1) – 1$
$= 3^k . 3 – 2k – 2 – 1$
= (4a + 2k + 1) . 3 – 2k – 3 ....[From (i)]
= 12a + 6k + 3 – 2k – 3
= 12a + 4k
= 4(3a + k)
= 4b, where b = (3a + k) ∈ N
∴ P(n) is tme for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is tme for all n ∈ N.
∴ $3^n – 2n – 1$ is divisible by 4, for all n ∈ N.

Question 13.
$5 + 5^2 + 5^3 + ... + 5^n = \frac{5}{4} (5^n – 1)$
Solution:
Let P(n) ≡ 5 + 52 + 53 +.....+ 5n = $\frac{5}{4}$ (5n – 1), for all n ∈ N.

Step I:
Put n = 1
L.H.S. = 5
R.H.S. = $\frac{5}{4} (5^1 – 1) = 5$
∴ L.H.S. = R.H.S.
∴ P(n) is true for n = 1.

Step II:
Let us assume that P(n) is true for n = k.
∴ $5 + 5^2 + 5^3 + ... + 5^k = \frac{5}{4} (5^k – 1)$ ....(i)

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., to prove that

$$5 + 5^2 + 5^3 + \ldots + 5^{k+1} = \frac{5}{4}(5^{k+1} - 1)$$

$$\begin{aligned}
\text{L.H.S.} &= 5 + 5^2 + 5^3 + \ldots + 5^{k+1} \\
&= 5 + 5^2 + 5^3 + \ldots + 5^k + 5^{k+1} \\
&= \frac{5}{4}(5^k - 1) + 5^{k+1} \qquad \text{...[From (i)]} \\
&= \frac{5.5^k - 5 + 4.5^{k+1}}{4} \\
&= \frac{5^{k+1} + 4.5^{k+1} - 5}{4} \\
&= \frac{5.5^{k+1} - 5}{4} \\
&= \frac{5}{4}(5^{k+1} - 1) \\
&= \text{R.H.S.}
\end{aligned}$$

∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.
∴ $5 + 5^2 + 5^3 + ... + 5^n = \frac{5}{4} (5^n – 1)$, for all n ∈ N.

Question 14.
$(\cos θ + i \sin θ)^n = \cos (nθ) + i \sin (nθ)$
Solution:
Let P(n) ≡ $(\cos θ + i \sin θ)^n = \cos nθ + i \sin nθ$, for all n ∈ N.
Step I:
Put n = 1
L.H.S. = $(\cos θ + i \sin θ)^1 = \cos θ + i \sin θ$
R.H.S. = cos[(1)θ] + i sin[(1)θ] = cos θ + i sin θ

∴ L.H.S. = R.H.S.
∴ P(n) is true for n = 1.

Step II:
Let us assume that P(n) is true for n = k.
∴ $(\cos \theta + i \sin \theta)^k = \cos k\theta + i \sin k\theta$ …….(i)

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., to prove that
$(\cos \theta + i \sin \theta)^{k+1} = \cos (k + 1)\theta + i \sin (k + 1)\theta$
L.H.S. = $(\cos \theta + i \sin \theta)^{k+1}$
= $(\cos \theta + i \sin \theta)^k \cdot (\cos \theta + i \sin \theta)$
= $(\cos k\theta + i \sin k\theta) \cdot (\cos \theta + i \sin \theta)$ ……[From (i)]
= $\cos k\theta \cos \theta + i \sin \theta \cos k\theta + i \sin k\theta \cos\theta – \sin k\theta \sin \theta$ ……[∵ $i^2 = -1$]
= $(\cos k\theta \cos \theta – \sin k \theta \sin \theta) + i(\sin k\theta \cos \theta + \cos k\theta \sin \theta)$
= $\cos(k\theta + \theta) + i \sin(k\theta + \theta)$
= $\cos(k + 1) \theta + i \sin (k + 1) \theta$
= R.H.S.
∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.
∴ $(\cos \theta + i \sin \theta)^n = \cos (n\theta) + i \sin (n\theta)$, for all n ∈ N.

Question 15.
Given that $t_{n+1} = 5 t_{n+4}$, $t_1 = 4$, prove by method of induction that $t_n = 5^n – 1$.
Solution:
Let the statement P(n) has L.H.S. a recurrence relation $t_{n+1} = 5 t_{n+4}$, $t_1 = 4$ and R.H.S. a general statement $t_n = 5^n – 1$.
Step I:
Put n = 1
L.H.S. = 4
R.H.S. = $5^1 – 1 = 4$
∴ L.H.S. = R.H.S.
∴ P(n) is true for n = 1.
Put n = 2
L.H.S. = $t_2 = 5t_1 + 4 = 24$
R.H.S. = $t_2 = 5^2 – 1 = 24$
∴ L.H.S. = R.H.S.
∴ P(n) is true for n = 2.

Step II:
Let us assume that P(n) is true for n = k.
∴ $t_{k+1} = 5 t_{k+4}$ and $t_k = 5^k – 1$

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., to prove that $t_{k+1} = 5^{k+1} – 1$
Since $t_{k+1} = 5 t_{k+4}$ and $t_k = 5^k – 1$ …..[From Step II]
$t_{k+1} = 5 (5^k – 1) + 4 = 5^{k+1} – 1$
∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.
∴ $t_n = 5^n – 1$, for all n ∈ N.

Question 16.
Prove by method of induction
$(1021)^n=(102n1) \forall n \in N$
Solution:

Let $P(n) \equiv \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$, for all $n \in N$.

Step I:

Put n = 1

$$\text{L.H.S.} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\text{R.H.S.} = \begin{bmatrix} 1 & 2(1) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

∴ L.H.S. = R.H.S.

∴ P(n) is true for n = 1.

Step II

Let us assume that P(n) is true for n = k.

$$\therefore \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix} \quad \ldots\text{(i)}$$

Step III:

We have to prove that P(n) is true for n = k + 1,

i.e., to prove that $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 2(k+1) \\ 0 & 1 \end{bmatrix}$

$$\text{L.H.S.} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{k+1}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \ldots\text{[From (i)]}$$

$$= \begin{bmatrix} 1 + 2k(0) & 1(2) + 2k(1) \\ 0(1) + 1(0) & 0(2) + 1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2k+2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2(k+1) \\ 0 & 1 \end{bmatrix}$$

= R.H.S.

∴ P(n) is true for n = k + 1.

Step IV:

From all the steps above, by the principle of mathematical induction, P(n) is true for all $n \in N$.

$$\therefore \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}, \text{ for all } n \in N.$$

Maharashtra State Board 11th Maths Solutions Chapter 4 Methods of Induction and Binomial Theorem Ex 4.2

Question 1.
Expand:
(i) $(\sqrt{3} + \sqrt{2})^4$
Solution:
Here, $a = \sqrt{3}$, $b = \sqrt{2}$ and $n = 4$.
Using binomial theorem,

$$\left(\sqrt{3}+\sqrt{2}\right)^4$$
$$= {}^4C_0\left(\sqrt{3}\right)^4\left(\sqrt{2}\right)^0 + {}^4C_1\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^1 + {}^4C_2\left(\sqrt{3}\right)^2\left(\sqrt{2}\right)^2 + {}^4C_3\left(\sqrt{3}\right)^1\left(\sqrt{2}\right)^3 + {}^4C_4\left(\sqrt{3}\right)^0\left(\sqrt{2}\right)^4$$

Now, ${}^4C_0 = {}^4C_4 = 1, {}^4C_1 = {}^4C_3 = 4,$

$${}^4C_2 = \frac{4\times3}{2\times1} = 6$$

∴ $(\sqrt{3} + \sqrt{2})^4 = 1(9)(1) + 4(3\sqrt{3})(\sqrt{2}) + 6(3)(2) + 4(\sqrt{3})(2\sqrt{2}) + 1(1)(4)$
$= 9 + 12\sqrt{6} + 36 + 8\sqrt{6} + 4$
$= 49 + 20\sqrt{6}$

(ii) $(\sqrt{5} - \sqrt{2})^5$
Solution:
Here, $a = \sqrt{5}$, $b = \sqrt{2}$ and $n = 5$.
Using binomial theorem,

$$\left(\sqrt{5}-\sqrt{2}\right)^5$$
$$= {}^5C_0\left(\sqrt{5}\right)^5\left(\sqrt{2}\right)^0 - {}^5C_1\left(\sqrt{5}\right)^4\left(\sqrt{2}\right)^1 + {}^5C_2\left(\sqrt{5}\right)^3\left(\sqrt{2}\right)^2 - {}^5C_3\left(\sqrt{5}\right)^2\left(\sqrt{2}\right)^3 + {}^5C_4\left(\sqrt{5}\right)^1\left(\sqrt{2}\right)^4 - {}^5C_5\left(\sqrt{5}\right)^0\left(\sqrt{2}\right)^5$$

Now, ${}^5C_0 = {}^5C_5 = 1, {}^5C_1 = {}^5C_4 = 5,$

$${}^5C_2 = {}^5C_3 = \frac{5\times4}{2\times1} = 10$$

$$\left(\sqrt{5}-\sqrt{2}\right)^5$$
$$= 1\left(25\sqrt{5}\right)(1) - 5(25)\left(\sqrt{2}\right) + 10\left(5\sqrt{5}\right)(2) - 10(5)\left(2\sqrt{2}\right) + 5\left(\sqrt{5}\right)(4) - 1(1)\left(4\sqrt{2}\right)$$
$$= 25\sqrt{5} - 125\sqrt{2} + 100\sqrt{5} - 100\sqrt{2} + 20\sqrt{5} - 4\sqrt{2}$$
$$= 145\sqrt{5} - 229\sqrt{2}$$

Question 2.
Expand:
(i) $(2x^2 + 3)^4$
Solution:
Here, $a = 2x^2$, $b = 3$ and $n = 4$.
Using binomial theorem,

$$(2x^2+3)^4$$
$$= {}^4C_0(2x^2)^4(3)^0 + {}^4C_1(2x^2)^3(3)^1 + {}^4C_2(2x^2)^2(3)^2 + {}^4C_3(2x^2)^1(3)^3 + {}^4C_4(2x)^0(3)^4$$

Now, ${}^4C_0 = {}^4C_4 = 1, {}^4C_1 = {}^4C_3 = 4,$

$${}^4C_2 = \frac{4\times3}{2\times1} = 6$$

$$(2x^2+3)^4 = 1(16x^8)(1) + 4(8x^6)(3) + 6(4x^4)(9) + 4(2x^2)(27) + 1(1)(81)$$
$$= 16x^8 + 96x^6 + 216x^4 + 216x^2 + 81$$

(ii) $\left(2x-\frac{1}{x}\right)^6$
Solution:
Here, $a = 2x$, $b = \frac{1}{x}$ and $n = 6$.

Using binomial theorem,

$$\left(2x-\frac{1}{x}\right)^6$$

$$= {}^6C_0(2x)^6\left(\frac{1}{x}\right)^0 - {}^6C_1(2x)^5\left(\frac{1}{x}\right)^1 + {}^6C_2(2x)^4\left(\frac{1}{x}\right)^2 - {}^6C_3(2x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(2x)^2\left(\frac{1}{x}\right)^4 - {}^6C_5(2x)^1\left(\frac{1}{x}\right)^5 + {}^6C_6\,(2x)^0\left(\frac{1}{x}\right)^6$$

Now, ${}^6C_0 = {}^6C_6 = 1$, ${}^6C_1 = {}^6C_5 = 6$,

${}^6C_2 = {}^6C_4 = \frac{6\times5}{2\times1} = 15$, ${}^6C_3 = \frac{6\times5\times4}{3\times2\times1} = 20$

$$\left(2x-\frac{1}{x}\right)^6$$

$$= 1(64x^6)\,(1) - 6(32x^4) + 15(16x^2) - 20(8) + 15\left(\frac{4}{x^2}\right) - 6\left(\frac{2}{x^4}\right) + 1(1)\left(\frac{1}{x^6}\right)$$

$$= 64x^6 - 192x^4 + 240x^2 - 160 + \frac{60}{x^2} - \frac{12}{x^4} + \frac{1}{x^6}$$

Question 3.
Find the value of
(i) $(\sqrt{3} + 1)^4 - (\sqrt{3} - 1)^4$
Solution:

$$\left(\sqrt{3}+1\right)^4$$

$$= {}^4C_0\left(\sqrt{3}\right)^4(1)^0 + {}^4C_1\left(\sqrt{3}\right)^3(1)^1 + {}^4C_2\left(\sqrt{3}\right)^2(1)^2 + {}^4C_3\left(\sqrt{3}\right)^1(1)^3 + {}^4C_4\left(\sqrt{3}\right)^0(1)^4$$

Now, ${}^4C_0 = {}^4C_4 = 1$, ${}^4C_1 = {}^4C_3 = 4$,

${}^4C_2 = \frac{4\times3}{2\times1} = 6$

$$\therefore \quad \left(\sqrt{3}+1\right)^4 = 1(9)(1) + 4\left(3\sqrt{3}\right)(1) + 6(3)\,(1) + 4\left(\sqrt{3}\right)(1) + 1(1)\,(1)$$

$$\therefore \quad \left(\sqrt{3}+1\right)^4 = 9 + 12\sqrt{3} + 18 + 4\sqrt{3} + 1 \quad \ldots\text{(i)}$$

Also, $\left(\sqrt{3}-1\right)^4$

$$= {}^4C_0\left(\sqrt{3}\right)^4(1)^0 - {}^4C_1\left(\sqrt{3}\right)^3(1)^1 + {}^4C_2\left(\sqrt{3}\right)^2(1)^2 - {}^4C_3\left(\sqrt{3}\right)^1(1)^3 + {}^4C_4\left(\sqrt{3}\right)^0(1)^4$$

$$= 1(9)(1) - 4\left(3\sqrt{3}\right)(1) + 6(3)(1) - 4\left(\sqrt{3}\right)(1) + 1(1)(1)$$

$$\left(\sqrt{3}-1\right)^4 = 9 - 12\sqrt{3} + 18 - 4\sqrt{3} + 1 \quad \ldots\text{(ii)}$$

Subtracting (ii) from (i), we get

$$\left(\sqrt{3}+1\right)^4 - \left(\sqrt{3}-1\right)^4$$

$$= (9 + 12\sqrt{3} + 18 + 4\sqrt{3} + 1) - (9 - 12\sqrt{3} + 18 - 4\sqrt{3} + 1)$$

$$= 24\sqrt{3} + 8\sqrt{3}$$

$$= 32\sqrt{3}$$

(ii) $(2 + \sqrt{5})^5 + (2 - \sqrt{5})^5$
Solution:

$$(2+\sqrt{5})^5$$
$$= {}^5C_0(2)^5\left(\sqrt{5}\right)^0 + {}^5C_1(2)^4\left(\sqrt{5}\right)^1 + {}^5C_2(2)^3\left(\sqrt{5}\right)^2 + {}^5C_3(2)^2\left(\sqrt{5}\right)^3 + {}^5C_4(2)^1\left(\sqrt{5}\right)^4 + {}^5C_5(2)^0\left(\sqrt{5}\right)^5$$

Now, ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$,

$${}^5C_2 = {}^5C_3 = \frac{5\times 4}{2\times 1} = 10$$

$$\therefore \quad (2+\sqrt{5})^5$$
$$= 1(32)(1) + 5(16)\left(\sqrt{5}\right) + 10(8)(5) + 10(4)\left(5\sqrt{5}\right) + 5(2)(25) + 1(1)(25\sqrt{5})$$

$$\therefore \quad (2+\sqrt{5})^5$$
$$= 32 + 80\sqrt{5} + 400 + 200\sqrt{5} + 250 + 25\sqrt{5} \quad \ldots\text{(i)}$$

