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Maharashtra State Board 12th Maths Solutions Chapter 4 Definite Integration Ex 4.1

I. Evaluate the following integrals as a limit of a sum.

Question 1.
$\int_1^3 (3x-4) \cdot dx$
Solution:
Let f(x) = 3x – 4, for 1 ≤ x ≤ 3
Divide the closed interval [1, 3] into n subintervals each of length h at the points
1, 1 + h, 1 + 2h, 1 + rh, ….., 1 + nh = 3
∴ nh = 2
∴ h = $\frac{2}{n}$ and as n → ∞, h → 0
Here, a = 1
∴ f(a + rh) = f(1 + rh)
= 3(1 + rh) – 4
= 3rh – 1

$$\because \int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{r=1}^{n} f(a+rh) \cdot h$$

$$\therefore \int_1^3 (3x-4)\,dx = \lim_{n \to \infty} \sum_{r=1}^{n} (3rh-1)h$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \left(3r \cdot \frac{2}{n} - 1\right) \cdot \frac{2}{n} \quad \ldots \left[\because h = \frac{2}{n}\right]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \left(\frac{12r}{n^2} - \frac{2}{n}\right)$$

$$= \lim_{n \to \infty} \left[\frac{12}{n^2} \sum_{r=1}^{n} r - \frac{2}{n} \sum_{r=1}^{n} 1\right]$$

$$= \lim_{n \to \infty} \left[\frac{12}{n^2} \cdot \frac{n(n+1)}{2} - \frac{2}{n} \cdot n\right]$$

$$= \lim_{n \to \infty} \left[6\left(\frac{n+1}{n}\right) - 2\right]$$

$$= \lim_{n \to \infty} \left[6\left(1 + \frac{1}{n}\right) - 2\right]$$

$$= 6(1+0) - 2 \quad \ldots \left[\because \lim_{n \to \infty} \frac{1}{n} = 0\right]$$

$$= 4.$$

Question 2.
$\int_0^4 x^2 dx$
Solution:
Let f(x) = x^2, for 0 ≤ x ≤ 4
Divide the closed interval [0, 4] into n subintervals each of length h at the points
0, 0 + h, 0 + 2h, ….., 0 + rh, ….., 0 + nh = 4
i.e. 0, h, 2h, ….., rh, ….., nh = 4
∴ h = $\frac{4}{n}$ as n → ∞, h → 0

Here, a = 0

$$\therefore f(a+rh) = f(0+rh) = f(rh) = r^2h^2$$

$$\because \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} f(a+rh)\cdot h$$

$$\therefore \int_0^4 x^2\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} r^2h^2\cdot h$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} r^2 \frac{64}{n^3} \qquad \ldots \left[\because h = \frac{4}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{n^3} \sum_{r=1}^{n} r^2\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{n^3}\cdot \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= \frac{64}{6}(1+0)(2+0) \qquad \ldots \left[\because \lim_{n\to\infty} \frac{1}{n} = 0\right]$$

$$= \frac{64}{3}.$$

Question 3.

$\int_0^2 e^x dx$

Solution:

Let f(x) = e^x, for 0 ≤ x ≤ 2

Divide the closed interval [0, 2] into n equal subntervals each of length h at the points

0, 0 + h, 0 + 2h, ….., 0 + rh, ….., 0 + nh = 2

i.e. 0, h, 2h, ….., rh, ….., nh = 2

∴ h = $\frac{2}{n}$ and as n → ∞, h → 0

Here, a = 0

$\therefore f(a+rh) = f(0+rh) = f(rh) = e^{rh}$

$\because \int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{r=1}^{n} f(a+rh)h$

$\therefore \int_0^2 e^x\,dx = \lim_{n \to \infty} \sum_{r=1}^{n} e^{rh} \cdot h$

$= \lim_{h \to 0} \left[h \sum_{r=1}^{n} e^{rh} \right]$ ... [as $n \to \infty$, $h \to 0$]

Now, $\sum_{r=1}^{n} e^{rh} = e^h + e^{2h} + \ldots + e^{nh}$

$= \frac{e^h[(e^h)^n - 1]}{e^h - 1} = \frac{e^h[e^{nh} - 1]}{e^h - 1}$

$= \frac{e^h \cdot (e^2 - 1)}{e^h - 1}$ ... [$\because h = \frac{2}{n}$ $\therefore nh = 2$]

$= (e^2 - 1) \frac{e^h}{e^h - 1}$

$\therefore \int_0^2 e^x\,dx = \lim_{h \to 0} \frac{h(e^2-1)\,e^h}{e^h - 1}$

$= (e^2 - 1) \lim_{h \to 0} \frac{e^h}{\left(\frac{e^h - 1}{h}\right)}$

$= (e^2 - 1) \frac{\lim_{h \to 0} e^h}{\lim_{h \to 0} \left(\frac{e^h - 1}{h}\right)} = (e^2 - 1) \frac{\lim_{h \to 0} e^h}{\lim_{h \to 0} \left(\frac{e^h - 1}{h}\right)}$

$= (e^2 - 1) \cdot \frac{e^0}{1}$ ... $\left[\because \lim_{h \to 0} \frac{e^{h-1}}{h} = 1\right]$

$= (e^2 - 1)$.

Question 4.
$\int_0^2 (3x^2 - 1)\,dx$
Solution:
Let $f(x) = 3x^2 - 1$, for $0 \le x \le 2$
Divide the closed interval [0, 2] into n subintervals each of length h at the points.
0, 0 + h, 0 + 2h, ….., 0 + rh, ……, 0 + nh = 2
i.e. 0, h, 2h, ….., rh, ….., nh = 2
$\therefore h = \frac{2}{n}$ and as $n \to \infty$, $h \to 0$
Here, a = 0
$\therefore f(a + rh) = f(0 + rh)$
$= f(rh)$
$= 3(rh)^2 - 1$

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$= 3r^2h^2 - 1$

$$\because \int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{r=1}^{n} f(a+rh)\cdot h$$

$$\therefore \int_0^2 (3x^2-1)\,dx = \lim_{n \to \infty} \sum_{r=1}^{n} (3r^2h^2-1)\cdot h$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \left(3r^2 \times \frac{4}{n^2} - 1\right)\cdot\frac{2}{n} \qquad \ldots \left[\because\ h = \frac{2}{n}\right]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \left(\frac{24r^2}{n^3} - \frac{2}{n}\right)$$

$$= \lim_{n \to \infty} \left[\frac{24}{n^3}\sum_{r=1}^{n} r^2 - \frac{2}{n}\sum_{r=1}^{n} 1\right]$$

$$= \lim_{n \to \infty} \left[\frac{24}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{2}{n}\cdot n\right]$$

$$= \lim_{n \to \infty} \left[4\cdot\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) - 2\right]$$

$$= \lim_{n \to \infty} \left[4\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 2\right]$$

$$= 4(1+0)(2+0) - 2 \qquad \ldots \left[\because\ \lim_{n \to \infty} \frac{1}{n} = 0\right]$$

$$= 8 - 2 = 6.$$

Question 5.
$\int_1^3 x^3\,dx$
Solution:
Let $f(x) = x^3$, for $1 \le x \le 3$.
Divide the closed interval [1, 3] into n equal su bintervals each of length h at the points
1, 1 + h, 1 + 2h, ……, 1 + rh, ……, 1 + nh = 3
∴ nh = 2
∴ $h = \frac{2}{n}$ and as $n \to \infty$, $h \to 0$
Here a = 1