Also, $(2-\sqrt{5})^5$
$$= {}^5C_0(2)^5\left(\sqrt{5}\right)^0 - {}^5C_1(2)^4\left(\sqrt{5}\right)^1 + {}^5C_2(2)^3\left(\sqrt{5}\right)^2 - {}^5C_3(2)^2\left(\sqrt{5}\right)^3 + {}^5C_4(2)^1\left(\sqrt{5}\right)^4 - {}^5C_5(2)^0\left(\sqrt{5}\right)^5$$
$$= 1(32)(1) - 5(16)\left(\sqrt{5}\right) + 10(8)(5) - 10(4)\left(5\sqrt{5}\right) + 5(2)(25) - 1(1)(25\sqrt{5})$$

$$\therefore \quad (2-\sqrt{5})^5$$
$$= 32 - 80\sqrt{5} + 400 - 200\sqrt{5} + 250 - 25\sqrt{5} \quad \ldots\text{(ii)}$$

Adding (i) and (ii), we get
$\therefore (2 + \sqrt{5})^5 + (2 - \sqrt{5})^5 = (32 + 80\sqrt{5} + 400 + 200\sqrt{5} + 250 + 25\sqrt{5}) + (32 - 80\sqrt{5} + 400 - 200\sqrt{5} + 250 - 25\sqrt{5})$
= 64 + 800 + 500
= 1364

Question 4.
Prove that:
(i) $(\sqrt{3} + \sqrt{2})^6 + (\sqrt{3} - \sqrt{2})^6 = 970$
Solution:

$$\left(\sqrt{3}+\sqrt{2}\right)^6$$
$$= {}^6C_0\left(\sqrt{3}\right)^6\left(\sqrt{2}\right)^0 + {}^6C_1\left(\sqrt{3}\right)^5\left(\sqrt{2}\right)^1 + {}^6C_2\left(\sqrt{3}\right)^4\left(\sqrt{2}\right)^2 + {}^6C_3\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3 + {}^6C_4\left(\sqrt{3}\right)^2\left(\sqrt{2}\right)^4 + {}^6C_5\left(\sqrt{3}\right)^1\left(\sqrt{2}\right)^5 + {}^6C_6\left(\sqrt{3}\right)^0\left(\sqrt{2}\right)^6$$

Now, ${}^6C_0 = {}^6C_6 = 1$, ${}^6C_1 = {}^6C_5 = 6$,

${}^6C_2 = {}^6C_4 = \frac{6\times 5}{2\times 1} = 15$, ${}^6C_3 = \frac{6\times 5\times 4}{3\times 2\times 1} = 20$

$\therefore \quad \left(\sqrt{3}+\sqrt{2}\right)^6$
$$= 1(27)(1) + 6\left(9\sqrt{3}\right)\left(\sqrt{2}\right) + 15(9)(2) + 20\left(3\sqrt{3}\right)\left(2\sqrt{2}\right) + 15(3)(4) + 6\left(\sqrt{3}\right)\left(4\sqrt{2}\right) + 1(1)(8)$$

$\therefore \quad \left(\sqrt{3}+\sqrt{2}\right)^6$
$$= 27 + 54\sqrt{6} + 270 + 120\sqrt{6} + 180 + 24\sqrt{6} + 8 \quad \ldots\text{(i)}$$

Also, $\left(\sqrt{3}-\sqrt{2}\right)^6$
$$= {}^6C_0\left(\sqrt{3}\right)^6\left(\sqrt{2}\right)^0 - {}^6C_1\left(\sqrt{3}\right)^5\left(\sqrt{2}\right)^1 + {}^6C_2\left(\sqrt{3}\right)^4\left(\sqrt{2}\right)^2 - {}^6C_3\left(\sqrt{3}\right)^3\left(\sqrt{2}\right)^3 + {}^6C_4\left(\sqrt{3}\right)^2\left(\sqrt{2}\right)^4 - {}^6C_5\left(\sqrt{3}\right)^1\left(\sqrt{2}\right)^5 + {}^6C_6\left(\sqrt{3}\right)^0\left(\sqrt{2}\right)^6$$
$$= 1(27)(1) - 6\left(9\sqrt{3}\right)\left(\sqrt{2}\right) + 15(9)(2) - 20\left(3\sqrt{3}\right)\left(2\sqrt{2}\right) + 15(3)(4) - 6\left(\sqrt{3}\right)\left(4\sqrt{2}\right) + 1(1)(8)$$

$\left(\sqrt{3}-\sqrt{2}\right)^6$
$$= 27 - 54\sqrt{6} + 270 - 120\sqrt{6} + 180 - 24\sqrt{6} + 8 \quad \ldots\text{(ii)}$$

Adding (i) and (ii), we get

$\left(\sqrt{3}+\sqrt{2}\right)^6 + \left(\sqrt{3}-\sqrt{2}\right)^6$
$$= (27 + 54\sqrt{6} + 270 + 120\sqrt{6} + 180 + 24\sqrt{6} + 8) + (27 - 54\sqrt{6} + 270 - 120\sqrt{6} + 180 - 24\sqrt{6} + 8)$$
$$= 54 + 540 + 360 + 16$$
$$= 970$$

(ii) $(\sqrt{5} + 1)^5 - (\sqrt{5} - 1)^5 = 352$

Solution:

$\left(\sqrt{5}+1\right)^5$

$= {}^5C_0\left(\sqrt{5}\right)^5(1)^0 + {}^5C_1\left(\sqrt{5}\right)^4(1)^1$

$+ {}^5C_2\left(\sqrt{5}\right)^3(1)^2 + {}^5C_3\left(\sqrt{5}\right)^2(1)^3$

$+ {}^5C_4\left(\sqrt{5}\right)^1(1)^4 + {}^5C_5\left(\sqrt{5}\right)^0(1)^5$

Now, ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$,

${}^5C_2 = {}^5C_3 = 10$

$\therefore \quad \left(\sqrt{5}+1\right)^5$

$= 1\left(25\sqrt{5}\right)(1) + 5(25)(1) + 10\left(5\sqrt{5}\right)(1)$

$+ 10(5)(1) + 5\sqrt{5}(1) + 1(1)(1)$

$\therefore \quad \left(\sqrt{5}+1\right)^5 = 25\sqrt{5} + 125 + 50\sqrt{5} + 50 + 5\sqrt{5} + 1 \quad \ldots(i)$

Also, $\left(\sqrt{5}-1\right)^5$

$= {}^5C_0\left(\sqrt{5}\right)^5(1)^0 - {}^5C_1\left(\sqrt{5}\right)^4(1)^1 + {}^5C_2\left(\sqrt{5}\right)^3(1)^2$

$- {}^5C_3\left(\sqrt{5}\right)^2(1)^3 + {}^5C_4\left(\sqrt{5}\right)^1(1)^4 - {}^5C_5\left(\sqrt{5}\right)^0(1)^5$

$= 1\left(25\sqrt{5}\right)(1) - 5(25)(1) + 10\left(5\sqrt{5}\right)(1)$

$- 10(5)(1) + 5\sqrt{5}(1) - 1(1)(1)$

$\left(\sqrt{5}-1\right)^5 = 25\sqrt{5} - 125 + 50\sqrt{5} - 50 + 5\sqrt{5} - 1 \quad \ldots(ii)$

Subtracting (ii) from (i), we get

$\left(\sqrt{5}+1\right)^5 - \left(\sqrt{5}-1\right)^5$

$= (25\sqrt{5} + 125 + 50\sqrt{5} + 50 + 5\sqrt{5} + 1)$

$- (25\sqrt{5} - 125 + 50\sqrt{5} - 50 + 5\sqrt{5} - 1)$

$= 250 + 100 + 2$

$= 352$

Question 5.
Using binomial theorem, find the value of
(i) $(102)^4$
Solution:

$(102)^4 = (100 + 2)^4$

$= {}^4C_0(100)^4(2)^0 + {}^4C_1(100)^3(2)^1$

$+ {}^4C_2(100)^2(2)^2 + {}^4C_3(100)^1(2)^3 + {}^4C_4(100)^0(2)^4$

Now, ${}^4C_0 = {}^4C_4 = 1$, ${}^4C_1 = {}^4C_3 = 4$, ${}^4C_2 = 6$

$\therefore \quad (102)^4 = 1(100000000)(1) + 4(1000000)(2)$

$+ 6(10000)(4) + 4(100)(8) + 1(1)(16)$

$= 100000000 + 8000000 + 240000 + 3200 + 16$

$= 108243216$

(ii) $(1.1)^5$
Solution:

$(1.1)^5 = (1 + 0.1)^5$

$= {}^5C_0(1)^5(0.1)^0 + {}^5C_1(1)^4(0.1)^1 + {}^5C_2(1)^3(0.1)^2$

$+ {}^5C_3(1)^2(0.1)^3 + {}^5C_4(1)^1(0.1)^4 + {}^5C_5(1)^0(0.1)^5$

Now, ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$,

${}^5C_2 = {}^5C_3 = 10$

$\therefore \quad (1.1)^5 = 1(1)(1) + 5(1)(0.1) + 10(1)(0.01)$

$+ 10(1)(0.001) + 5(1)(0.0001) + 1(1)(0.00001)$

$= 1 + 0.5 + 0.1 + 0.01 + 0.0005 + 0.00001$

$= 1.61051$

Question 6.
Using binomial theorem, find the value of
(i) $(9.9)^3$
Solution:

$(9.9)^3 = (10 - 0.1)^3$
$= {}^3C_0 (10)^3(0.1)^0 - {}^3C_1 (10)^2(0.1)^1 + {}^3C_2 (10)^1(0.1)^2 - {}^3C_3 (10)^0(0.1)^3$
Now, ${}^3C_0 = {}^3C_3 = 1$, ${}^3C_1 = {}^3C_2 = 3$
$\therefore \quad (9.9)^3 = 1(1000)(1) - 3(100)(0.1) + 3(10)(0.01) - 1(1)(0.001)$
$= 1000 - 30 + 0.3 - 0.001$
$= 1000.3 - 30.001$
$= 970.299$

(ii) $(0.9)^4$
Solution:

$(0.9)^4 = (1 - 0.1)^4$
$= {}^4C_0 (1)^4 (0.1)^0 - {}^4C_1 (1)^3(0.1)^1 + {}^4C_2(1)^2(0.1)^2 - {}^4C_3 (1)^1(0.1)^3 + {}^4C_4 (1)^0(0.1)^4$
Now, ${}^4C_0 = {}^4C_4 = 1$, ${}^4C_1 = {}^4C_3 = 4$, ${}^4C_2 = 6$
$\therefore \quad (0.9)^4 = 1(1)(1) - 4(1)(0.1) + 6(1)(0.01) - 4(1)(0.001) + 1(1)(0.0001)$
$= 1 - 0.4 + 0.06 - 0.004 + 0.0001$
$= 1.0601 - 0.404$
$= 0.6561$

Question 7.
Without expanding, find the value of
(i) $(x + 1)^4 - 4(x + 1)^3 (x - 1) + 6(x + 1)^2 (x - 1)^2 - 4(x + 1) (x - 1)^3 + (x - 1)^4$
Solution:

Let $x + 1 = a$ and $x - 1 = b$
We notice that 1, 4, 6, 4, 1 are the values of ${}^4C_0$, ${}^4C_1$, ${}^4C_2$, ${}^4C_3$, ${}^4C_4$ respectively.
$\therefore$ The given expression becomes
${}^4C_0\, a^4b^0 - {}^4C_1\, a^3 b + {}^4C_2\, a^2 b^2 - {}^4C_3\, a b^3 + {}^4C_4 a^0b^4$
$= (a - b)^4$
$= [x + 1 - (x - 1)]^4 = (x + 1 - x + 1)^4$
$= 2^4$
$= 16$

(ii) $(2x - 1)^4 + 4(2x - 1)^3 (3 - 2x) + 6(2x - 1)^2 (3 - 2x)^2 + 4(2x - 1)^1 (3 - 2x)^3 + (3 - 2x)^4$
Solution:

Let $2x - 1 = a$ and $3 - 2x = b$
We notice that 1, 4, 6, 4, 1 are the values of ${}^4C_0$, ${}^4C_1$, ${}^4C_2$, ${}^4C_3$, ${}^4C_4$ respectively.
$\therefore$ The given expression becomes
${}^4C_0\, a^4b^0 + {}^4C_1\, a^3b + {}^4C_2\, a^2 b^2 + {}^4C_3\, ab^3 + {}^4C_4\, a^0b^4$
$= (a + b)^4$
$= (2x - 1 + 3 - 2x)^4 = 2^4$
$= 16$

Question 8.
Find the value of $(1.02)^6$, correct upto four places of decimals.

Solution:

$$(1.02)^6 = (1 + 0.02)^6$$
$$= {}^6C_0 (1)^6 (0.02)^0 + {}^6C_1 (1)^5 (0.02)^1 + {}^6C_2 (1)^4 (0.02)^2 + {}^6C_3 (1)^3 (0.02)^3 + {}^6C_4 (1)^2 (0.02)^4 + {}^6C_5 (1)^1 (0.02)^5 + {}^6C_6 (1)^0 (0.02)^6$$

Now, ${}^6C_0 = {}^6C_6 = 1$, ${}^6C_1 = {}^6C_5 = 6$,

$${}^6C_2 = {}^6C_4 = \frac{6\times5}{2\times1} = 15,\ {}^6C_3 = \frac{6\times5\times4}{3\times2\times1} = 20$$

$$\therefore\ (1.02)^6 = 1(1)(1) + 6(1)(0.02) + 15(1)(0.0004) + 20(1)(0.000008) + \ldots$$
$$= 1 + 0.12 + 0.0060 + 0.000160 + \ldots$$
$$= 1.12616$$
$$= 1.1262, \text{ correct upto 4 decimal places.}$$

Question 9.

Find the value of $(1.01)^5$, correct upto three places of decimals.

Solution:

$$(1.01)^5 = (1 + 0.01)^5$$
$$= {}^5C_0 (1)^5 (0.01)^0 + {}^5C_1 (1)^4 (0.01)^1 + {}^5C_2 (1)^3 (0.01)^2 + {}^5C_3 (1)^2 (0.01)^3 + {}^5C_4 (1)^1 (0.01)^4 + {}^5C_5 (1)^0 (0.01)^5$$

Now, ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$,

$${}^5C_2 = {}^5C_3 = \frac{5\times4}{2\times1} = 10$$

$$\therefore\ (1.01)^5 = 1(1)(1) + 5(1)(0.01) + 10(1)(0.\ 0001) + 10(1)(0.000001) + \ldots$$
$$= 1 + 0.05 + 0.001 + 0.00001 + \ldots$$
$$= 1.05101$$
$$= 1.051, \text{ correct upto 3 decimal places.}$$

Question 10.

Find the value of $(0.9)^6$, correct upto four places of decimals.

Solution:

$$(0.9)^6 = (1 - 0.1)^6$$
$$= {}^6C_0 (1)^6 (0.1)^0 - {}^6C_1 (1)^5 (0.1)^1 + {}^6C_2 (1)^4 (0.1)^2 - {}^6C_3 (1)^3 (0.1)^3 + {}^6C_4 (1)^2 (0.1)^4 - {}^6C_5 (1)^1 (0.1)^5 + {}^6C_6 (1)^0 (0.1)^6$$

Now, ${}^6C_0 = {}^6C_6 = 1$, ${}^6C_1 = {}^6C_5 = 6$,

$${}^6C_2 = {}^6C_4 = \frac{6\times5}{2\times1} = 15,\ {}^6C_3 = \frac{6\times5\times4}{3\times2\times1} = 20$$

$$\therefore\ (0.9)^6 = 1(1)(1) - 6(1)(0.1) + 15(1)(0.01) - 20(1)(0.001) + 15(1)(0.0001) - 6(1)(0.00001) + 1(1)(0.000001)$$
$$= 1 - 0.6 + 0.15 - 0.02 + 0.0015 - 0.00006 + 0.000001$$
$$= 0.531441$$
$$= 0.5314, \text{ correct upto 4 decimal places.}$$

Maharashtra State Board 11th Maths Solutions Chapter 4 Methods of Induction and Binomial Theorem Ex 4.3

Question 1.
In the following expansions, find the indicated term.
(i) $\left(2x^2+\frac{3}{2x}\right)^8$, 3rd term
Solution:

Here, $a = 2x^2$, $b = \frac{3}{2x}$, $n = 8$.
For 3^{rd} term, $r = 2$
We have $t_{r+1} = {}^nC_r\, a^{n-r}.b^r$

$$\therefore \quad t_3 = {}^8C_2 (2x^2)^{8-2}\left(\frac{3}{2x}\right)^2 = \frac{8!}{2!\,6!}\times\left(2x^2\right)^6\times\left(\frac{3}{2x}\right)^2$$
$$= \frac{8\times7}{2\times1}\times 64x^{12}\times\frac{9}{4x^2}$$
$$= 28\times 64x^{12}\times\frac{9}{4x^2}$$
$$= 4032\,x^{10}$$

$\therefore$ 3^{rd} term in the expansion of $\left(2x^2+\frac{3}{2x}\right)^8$ is $4032\,x^{10}$.

(ii) $\left(x^2-\frac{4}{x^3}\right)^{11}$, 5th term
Solution:

Here, $a = x^2$, $b = \frac{-4}{x^3}$, $n = 11$.
For 5^{th} term, $r = 4$
We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$\therefore \quad t_5 = {}^{11}C_4 (x^2)^{11-4}\left(\frac{-4}{x^3}\right)^4 = \frac{11!}{4!\,7!}\times(x^2)^7\times\left(\frac{-4}{x^3}\right)^4$$
$$= \frac{11\times10\times9\times8}{4\times3\times2\times1}\times x^{14}\times\frac{256}{x^{12}}$$
$$= 330\times256\times x^2$$
$$= 84480\,x^2$$

$\therefore$ 5^{th} term in the expansion of $\left(x^2-\frac{4}{x^3}\right)^{11}$ is $84480\,x^2$.

(iii) $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$, 7th term

Solution:

Here, $a = \frac{4x}{5}$, $b = \frac{-5}{2x}$, $n = 9$.

For 7^{th} term, $r = 6$

We have $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

$$\therefore \quad t_7 = {}^9C_6 \left(\frac{4x}{5}\right)^{9-6} \left(\frac{-5}{2x}\right)^6 = \frac{9!}{6!\,3!} \times \left(\frac{4x}{5}\right)^3 \times \left(\frac{-5}{2x}\right)^6$$

$$= \frac{9\times8\times7}{3\times2\times1} \times \frac{4^3.x^3}{5^3} \times \frac{5^6}{2^6.x^6}$$

$$= 84 \times 64 \times \frac{5^3}{64.x^3}$$

$$= \frac{84\times125}{x^3}$$

$$= \frac{10500}{x^3}$$

$\therefore$ 7^{th} term in the expansion of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$ is $\frac{10500}{x^3}$.

(iv) In $\left(\frac{1}{3}+a^2\right)^{12}$, 9th term

Solution:

Here, $a = \frac{1}{3}$, $b = a^2$, $n = 12$.

For 9^{th} term, $r = 8$

We have $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

$$\therefore \quad t_9 = {}^{12}C_8 \left(\frac{1}{3}\right)^{12-8} (a^2)^8 = \frac{12!}{8!\,4!} \times \left(\frac{1}{3}\right)^4 \times a^{16}$$

$$= \frac{12\times11\times10\times9}{4\times3\times2\times1} \times \frac{1}{81} \times a^{16}$$

$$= \frac{55}{9}a^{16}$$

$\therefore$ 9^{th} term in the expansion of $\left(\frac{1}{3}+a^2\right)^{12}$ is $\frac{55}{9}a^{16}$.

(v) In $\left(3a+\frac{4}{a}\right)^{13}$, 10th term

Solution:

Here, $a = 3a$, $b = \frac{4}{a}$, $n = 13$.

For 10^{th} term, $r = 9$

We have $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

$$\therefore \quad t_{10} = {}^{13}C_9\, (3a)^{13-9} \left(\frac{4}{a}\right)^9 = \frac{13!}{9!\,4!} \times (3a)^4 \times \left(\frac{4}{a}\right)^9$$

$$= \frac{13\times12\times11\times10}{4\times3\times2\times1} \times 81a^4 \times \frac{4^9}{a^9}$$

$$= \frac{143\times5\times81\times256\times256\times4}{a^5}$$

$$= \frac{15182069760}{a^5}$$

$\therefore$ 10^{th} term in the expansion of $\left(3a+\frac{4}{a}\right)^{13}$ is $\frac{15182069760}{a^5}$.

Question 2.
In the following expansions, find the indicated coefficients.
(i) x^3 in $\left(x^2+\frac{3\sqrt{2}}{x}\right)^9$
Solution:

Here, $a = x^2$, $b = \frac{3\sqrt{2}}{x}$, $n = 9$.

We have $t_{r+1} = {}^nC_r a^{n-r} b^r$

$$= {}^9C_r (x^2)^{9-r}\left(\frac{3\sqrt{2}}{x}\right)^r$$

$$= {}^9C_r x^{18-2r} (3\sqrt{2})^r . x^{-r}$$

$$= {}^9C_r \left(3\sqrt{2}\right)^r x^{18-3r}$$

To get the coefficient of x^3, we must have
$x^{18-3r} = x^3$

$\therefore$ $18 - 3r = 3$

$\therefore$ $15 = 3r$

$\therefore$ $r = 5$

$\therefore$ Coefficient of x^3 $= {}^9C_5\left(3\sqrt{2}\right)^5$

$$= \frac{9!}{5!\,4!}\left(3\sqrt{2}\right)^5$$

$$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times 243 \times 4\sqrt{2}$$

$$= 122472\sqrt{2}$$

(ii) x^8 in $\left(2x^5-\frac{5}{x^3}\right)^8$
Solution:

Here, $a = 2x^5$, $b = \frac{-5}{x^3}$, $n = 8$.

We have $t_{r+1} = {}^nC_r a^{n-r} b^r$

$$= {}^8C_r (2x^5)^{8-r}\left(\frac{-5}{x^3}\right)^r$$

$$= {}^8C_r (2)^{8-r} x^{40-5r} (-5)^r x^{-3r}$$

$$= {}^8C_r (2)^{8-r} (-5)^r x^{40-8r}$$

To get the coefficient of x^8, we must have
$x^{40-8r} = x^8$

$\therefore$ $40 - 8r = 8$

$\therefore$ $8r = 32$

$\therefore$ $r = 4$

$\therefore$ Coefficient of x^8 $= {}^8C_4 (2)^4 (-5)^4$

$$= \frac{8!}{4!\,4!}(2)^4(-5)^4$$

$$= \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} \times 16 \times 625$$

$$= 700000$$

(iii) x9 in $\left(\frac{1}{x}+x^2\right)^{18}$

Solution:

Here, $a = \frac{1}{x}$, $b = x^2$, $n = 18$.

We have $t_{r+1} = {}^nC_r\, a^{n-r}.b^r$

$= {}^{18}C_r \left(\frac{1}{x}\right)^{18-r} (x^2)^r$

$= {}^{18}C_r\, (x^{-1})^{18-r}.x^{2r}$

$= {}^{18}C_r\, x^{r-18}.\, x^{2r}$

$= {}^{18}C_r\, x^{3r-18}$

To get the coefficient of x^9, we must have

$x^{3r-18} = x^9$

$\therefore$ $3r - 18 = 9$

$\therefore$ $3r = 27$

$\therefore$ $r = 9$

$\therefore$ Coefficient of x^9

$= {}^{18}C_9$

$= \frac{18!}{9!9!}$

$= \frac{18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11 \times 10}{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$

$= 48620$

(iv) x-3 in $\left(x-\frac{1}{2x}\right)^5$

Solution:

Here, $a = x$, $b = \frac{-1}{2x}$, $n = 5$.

We have $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

$= {}^5C_r (x)^{5-r} \left(\frac{-1}{2x}\right)^r$

$= {}^5C_r\, x^{5-r} \left(\frac{-1}{2}\right)^r x^{-r}$

$= {}^5C_r \left(\frac{-1}{2}\right)^r x^{5-2r}$

To get the coefficient of x^{-3}, we must have

$x^{5-2r} = x^{-3}$

$\therefore$ $5 - 2r = -3$

$\therefore$ $2r = 8$

$\therefore$ $r = 4$

$\therefore$ Coefficient of x^{-3} $= {}^5C_4 \left(\frac{-1}{2}\right)^4$

$= \frac{5!}{4!1!} \times \frac{1}{16} = 5 \times \frac{1}{16}$

$= \frac{5}{16}$

(v) x^{-20} in $\left(x^3-\frac{1}{2x^2}\right)^{15}$

Solution:

Here, $a = x^3$, $b = \frac{-1}{2x^2}$, $n = 15$.

We have $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

$$= {}^{15}C_r\, (x^3)^{15-r}\left(\frac{-1}{2x^2}\right)^r$$

$$= {}^{15}C_r\, x^{45-3r}\left(\frac{-1}{2}\right)^r x^{-2r}$$

$$= {}^{15}C_r\left(\frac{-1}{2}\right)^r x^{45-5r}$$

To get the coefficient of x^{-20}, we must have

$x^{45-5r} = x^{-20}$

$\therefore$ $45 - 5r = -20$

$\therefore$ $5r = 65$

$\therefore$ $r = 13$

$\therefore$ Coefficient of $x^{-20} = {}^{15}C_{13}\left(\frac{-1}{2}\right)^{13}$

$$= \frac{15!}{13!\,2!}\left(\frac{-1}{2}\right)^{13}$$

$$= \frac{15\times 14}{2\times 1}\times\left(\frac{-1}{8192}\right)$$

$$= \frac{-105}{8192}$$

Question 3.

Find the constant term (term independent of x) in the expansion of

(i) $\left(2x+\frac{1}{3x^2}\right)^9$

Solution:

Here, $a = 2x$, $b = \frac{1}{3x^2}$, $n = 9$.

We have $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

$$= {}^9C_r\, (2x)^{9-r}\left(\frac{1}{3x^2}\right)^r$$

$$= {}^9C_r(2)^{9-r} x^{9-r}\left(\frac{1}{3}\right)^r x^{-2r}$$

$$= {}^9C_r\, (2)^{9-r}\left(\frac{1}{3}\right)^r x^{9-3r}$$

To get the term independent of x, we must have

$x^{9-3r} = x^0$

$\therefore$ $9 - 3r = 0$

$\therefore$ $r = 3$

$\therefore$ The term independent of $x = {}^9C_3\, (2)^6\left(\frac{1}{3}\right)^3$

$$= \frac{9!}{3!\,6!}(2)^6\left(\frac{1}{3}\right)^3$$

$$= \frac{9\times 8\times 7}{3\times 2\times 1}\times\frac{64}{27}$$

$$= \frac{1792}{9}$$

(ii) $\left(x-\frac{2}{x^2}\right)^{15}$

Solution:

Here, $a = x$, $b = \frac{-2}{x^2}$, $n = 15$.

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$= {}^{15}C_r\,(x)^{15-r}\left(\frac{-2}{x^2}\right)^r$$

$$= {}^{15}C_r\, x^{15-r}\,(-2)^r\, x^{-2r}$$

$$= {}^{15}C_r\,(-2)^r\, x^{15-3r}$$

To get the term independent of x, we must have

$x^{15-3r} = x^0$

$\therefore$ $15 - 3r = 0$

$\therefore$ $r = 5$

$\therefore$ The term independent of x

$= {}^{15}C_5\,(-2)^5$

$= \frac{15!}{5!\,10!}(-2)^5$

$= \frac{15\times14\times13\times12\times11}{5\times4\times3\times2\times1} \times (-32)$

$= -96096$

(iii) $\left(\sqrt{x}-\frac{3}{x^2}\right)^{10}$

Solution:

Here, $a = \sqrt{x}$, $b = \frac{-3}{x^2}$, $n = 10$.

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$= {}^{10}C_r\,(\sqrt{x})^{10-r}\left(\frac{-3}{x^2}\right)^r$$

$$= {}^{10}C_r\,(x)^{\frac{10-r}{2}}\,(-3)^r\, x^{-2r}$$

$$= {}^{10}C_r\,(-3)^r\, x^{\frac{10-r}{2}}\, x^{-2r}$$

$$= {}^{10}C_r\,(-3)^r\, x^{\frac{10-5r}{2}}$$

To get the term independent of x, we must have

$x^{\frac{10-5r}{2}} = x^0$

$\therefore$ $\frac{10-5r}{2} = 0$

$\therefore$ $10 - 5r = 0$

$\therefore$ $r = 2$

$\therefore$ The term independent of x $= {}^{10}C_2\,(-3)^2$

$$= \frac{10!}{2!\,8!}(-3)^2$$

$$= \frac{10\times9}{2\times1}\times 9$$

$$= 405$$

(iv) $\left(x^2-\frac{1}{x}\right)^9$

Solution:

Here, $a = x^2$, $b = \frac{-1}{x}$, $n = 9$.

We have $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

$= {}^9C_r\, (x^2)^{9-r}\left(\frac{-1}{x}\right)^r$

$= {}^9C_r\, x^{18-2r} (-1)^r . x^{-r}$

$= {}^9C_r\, (-1)^r x^{18-3r}$

To get the term independent of x, we must have $x^{18-3r} = x^0$

$\therefore$ $18 - 3r = 0$

$\therefore$ $r = 6$

$\therefore$ The term independent of x $= {}^9C_6 (-1)^6$

$= \frac{9!}{6!\,3!} \times 1$

$= \frac{9 \times 8 \times 7}{3 \times 2 \times 1}$

$= 84$

(v) $\left(2x^2-\frac{5}{x}\right)^9$

Solution:

Here, $a = 2x^2$, $b = \frac{-5}{x}$, $n = 9$.

We have $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

$= {}^9C_r (2x^2)^{9-r}\left(\frac{-5}{x}\right)^r$

$= {}^9C_r\, 2^{9-r} x^{18-2r} (-5)^r x^{-r}$

$= {}^9C_r\, 2^{9-r} (-5)^r x^{18-3r}$

To get the term independent of x, we must have $x^{18-3r} = x^0$

$\therefore$ $18 - 3r = 0$

$\therefore$ $r = 6$

$\therefore$ The term independent of x

$= {}^9C_6\, 2^3 (-5)^6$

$= \frac{9!}{6!\,3!} (2)^3 (-5)^6$

$= \frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times 8 \times 15625$

$= 10500000$

Question 4.
Find the middle terms in the expansion of

(i) $\left(\frac{x}{y}+\frac{y}{x}\right)^{12}$

Solution:

Here, $a = \frac{x}{y}$, $b = \frac{y}{x}$, $n = 12$.

Now, n is even.

$\therefore$ $\frac{n+2}{2} = \frac{12+2}{2} = 7$

$\therefore$ Middle term is t_7, for which $r = 6$.

We have $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

$\therefore$ $t_7 = {}^{12}C_6 \left(\frac{x}{y}\right)^6 . \left(\frac{y}{x}\right)^6 = \frac{12!}{6!\,6!}$

$= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 924$

$\therefore$ Middle term is 924.

(ii) (X2+1x)7

Solution:

Here, $a = x^2$, $b = \frac{1}{x}$, $n = 7$.

Now, n is odd.

$\therefore \quad \frac{n+1}{2} = \frac{7+1}{2} = 4, \; \frac{n+3}{2} = \frac{7+3}{2} = 5$

$\therefore$ Middle terms are t_4 and t_5, for which $r = 3$ and $r = 4$ respectively.