$$\therefore\ f(a+rh)=f(1+rh)=(1+rh)^3$$

$$=1+3rh+3r^2h^2+r^3h^3$$

$$\because\ \int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{r=1}^{n} f(a+rh)\cdot h$$

$$\therefore\ \int_1^3 x^3\,dx=\lim_{n\to\infty}\sum_{r=1}^{n}(1+3rh+3r^2h^2+r^3h^3)\cdot h$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}(h+3rh^2+3r^2h^3+r^3h^4)$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\left[\frac{2}{n}+3r\left(\frac{2}{n}\right)^2+3r^2\left(\frac{2}{n}\right)^3+r^3\left(\frac{2}{n}\right)^4\right]$$

$$\ldots\left[\because\ h=\frac{2}{n}\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\left[\frac{2}{n}+\frac{12r}{n^2}+\frac{24r^2}{n^3}+\frac{16r^3}{n^4}\right]$$

$$=\lim_{n\to\infty}\left[\frac{2}{n}\sum_{r=1}^{n}1+\frac{12}{n^2}\sum_{r=1}^{n}r+\frac{24}{n^3}\sum_{r=1}^{n}r^2+\frac{16}{n^4}\sum_{r=1}^{n}r^3\right]$$

$$=\lim_{n\to\infty}\left[\frac{2}{n}\cdot n+\frac{12}{n^2}\cdot\frac{n(n+1)}{2}+\frac{24}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$=\lim_{n\to\infty}\left[2+6\left(\frac{n+1}{n}\right)+4\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)+4\left(\frac{n+1}{n}\right)^2\right]$$

$$=\lim_{n\to\infty}\left[2+6\left(1+\frac{1}{n}\right)+4\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+4\left(1+\frac{1}{n}\right)^2\right]$$

$$=[2+6(1+0)+4(1+0)(2+0)+4(1+0)^2]$$

$$\ldots\left[\because\ \lim_{n\to\infty}\frac{1}{n}=0\right]$$

$$=2+6+8+4=20.$$

Maharashtra State Board 12th Maths Solutions Chapter 4 Definite Integration Ex 4.2

I. Evaluate:

Question 1.
∫91x+1x√⋅ dx
Solution:

$$\int_1^9 \frac{x+1}{\sqrt{x}}\,dx = \int_1^9 \left(\frac{x}{\sqrt{x}}+\frac{1}{\sqrt{x}}\right)dx$$

$$= \int_1^9 x^{\frac{1}{2}}\,dx + \int_1^9 x^{-\frac{1}{2}}\,dx$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_1^9 + \left[\frac{x^{1/2}}{1/2}\right]_1^9$$

$$= \frac{2}{3}[9^{\frac{3}{2}} - 1^{\frac{3}{2}}] + 2[9^{\frac{1}{2}} - 1^{\frac{1}{2}}]$$

$$= \frac{2}{3}[(3^2)^{\frac{3}{2}} - 1] + 2[3-1]$$

$$= \frac{2}{3}[27-1] + 4$$

$$= \frac{52}{3} + 4 = \frac{64}{3}.$$

Question 2.
∫321x2+5x+6⋅ dx
Solution:

$$\int_2^3 \frac{1}{x^2+5x+6}\,dx = \int_2^3 \frac{1}{(x+2)(x+3)}\,dx$$

$$= \int_2^3 \frac{(x+3)-(x+2)}{(x+2)(x+3)}\,dx$$

$$= \int_2^3 \left[\frac{1}{x+2} - \frac{1}{x+3}\right]dx$$

$$= [\log(x+2) - \log(x+3)]_2^3$$

$$= \left[\log\left|\frac{x+2}{x+3}\right|\right]_2^3$$

$$= \log\left(\frac{3+2}{3+3}\right) - \log\left(\frac{2+2}{2+3}\right) = \log\frac{5}{6} - \log\frac{4}{5}$$

$$= \log\left(\frac{5}{6}\times\frac{5}{4}\right) = \log\left(\frac{25}{24}\right).$$

Question 3.
∫π/40cot2⋅ dx
Solution:

$$\int_0^{\pi/4} \cot^2 x\,dx = \int_0^{\pi/4} (\operatorname{cosec}^2 x - 1)\,dx$$

$$= \int_0^{\pi/4} \operatorname{cosec}^2 x\,dx - \int_0^{\pi/4} dx$$

$$= [-\cot x]_0^{\pi/4} - [x]_0^{\pi/4}$$

$$= \left[\left(-\cot\frac{\pi}{4}\right) - (-\cot 0)\right] - \left[\frac{\pi}{4} - 0\right]$$

$$= -1 + \cot 0 - \frac{\pi}{4}$$

The integral does not exist since cot 0 is not defined.

Question 4.
∫π/4−π/411−sinx⋅ dx
Solution:

$$\int_{-\pi/4}^{\pi/4} \frac{1}{1-\sin x}\,dx$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}\,dx$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1+\sin x}{1-\sin^2 x}\,dx = \int_{-\pi/4}^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx$$

$$= \int_{-\pi/4}^{\pi/4} \left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)dx$$

$$= \int_{-\pi/4}^{\pi/4} (\sec^2 x + \sec x \tan x)\,dx$$

$$= [\tan x + \sec x]_{-\pi/4}^{\pi/4}$$

$$= \left(\tan\frac{\pi}{4}+\sec\frac{\pi}{4}\right)-\left[\tan\left(-\frac{\pi}{4}\right)+\sec\left(-\frac{\pi}{4}\right)\right]$$

$$= (1+\sqrt{2})-\left(-\tan\frac{\pi}{4}+\sec\frac{\pi}{4}\right)$$

$$= (1+\sqrt{2})-(-1+\sqrt{2})$$

$$= 1+\sqrt{2}+1-\sqrt{2} = 2.$$

Question 5.
∫5312x+3√−2x−3√⋅ dx
Solution:

$$\int_3^5 \frac{1}{\sqrt{2x+3}-\sqrt{2x-3}}\,dx$$

$$= \int_3^5 \frac{1}{\sqrt{2x+3}-\sqrt{2x-3}} \times \frac{\sqrt{2x+3}+\sqrt{2x-3}}{\sqrt{2x+3}+\sqrt{2x-3}}\,dx$$

$$= \int_3^5 \frac{\sqrt{2x+3}+\sqrt{2x-3}}{(2x+3)-(2x-3)}\,dx$$

$$= \frac{1}{6}\int_3^5 (2x+3)^{\frac{1}{2}}dx + \frac{1}{6}\int_3^5 (2x-3)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{6}\left[\frac{(2x+3)^{3/2}}{2\left(\frac{3}{2}\right)}\right]_3^5 + \frac{1}{6}\left[\frac{(2x-3)^{3/2}}{2\left(\frac{3}{2}\right)}\right]_3^5$$

$$= \frac{1}{18}[(10+3)^{3/2}-(6+3)^{3/2}] + \frac{1}{18}[(10-3)^{3/2}-(6-3)^{3/2}]$$

$$= \frac{1}{18}[13\sqrt{13}-9\sqrt{9}] + \frac{1}{18}[7\sqrt{7}-3\sqrt{3}]$$

$$= \frac{1}{18}(13\sqrt{13}-27+7\sqrt{7}-3\sqrt{3})$$

$$= \frac{1}{18}(13\sqrt{13}+7\sqrt{7}-3\sqrt{3}-27).$$

Question 6.