We have $t_{r+1} = {}^nC_r a^{n-r} b^r$

$$\therefore \quad t_4 = {}^7C_3 (x^2)^4 \left(\frac{1}{x}\right)^3 = \frac{7!}{3!4!}\left(x^8\right) \times \frac{1}{x^3}$$

$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1}\left(x^8\right) \times \frac{1}{x^3}$$

$$= 35x^5$$

$$\text{and } t_5 = {}^7C_4 (x^2)^3 \left(\frac{1}{x}\right)^4 = \frac{7!}{4!3!}\left(x^6\right) \times \frac{1}{x^4}$$

$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1}\left(x^6\right) \times \frac{1}{x^4}$$

$$= 35x^2$$

$\therefore$ Middle terms are $35x^5$ and $35x^2$.

(iii) (X2−2x)8

Solution:

Here, $a = x^2$, $b = \frac{-2}{x}$, $n = 8$.

Now, n is even.

$\therefore \quad \frac{n+2}{2} = \frac{8+2}{2} = 5$

$\therefore$ Middle term is t_5, for which $r = 4$

We have $t_{r+1} = {}^nC_r a^{n-r} b^r$

$$\therefore \quad t_5 = {}^8C_4 \left(x^2\right)^4 \left(\frac{-2}{x}\right)^4$$

$$= \frac{8!}{4!\,4!}\left(x^8\right)\left(\frac{16}{x^4}\right)$$

$$= \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} \times \frac{x^8}{x^4} \times 16$$

$$= 1120x^4$$

$\therefore$ Middle term is $1120x^4$.

(iv) (xa−ax)10

Solution:

Here, $a = \frac{x}{a}$, $b = \frac{-a}{x}$, $n = 10$.

Now, n is even.

$\therefore \quad \frac{n+2}{2} = \frac{10+2}{2} = 6$

$\therefore$ Middle term is t_6, for which $r = 5$.

We have $t_{r+1} = {}^nC_r a^{n-r} b^r$

$$\therefore \quad t_6 = {}^{10}C_5 \cdot \left(\frac{x}{a}\right)^5 \cdot \left(-\frac{a}{x}\right)^5$$

$$= {}^{10}C_5 \times -1$$

$$= \frac{-10!}{5!5!}$$

$$= -\frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}$$

$$= -252$$

$\therefore$ Middle term is -252.

(v) $(x^4-\frac{1}{x^3})^{11}$
Solution:

Here, $a = x^4$, $b = \frac{-1}{x^3}$, $n = 11$.

Now, n is odd.

$\therefore \quad \frac{n+1}{2} = \frac{11+1}{2} = 6, \ \frac{n+3}{2} = \frac{11+3}{2} = 7$

$\therefore$ Middle terms are t_6 and t_7; for which $r = 5$ and $r = 6$ respectively.

We have $t_{r+1} = {}^nC_r \, a^{n-r} b^r$

$$\therefore \quad t_6 = {}^{11}C_5 \left(x^4\right)^6 \left(\frac{-1}{x^3}\right)^5$$

$$= \frac{11!}{5!6!} \left(x^{24}\right) \times \left(\frac{-1}{x^{15}}\right)$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1} \left(-x^9\right)$$

$$= -462x^9$$

$$\text{and } t_7 = {}^{11}C_6 \left(x^4\right)^5 \left(\frac{-1}{x^3}\right)^6$$

$$= \frac{11!}{6!5!} \left(x^4\right)^5 \left(\frac{-1}{x^3}\right)^6$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1} \left(x^{20}\right) \times \frac{1}{x^{18}}$$

$$= 462x^2$$

$\therefore$ Middle terms are $-462x^9$ and $462x^2$.

Question 5.
In the expansion of $(k + x)^8$, the coefficient of x^5 is 10 times the coefficient of x^6. Find the value of k.
Solution:

Coefficient of x^5 in $(k + x)^8 = {}^8C_5 \ k^{8-5}$
$= {}^8C_5 \, k^3$

Coefficient of x^6 in $(k + x)^8 = {}^8C_6 \ k^2$

The coefficient of x^5 is 10 times the coefficient of x^6.

$\therefore \quad {}^8C_5 \, k^3 = 10 \left({}^8C_6 \, k^2\right)$

$\therefore \quad \frac{8 \times 7 \times 6}{3 \times 2 \times 1} k = 10 \left(\frac{8 \times 7}{2 \times 1}\right)$

$\therefore \quad k = 5$

Question 6.
Find the term containing x^6 in the expansion of $(2 - x)(3x + 1)^9$.

Solution:

$(2-x)(3x+1)^9 = 2(3x+1)^9 - x(3x+1)^9$

Consider $(3x+1)^9$

Here, $a = 3x$, $b = 1$, $n = 9$

We have $t_{r+1} = {}^nC_r\, a^{n-r}.\, b^r$

$= {}^9C_r\,(3x)^{9-r}.(1)^r$

$= {}^9C_r\, 3^{9-r}\, x^{9-r}$

To get the coefficient of x^6 in $2(3x+1)^9$, we must have

$x^{9-r} = x^6$

$\therefore$ $9 - r = 6$

$\therefore$ $r = 3$ ...(i)

Also, to get the coefficient of x^6 in $x(3x+1)^9$, we must have

$x.x^{9-r} = x^6$

$\therefore$ $x^{9-r} = x^5$

$\therefore$ $9 - r = 5$

$\therefore$ $r = 4$ ...(ii)

$\therefore$ The term containing x^6 in the expansion of $2(3x+1)^9 - x(3x+1)^9$

$= 2\,{}^9C_r\, 3^{9-r} - {}^9C_r\, 3^{9-r}$

$= 2\,{}^9C_3\, 3^{9-3} - {}^9C_4\, 3^{9-4}$ ...[From (i) and (ii)]

$= 2 \times 84 \times (3^6) - 126 \times 3^5$

$= 2 \times 3^5\,(3 \times 84 - 63)$

$= 2 \times 243(252 - 63)$

$= 486 \times 189 = 91854$

Question 7.
The coefficient of x^2 in the expansion of $(1 + 2x)^m$ is 112. Find m.
Solution:
The coefficient of x^2 in $(1 + 2x)^m = {}^mC_2\,(2^2)$
Given that the coefficient of $x^2 = 112$
$\therefore {}^mC_2\,(4) = 112$
$\therefore {}^mC_2 = 28$
$\therefore \frac{m!}{2!(m-2)!} = 28$
$\therefore \frac{m(m-1)(m-2)!}{2 \times (m-2)!} = 28$
$\therefore m(m-1) = 56$
$\therefore m^2 - m - 56 = 0$
$\therefore (m-8)(m+7) = 0$
As m cannot be negative.
$\therefore m = 8$

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Question 1.
State, by writing the first four terms, the expansion of the following, where $|x| < 1$.
(i) $(1 + x)^{-4}$

Solution:

$$(1+x)^{-4}$$
$$= 1 + (-4)x + \frac{(-4)(-4-1)}{2!}x^2 + \frac{(-4)(-4-1)(-4-2)}{3!}x^3 + \dots$$
$$= 1 + (-4)x + \frac{(-4)(-5)}{2}x^2 + \frac{(-4)(-5)(-6)}{6}x^3 + \dots$$
$$= 1 - 4x + 10x^2 - 20x^3 + \dots$$

(ii) $(1 - x)^{1/3}$

Solution:

$$(1-x)^{\frac{1}{3}}$$
$$= 1 - \frac{1}{3}x + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2!}x^2 - \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3 + \dots$$
$$= 1 - \frac{x}{3} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}x^2 \quad \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}x^3 + \dots$$
$$= 1 - \frac{x}{3} - \frac{1}{9}x^2 - \frac{5}{81}x^3 - \dots$$

(iii) $(1 - x^2)^{-3}$

Solution:

$$(1-x^2)^{-3}$$
$$= 1 + (-3)(-x^2) + \frac{(-3)(-3-1)}{2!}(-x^2)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}(-x^2)^3 + \dots$$
$$= 1 + 3x^2 + \frac{(-3)(-4)}{2}x^4 - \frac{(-3)(-4)(-5)}{6}x^6 + \dots$$
$$= 1 + 3x^2 + 6x^4 + 10x^6 + \dots$$

(iv) $(1 + x)^{-1/5}$

Solution:

$$(1+x)^{-1/5}$$
$$= 1 + \left(\frac{-1}{5}\right)x + \frac{\left(\frac{-1}{5}\right)\left(\frac{-1}{5}-1\right)}{2!}x^2 + \frac{\left(\frac{-1}{5}\right)\left(\frac{-1}{5}-1\right)\left(\frac{-1}{5}-2\right)}{3!}x^3 + \dots$$
$$= 1 - \frac{x}{5} + \frac{\left(\frac{-1}{5}\right)\left(\frac{-6}{5}\right)}{2}x^2 + \frac{\left(\frac{-1}{5}\right)\left(\frac{-6}{5}\right)\left(\frac{-11}{5}\right)}{6}x^3 + \dots$$
$$= 1 - \frac{x}{5} + \frac{3x^2}{25} - \frac{11x^3}{125} + \dots$$

(v) $(1 + x^2)^{-1}$
Solution:

$$(1+x^2)^{-1}$$
$$= 1 + (-1)x^2 + \frac{(-1)(-1-1)}{2!}(x^2)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}(x^2)^3 + \ldots$$
$$= 1 - x^2 + \frac{(-1)(-2)}{2}x^4 + \frac{(-1)(-2)(-3)}{6}x^6 + \ldots$$
$$= 1 - x^2 + x^4 - x^6 + \ldots$$

Question 2.
State by writing first four terms, the expansion of the following, where $|b| < |a|$.
(i) $(a - b)^{-3}$
Solution:
$(a - b)^{-3} = \left[a\left(1-\frac{b}{a}\right)\right]^{-3}$

$$= a^{-3}\left(1-\frac{b}{a}\right)^{-3}$$
$$= a^{-3}\left[1+(-3)\left(-\frac{b}{a}\right)+\frac{(-3)(-3-1)}{2!}\left(-\frac{b}{a}\right)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}\left(-\frac{b}{a}\right)^3+\ldots\right]$$
$$= a^{-3}\left[1+\frac{3b}{a}+\frac{(-3)(-4)}{2}\cdot\frac{b^2}{a^2}-\frac{(-3)(-4)(-5)}{6}\cdot\frac{b^3}{a^3}+\ldots\right]$$
$$= a^{-3}\left[1+\frac{3b}{a}+\frac{6b^2}{a^2}+\frac{10b^3}{a^3}+\ldots\right]$$

(ii) $(a + b)^{-4}$
Solution:

$$(a+b)^{-4}$$
$$= \left[a\left(1+\frac{b}{a}\right)\right]^{-4}$$
$$= a^{-4}\left(1+\frac{b}{a}\right)^{-4}$$
$$= a^{-4}\left[1+(-4)\frac{b}{a}+\frac{(-4)(-4-1)}{2!}\left(\frac{b}{a}\right)^2 + \frac{(-4)(-4-1)(-4-2)}{3!}\left(\frac{b}{a}\right)^3+\ldots\right]$$
$$= a^{-4}\left[1-\frac{4b}{a}+\frac{(-4)(-5)}{2}\cdot\frac{b^2}{a^2}+\frac{(-4)(-5)(-6)}{6}\cdot\frac{b^3}{a^3}+\ldots\right]$$
$$= a^{-4}\left[1-\frac{4b}{a}+\frac{10b^2}{a^2}-\frac{20b^3}{a^3}+\ldots\right]$$

(iii) $(a + b)^{1/4}$

Solution:

$$(a+b)^{1/4}$$

$$= \left[a\left(1+\frac{b}{a}\right)\right]^{\frac{1}{4}}$$

$$= a^{\frac{1}{4}}\left(1+\frac{b}{a}\right)^{\frac{1}{4}}$$

$$= a^{\frac{1}{4}}\left[1+\left(\frac{1}{4}\right)\frac{b}{a}+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}\left(\frac{b}{a}\right)^2+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{3!}\left(\frac{b}{a}\right)^3+\dots\right]$$

$$= a^{\frac{1}{4}}\left[1+\frac{b}{4a}+\frac{\frac{1}{4}\left(\frac{-3}{4}\right)}{2}\cdot\frac{b^2}{a^2}+\frac{\frac{1}{4}\left(\frac{-3}{4}\right)\left(\frac{-7}{4}\right)}{6}\cdot\frac{b^3}{a^3}+\dots\right]$$

$$= a^{\frac{1}{4}}\left[1+\frac{b}{4a}-\frac{3b^2}{32a^2}+\frac{7b^3}{128a^3}-\dots\right]$$

(iv) $(a – b)^{-1/4}$

Solution:

$(a – b)^{-1/4} = [a(1-ba)]^{-14}$

$$= a^{\frac{-1}{4}}\left(1-\frac{b}{a}\right)^{\frac{-1}{4}}$$

$$= a^{\frac{-1}{4}}\left[1+\left(-\frac{1}{4}\right)\left(-\frac{b}{a}\right)+\frac{\left(-\frac{1}{4}\right)\left(\frac{-1}{4}-1\right)}{2!}\left(-\frac{b}{a}\right)^2+\frac{\left(\frac{-1}{4}\right)\left(\frac{-1}{4}-1\right)\left(\frac{-1}{4}-2\right)}{3!}\left(-\frac{b}{a}\right)^3+\dots\right]$$

$$= a^{\frac{-1}{4}}\left[1+\frac{b}{4a}+\frac{\left(\frac{-1}{4}\right)\left(\frac{-5}{4}\right)}{2}\cdot\frac{b^2}{a^2}-\frac{\left(\frac{-1}{4}\right)\left(\frac{-5}{4}\right)\left(\frac{-9}{4}\right)}{6}\cdot\frac{b^3}{a^3}+\dots\right]$$

$$= a^{\frac{-1}{4}}\left[1+\frac{b}{4a}+\frac{5b^2}{32a^2}+\frac{15b^3}{128a^3}+\dots\right]$$

(v) $(a + b)^{-1/3}$

Solution:

$$(a+b)^{-1/3} = \left[a\left(1+\frac{b}{a}\right)\right]^{\frac{-1}{3}}$$

$$= a^{\frac{-1}{3}}\left(1+\frac{b}{a}\right)^{\frac{-1}{3}}$$

$$= a^{\frac{-1}{3}}\left[1+\left(\frac{-1}{3}\right)\frac{b}{a}+\frac{\left(\frac{-1}{3}\right)\left(-\frac{1}{3}-1\right)}{2!}\left(\frac{b}{a}\right)^2 + \frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)\left(\frac{-1}{3}-2\right)}{3!}\left(\frac{b}{a}\right)^3+\ldots\right]$$

$$= a^{\frac{-1}{3}}\left[1-\frac{b}{3a}+\frac{\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)}{2}\cdot\frac{b^2}{a^2} + \frac{\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)\left(\frac{-7}{3}\right)}{6}\cdot\frac{b^3}{a^3}+\ldots\right]$$

$$= a^{\frac{-1}{3}}\left[1-\frac{b}{3a}+\frac{2b^2}{9a^2}-\frac{14b^3}{81a^3}+\ldots\right]$$

Question 3.

Simplify the first three terms in the expansion of the following:

(i) $(1 + 2x)^{-4}$

Solution:

$$(1+2x)^{-4}$$

$$= 1+(-4)(2x)+\frac{(-4)(-4-1)}{2!}(2x)^2+\ldots$$

$$= 1-8x+\frac{(-4)(-5)}{2}(4x^2)+\ldots$$

$$= 1-8x+40x^2+\ldots$$

(ii) $(1 + 3x)^{-1/2}$

Solution:

$$(1+3x)^{\frac{-1}{2}}$$

$$= 1+\left(\frac{-1}{2}\right)(3x)+\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}(3x)^2+\ldots$$

$$= 1-\frac{3x}{2}+\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2}(9x^2)+\ldots$$

$$= 1-\frac{3x}{2}+\frac{27}{8}x^2+\ldots$$

(iii) $(2 - 3x)^{1/3}$

Solution:

$$(2-3x)^{\frac{1}{3}}$$

$$= \left[2\left(1-\frac{3x}{2}\right)\right]^{\frac{1}{3}}$$

$$= 2^{\frac{1}{3}}\left(1-\frac{3x}{2}\right)^{\frac{1}{3}}$$

$$= 2^{\frac{1}{3}}\left[1-\left(\frac{1}{3}\right)\left(\frac{3x}{2}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2!}\left(\frac{3x}{2}\right)^2-\ldots\right]$$

$$= 2^{\frac{1}{3}}\left[1-\frac{x}{2}+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2}\cdot\left(\frac{9x^2}{4}\right)-\ldots\right]$$

$$= 2^{\frac{1}{3}}\left[1-\frac{x}{2}-\frac{x^2}{4}-\ldots\right]$$

(iv) $(5 + 4x)^{-1/2}$

Solution:

$$(5+4x)^{\frac{-1}{2}}$$

$$= \left[5\left(1+\frac{4}{5}x\right)\right]^{\frac{-1}{2}}$$

$$= 5^{\frac{-1}{2}}\left(1+\frac{4x}{5}\right)^{\frac{-1}{2}}$$

$$= 5^{\frac{-1}{2}}\left[1+\left(\frac{-1}{2}\right)\left(\frac{4x}{5}\right)+\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}\left(\frac{4x}{5}\right)^2+\ldots\right]$$

$$= 5^{\frac{-1}{2}}\left[1-\frac{2x}{5}+\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2}\left(\frac{16x^2}{25}\right)+\ldots\right]$$

$$= 5^{\frac{-1}{2}}\left[1-\frac{2x}{5}+\frac{6x^2}{25}+\ldots\right]$$

(v) $(5 - 3x)^{-1/3}$

Solution:

$$(5-3x)^{\frac{-1}{3}}$$

$$= \left[5\left(1-\frac{3x}{5}\right)\right]^{\frac{-1}{3}}$$

$$= 5^{\frac{-1}{3}}\left(1-\frac{3x}{5}\right)^{\frac{-1}{3}}$$

$$= 5^{\frac{-1}{3}}\left[1+\left(\frac{-1}{3}\right)\left(-\frac{3x}{5}\right)+\frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)}{2!}\left(-\frac{3x}{5}\right)^2+\ldots\right]$$

$$= 5^{\frac{-1}{3}}\left[1+\frac{x}{5}+\frac{\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)}{2}\left(\frac{9x^2}{25}\right)+\ldots\right]$$

$$= 5^{\frac{-1}{3}}\left[1+\frac{x}{5}+\frac{2x^2}{25}+\ldots\right]$$

Question 4.

Use the binomial theorem to evaluate the following upto four places of decimals.

(i) √99

Solution:

$$\sqrt{99} = (99)^{\frac{1}{2}}$$

$$= (100-1)^{\frac{1}{2}}$$

$$= \left[100\left(1-\frac{1}{100}\right)\right]^{\frac{1}{2}}$$

$$= (100)^{\frac{1}{2}}\ (1-0.01)^{\frac{1}{2}}$$

$$= 10\left[1-\frac{1}{2}(0.01)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(0.01)^2-\ldots\right]$$

$$= 10\left[1-\frac{1}{2}(0.01)+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)}{2}(0.01)^2-\ldots\right]$$

$$= 10\left[1-0.005-\frac{0.0001}{8}-\ldots\right]$$

= 10 [1 – 0.005 – 0.0000125 – ……]

= 10(0.9949875)

= 9.94987 5

= 9.9499

(ii) 126---√$_3$

Solution:

$$\sqrt[3]{126}$$

$$= (125+1)^{\frac{1}{3}}$$

$$= \left[125\left(1+\frac{1}{125}\right)\right]^{\frac{1}{3}}$$

$$= (125)^{\frac{1}{3}}\left(1+\frac{1}{125}\right)^{\frac{1}{3}}$$

$$= 5(1+0.008)^{\frac{1}{3}}$$

$$= 5\left[1+\frac{1}{3}(0.008)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(0.008)^2+\ldots\right]$$

$$= 5\left[1+\frac{1}{3}(0.008)+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2}(0.008)^2+\ldots\right]$$

$$= 5\left[1+0.00266-\frac{0.000064}{9}\right]$$

$$= 5\ (1.00266 - 0.000007)$$

$$= 5\ (1.002653)$$

$$= 5.013265$$

$$= 5.0133$$

(iii) 16.08----√$_4$

Solution:

$$\sqrt[4]{16.08}$$

$$= (16+0.08)^{1/4}$$

$$= [16(1+0.005)]^{1/4}$$

$$= 16^{\frac{1}{4}}(1+0.005)^{1/4}$$

$$= 2\left[1+\frac{1}{4}(0.005)+\frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2!}(0.005)^2+\ldots\right]$$

$$= 2\left[1+\frac{1}{4}(0.005)+\frac{\frac{1}{4}\left(\frac{-3}{4}\right)}{2}(0.005)^2+\ldots\right]$$

$$= 2\left[1+0.00125-\frac{0.000075}{32}+\ldots\right]$$

$$= 2\ [1 + 0.00125 - 0.0000023 + \ldots]$$

$$= 2\ (1.0012477)$$

$$= 2.0024954$$

$$= 2.0025$$

(iv) $(1.02)^{-5}$
Solution:

$$(1.02)^{-5}$$
$$= (1 + 0.02)^{-5}$$
$$= 1 + (-5)(0.02) + \frac{(-5)(-5-1)}{2!}(0.02)^2 + \frac{(-5)(-5-1)(-5-2)}{3!}(0.02)^3 + \dots$$
$$= 1 + (-5)(0.02) + \frac{(-5)(-6)}{2}(0.02)^2 + \frac{(-5)(-6)(-7)}{6}(0.02)^3 + \dots$$
$$= 1 - 0.1 + 0.006 - 0.00028 + \dots$$
$$= 0.90572$$
$$= 0.9057$$

(v) $(0.98)^{-3}$
Solution:

$$(0.98)^{-3}$$
$$= (1 - 0.02)^{-3}$$
$$= 1 + (-3)(-0.02) + \frac{(-3)(-3-1)}{2!}(-0.02)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}(-0.02)^3 + \dots$$
$$= 1 + 0.06 + \frac{(-3)(-4)}{2}(0.02)^2 - \frac{(-3)(-4)(-5)}{6}(0.02)^3 + \dots$$
$$= 1 + 0.06 + 0.0024 + 0.00008 + \dots$$
$$= 1.06248$$
$$= 1.0625$$

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Question 1.
Show that $C_0 + C_1 + C_2 + \dots + C_8 = 256$
Solution:
Since $C_0 + C_1 + C_2 + C_3 + \dots + C_n = 2^n$
Putting n = 8, we get
$C_0 + C_1 + C_2 + \dots + C_8 = 2^8$
$\therefore C_0 + C_1 + C_2 + \dots + C_8 = 256$

Question 2.
Show that $C_0 + C_1 + C_2 + \dots + C_9 = 512$
Solution:
Since $C_0 + C_1 + C_2 + C_3 + \dots + C_n = 2^n$
Putting n = 9, we get
$C_0 + C_1 + C_2 + \dots + C_9 = 2^9$
$\therefore C_0 + C_1 + C_2 + \dots + C_9 = 512$

Question 3.
Show that $C_1 + C_2 + C_3 + \ldots + C_7 = 127$
Solution:
Since $C_0 + C_1 + C_2 + C_3 + \ldots + C_n = 2^n$
Putting n = 7, we get
$C_0 + C_1 + C_2 + \ldots + C_7 = 2^7$
$\therefore C_0 + C_1 + C_2 + \ldots + C_7 = 128$
But, $C_0 = 1$
$\therefore 1 + C_1 + C_2 + \ldots + C_7 = 128$
$\therefore C_1 + C_2 + \ldots + C_7 = 128 - 1 = 127$

Question 4.
Show that $C_1 + C_2 + C_3 + \ldots + C_6 = 63$
Solution:
Since $C_0 + C_1 + C_2 + C_3 + \ldots + C_n = 2^n$
Putting n = 6, we get
$C_0 + C_1 + C_2 + \ldots + C_6 = 2^6$
$\therefore C_0 + C_1 + C_2 + \ldots + C_6 = 64$
But, $C_0 = 1$
$\therefore 1 + C_1 + C_2 + \ldots + C_6 = 64$
$\therefore C_1 + C_2 + \ldots + C_6 = 64 - 1 = 63$

Question 5.
Show that $C_0 + C_2 + C_4 + C_6 + C_8 = C_1 + C_3 + C_5 + C_7 = 128$
Solution:
Since $C_0 + C_1 + C_2 + C_3 + \ldots + C_n = 2^n$
Putting n = 8, we get
$C_0 + C_1 + C_2 + C_3 + \ldots + C_8 = 2^8$
But, sum of even coefficients = sum of odd coefficients
$\therefore C_0 + C_2 + C_4 + C_6 + C_8 = C_1 + C_3 + C_5 + C_7$
Let $C_0 + C_2 + C_4 + C_6 + C_8 = C_1 + C_3 + C_5 + C_7 = k$
Now, $C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8 = 256$
$\therefore (C_0 + C_2 + C_4 + C_6 + C_8) + (C_1 + C_3 + C_5 + C_7) = 256$
$\therefore k + k = 256$
$\therefore 2k = 256$
$\therefore k = 128$
$\therefore C_0 + C_2 + C_4 + C_6 + C_8 = C_1 + C_3 + C_5 + C_7 = 128$

Question 6.
Show that $C_1 + C_2 + C_3 + \ldots + C_n = 2^n - 1$
Solution:
Since $C_0 + C_1 + C_2 + C_3 + \ldots + C_n = 2^n$
But, $C_0 = 1$
$\therefore 1 + C_1 + C_2 + C_3 + \ldots + C_n = 2^n$
$\therefore C_1 + C_2 + C_3 + \ldots + C_n = 2^n - 1$

Question 7.
Show that $C_0 + 2C_1 + 3C_2 + 4C_3 + \ldots + (n + 1)C_n = (n + 2) 2^{n-1}$
Solution:

Since $C_0 + C_1 + C_2 + C_3 + \ldots + C_n = 2^n$

$$\begin{aligned}
&C_0 + 2C_1 + 3C_2 + 4C_3 + \ldots + (n+1)C_n \\
&= (C_0 + C_1 + C_2 + C_3 + \ldots + C_n) \\
&\qquad + (C_1 + 2C_2 + 3C_3 + \ldots + nC_n) \\
&= 2^n + (C_1 + 2C_2 + 3C_3 + \ldots + nC_n) \\
&= 2^n + \left[n + \frac{2n(n-1)}{2!} + \frac{3n(n-1)(n-2)}{3!} + \ldots + n(1)\right] \\
&= 2^n + n\left[1 + \frac{(n-1)}{1!} + \frac{(n-1)(n-2)}{2!} + \ldots + 1\right] \\
&= 2^n + n\left[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1}\right] \\
&= 2^n + n.\, 2^{n-1} \\
&= 2^{n-1}.2 + n.2^{n-1} \\
&= (2+n).2^{n-1} \\
&= (n+2).2^{n-1}
\end{aligned}$$

Maharashtra State Board 11th Maths Solutions Chapter 4 Methods of Induction and Binomial Theorem Miscellaneous Exercise 4

(I) Select the correct answers from the given alternatives.

Question 1.
The total number of terms in the expression of $(x + y)^{100} + (x - y)^{100}$ after simplification is:
(A) 50
(B) 51
(C) 100
(D) 202
Answer:
(B) 51
Hint:

$$(x+y)^{100}$$
$$= x^{100} + {}^{100}C_1\, x^{99}\, y + {}^{100}C_2\, x^{98}\, y^2 + \ldots + y^{100}$$
$$(x-y)^{100}$$
$$= x^{100} - {}^{100}C_1\, x^{99}\, y + {}^{100}C_2\, x^{98}\, y^2 - \ldots + y^{100}$$
$$(x+y)^{100} + (x-y)^{100}$$
$$= 2\underbrace{\left[x^{100} + {}^{100}C_2\, x^{98}\, y^2 + \ldots + y^{100}\right]}_{51 \text{ terms}}$$

Question 2.
The middle term in the expansion of $(1 + x)^{2n}$ will be:
(A) $(n - 1)^{th}$
(B) n^{th}
(C) $(n + 1)^{th}$
(D) $(n + 2)^{th}$
Answer:
(C) $(n + 1)^{th}$
Hint:
$(1 + x)^{2n}$ has $(2n + 1)$ terms.
$\therefore$ $(n + 1)^{th}$ term is the middle term.

Question 3.
In the expansion of $(x^2 - 2x)^{10}$, the coefficient of x^{16} is
(A) -1680
(B) 1680
(C) 3360
(D) 6720
Answer:
(C) 3360
Hint:
$(x^2 - 2x)^{10} = x^{10} (x - 2)^{10}$
To get the coefficient of x^{16} in $(x^2 - 2x)^{10}$,
we need to check coefficient of x^6 in $(x - 2)^{10}$
$\therefore$ Required coefficient $= {}^{10}C_6 (-2)^4$
$= 210 \times 16$
$= 3360$

Question 4.
The term not containing x in expansion of $(1 - x)^2 \left(x + \frac{1}{x}\right)^{10}$ is
(A) ${}^{11}C_5$
(B) ${}^{10}C_5$
(C) ${}^{10}C_4$
(D) ${}^{10}C_7$
Answer:
(A) ${}^{11}C_5$

Hint:

$$(1-x)^2\left(x+\frac{1}{x}\right)^{10}$$

$$= \left(1-2x+x^2\right) \underbrace{\left(x+\frac{1}{x}\right)^{10}}_{\text{This expansion has even indices of } x}$$

Term not containing x = A + B,

where $A = {}^{10}C_r x^{10-r}\left(\frac{1}{x}\right)^r$ for r such that

$10 - r - r = 0$, i.e., $r = 5$

$A = {}^{10}C_5$

and $B = {}^{10}C_r\, x^{10-r}\left(\frac{1}{x}\right)^r$

for r such that $10 - r - r = -2$

i.e., $r = 6$

$B = {}^{10}C_6$

Required coefficient = $A + B = {}^{10}C_5 + {}^{10}C_6$

$= {}^{10}C_5 + {}^{10}C_4 = {}^{11}C_5$

Question 5.
The number of terms in expansion of $(4y + x)^8 - (4y - x)^8$ is
(A) 4
(B) 5
(C) 8
(D) 9
Answer:
(A) 4
Hint:

$$(4y+x)^8 = (4y)^8 + {}^8C_1(4y)^7 x + \ldots + (x)^8$$

$$(4y-x)^8 = (4y)^8 - {}^8C_1(4y)^7 x + \ldots + (x)^8$$

$$(4y+x)^8 - (4y-x)^8$$

$$= 2\left[{}^8C_1(4y)^7 x + {}^8C_3(4y)^5 x^3 + {}^8C_5(4y)^3 x^5 + {}^8C_7(4y)x^7\right]$$