∫10x2−2x2+1 · dx

Solution:

$$\int_0^1 \frac{x^2-2}{x^2+1}\,dx = \int_0^1 \frac{(x^2+1)-3}{x^2+1}\,dx$$

$$= \int_0^1 \left(1 - \frac{3}{x^2+1}\right) dx$$

$$= \left[x - 3\tan^{-1}x\right]_0^1$$

$$= (1 - 3\tan^{-1}1) - (0 - 3\tan^{-1}0)$$

$$= 1 - 3\left(\frac{\pi}{4}\right) - 0 = 1 - \frac{3\pi}{4}.$$

Question 7.

∫π/40sin4xsin3x · dx

Solution:

$$\int_0^{\pi/4} \sin 4x \sin 3x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/4} 2\sin 4x \sin 3x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/4} [\cos(4x-3x) - \cos(4x+3x)]\,dx$$

$$= \frac{1}{2}\int_0^{\pi/4} \cos x\,dx - \frac{1}{2}\int_0^{\pi/4} \cos 7x\,dx$$

$$= \frac{1}{2}[\sin x]_0^{\pi/4} - \frac{1}{2}\left[\frac{\sin 7x}{7}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{4} - \sin 0\right] - \frac{1}{14}\left[\sin\frac{7\pi}{4} - \sin 0\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} - 0\right] - \frac{1}{14}\left[\sin\left(2\pi - \frac{\pi}{4}\right) - 0\right]$$

$$= \frac{1}{2\sqrt{2}} - \frac{1}{14}\left(-\sin\frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}} + \frac{1}{14\sqrt{2}} = \frac{7+1}{14\sqrt{2}}$$

$$= \frac{4}{7\sqrt{2}}.$$

Question 8.

∫π/401+sin2x−−−−−−−−√ · dx

Solution:

$$\int_0^{\pi/4} \sqrt{1+\sin 2x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x}\, dx$$

$$= \int_0^{\pi/4} \sqrt{(\sin x + \cos x)^2}\, dx$$

$$= \int_0^{\pi/4} (\sin x + \cos x)\, dx = \int_0^{\pi/4} \sin x\, dx + \int_0^{\pi/4} \cos x\, dx$$

$$= [-\cos x]_0^{\pi/4} + [\sin x]_0^{\pi/4}$$

$$= \left[-\cos\frac{\pi}{4} - (-\cos 0)\right] + \left[\sin\frac{\pi}{4} - \sin 0\right]$$

$$= -\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} - 0 = 1.$$

Question 9.

$\int_0^{\pi/4} \sin^4 x \cdot dx$

Solution:

$$\text{Consider } \sin^4 x = (\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2$$

$$= \frac{1}{4}[1 - 2\cos 2x + \cos^2 2x]$$

$$= \frac{1}{4}\left[1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right]$$

$$= \frac{1}{4}\left[\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right]$$

$$\therefore \int_0^{\pi/4} \sin^4 x\, dx$$

$$= \frac{1}{4}\int_0^{\pi/4} \left[\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right] dx$$

$$= \frac{3}{8}\int_0^{\pi/4} 1 dx - \frac{1}{2}\int_0^{\pi/4} \cos 2x\, dx + \frac{1}{8}\int_0^{\pi/4} \cos 4x\, dx$$

$$= \frac{3}{8}[x]_0^{\pi/4} - \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} + \frac{1}{8}\left[\frac{\sin 4x}{4}\right]_0^{\pi/4}$$

$$= \frac{3}{8}\left[\frac{\pi}{4} - 0\right] - \frac{1}{4}\left[\sin\frac{\pi}{2} - \sin 0\right] + \frac{1}{32}[\sin \pi - \sin 0]$$

$$= \frac{3\pi}{32} - \frac{1}{4}[1-0] + \frac{1}{32}(0-0)$$

$$= \frac{3\pi}{32} - \frac{1}{4} = \frac{3\pi - 8}{32}.$$

Question 10.

$\int_{-4}^{2} \frac{1}{x^2+4x+13} \cdot dx$

Solution:

$$\int_{-4}^{2} \frac{1}{x^2+4x+13}\,dx$$

$$= \int_{-4}^{2} \frac{1}{x^2+4x+4+9}\,dx = \int_{-4}^{2} \frac{1}{(x+2)^2+3^2}\,dx$$

$$= \left[\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)\right]_{-4}^{2}$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{2+2}{3}\right) - \frac{1}{3}\tan^{-1}\left(\frac{-4+2}{3}\right)$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{4}{3}\right) - \frac{1}{3}\tan^{-1}\left(-\frac{2}{3}\right)$$

$$= \frac{1}{3}\left[\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{2}{3}\right] \quad \ldots [\because \tan^{-1}(-x) = -\tan^{-1}x]$$

Question 11.

∫4014x−x2√⋅ dx

Solution:

$$\int_{0}^{4} \frac{1}{\sqrt{4x-x^2}}\,dx$$

$$= \int_{0}^{4} \frac{1}{\sqrt{4-(x^2-4x+4)}}\,dx$$

$$= \int_{0}^{4} \frac{1}{\sqrt{2^2-(x-2)^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x-2}{2}\right)\right]_{0}^{4} = \sin^{-1}\left(\frac{4-2}{2}\right) - \sin^{-1}\left(\frac{0-2}{2}\right)$$

$$= \sin^{-1}1 - \sin^{-1}(-1)$$

$$= 2\sin^{-1}1 \quad \ldots [\because \sin^{-1}(-x) = -\sin^{-1}x]$$

$$= 2\left(\frac{\pi}{2}\right) = \pi.$$

Question 12.

∫1013+2x−x2√⋅ dx

Solution:

$$\int_{0}^{1} \frac{1}{\sqrt{3+2x-x^2}}\,dx$$

$$= \int_{0}^{1} \frac{1}{\sqrt{3-(x^2-2x+1)+1}}\,dx$$

$$= \int_{0}^{1} \frac{1}{\sqrt{(2)^2-(x-1)^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x-1}{2}\right)\right]_{0}^{1}$$

$$= \sin^{-1}(0) - \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= 0 - \sin^{-1}\left(-\sin\frac{\pi}{6}\right)$$

$$= -\sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right] = -\left(-\frac{\pi}{6}\right) = \frac{\pi}{6}.$$

Question 13.
∫π/20x⋅ sinx⋅ dx
Solution:

$$\int_0^{\pi/2} x\sin x\, dx$$

$$= [x\int \sin x\, dx]_0^{\pi/2} - \int_0^{\pi/2} \left[\frac{d}{dx}(x)\int \sin x\, dx\right]dx$$

$$= [x(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} 1\cdot(-\cos x)\, dx$$

$$= -[x\cos x]_0^{\pi/2} + \int_0^{\pi/2} \cos x\, dx$$

$$= -\left[\frac{\pi}{2}\cos\frac{\pi}{2} - 0\right] + [\sin x]_0^{\pi/2}$$

$$= 0 + \left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= 1.$$

Question 14.
∫10x⋅ tan−1x⋅ dx
Solution:

$$\text{Let } I = \int_0^1 x\tan^{-1} x\, dx$$

$$= \int_0^1 (\tan^{-1} x)(x)\, dx$$

$$= \left[(\tan^{-1} x)\int x\, dx\right]_0^1 - \int_0^1 \left[\frac{d}{dx}(\tan^{-1} x)\cdot\int x\, dx\right]dx$$