$\therefore$ Number of terms = 4

Question 6.
The value of ${}^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 + \ldots + {}^{14}C_{11}$ is
(A) $2^{14} - 1$
(B) $2^{14} - 14$
(C) 2^{12}
(D) $2^{13} - 14$
Answer:
(D) $2^{13} - 14$
Hint:

$${}^{14}C_1 + {}^{14}C_3 + \ldots + {}^{14}C_{13} = 2^{14-1} = 2^{13}$$

$${}^{14}C_1 + {}^{14}C_3 + \ldots + {}^{14}C_{11} + {}^{14}C_{13} = 2^{13}$$

$${}^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 + \ldots + {}^{14}C_{11} = 2^{13} - {}^{14}C_{13}$$

$$= 2^{13} - 14$$

Question 7.
The value of ${}^{11}C_2 + {}^{11}C_4 + {}^{11}C_6 + {}^{11}C_8$ is equal to
(A) $2^{10} - 1$
(B) $2^{10} - 11$
(C) $2^{10} + 12$
(D) $2^{10} - 12$
Answer:
(D) $2^{10} - 12$

Hint:

$$^{11}C_0 + {}^{11}C_2 + \dots + {}^{11}C_8 + {}^{11}C_{10} = 2^{11-1} = 2^{10}$$

$$^{11}C_2 + {}^{11}C_4 + {}^{11}C_6 + {}^{11}C_8$$

$$= 2^{10} - \left({}^{11}C_0 + {}^{11}C_{10}\right)$$

$$= 2^{10} - (1 + 11)$$

$$= 2^{10} - 12$$

Question 8.
In the expansion of $(3x + 2)^4$, the coefficient of the middle term is
(A) 36
(B) 54
(C) 81
(D) 216
Answer:
(D) 216
Hint:
$(3x + 2)^4$ has 5 terms.
∴ $(3x + 2)^4$ has 3rd term as the middle term.
The coefficient of the middle term

$$= {}^4C_2 \, (3)^{4-2} \, (2)^2 \qquad \dots \left[\begin{array}{l} T_{r+1} = {}^nC_r (a)^{n-r} (b)^r \\ \text{in } (a+b)^n \\ n = 4, r = 2 \end{array}\right]$$

= 6 × 9 × 4
= 216

Question 9.
The coefficient of the 8th term in the expansion of $(1 + x)^{10}$ is:
(A) 7
(B) 120
(C) $^{10}C_8$
(D) 210
Answer:
(B) 120
Hint:
r = 7
$t_8 = {}^{10}C_7 x^7 = {}^{10}C_3 x^7$
∴ Coefficient of 8th term = $^{10}C_3$ = 120

Question 10.
If the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are the same, then the value of a is
(A) -79
(B) -97
(C) 79
(D) 97
Answer:
(D) 97

Hint:

The coefficient of x^2 in $(3 + ax)^9$

$= {}^9C_2 (3)^{9-2} (a)^2$

The coefficient of x^3 in $(3 + ax)^9$

$= {}^9C_3 (3)^{9-3} (a)^3$

Coefficients of x^2 and x^3 are equal.

$${}^9C_2\ 3^7\ a^2 = {}^9C_3\ 3^6\ a^3$$

$$\frac{3(3)}{7} = a \qquad \ldots\left[\frac{{}^9C_2}{{}^9C_3} = \frac{\frac{9\times 8}{2\times 1}}{\frac{9\times 8\times 7}{3\times 2\times 1}} = \frac{3}{7}\right]$$

$$a = \frac{9}{7}$$

(II) Answer the following.

Question 1.
Prove by the method of induction, for all n ∈ N.
(i) 8 + 17 + 26 + ….. + (9n – 1) = n2 (9n + 7)
Solution:
Let P(n) ≡ 8 + 17 + 26 +…..+(9n – 1) = n2 (9n + 7), for all n ∈ N.
Step I:
Put n = 1
L.H.S. = 8
R.H.S. = 12 [9(1) + 7] = 8
∴ L.H.S. = R.H.S.
∴ P(n) is true for n = 1.

Step II:
Let us assume that P(n) is true for n = k.
∴ 8 + 17 + 26 +…..+ (9k – 1) = k2 (9k + 7) ……(i)

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., 8 + 17 + 26 + …… + [9(k + 1) – 1]

$$= \frac{(k+1)}{2}[9(k+1)+7]$$

$$= \frac{(k+1)}{2}(9k+16)$$

$$\begin{aligned}
\text{L.H.S.} &= 8 + 17 + 26 + \ldots + [9(k+1) - 1] \\
&= 8 + 17 + 26 + \ldots + (9k-1) + (9k+8) \\
&= \frac{k}{2}(9k+7) + (9k+8) \qquad \ldots\text{[From (i)]} \\
&= \frac{9k^2 + 7k + 18k + 16}{2} \\
&= \frac{9k^2 + 25k + 16}{2} \\
&= \frac{9k^2 + 9k + 16k + 16}{2} \\
&= \frac{9k(k+1) + 16(k+1)}{2} \\
&= \frac{(k+1)}{2}(9k+16) \\
&= \text{R.H.S.}
\end{aligned}$$

∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.
∴ 8 + 17 + 26 +…..+ (9n – 1) = n2 (9n + 7) for all n ∈ N.

(ii) 12 + 42 + 72 + …… + (3n – 2)2 = n2 (6n2 – 3n – 1)
Solution:

Let P(n) = $1^2 + 4^2 + 7^2 + \ldots + (3n-2)^2 = \frac{n}{2}(6n^2 - 3n - 1)$, for all n ∈ N.

Step I:

Put n = 1

L.H.S.= $1^2 = 1$

R.H.S.= $\frac{1}{2}[6(1)^2 - 3(1) - 1] = 1$

∴ L.H.S. = R.H.S.

∴ P(n) is true for n = 1.

Step II:

Let us assume that P(n) is true for n = k.

∴ $1^2 + 4^2 + 7^2 + \ldots + (3k-2)^2 = \frac{k}{2}(6k^2 - 3k - 1)$ ……(i)

Step III:

We have to prove that P(n) is true for n = k + 1,

i.e., to prove that

$$1^2 + 4^2 + 7^2 + \ldots + [3(k+1) - 2]^2$$
$$= \frac{(k+1)}{2}[6(k+1)^2 - 3(k+1) - 1]$$
$$= \frac{(k+1)}{2}(6k^2 + 12k + 6 - 3k - 3 - 1)$$
$$= \frac{(k+1)}{2}(6k^2 + 9k + 2)$$

$$\text{L.H.S.} = 1^2 + 4^2 + 7^2 + \ldots + [3(k+1) - 2]^2$$
$$= 1^2 + 4^2 + 7^2 + \ldots + (3k-2)^2 + (3k+1)^2$$
$$= \frac{k}{2}(6k^2 - 3k - 1) + (3k+1)^2 \quad \ldots\text{[From (i)]}$$
$$= \frac{(6k^3 - 3k^2 - k) + 2(9k^2 + 6k + 1)}{2}$$
$$= \frac{6k^3 + 15k^2 + 11k + 2}{2}$$
$$= \frac{(k+1)(6k^2 + 9k + 2)}{2}$$
$$= \text{R.H.S.}$$

∴ P(n) is true for n = k + 1.

Step IV:

From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.

∴ $1^2 + 4^2 + 7^2 + \ldots + (3n-2)^2 = \frac{n}{2}(6n^2 - 3n - 1)$ for all n ∈ N.

(iii) $2 + 3.2 + 4.2^2 + \ldots + (n+1)2^{n-1} = n \cdot 2^n$

Solution:

Let P(n) ≡ $2 + 3.2 + 4.2^2 + \ldots + (n+1)2^{n-1} = n.2^n$, for all n ∈ N.

Step I:

Put n = 1

L.H.S. = 2

R.H.S. = $1(2^1) = 2$

∴ L.H.S. = R.H.S.

∴ P(n) is true for n = 1.

Step II:

Let us assume that P(n) is true for n = k.

∴ $2 + 3.2 + 4.2^2 + \ldots + (k+1)2^{k-1} = k.2^k$ …..(i)

Step III:

We have to prove that P(n) is true for n = k + 1,

i.e., to prove that

$2 + 3.2 + 4.2^2 + \ldots + (k+2)2^k = (k+1)2^{k+1}$

$$\begin{aligned}
\text{L.H.S.} &= 2 + 3.2 + 4.2^2 + \ldots + (k+2)2^k \\
&= 2 + 3.2 + 4.2^2 + \ldots + (k+1)2^{k-1} \\
&\qquad\qquad\qquad\qquad + (k+2)2^k \\
&= k.2^k + (k+2)2^k \qquad \ldots\text{[From (i)]} \\
&= (k + k + 2)2^k \\
&= (2k+2)2^k \\
&= (k+1).2.2^k \\
&= (k+1)2^{k+1} \\
&= \text{R.H.S.}
\end{aligned}$$

∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.
∴ $2 + 3.2 + 4.2^2 + \ldots\ldots + (n + 1) 2^{n-1} = n.2^n$ for all n ∈ N.

(iv) 13.4.5+24.5.6+35.6.7+…+n(n+2)(n+3)(n+4) = n(n+1)6(n+3)(n+4)
Solution:

$$\text{Let } P(n) \equiv \frac{1}{3.4.5} + \frac{2}{4.5.6} + \frac{3}{5.6.7} + \ldots + \frac{n}{(n+2)(n+3)(n+4)}$$

$$= \frac{n(n+1)}{6(n+3)(n+4)}, \text{ for all } n \in N.$$

Step I:

Put n = 1

$$\text{L.H.S.} = \frac{1}{3.4.5} = \frac{1}{60}$$

$$\text{R.H.S.} = \frac{1(1+1)}{6(1+3)(1+4)} = \frac{2}{120} = \frac{1}{60}$$

∴ L.H.S. = R.H.S.

∴ P(n) is true for n = 1.

Step II:

Let us assume that P(n) is true for n = k.

$\therefore \quad \frac{1}{3.4.5}+\frac{2}{4.5.6}+\frac{3}{5.6.7}+\ldots+\frac{k}{(k+2)(k+3)(k+4)}$

$=\frac{k(k+1)}{6(k+3)(k+4)} \quad \ldots(i)$

Step III:

We have to prove that P(n) is true for n = k+1,

i.e., to prove that

$\frac{1}{3.4.5}+\frac{2}{4.5.6}+\frac{3}{5.6.7}+\ldots+\frac{k+1}{(k+3)(k+4)(k+5)}$

$=\frac{(k+1)(k+2)}{6(k+4)(k+5)}$

L.H.S. $=\frac{1}{3.4.5}+\frac{2}{4.5.6}+\frac{3}{5.6.7}+\ldots$

$+\frac{k+1}{(k+3)(k+4)(k+5)}$

$=\frac{1}{3.4.5}+\frac{2}{4.5.6}+\frac{3}{5.6.7}+\ldots$

$+\frac{k}{(k+2)(k+3)(k+4)}+\frac{k+1}{(k+3)(k+4)(k+5)}$

$=\frac{k(k+1)}{6(k+3)(k+4)}+\frac{k+1}{(k+3)(k+4)(k+5)}$

...[From (i)]

$=\frac{k+1}{(k+3)(k+4)}\left[\frac{k}{6}+\frac{1}{k+5}\right]$

$=\frac{(k+1)}{(k+3)(k+4)}\left[\frac{k^2+5k+6}{6(k+5)}\right]$

$=\frac{(k+1)(k+2)(k+3)}{6(k+3)(k+4)(k+5)}$

$=\frac{(k+1)(k+2)}{6(k+4)(k+5)}$

= R.H.S.

$\therefore$ P(n) is true for n = k + 1.

Step IV:

From all the steps above, by the principle of mathematical induction, P(n) is true for all $n \in N$.

$\therefore \quad \frac{1}{3.4.5}+\frac{2}{4.5.6}+\frac{3}{5.6.7}+\ldots$

$+\frac{n}{(n+2)(n+3)(n+4)}=\frac{n(n+1)}{6(n+3)(n+4)}$

for all $n \in N$.

Question 2.
Given that $t_{n+1} = 5t_n - 8$, $t_1 = 3$, prove by method of induction that $t_n = 5^{n-1} + 2$.
Solution:
Let the statement P(n) has L.H.S. a recurrence relation $t_{n+1} = 5t_n - 8$, $t_1 = 3$
and R.H.S. a general statement $t_n = 5^{n-1} + 2$.
Step I:
Put n = 1
L.H.S. = 3
R.H.S. = $5^{1-1} + 2 = 1 + 2 = 3$
∴ L.H.S. = R.H.S.
∴ P(n) is true for n = 1.
Put n = 2
L.H.S = $t_2 = 5t_1 - 8 = 5(3) - 8 = 7$
R.H.S. = $t_2 = 5^{2-1} + 2 = 5 + 2 = 7$
∴ L.H.S. = R.H.S.
∴ P(n) is tme for n = 2.

Step II:
Let us assume that P(n) is true for n = k.
∴ $t_{k+1} = 5t_k - 8$ and $t_k = 5^{k-1} + 2$

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., to prove that
$t_{k+1} = 5^{k+1-1} + 2 = 5^k + 2$
$t_{k+1} = 5t_k - 8$ and $t_k = 5^{k-1} + 2$ ……[From Step II]
∴ $t_{k+1} = 5(5^{k-1} + 2) - 8 = 5^k + 2$
∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.
∴ $t_n = 5^{n-1} + 2$, for all n ∈ N.

Question 3.
Prove by method of induction
$\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}^n = \begin{bmatrix} 2n+1 & -4n \\ n & -2n+1 \end{bmatrix}$, ∀ n ∈ N.
Solution:

Let $P(n) \equiv \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}^n = \begin{bmatrix} 2n+1 & -4n \\ n & -2n+1 \end{bmatrix}$, for all $n \in N$.

Step I:
Put n = 1

$$\text{L.H.S.} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}^1 = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$\text{R.H.S.} = \begin{bmatrix} 2(1)+1 & -4(1) \\ 1 & -2(1)+1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

∴ P(n) is true for n = 1.

Step II:
Let us assume that P(n) is true for n = k.

$$\therefore \quad \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}^k = \begin{bmatrix} 2k+1 & -4k \\ k & -2k+1 \end{bmatrix} \quad \ldots(i)$$

Step III:
We have to prove that P(n) is true for n = k + 1,
i.e., to prove that

$$\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}^{k+1} = \begin{bmatrix} 2(k+1)+1 & -4(k+1) \\ k+1 & -2(k+1)+1 \end{bmatrix}$$

$$\text{L.H.S.} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}^k \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2k+1 & -4k \\ k & -2k+1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \ldots\text{[From (i)]}$$

$$= \begin{bmatrix} 6k+3-4k & -8k-4+4k \\ 3k-2k+1 & -4k+2k-1 \end{bmatrix}$$

$$= \begin{bmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(k+1)+1 & -4(k+1) \\ k+1 & -2(k+1)+1 \end{bmatrix}$$

= R.H.S.

∴ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all n ∈ N.
∴ $\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}^n = \begin{bmatrix} 2n+1 & -4n \\ n & -2n+1 \end{bmatrix}$, ∀ n ∈ N.

Question 4.
Expand $(3x^2 + 2y)^5$
Solution:

Here, a = $3x^2$, b = 2y, n = 5.
Using binomial theorem,

$$(3x^2+2y)^5$$
$$= {}^5C_0(3x^2)^5(2y)^0 + {}^5C_1(3x^2)^4(2y)^1 + {}^5C_2(3x^2)^3(2y)^2 + {}^5C_3(3x^2)^2(2y)^3 + {}^5C_4(3x^2)^1(2y)^4 + {}^5C_5(3x^2)^0(2y)^5$$

Now, ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$,

$${}^5C_2 = {}^5C_3 = \frac{5\times 4}{2\times 1} = 10$$

$$(3x^2+2y)^5$$
$$= 1(243x^{10})(1) + 5(81x^8)(2y) + 10(27x^6).(4y^2) + 10.(9x^4)(8y^3) + 5(3x^2)(16y^4) + 1(1)(32y^5)$$
$$= 243x^{10} + 810x^8y + 1080x^6y^2 + 720x^4y^3 + 240x^2y^4 + 32y^5$$

Question 5.
Expand $\left(\frac{2x}{3}-\frac{3}{2x}\right)^4$
Solution:

Here, a = $\frac{2x}{3}$, b = $\frac{3}{2x}$, n = 4.