$$= \left[\frac{x^2 \tan^{-1} x}{2}\right]_0^1 - \int_0^1 \frac{1}{1+x^2}\cdot\frac{x^2}{2}\, dx$$

$$= \left(\frac{1^2 \tan^{-1} 1}{2} - 0\right) - \frac{1}{2}\int_0^1 \frac{1+x^2-1}{1+x^2}\, dx$$

$$= \frac{\pi/4}{2} - \frac{1}{2}\int_0^1 \left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\left[x - \tan^{-1}(x)\right]_0^1$$

$$= \frac{\pi}{8} - \frac{1}{2}[(1 - \tan^{-1} 1) - 0]$$

$$= \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}.$$

Question 15.
∫∞0x⋅ e−x⋅ dx
Solution:

$$\int_0^{\infty} xe^{-x}\,dx$$

$$=[x\int e^{-x}\,dx]_0^{\infty}-\int_0^{\infty}\left[\frac{d}{dx}(x)\int e^{-x}\,dx\right]dx$$

$$=\left[x\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty}-\int_0^{\infty}1\cdot\frac{e^{-x}}{(-1)}\,dx$$

$$=\left[-\frac{x}{e^x}\right]_0^{\infty}+\int_0^{\infty}e^{-x}dx$$

$$=\left[-\frac{x}{e^x}\right]_0^{\infty}+[-e^x]_0^{\infty}$$

$$=[0-(-0)]+[0-(-1)]$$

$$=1. \qquad \ldots\left[\because e^0=1,\ e^{-x}=0,\ \text{when } x=\infty\right]$$

II. Evaluate:

Question 1.
∫12√0sin-1x(1-x2)32 · dx
Solution:

$$\text{Let } I=\int_0^{1/\sqrt{2}}\frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx=\int_0^{1/\sqrt{2}}\frac{\sin^{-1}x}{(1-x^2)\sqrt{1-x^2}}\,dx$$

$$\text{Put } \sin^{-1}x=t \qquad \therefore \frac{1}{\sqrt{1-x^2}}\,dx=dt$$

Also, $x=\sin t$

When $x=\frac{1}{\sqrt{2}}$, $t=\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$

When $x=0$, $t=\sin^{-1}0=0$

$$\therefore\ I=\int_0^{\pi/4}\frac{t}{1-\sin^2 t}\,dt=\int_0^{\pi/4}\frac{t}{\cos^2 t}\,dt=\int_0^{\pi/4}t\sec^2 t\,dt$$

$$=[t\int\sec^2 t\,dt]_0^{\pi/4}-\int_0^{\pi/4}\left[\frac{d}{dt}(t)\int\sec^2 t\,dt\right]dt$$

$$=[t\tan t]_0^{\pi/4}-\int_0^{\pi/4}1\cdot\tan t\,dt$$

$$=\left[\frac{\pi}{4}\tan\frac{\pi}{4}-0\right]-[\log|\sec t|]_0^{\pi/4}$$

$$=\frac{\pi}{4}-\left[\log\left(\sec\frac{\pi}{4}\right)-\log(\sec 0)\right]$$

$$=\frac{\pi}{4}-[\log\sqrt{2}-\log 1]$$

$$=\frac{\pi}{4}-\frac{1}{2}\log 2. \qquad \ldots[\because \log 1=0]$$

Question 2.

∫π/40sec2x3tan2x+4tanx+1 · dx

Solution:

$$\text{Let } I=\int_0^{\pi/4}\frac{\sec^2 x}{3\tan^2 x+4\tan x+1}\,dx$$

Put $\tan x=t$ $\quad\therefore\ \sec^2 x\,dx=dt$

When $x=0,\ t=\tan 0=0$

When $x=\frac{\pi}{4},\ t=\tan\frac{\pi}{4}=1$

$$\therefore\ I=\int_0^1\frac{dt}{3t^2+4t+1}=\frac{1}{3}\int_0^1\frac{dt}{t^2+\frac{4}{3}t+\frac{1}{3}}$$

$$=\frac{1}{3}\int_0^1\frac{dt}{t^2+\frac{4t}{3}+\frac{4}{9}-\frac{4}{9}+\frac{1}{3}}=\frac{1}{3}\int_0^1\frac{dt}{\left(t+\frac{2}{3}\right)^2-\left(\frac{1}{3}\right)^2}$$

$$=\frac{1}{3}\,\frac{1}{2\left(\frac{1}{3}\right)}\left[\log\left|\frac{t+\frac{2}{3}-\frac{1}{3}}{t+\frac{2}{3}+\frac{1}{3}}\right|\right]_0^1$$

$$=\frac{1}{2}\left[\log\left(\frac{1+\frac{1}{3}}{1+1}\right)-\log\left(\frac{0+\frac{1}{3}}{0+1}\right)\right]$$

$$=\frac{1}{2}\left[\log\left(\frac{2}{3}\right)-\log\left(\frac{1}{3}\right)\right]=\frac{1}{2}\log 2.$$

Question 3.

∫4π0sin2xsin4x+cos4x · dx

Solution:

$$\text{Let } I = \int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$$

$$= \int_0^{\pi/4} \frac{2 \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$$

Dividing each term by $\cos^4 x$, we get

$$I = \int_0^{\pi/4} \frac{2\, \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x}}{\frac{\sin^4 x}{\cos^4 x} + 1}\, dx$$

$$= \int_0^{\pi/4} \frac{2 \tan x \sec^2 x}{(\tan^2 x)^2 + 1}\, dx$$

Put $\tan^2 x = t \quad \therefore\ 2 \tan x \sec^2 x\, dx = dt$

When $x = 0$, $t = \tan^2 0 = 0$

When $x = \frac{\pi}{4}$, $t = \tan^2 \frac{\pi}{4} = 1$

$$\therefore\ I = \int_0^1 \frac{dt}{1+t^2} = [\tan^{-1} t]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

Question 4.

∫2π0cosx----√⋅ sin3x⋅ dx

Solution:

$$\text{Let } I = \int_0^{2\pi} \sqrt{\cos x}\, \sin^3 x\, dx$$

$$= \int_0^{2\pi} \sqrt{\cos x}\, \sin^2 x \sin x\, dx$$

$$= \int_0^{2\pi} \sqrt{\cos x}\, (1 - \cos^2 x) \sin x\, dx$$

Put $\cos x = t \quad \therefore\ -\sin x\, dx = dt$

$\therefore\ \sin x\, dx = -dt$

When $x = 0$, $t = \cos 0 = 1$

When $x = 2\pi$, $t = \cos 2\pi = 1$

$$\therefore\ I = \int_1^1 \sqrt{t}(1-t^2)(-dt) = 0. \qquad \ldots \left[\because \int_a^a f(x)\,dx = 0\right]$$

Question 5.

∫π/2015+4cosx⋅ dx

Solution:

$$\text{Let } I=\int_0^{\pi/2} \frac{1}{5+4\cos x}\,dx$$

Put $\tan\left(\frac{x}{2}\right)=t \quad \therefore\ x=2\tan^{-1}t \quad \therefore\ dx=\frac{2dt}{1+t}$

and $\cos x=\frac{1-t^2}{1+t^2}$

When $x=\frac{\pi}{2}$, $t=\tan\left(\frac{\pi}{4}\right)=1$

When $x=0$, $t=\tan 0=0$

$$\therefore\ I=\int_0^1 \frac{\left(\frac{2dt}{1+t^2}\right)}{5+4\left(\frac{1-t^2}{1+t^2}\right)}=\int_0^1 \frac{2dt}{5(1+t^2)+4(1-t^2)}$$

$$=2\int_0^1 \frac{1}{t^2+9}\,dt$$

Question 6.

∫π/40cosx4−sin2x⋅ dx

Solution:

$$\text{Let } I=\int_0^{\pi/4} \frac{\cos x}{4-\sin^2 x}\,dx$$

Put $\sin x=t \quad \therefore\ \cos x\,dx=dt$

When $x=\frac{\pi}{4}$, $t=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

When $x=0$, $t=\sin 0=0$.

$$\therefore\ I=\int_0^{1/\sqrt{2}} \frac{dt}{2^2-t^2}$$

$$=\left[\frac{1}{2(2)}\log\left|\frac{2+t}{2-t}\right|\right]_0^{1/\sqrt{2}}$$

$$=\frac{1}{4}\left[\log\left(\frac{2+\frac{1}{\sqrt{2}}}{2-\frac{1}{\sqrt{2}}}\right)-\log\left(\frac{2+0}{2-0}\right)\right]$$

$$=\frac{1}{4}\left[\log\left(\frac{2\sqrt{2}+1}{2\sqrt{2}-1}\right)-\log 1\right]$$

$$=\frac{1}{4}\log\left(\frac{2\sqrt{2}+1}{2\sqrt{2}-1}\right). \qquad \ldots [\because\ \log 1=0]$$

Question 7.

∫π/20cosX(1+sinx)(2+sinx)⋅ dx

Solution:

$$\text{Let } I=\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)}\, dx$$

Put $\sin x = t \quad \therefore \cos x\, dx = dt$

When $x=\frac{\pi}{2}$, $t=\sin\frac{\pi}{2}=1$

When $x=0$, $t=\sin 0=0$

$$\therefore I=\int_0^1 \frac{dt}{(1+t)(2+t)}=\int_0^1 \frac{(2+t)-(1+t)}{(1+t)(2+t)}\, dt$$

$$=\int_0^1 \left[\frac{1}{1+t}-\frac{1}{2+t}\right] dt$$

$$=\int_0^1 \frac{1}{1+t}\, dt-\int_0^1 \frac{1}{2+t}\, dt$$

$$=[\log|1+t|]_0^1-[\log|2+t|]_0^1$$

$$=[\log(1+1)-\log(1+0)]-[\log(2+1)-\log(2+0)]$$

$$=\log 2-\log 3+\log 2 \qquad \ldots [\because \log 1=0]$$

$$=\log\left(\frac{2\times 2}{3}\right)=\log\left(\frac{4}{3}\right).$$

Question 8.
$\int_{-1}^{1} \frac{1}{a^2e^x+b^2e^{-x}}\, dx$
Solution:

$$\text{Let } I=\int_{-1}^{1} \frac{1}{a^2e^x+b^2e^{-x}}\, dx$$

Multiplying each term by e^x, we get

$$I=\int_{-1}^{1} \frac{e^x}{a^2(e^x)^2+b^2}\, dx$$

Put $e^x=t \quad \therefore e^x\, dx=dt$

When $x=1$, $t=e$

When $x=-1$, $t=e^{-1}=\frac{1}{e}$

$$\therefore I=\int_{1/e}^{e} \frac{dt}{a^2t^2+b^2}=\int_{1/e}^{e} \frac{dt}{(at)^2+b^2}$$

$$=\left[\frac{1}{a}\cdot\frac{1}{b}\tan^{-1}\left(\frac{at}{b}\right)\right]_{1/e}^{e}$$

$$=\frac{1}{ab}\tan^{-1}\left(\frac{ae}{b}\right)-\frac{1}{ab}\tan^{-1}\left(\frac{a}{be}\right)$$

$$=\frac{1}{ab}\left[\tan^{-1}\left(\frac{ae}{b}\right)-\tan^{-1}\left(\frac{a}{be}\right)\right].$$

Question 9.
$\int_0^{\pi} \frac{1}{3+2\sin x+\cos x}\, dx$
Solution:

$$\text{Let } I = \int_0^{\pi} \frac{1}{3 + 2\sin x + \cos x}\, dx$$

Put $\tan\frac{x}{2} = t \quad \therefore\ x = 2\tan^{-1} t$

$\therefore\ dx = \frac{2dt}{1+t^2}$ and $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

When $x = 0$, $t = \tan 0 = 0$ When $x = \pi$, $t = \tan\frac{\pi}{2} = \infty$

$$\therefore\ I = \int_0^{\infty} \frac{1}{3 + 2\left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^{\infty} \frac{1+t^2}{3 + 3t^2 + 4t + 1 - t^2} \cdot \frac{2dt}{1+t^2}$$

$$= 2\int_0^{\infty} \frac{1}{2t^2 + 4t + 4}\, dt$$

$$= \frac{2}{2}\int_0^{\infty} \frac{1}{t^2 + 2t + 2}\, dt = \int_0^{\infty} \frac{1}{(t^2 + 2t + 1) + 1}\, dt$$

$$= \int_0^{\infty} \frac{1}{(t+1)^2 + (1)^2}\, dt = \frac{1}{1}\left[\tan^{-1}\left(\frac{t+1}{1}\right)\right]_0^{\infty}$$

$$= [\tan^{-1}(t+1)]_0^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}1 \qquad = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Question 10.
∫π/40sec4x⋅ dx
Solution:

$$\text{Let } I = \int_0^{\pi/4} \sec^4 x\, dx$$

$$= \int_0^{\pi/4} \sec^2 x \cdot \sec^2 x\, dx$$

$$= \int_0^{\pi/4} (1 + \tan^2 x)\sec^2 x\, dx$$

Put $\tan x = t \quad \therefore\ \sec^2 x\, dx = dt$

When $x = 0$, $t = \tan 0 = 0$

When $x = \frac{\pi}{4}$, $t = \tan\frac{\pi}{4} = 1$

$$\therefore\ I = \int_0^1 (1 + t^2)\, dt = \left[t + \frac{t^3}{3}\right]_0^1$$

$$= 1 + \frac{1}{3} - 0 = \frac{4}{3}.$$

Question 11.
∫101−x1+x−−−√⋅ dx
Solution:

$$\text{Let } I=\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$$

Put $x=\cos\theta \quad \therefore\ dx=-\sin\theta\,d\theta$

When $x=0$, $\cos\theta=0=\cos\frac{\pi}{2} \quad \therefore\ \theta=\frac{\pi}{2}$

When $x=1$, $\cos\theta=1=\cos 0 \quad \therefore\ \theta=0$

$$\therefore\ I=\int_{\pi/2}^{0} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\cdot(-\sin\theta)\,d\theta$$

$$=\int_{\pi/2}^{0} \sqrt{\frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}}\left(-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)d\theta$$

$$=\int_{\pi/2}^{0} \frac{\sin(\theta/2)}{\cos(\theta/2)}\left[-2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right]d\theta$$

$$=\int_{\pi/2}^{0} -2\sin^2\left(\frac{\theta}{2}\right)d\theta$$

$$=-\int_{\pi/2}^{0} (1-\cos\theta)\,d\theta$$

$$=-[\theta-\sin\theta]_{\pi/2}^{0}$$

$$=-\left[(0-\sin 0)-\left(\frac{\pi}{2}-\sin\frac{\pi}{2}\right)\right]$$

$$=-\left[0-\frac{\pi}{2}+1\right]=\frac{\pi}{2}-1.$$

Question 12.