Using binomial theorem, $\left(\frac{2x}{3}-\frac{3}{2x}\right)^4$

$$= {}^4C_0\left(\frac{2x}{3}\right)^4\left(\frac{3}{2x}\right)^0 - {}^4C_1\left(\frac{2x}{3}\right)^3\left(\frac{3}{2x}\right)^1 + {}^4C_2\left(\frac{2x}{3}\right)^2\left(\frac{3}{2x}\right)^2 - {}^4C_3\left(\frac{2x}{3}\right)^1\left(\frac{3}{2x}\right)^3 + {}^4C_4\left(\frac{2x}{3}\right)^0\left(\frac{3}{2x}\right)^4$$

Now, ${}^4C_0 = {}^4C_4 = 1$, ${}^4C_1 = {}^4C_3 = 4$,

$${}^4C_2 = \frac{4\times 3}{2\times 1} = 6,$$

$$\left(\frac{2x}{3}-\frac{3}{2x}\right)^4 = 1\left(\frac{16x^4}{81}\right)(1) - 4\left(\frac{8x^3}{27}\right)\left(\frac{3}{2x}\right) + 6\left(\frac{4x^2}{9}\right)\left(\frac{9}{4x^2}\right) - 4\left(\frac{2x}{3}\right)\left(\frac{27}{8x^3}\right) + 1(1)\left(\frac{81}{16x^4}\right)$$
$$= \frac{16x^4}{81} - \frac{16x^2}{9} + 6 - \frac{9}{x^2} + \frac{81}{16x^4}$$

Question 6.
Find third term in the expansion of $\left(9x^2-\frac{y^3}{6}\right)^4$
Solution:

Here, a = $9x^2$, b = $\frac{-y^3}{6}$, n = 4.
For 3[rd] term, r = 2
We have $t_{r+1} = {}^nC_r\, a^{n-r}.b^r$

$$\therefore \quad t_3 = {}^4C_2(9x^2)^{4-2}\left(\frac{-y^3}{6}\right)^2$$
$$= \frac{4!}{2!2!}(81x^4)\frac{y^6}{36}$$
$$= \frac{4\times 3}{2\times 1}\times 81x^4\times\frac{y^6}{36} = \frac{27}{2}x^4y^6$$

$\therefore$ 3[rd] term in the expansion of $\left(9x^2-\frac{y^3}{6}\right)^4$ is

$\frac{27}{2}x^4y^6$.

Question 7.
Find tenth term in the expansion of $\left(2x^2+\frac{1}{x}\right)^{12}$
Solution:

Here, $a = 2x^2$, $b = \frac{1}{x}$, $n = 12$.

For 10^{th} term, $r = 9$

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$t_{10} = {}^{12}C_9\,(2x^2)^{12-9}\left(\frac{1}{x}\right)^9 = \frac{12!}{9!3!}\left(8x^6\right)\left(\frac{1}{x^9}\right)$$

$$= \frac{12\times11\times10}{3\times2\times1}\times\frac{8x^6}{x^9}$$

$$= \frac{1760}{x^3}$$

10^{th} term in the expansion of $\left(2x^2+\frac{1}{x}\right)^{12}$ is $\frac{1760}{x^3}$.

Question 8.
Find the middle term(s) in the expansion of
(i) $\left(\frac{2a}{3}-\frac{3}{2a}\right)^6$
Solution:
Here, a = $\frac{2a}{3}$, b = $-\frac{3}{2a}$, n = 6.
Now, n is even.
$\therefore \frac{n+2}{2}=\frac{6+2}{2}=4$
$\therefore$ Middle term is t_4, for which r = 3.

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$t_4 = {}^6C_3\left(\frac{2a}{3}\right)^3\left(\frac{-3}{2a}\right)^3$$

$$= \frac{6!}{3!3!}\times(-1)$$

$$= \frac{-6\times5\times4\times3!}{3\times2\times1\times3!}$$

$$= -20$$

$\therefore$ The Middle term is -20.

(ii) $\left(x-\frac{1}{2y}\right)^{10}$
Solution:
Here, a = x, b = $-\frac{1}{2y}$, n = 10.
Now, n is even.
$\therefore \frac{n+2}{2}=\frac{10+2}{2}=6$
$\therefore$ Middle term is t_6, for which r = 5

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$\therefore \quad t_6 = {}^{10}C_5\,(x)^5\left(\frac{-1}{2y}\right)^5 = \frac{10!}{5!\,5!}(x)^5\left(\frac{-1}{32y^5}\right)$$

$$= -\frac{10\times9\times8\times7\times6\times5!}{5\times4\times3\times2\times1\times5!}\times\frac{x^5}{32y^5}$$

$$= -\frac{63x^5}{8y^5}$$

$\therefore$ Middle term is $-\frac{63x^5}{8y^5}$.

(iii) $(x^2 + 2y^2)^7$
Solution:
Here, a = x^2, b = $2y^2$, n = 7.
Now, n is odd.

∴ $\frac{n+1}{2}=\frac{7+1}{2}=4, \frac{n+3}{2}=\frac{7+3}{2}=5$

∴ Middle terms are t_4 and t_5, for which r = 3 and r = 4 respectively.

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$t_4 = {}^7C_3\,(x^2)^4\,(2y^2)^3$$
$$= \frac{7!}{3!4!}(x^8)(8y^6)$$
$$= \frac{7\times6\times5\times4!}{3\times2\times1\times4!}\times 8x^8y^6$$
$$= 280x^8y^6$$

and $t_5 = {}^7C_4\,(x^2)^3.\,(2y^2)^4$
$$= \frac{7!}{4!3!}(x^6).(16y^8)$$
$$= \frac{7\times6\times5\times4!}{4!\times3\times2\times1}\times16x^6y^8$$
$$= 560x^6y^8$$

∴ Middle terms are $280x^8y^6$ and $560x^6y^8$.

(iv) $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$

Solution:

Here, $a = \frac{3x^2}{2}$, $b = \frac{-1}{3x}$, n = 9.

Now, n is odd.

$\frac{n+1}{2} = \frac{9+1}{2} = 5$, $\frac{n+3}{2} = \frac{9+3}{2} = 6$

Middle terms are t_5 and t_6, for which r = 4 and r = 5 respectively.

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$t_5 = {}^9C_4\left(\frac{3x^2}{2}\right)^5\left(\frac{-1}{3x}\right)^4$$
$$= \frac{9!}{4!5!}\times\frac{243\times x^{10}}{32}\times\frac{1}{81x^4}$$
$$= \frac{9\times8\times7\times6\times5!}{4\times3\times2\times1\times5!}\times\frac{3x^6}{32}$$
$$= \frac{189}{16}x^6$$

and $t_6 = {}^9C_5\left(\frac{3x^2}{2}\right)^4\left(\frac{-1}{3x}\right)^5$
$$= \frac{9!}{4!5!}\times\left(\frac{81x^8}{16}\right)\times\left(\frac{-1}{243x^5}\right)$$
$$= \frac{9\times8\times7\times6\times5!}{4\times3\times2\times1\times5!}\left(-\frac{x^3}{16\times3}\right)$$
$$= \frac{-126x^3}{16\times3}$$
$$= \frac{-21}{8}x^3$$

Middle terms are $\frac{189x^6}{16}$ and $\frac{-21x^3}{8}$.

Question 9.
Find the coefficients of

(i) x^6 in the expantion of $\left(3x^2-\frac{1}{3x}\right)^9$

Solution:

Here, $a = 3x^2$, $b = \frac{-1}{3x}$, $n = 9$.

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$= {}^9C_r\,(3x^2)^{9-r}\left(\frac{-1}{3x}\right)^r$$

$$= {}^9C_r\,(3)^{9-r}.x^{18-2r}\left(\frac{-1}{3}\right)^r.\,x^{-r}$$

$$= {}^9C_r\,(3)^{9-r}\left(\frac{-1}{3}\right)^r x^{18-3r}$$

To get the coefficient of x^6, we must have

$x^{18-3r} = x^6$

$18 - 3r = 6$

$3r = 12$

$r = 4$

coefficient of x^6

$= {}^9C_4\,(3)^5\left(\frac{-1}{3}\right)^4$

$= \frac{9!}{4!\,5!}(3)^5\left(\frac{1}{3^4}\right)$

$= \frac{9\times8\times7\times6\times5!}{4\times3\times2\times1\times5!}\times 3$

$= 378$

(ii) x^{60} in the expansion of $\left(\frac{1}{x^2}+x^4\right)^{18}$

Solution:

Here, $a = \frac{1}{x^2}$, $b = x^4$, $n = 18$.

We have, $t_{r+1} = {}^nC_r\, a^{n-r}.b^r$

$$= {}^{18}C_r\left(\frac{1}{x^2}\right)^{18-r}(x^4)^r$$

$$= {}^{18}C_r\,x^{2r-36}.x^{4r}$$

$$= {}^{18}C_r\,x^{6r-36}$$

To get the coefficient of x^{60}, we must have

$x^{6r-36} = x^{60}$

$6r - 36 = 60$

$6r = 96$

$r = 16$

Coefficient of x^{60}

$= {}^{18}C_{16} = \frac{18!}{16!\,2!} = \frac{18\times17\times16!}{16!\times2\times1}$

$= 153$

Question 10.

Find the constant term in the expansion of

(i) $\left(\frac{4x^2}{3}+\frac{3}{2x}\right)^9$

Solution:

Here, $a = \frac{4x^2}{3}$, $b = \frac{3}{2x}$, $n = 9$.

We have $t_{r+1} = {}^nC_r\, a^{n-r}.b^r$

$$= {}^9C_r \left(\frac{4x^2}{3}\right)^{9-r} \left(\frac{3}{2x}\right)^r$$

$$= {}^9C_r \left(\frac{4}{3}\right)^{9-r} \left(\frac{3}{2}\right)^r .x^{18-2r}.x^{-r}$$

$$= {}^9C_r \left(\frac{4}{3}\right)^{9-r} . \left(\frac{3}{2}\right)^r .x^{18-3r}$$

To get the constant term, we must have

$x^{18-3r} = x^0$

$18 - 3r = 0$

$r = 6$

Constant term $= {}^9C_6 \left(\frac{4}{3}\right)^3 \left(\frac{3}{2}\right)^6$

$$= \frac{9!}{6!\,3!} \frac{64}{27} \times \frac{729}{64}$$

$$= \frac{9\times8\times7\times6!}{6!\times3\times2\times1} \times 27$$

$$= 2268$$

(ii) $\left(2x^2 - \frac{1}{x}\right)^{12}$

Solution:

Here, $a = 2x^2$, $b = \frac{-1}{x}$, $n = 12$.

We have $t_{r+1} = {}^nC_r\, a^{n-r}.b^r$

$$= {}^{12}C_r\, (2x^2)^{12-r} \left(\frac{-1}{x}\right)^r$$

$$= {}^{12}C_r\, (2)^{12-r}.(-1)^r.\, x^{24-2r}.x^{-r}$$

$$= {}^{12}C_r\, (2)^{12-r} (-1)^r x^{24-3r}$$

To get the constant term, we must have

$x^{24-3r} = x^0$

$24 - 3r = 0$

$r = 8$

Constant term $= {}^{12}C_8 (2)^4 (-1)^8$

$$= \frac{12!}{8!\,4!} \times 16 \times 1$$

$$= \frac{12\times11\times10\times9\times8!}{4\times3\times2\times1\times8!} \times 16$$

$$= 7920$$

Question 11.

Prove by method of induction

(i) $\log_a x^n = n \log_a x$, $x > 0$, $n \in N$

Solution:

Let $P(n) \equiv \log_a x^n = n \log_a x$, for all $n \in N$.

Step I:

Put n = 1

L.H.S. $= \log_a (x^1) = \log_a x$

R.H.S. $= 1.(\log_a x) = \log_a x$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ P(n) is true for n = 1.

Step II:

Let us assume that P(n) is true for n = k.

$\therefore \log_a x^k = k.\log_a x \quad \ldots(i)$

Step III:

We have to prove that P(n) is true for n = k + 1,

i.e., to prove that

$\log_a x^{k+1} = (k + 1).\log_a x$

L.H.S. $= \log_a (x^{k+1})$

$= \log_a (x^k.x) = \log_a x^k + \log_a x$

$= k.\log_a x + \log_a x \quad \ldots$[From (i)]

$= (k + 1).\log_a x$

= R.H.S.

$\therefore$ P(n) is true for n = k + 1.

Step IV:

From all the steps above, by the principle of mathematical induction, P(n) is true for all $n \in N$.

$\therefore \log_a x^n = n \log_a x, x > 0, n \in N$.

(ii) $15^{2n-1} + 1$ is divisible by 16, for all $n \in N$.
Solution:
$15^{2n-1} + 1$ is divisible by 16, if and only if $(15^{2n-1} + 1)$ is is a multiple of 16.
Let $P(n) \equiv 15^{2n-1} + 1 = 16m$, where $m \in N$.

Step I:

Put n = 1

$\therefore 15^{2n-1} + 1 = 15^{2(1)-1} + 1 = 15 + 1 = 16 = 16(1)$

$\therefore 15^{2n-1} + 1$ is a multiple of 16.

$\therefore$ P(n) is true for n = 1.

Step II:

Let us assume that P(n) is true for n = k.

$\therefore 15^{2k-1} + 1 = 16a$, where $a \in N$

$\therefore 15^{2k-1} = 16a - 1$

$\therefore \frac{15^{2k}}{15} = 16a - 1$

$\therefore 15^{2k} = 15.(16a - 1) \quad \ldots(i)$

Step III:

We have to prove that P(n) is true for n = k+1,

i.e., to prove that

$15^{2(k+1)-1} + 1 = 16b$, where $b \in N$.

$\therefore 15^{2(k+1)-1} + 1$

$= 15^{2k+1} + 1$

$= 15^{2k}.15 + 1$

$= 15(16a - 1) \times 15 + 1 \quad$ ...[From (i)]

$= 225 \times 16a - 225 + 1$

$= 225 \times 16a - 224$

$= 16(225a - 14)$

$= 16b$, where $b = (225a - 14) \in N$.

$\therefore$ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all $n \in N$.
$\therefore 15^{2n-1} + 1$ is divisible by 16, for all $n \in N$.

(iii) $5^{2n} - 2^{2n}$ is divisible by 3, for all $n \in N$.
Solution:

$5^{2n} - 2^{2n}$ is divisible by 3, if and only if $(5^{2n} - 2^{2n})$ is a multiple of 3.
Let $P(n) \equiv (5^{2n} - 2^{2n}) = 3m$, where $m \in N$.

Step I:
Put n = 1
$\therefore$ $5^{2n} - 2^{2n} = 5^2 - 2^2 = 25 - 4 = 21 = 3 \times 7$
$\therefore$ $5^{2n} - 2^{2n}$ is a multiple of 3.
$\therefore$ P(n) is true for n = 1.

Step II:
Let us assume that P(n) is true for n = k.
$\therefore$ $5^{2k} - 2^{2k} = 3a$, where $a \in N$
$\therefore$ $5^{2k} = 3a + 2^{2k}$ ...(i)

Step III:
We have to prove that P(n) is true for n = k+1,
i.e., to prove that
$5^{2(k+1)} - 2^{2(k+1)} = 3b$, where $b \in N$.
$\therefore$ $5^{2(k+1)} - 2^{2(k+1)}$
$= 5^{2k+2} - 2^{2k+2}$
$= 5^{2k}.\, 5^2 - 2^{2k}.\, 2^2$
$= (3a + 2^{2k}).\, 25 - 4.2^{2k}$ ...[From (i)]
$= 75a + 25.2^{2k} - 4.2^{2k}$
$= 75a + 21.2^{2k}$
$= 3(25a + 7.2^{2k})$
$= 3b$, where $b = (25a + 7.2^{2k}) \in N$
$\therefore$ P(n) is true for n = k + 1.