$\int_0^\pi \sin^3x(1+2\cos x)(1+\cos x)^2\cdot dx$

Solution:

$$\text{Let } I=\int_0^\pi \sin^3x\,(1+2\cos x)\,(1+\cos x)^2\,dx$$

$$=\int_0^\pi \sin^2x\,(1+2\cos x)\,(1+\cos x)^2\cdot\sin x\,dx$$

$$=\int_0^\pi (1-\cos^2x)(1+2\cos x)(1+\cos x)^2\cdot\sin x\,dx$$

Put $\cos x=t \quad \therefore\ -\sin x\,dx=dt.$

$\therefore\ \sin x\,dx=-dt$

When $x=0$, $t=\cos 0=1$

When $x=\pi$, $t=\cos\pi=-1$

$$\therefore\ I=\int_1^{-1} (1-t^2)(1+2t)(1+t)^2(-dt)$$

$$= -\int_{1}^{-1} (1+2t-t^2-2t^3)(1+2t+t^2)\,dt$$

$$= -\int_{1}^{-1} (1+2t-t^2-2t^3+2t+4t^2-2t^3-4t^4+ t^2+2t^3-t^4-2t^5)\,dt$$

$$= -\int_{1}^{-1} (1+4t+4t^2-2t^3-5t^4-2t^5)\,dt$$

$$= -\left[t+4\left(\frac{t^2}{2}\right)+4\left(\frac{t^3}{3}\right)-2\left(\frac{t^4}{4}\right)-5\left(\frac{t^5}{5}\right)-2\left(\frac{t^6}{6}\right)\right]_1^{-1}$$

$$= -\left[t+2t^2+\frac{4}{3}t^3-\frac{1}{2}t^4-t^5-\frac{1}{3}t^6\right]_1^{-1}$$

$$= -\left[\left(-1+2-\frac{4}{3}-\frac{1}{2}+1-\frac{1}{3}\right)- \left(1+2+\frac{4}{3}-\frac{1}{2}-1-\frac{1}{3}\right)\right]$$

$$= -\left[-1+2-\frac{4}{3}-\frac{1}{2}+1-\frac{1}{3}-1-2-\frac{4}{3}+\frac{1}{2}+1+\frac{1}{3}\right]$$

$$= -\left[-\frac{8}{3}\right]=\frac{8}{3}.$$

Question 13.

$\int_0^{\pi/2} \sin 2x \cdot \tan^{-1}(\sin x) \cdot dx$

Solution:

$$\text{Let } I=\int_0^{\pi/2} \sin 2x\cdot\tan^{-1}(\sin x)\,dx$$

$$= \int_0^{\pi/2} \tan^{-1}(\sin x)\cdot(2\sin x\cos x)\,dx$$

Put $\sin x = t \quad \therefore \cos x\,dx = dt$

When $x=0,\ t=\sin 0=0.$ When $x=\frac{\pi}{2},\ t=\sin\frac{\pi}{2}=1$

$$\therefore\ I=\int_0^1 (\tan^{-1}t)(2t)\,dt$$

$$=\left[\tan^{-1}t\int 2t\,dt\right]_0^1-\int_0^1\left(\frac{d}{dt}(\tan^{-1}t)\int 2t\,dt\right)dt$$

$$=\left[(\tan^{-1}t)(t^2)\right]_0^1-\int_0^1\frac{1}{1+t^2}\cdot t^2\,dt$$

$$=\left[t^2\tan^{-1}t\right]_0^1-\int_0^1\frac{(1+t^2)-1}{1+t^2}\,dt$$

$$=[1\cdot\tan^{-1}1-0]-\int_0^1\left(1-\frac{1}{1+t^2}\right)dt$$

$$=\frac{\pi}{4}-\left[t-\tan^{-1}t\right]_0^1$$

$$=\frac{\pi}{4}-[(1-\tan^{-1}1)-0]$$

$$=\frac{\pi}{4}-1+\frac{\pi}{4}=\frac{\pi}{2}-1.$$

Question 14.

$\int_{1/\sqrt{2}}^{1} \frac{e^{\cos^{-1}x} \sin^{-1}x}{\sqrt{1-x^2}} \cdot dx$

Solution:

$$\text{Let } I = \int_{1/\sqrt{2}}^{1} \frac{e^{\cos^{-1}x} \sin^{-1}x}{\sqrt{1-x^2}}\, dx$$

Put $\sin^{-1}x = t \qquad \therefore \frac{1}{\sqrt{1-x^2}}\, dx = dt$

When $x = 1$, $t = \sin^{-1} 1 = \frac{\pi}{2}$

When $x = \frac{1}{\sqrt{2}}$, $t = \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

Also, $\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x = \frac{\pi}{2} - t$

$$\therefore\ I = \int_{\pi/4}^{\pi/2} e^{\frac{\pi}{2}-t} \cdot t\, dt$$

$$= e^{\frac{\pi}{2}} \int_{\pi/4}^{\pi/2} te^{-t}\, dt$$

$$= e^{\frac{\pi}{2}} \left\{ [t \int e^{-t}\, dt]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \left[\frac{d}{dt}(t) \int e^{-t} dt\right] dt \right\}$$

$$= e^{\frac{\pi}{2}} \{ [-te^{-t}]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} (1)(-e^{-t})\, dt \}$$

$$= e^{\frac{\pi}{2}} \left\{ \frac{-\pi}{2} e^{-\frac{\pi}{2}} + \frac{\pi}{4} e^{-\frac{\pi}{4}} + \int_{\pi/4}^{\pi/2} e^{-t}\, dt \right\}$$

$$= -\frac{\pi}{2} e^{0} + \frac{\pi}{4} e^{\frac{\pi}{2}-\frac{\pi}{4}} + e^{\frac{\pi}{2}} [-e^{-t}]^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{2} + \frac{\pi}{4} e^{\frac{\pi}{4}} + e^{\frac{\pi}{2}} [-e^{-\frac{\pi}{2}} + e^{\frac{-\pi}{4}}]$$

$$= -\frac{\pi}{2} + e^{\frac{\pi}{4}} \frac{\pi}{4} - e^{0} + e^{\frac{\pi}{2}-\frac{\pi}{4}}$$

$$= -\frac{\pi}{2} + e^{\frac{\pi}{4}} \frac{\pi}{4} - 1 + e^{\frac{\pi}{4}}$$

$$= e^{\frac{\pi}{4}} \left(\frac{\pi}{4} + 1\right) - \left(\frac{\pi}{2} + 1\right).$$

Question 15.

$\int_{2}^{3} \frac{\cos(\log x)}{x} \cdot dx$

Solution:

$$\text{Let } I=\int_1^3 \frac{\cos(\log x)}{x}\,dx$$

$$=\int_1^3 \cos(\log x)\cdot\frac{1}{x}\,dx$$

Put $\log x=t \quad \therefore \frac{1}{x}\,dx=dt$

When $x=1,\ t=\log 1=0$

When $x=3,\ t=\log 3$

$$\therefore\ I=\int_0^{\log 3} \cos t\,dt=[\sin t]_0^{\log 3}$$

$$=\sin(\log 3)-\sin 0=\sin(\log 3).$$

III. Evaluate:

Question 1.