Step IV:
From all the steps above, by the principle of mathematical induction, P(n) is true for all $n \in N$.
$\therefore$ $5^{2n} - 2^{2n}$ is divisible by 3, for all $n \in N$.

Question 12.
If the coefficient of x^{16} in the expansion of $(x^2 + ax)^{10}$ is 3360, find a.
Solution:

Here, $a = x^2$, $b = ax$, $n = 10$.
We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
$= {}^{10}C_r\, (x^2)^{10-r}\, (ax)^r$
$= {}^{10}C_r\, a^r\, x^{20-2r}\, x^r$
$= {}^{10}C_r\, a^r\, x^{20-r}$
To get the coefficient of x^{16}, we must have
$x^{20-r} = x^{16}$
$\therefore$ $20 - r = 16$
$\therefore$ $r = 4$
$\therefore$ Coefficient of $x^{16} = {}^{10}C_4.a^4$
Given, coefficient of $x^{16} = 3360$
$\therefore$ ${}^{10}C_4 a^4 = 3360$
$\therefore$ $\frac{10!}{4!\,6!}\, a^4 = 3360$
$\therefore$ $\frac{10\times9\times8\times7\times6!}{4\times3\times2\times1\times6!} a^4 = 3360$
$\therefore$ $210.\, a^4 = 3360$
$\because$ $a^4 = 16$
$\therefore$ $a = \pm 2$

Question 13.
If the middle term in the expansion of $(x+bx)^6$ is 160, find b.
Solution:

Here, $a = x$, $b = \frac{b}{x}$, $n = 6$.

Now, n is even.

$\therefore \frac{n+2}{2} = \frac{6+2}{2} = 4$

$\therefore$ Middle term is t_4, for which $r = 3$

$\therefore t_4 = 160$

We have $t_{r+1} = {}^nC_r a^{n-r}.b^r$

$\therefore t_4 = {}^6C_3 (x)^3 \left(\frac{b}{x}\right)^3$

$\therefore 160 = \frac{6!}{3!3!} (x)^3 \left(\frac{b^3}{x^3}\right)$

$\therefore 160 = \frac{6\times5\times4\times3!}{3\times2\times1\times3!} \times b^3$

$\therefore 160 = 20b^3$

$\therefore 8 = b^3$

$\therefore b = 2$

Question 14.
If the coefficients of x^2 and x^3 in theexpansion of $(3 + kx)^9$ are equal, find k.
Solution:

Here, $a = 3$, $b = kx$, $n = 9$

We have $t_{r+1} = {}^nC_r a^{n-r} b^r$

$= {}^9C_r (3)^{9-r} (kx)^r$

$= {}^9C_r (3)^{9-r} k^r(x)^r$

For coefficient of x^2, $r = 2$

For coefficient of x^3, $r = 3$

But, coefficient of x^2 = coefficient of x^3.

$\therefore {}^9C_2 (3)^7 k^2 = {}^9C_3 3^6 k^3$

$\therefore \frac{9!}{2!\,7!} (3)^7 k^2 = \frac{9!}{3!\,6!} (3)^6 k^3$

$\therefore \frac{9\times8\times7!}{2\times1\times7!} (3)^7 k^2 = \frac{9\times8\times7\times6!}{3\times2\times1\times6!} (3)^6.k^3$

$\therefore 36 \times 3^7 \times k^2 = 84 \times 3^6 \times k^3$

$\therefore \frac{36}{84} \times 3 = k$

$\therefore k = \frac{9}{7}$

Question 15.
If the constant term in the expansion of $\left(x^3+\frac{k}{x^8}\right)^{11}$ is 1320, find k.

Solution:

Here, $a = x^3$, $b = \frac{k}{x^8}$, $n = 11$ and

constant term = 1320.

We have $t_{r+1} = {}^nC_r\, a^{n-r} . b^r$

$= {}^{11}C_r (x^3)^{11-r} . \left(\frac{k}{x^8}\right)^r$

$= {}^{11}C_r\, k^r x^{33-3r} x^{-8r}$

$= {}^{11}C_r\, k^r x^{33-11r}$

To get the constant term, we must have

$x^{33-11r} = x^0$

$\therefore$ $33 - 11r = 0$

$\therefore$ $r = 3$

$\therefore$ Constant term $= {}^{11}C_3 . k^3$

$\therefore$ $1320 = \frac{11!}{3!8!} . k^3$

$\therefore$ $1320 = \frac{11 \times 10 \times 9 \times 8!}{3 \times 2 \times 1 \times 8!} . k^3$

$\therefore$ $1320 = 165\, k^3$

$\therefore$ $k^3 = 8$

$\therefore$ $k = 2$

Question 16.

Show that there is no term containing x^6 in the expansion of $\left(x^2 - \frac{3}{x}\right)^{11}$.

Solution:

Here, $a = x^2$, $b = \frac{-3}{x}$, $n = 11$.

We have $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

$= {}^{11}C_r (x^2)^{11-r} \left(\frac{-3}{x}\right)^r$

$= {}^{11}C_r\, x^{22-2r} . (-3)^r (x^{-r})$

$= {}^{11}C_r (-3)^r x^{22-3r}$

To get the term x^6, we must have

$x^{22-3r} = x^6$

$\therefore$ $22 - 3r = 6$

$\therefore$ $3r = 16$

$\therefore$ $r = \frac{16}{3}$, which is not possible

$\therefore$ There is no term containing x^6 in the expansion of $\left(x^2 - \frac{3}{x}\right)^{11}$

Question 17.

Show that there is no constant term in the expansion of $\left(2x - \frac{x^2}{4}\right)^9$

Solution:

Here, $a = 2x$, $b = \frac{-x^2}{4}$, $n = 9$.

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$= {}^9C_r\,(2x)^{9-r}\left(-\frac{x^2}{4}\right)^r$$

$$= {}^9C_r\,(2)^{9-r}\left(-\frac{1}{4}\right)^r x^{9-r}\,x^{2r}$$

$$= {}^9C_r\,2^{9-r}\left(-\frac{1}{4}\right)^r x^{9+r}$$

To get the constant term, we must have

$x^{9+r} = x^0$

$\therefore$ $9 + r = 0$

$\therefore$ $r = -9$, which is not possible

$\therefore$ There is no constant term in the expansion of $\left(2x - \frac{x^2}{4}\right)^9$.

Question 18.

State, first four terms in the expansion of $\left(1-\frac{2x}{3}\right)^{-1/2}$

Solution:

$$\left(1-\frac{2x}{3}\right)^{\frac{-1}{2}} = 1 + \left(-\frac{1}{2}\right)\left(-\frac{2x}{3}\right) + \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}\left(-\frac{2x}{3}\right)^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}\left(-\frac{2x}{3}\right)^3 + \dots$$

$$= 1 + \left(-\frac{1}{2}\right)\left(-\frac{2x}{3}\right) + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2}\left(\frac{2x}{3}\right)^2 - \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{6}\left(\frac{2x}{3}\right)^3 + \dots$$

$$= 1 + \frac{x}{3} + \frac{3}{8} \times \frac{4x^2}{9} + \frac{15}{8\times 6} \times \frac{8x^3}{27} + \dots$$

$$= 1 + \frac{x}{3} + \frac{x^2}{6} + \frac{5x^3}{54} + \dots$$

Question 19.

State, first four terms in the expansion of $(1-x)^{-1/4}$.

Solution:

$$(1-x)^{-1/4} = 1 + \left(\frac{-1}{4}\right)(-x) + \frac{\left(\frac{-1}{4}\right)\left(\frac{-1}{4}-1\right)}{2!}(-x)^2 + \frac{\left(\frac{-1}{4}\right)\left(\frac{-1}{4}-1\right)\left(\frac{-1}{4}-2\right)}{3!}(-x)^3 + \dots$$

$$= 1 + \left(\frac{-1}{4}\right)(-x) + \frac{\left(\frac{-1}{4}\right)\left(\frac{-5}{4}\right)}{2}x^2 - \frac{\left(\frac{-1}{4}\right)\left(\frac{-5}{4}\right)\left(\frac{-9}{4}\right)}{6}x^3 + \dots$$

$$= 1 + \frac{x}{4} + \frac{5x^2}{32} + \frac{15x^3}{128} + \dots$$

Question 20.

State, first three terms in the expansion of $(5+4x)^{-1/2}$

Solution:

$$(5+4x)^{-1/2} = \left[5\left(1+\frac{4}{5}x\right)\right]^{\frac{-1}{2}} = 5^{\frac{-1}{2}}\left(1+\frac{4x}{5}\right)^{\frac{-1}{2}}$$

$$= 5^{\frac{-1}{2}}\left[1+\left(\frac{-1}{2}\right)\frac{4x}{5}+\frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}\left(\frac{4x}{5}\right)^2+\ldots\right]$$

$$= 5^{\frac{-1}{2}}\left[1+\left(\frac{-1}{2}\right)\frac{4x}{5}+\frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{2}\left(\frac{16x^2}{25}\right)+\ldots\right]$$

$$= 5^{\frac{-1}{2}}\left[1-\frac{2x}{5}+\frac{6x^2}{25}-\ldots\right]$$

Question 21.

Using the binomial theorem, find the value of $\sqrt[3]{995}$ upto four places of decimals.

Solution:

$$\sqrt[3]{995} = (1000-5)^{\frac{1}{3}} = \left[1000\left(1-\frac{5}{1000}\right)\right]^{\frac{1}{3}}$$

$$= (1000)^{\frac{1}{3}}(1-0.005)^{\frac{1}{3}}$$

$$= 10\left[1-\frac{1}{3}(0.005)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(0.005)^2-\ldots\right]$$

$$= 10\left[1-\frac{1}{3}(0.005)+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2}(0.005)^2-\ldots\right]$$

$$= 10\left[1-0.00166-\frac{0.000025}{9}-\ldots\right]$$

$$= 10(1 - 0.00166 - 0.0000028 - \ldots)$$

$$= 10(0.9983372)$$

$$= 9.9833 \text{ upto 4 decimal places.}$$

Question 22.

Find approximate value of $\frac{1}{4.08}$ upto four places of decimals.

Solution:

$$\frac{1}{4.08} = (4.08)^{-1} = (4+0.08)^{-1} = [4(1+0.02)]^{-1}$$

$$= 4^{-1}(1+0.02)^{-1}$$

$$= \frac{1}{4}\left[1+(-1)(0.02)+\frac{(-1)(-1-1)}{2!}(0.02)^2+\ldots\right]$$

$$= \frac{1}{4}\left[1+(-1)(0.02)+\frac{(-1)(-2)}{2}(0.02)^2+\ldots\right]$$

$$= \frac{1}{4}(1 - 0.02 + 0.0004 + \ldots)$$

$$= \frac{0.9804}{4}$$

$$= 0.2451$$

Question 23.

Find the term independent of x in the expansion of $(1 - x^2)\left(x+\frac{2}{x}\right)^6$.

Solution:

$$(1-x^2)\left(x+\frac{2}{x}\right)^6 = \left(x+\frac{2}{x}\right)^6 - x^2\left(x+\frac{2}{x}\right)^6$$

Consider $\left(x+\frac{2}{x}\right)^6$

Here, $a = x$, $b = \frac{2}{x}$, $n = 6$.

We have $t_{r+1} = {}^nC_r\, a^{n-r} . b^r$

$= {}^6C_r\, x^{6-r}\left(\frac{2}{x}\right)^r$

$= {}^6C_r\, x^{6-r}\, 2^r . x^{-r}$

$= {}^6C_r\, x^{6-2r}\, 2^r$

To get the term independent of x in the expansion of $\left(x+\frac{2}{x}\right)^6$, we must have

$x^{6-2r} = x^0$

$\therefore$ $6 - 2r = 0$

$\therefore$ $r = 3$

To get the term independent of x in the expansion of $x^2\left(x+\frac{2}{x}\right)^6$, we must have

$x^2 \cdot x^{6-2r} = x^0$

$\therefore$ $8 - 2r = 0$

$\therefore$ $r = 4$

$\therefore$ Term independent of x in the expansion of

$(1-x^2)\left(x+\frac{2}{x}\right)^6 = {}^6C_r\, 2^r - {}^6C_r\, 2^r$

$= {}^6C_3\, 2^3 - {}^6C_4\, 2^4$

$= 20\,(8) - 15\,(16)$

$= 160 - 240$

$= -80$

Question 24.
$(a + bx)(1 - x)^6 = 3 - 20x + cx^2 + \ldots$, then find a, b, c.
Solution:

Consider $(a + bx)(1-x)^6$

$= a(1-x)^6 + bx(1-x)^6$

$= a(1 - {}^6C_1 x + {}^6C_2 x^2 - {}^6C_3 x^3 + \ldots)$
$+ bx(1 - {}^6C_1 x + {}^6C_2 x^2 - {}^6C_3 x^3 + \ldots)$

$= a(1 - 6x + 15x^2 - 20x^3 + \ldots)$
$+ bx(1 - 6x + 15x^2 - 20x^3 + \ldots)$

$= a + (b - 6a)x + (15a - 6b)x^2 + \ldots$ $\ldots$(i)

Since $(a + bx)(1-x)^6 = 3 - 20x + cx^2 + \ldots$

$a + (b - 6a)x + (15a - 6b)x^2 + \ldots$
$= 3 - 20x + cx^2 + \ldots$ $\ldots$[From (i)]

Equating both sides, we get

$a = 3$, $b - 6a = -20$, $15a - 6b = c$

$a = 3$, $b = -2$, $c = 15(3) - 6(-2) = 57$

Question 25.
The 3rd term of $(1 + x)^n$ is $36x^2$. Find 5th term.
Solution:

Here, a = 1, b = x.
For 3^{rd} term, r = 2
We have $t_{r+1} = {}^nC_r\, a^{n-r}.b^r$

$\therefore \quad t_3 = {}^nC_2 (1)^{n-2}. x^2$

$\therefore \quad 36x^2 = {}^nC_2. x^2$

$\therefore \quad {}^nC_2 = 36$

$\therefore \quad \frac{n(n-1)}{2} = 36$

$\therefore \quad n^2 - n - 72 = 0$

$\therefore \quad (n-9)(n+8) = 0$

$\therefore \quad n = 9$

Now, $t_{r+1} = {}^nC_r\, a^{n-r}.b^r$
For 5^{th} term r = 4

$\therefore \quad t_5 = {}^9C_4\, 1^{9-4}.x^4$

$\therefore \quad t_5 = {}^9C_4\, x^4 = 126x^4$

Question 26.
Suppose $(1 + kx)^n = 1 - 12x + 60x^2 - \ldots\ldots$ find k and n.
Solution:

$(1 + kx)^n = 1 - 12x + 60x^2 - \ldots$

$\therefore \quad 1 + (nk)x + \frac{n(n-1)}{2} k^2x^2 + \ldots$

$= 1 - 12x + 60x^2 + \ldots$

Equating the coefficients on both sides, we get

$nk = -12 \qquad \ldots(i)$

and $\frac{n(n-1)}{2} k^2 = 60 \qquad \ldots(ii)$

$\therefore \quad n^2k^2 - nk^2 = 120$

$\therefore \quad 144 - (nk)k = 120 \qquad \ldots$[From (i)]

$\therefore \quad 144 - 120 = -12k \qquad \ldots$[From (i)]

$\therefore \quad k = -\frac{24}{12} = -2$

Substituting the value of k in equation (i), we get

$n = \frac{-12}{k} = \frac{-12}{-2} = 6$