$\int_0^a \frac{1}{x+\sqrt{a^2-x^2}}\,dx$

Solution:

$$\text{Let } I = \int_0^a \frac{1}{x+\sqrt{a^2-x^2}}\,dx$$

Put $x = a\sin\theta \quad \therefore\ dx = a\cos\theta\,d\theta$

and $\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$

$= \sqrt{a^2\cos^2\theta} = a\cos\theta$

When $x = 0$, $a\sin\theta = 0 \quad \therefore\ \theta = 0$

When $x = a$, $a\sin\theta = a \quad \therefore\ \theta = \frac{\pi}{2}$

$$\therefore\ I = \int_0^{\pi/2} \frac{a\cos\theta\,d\theta}{a\sin\theta + a\cos\theta}$$

$$\therefore\ I = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta}\,d\theta \quad \ldots (1)$$

We use the property, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Hence in I, we change θ by $(\pi/2)-\theta$.

$$\therefore\ I = \int_0^{\pi/2} \frac{\cos\ [(\pi/2)-\theta]}{\sin\ [(\pi/2)-\theta] + \cos\ [(\pi/2)-\theta]}\,d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\ \theta}{\cos\ \theta + \sin\ \theta}\,d\theta \quad \ldots (2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\cos\ \theta}{\sin\ \theta + \cos\ \theta}\,d\theta + \int_0^{\pi/2} \frac{\sin\ \theta}{\cos\ \theta + \sin\ \theta}\,d\theta$$

$$= \int_0^{\pi/2} \frac{\cos\ \theta + \sin\ \theta}{\cos\ \theta + \sin\ \theta}\,d\theta = \int_0^{\pi/2} 1\ d\theta = [\theta]_0^{\pi/2}$$

$= (\pi/2) - 0 = \pi/2$

$\therefore\ I = \pi/4.$

Question 2.

$\int_0^{\pi/2} \log \tan x \cdot dx$

Solution:

Let $I = \int_0^{\pi/2} \log(\tan x)\,dx$

We use the property, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Here, $a = \frac{\pi}{2}$. Hence changing x by $\frac{\pi}{2} - x$, we get

$$I = \int_0^{\pi/2} \log\left[\tan\left(\frac{\pi}{2} - x\right)\right] dx$$

$$= \int_0^{\pi/2} \log(\cot x)\,dx = \int_0^{\pi/2} \log\left(\frac{1}{\tan x}\right) dx$$

$$= \int_0^{\pi/2} \log(\tan x)^{-1}\,dx = \int_0^{\pi/2} -\log(\tan x)\,dx$$

$$= -\int_0^{\pi/2} \log(\tan x)\,dx = -I$$

$\therefore\ 2I = 0 \qquad \therefore\ I = 0.$

Question 3.
$\int_0^1 \log\left(\frac{1}{x} - 1\right) \cdot dx$
Solution:

Let $I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx = \int_0^1 \log\left(\frac{1-x}{x}\right) dx$

$$= \int_0^1 [\log(1-x) - \log x]\,dx \qquad \ldots (1)$$

We use the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Here, $a = 1$

Hence in I, changing x to $1 - x$, we get

$$I = \int_0^1 [\log|1-(1-x)| - \log(1-x)]\,dx$$

$$= \int_0^1 [\log x - \log(1-x)]\,dx$$

$$= -\int_0^1 [\log(1-x) - \log x]\,dx = -I \qquad \ldots \text{[By (1)]}$$

$\therefore\ 2I = 0 \qquad \therefore\ I = 0.$

Question 4.
$\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cdot \cos x} \cdot dx$
Solution:

$$\text{Let } I=\int_0^{\pi/2} \frac{\sin x-\cos x}{1+\sin x \cos x}\,dx$$

We use the property, $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$.

Here $a=\frac{\pi}{2}$. Hence In I, we change x by $\frac{\pi}{2}-x$.

$$\therefore\ I=\int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}$$

$$=\int_0^{\pi/2} \frac{\cos x-\sin x}{1+\cos x \sin x}\,dx$$

$$=-\int_0^{\pi/2} \frac{\sin x-\cos x}{1+\sin x \cos x}\,dx=-I$$

$\therefore\ 2I=0 \qquad \therefore\ I=0.$

Question 5.
$\int_0^3 x^2(3-x)^{\frac{5}{2}}\,dx$
Solution:

$$\text{Let } I=\int_0^3 x^2(3-x)^{5/2}dx$$

We use the property $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

Here, $a=3$

Hence in I, changing x to $3-x$, we get

$$I=\int_0^3 (3-x)^2[3-(3-x)]^{5/2}dx=\int_0^3 (9-6x+x^2)\,x^{5/2}\,dx$$

$$=\int_0^3 [9x^{5/2}-6x^{7/2}+x^{9/2}]\,dx$$

$$=9\int_0^3 x^{5/2}dx-6\int_0^3 x^{7/2}dx+\int_0^3 x^{9/2}dx$$

$$=9\left[\frac{x^{7/2}}{7/2}\right]_0^3-6\left[\frac{x^{9/2}}{9/2}\right]_0^3+9\left[\frac{x^{11/2}}{11/2}\right]_0^3$$

$$=9\left[\frac{2\cdot 3^{7/2}}{7}-0\right]-6\left[\frac{2\cdot 3^{9/2}}{9}-0\right]+\left[\frac{2}{11}\cdot 3^{11/2}-0\right]$$

$$=\frac{18}{7}3^{7/2}-\frac{2\cdot 6}{9}\cdot 3^{\frac{7}{2}}\cdot 3+\frac{2}{11}\cdot 3^{\frac{7}{2}}\cdot 3^2$$

$$=2(3)^{7/2}\left[\frac{9}{7}-2+\frac{9}{11}\right]=2(3)^{\frac{7}{2}}\left[\frac{99-154+63}{77}\right]$$

$$=2(3)^{7/2}\times\frac{8}{77}=\frac{16}{77}(3)^{7/2}$$

Question 6.
$\int_3^{-3} \frac{x^3}{9-x^2}\,dx$

Solution:

$$\text{Let } I=\int_{-3}^{3} \frac{x^3}{9-x^2}\,dx$$

$$\text{Let } f(x)=\frac{x^3}{9-x^2}$$

$$\therefore f(-x)=\frac{(-x)^3}{9-(-x)^2}=\frac{-x^3}{9-x^2}=-f(x)$$

$\therefore f$ is an odd function.

$$\therefore \int_{-3}^{3} f(x)\,dx=0, \text{ i.e. } \int_{-3}^{3} \frac{x^3}{9-x^2}\,dx=0.$$

Question 7.

∫π/2−π/2log(2+sinx2−sinx)⋅ dx

Solution:

$$\text{Let } I=\int_{-\pi/2}^{\pi/2} \log\left(\frac{2-\sin x}{2+\sin x}\right)dx$$

$$\text{Let } f(x)=\log\left(\frac{2-\sin x}{2+\sin x}\right)$$

$$\therefore f(-x)=\log\left[\frac{2-\sin(-x)}{2+\sin(-x)}\right]=\log\left(\frac{2+\sin x}{2-\sin x}\right)$$

$$=-\log\left(\frac{2-\sin x}{2+\sin x}\right)=-f(x)$$

$\therefore f$ is an odd function.

$$\therefore \int_{-\pi/2}^{\pi/2} f(x)\,dx=0$$

$$\therefore \int_{-\pi/2}^{\pi/2} \log\left(\frac{2-\sin x}{2+\sin x}\right)dx=0.$$

Question 8.

∫π/4−π/4x+π42−cos2x⋅ dx

Solution:

$$\text{Let } I=\int_{-\pi/4}^{\pi/4} \frac{x+\frac{\pi}{4}}{2-\cos 2x}\,dx$$

$$=\int_{-\pi/4}^{\pi/4}\left[\frac{x}{2-\cos 2x}+\frac{\pi/4}{2-\cos 2x}\right]$$

$$=\int_{-\pi/4}^{\pi/4} \frac{x}{2-\cos 2x}\,dx+\frac{\pi}{4}\int_{-\pi/4}^{\pi/4} \frac{1}{2-\cos 2x}\,dx$$

$$=I_1+\frac{\pi}{4}I_2 \qquad \ldots (1)$$

$$\text{Let } f(x)=\frac{x}{2-\cos 2x}$$

$$\therefore f(-x)=\frac{-x}{2-\cos[2(-x)]}=\frac{-x}{2-\cos 2x}=-f(x)$$

$\therefore f$ is an odd function

$$\therefore \int_{-\pi/4}^{\pi/4} f(x)\,dx=0$$

$$\text{i.e. } \int_{-\pi/4}^{\pi/4} \frac{x}{2-\cos 2x}\,dx=0, \text{ i.e. } I_1=0 \qquad \ldots (2)$$

In I_2, put $\tan x = t$

$\therefore\ x = \tan^{-1} t \qquad \therefore\ dx = \frac{1}{1+t^2}\,dt$

and $\cos 2x = \frac{1-t^2}{1+t^2}$

When $x = -\frac{\pi}{4}$, $t = \tan\left(-\frac{\pi}{4}\right) = -1$

When $x = \frac{\pi}{4}$, $t = \tan\frac{\pi}{4} = 1$.

$$\therefore\ I_2 = \int_{-1}^{1} \frac{1}{2-\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{1}{(1+t^2)}\,dt$$

$$= \int_{-1}^{1} \frac{1}{2(1+t^2)-(1-t^2)}\,dt$$

$$= \int_{-1}^{1} \frac{1}{3t^2+1}\,dt = \int_{-1}^{1} \frac{1}{(\sqrt{3}\ t)^2+1}$$

$$= \left[\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}t}{1}\right)\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}(-\sqrt{3})\right]$$

$$= \frac{1}{\sqrt{3}}\left[\tan^{-1}\sqrt{3} + \tan^{-1}\sqrt{3}\right]$$

$$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{3}+\frac{\pi}{3}\right] = \frac{2\pi}{3\sqrt{3}} \qquad \ldots (3)$$

From (1), (2) and (3), we get

$$I = 0 + \frac{\pi}{4}\left[\frac{2\pi}{3\sqrt{3}}\right] = \frac{\pi^2}{6\sqrt{3}}.$$

Question 9.

$\int_{-\pi/4}^{\pi/4} x^3 \cdot \sin^4 x \cdot dx$

Solution:

Let I = $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^4 x \cdot dx$

Let f(x) = $x^3 \sin^4 x$

$\therefore$ f(–x) = $(-x)^3 \sin^4(-x)$

= $-x^3 \sin^4 x$

= $-f(x)$

$\therefore$ f is an odd function.

$\therefore \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} f(x) \cdot dx = 0$, i.e. $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^4 x \cdot dx$ = 0.

Question 10.

$\int_0^1 \frac{\log(x+1)}{x^2+1} \cdot dx$

Solution:

$$\text{Let } I = \int_0^1 \frac{\log\ (x+1)}{x^2+1}\,\mathrm{dx}$$

Put $x = \tan\,\theta$. $\therefore\ dx = \sec^2\theta\ d\theta$ and

$x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$

When $x = 0$, $\tan\,\theta = 0 \quad \therefore\ \theta = 0$

When $x = 1$, $\tan\,\theta = 1 \quad \therefore\ \theta = \pi/4$

$$\therefore\ I = \int_0^{\pi/4} \frac{\log\ (\tan\,\theta + 1)}{\sec^2\theta}\cdot\sec^2\theta\ d\theta$$

$$= \int_0^{\pi/4} \log\ (1 + \tan\,\theta)\ d\theta \qquad \ldots (1)$$

We use the property, $\int_0^a f(x)\ dx = \int_0^a f(a-x)\ dx$.

Here, $a = \frac{\pi}{4}$. Hence changing θ by $\frac{\pi}{4} - \theta$, we have,

$$I = \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right] d\theta$$

$$= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan\,\theta}{1 + \tan\,\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{1 + \tan\,\theta + 1 - \tan\,\theta}{1 + \tan\,\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan\,\theta}\right) d\theta$$

$$= \int_0^{\pi/4} [\log 2 - \log\ (1 + \tan\,\theta)]\ d\theta$$

$$= \log 2 \int_0^{\pi/4} 1\ d\theta - \int_0^{\pi/4} \log\ (1 + \tan\,\theta)\ d\theta$$

$$= (\log 2)\ [\theta]_0^{\pi/4} - I = \frac{\pi}{4}\log 2 - I$$

$$\therefore\ 2I = \frac{\pi}{4}\log 2 \qquad \therefore\ I = \frac{\pi}{8}\log 2.$$

Question 11.
∫1−1x3+2x2+4√⋅ dx
Solution:

$$\text{Let } I = \int_{-1}^{1} \frac{x^3 + 2}{\sqrt{x^2 + 4}}\, dx$$

$$= \int_{-1}^{1} \left[\frac{x^3}{\sqrt{x^2+4}} + \frac{2}{\sqrt{x^2+4}}\right] dx$$

$$= \int_{-1}^{1} \frac{x^3}{\sqrt{x^2+4}}\, dx + 2\int \frac{1}{\sqrt{x^2+4}}\, dx$$

$$= I_1 + 2I_2 \qquad \dots (1)$$

$$\text{Let } f(x) = \frac{x^3}{\sqrt{x^2+4}}$$

$$\therefore\ f(-x) = \frac{(-x)^3}{\sqrt{(-x)^2+4}} = \frac{-x^3}{\sqrt{x^2+4}} = -f(x)$$

$\therefore$ f is an odd function.

$$\therefore\ \int_{-1}^{1} f(x)\, dx = 0, \text{ i.e. } I_1 = \int_{-1}^{1} \frac{x^3}{\sqrt{x^2+4}}\, dx = 0 \qquad \dots (2)$$

$$\because\ (-x)^2 = x^2$$

$$\therefore\ \frac{1}{\sqrt{x^2+4}} \text{ is an even function.}$$

$$\therefore\ \int_{-1}^{1} f(x)dx = 2\int_{0}^{1} f(x)dx$$

$$\therefore\ I_2 = 2\int_{0}^{1} \frac{1}{\sqrt{x^2+4}}\, dx$$

$$= 2\left[\log\left(x + \sqrt{x^2+4}\right)\right]_0^1$$

$$= 2\left[\log\left(1 + \sqrt{1+4}\right) - \log\left(0 + \sqrt{0+4}\right)\right]$$

$$= 2\left[\log\left(\sqrt{5}+1\right) - \log 2\right]$$

$$= 2\log\left(\frac{\sqrt{5}+1}{2}\right) \qquad \dots (3)$$

From (1), (2) and (3), we get

$$I = 0 + 2\left[2\log\left(\frac{\sqrt{5}+1}{2}\right)\right] = 4\log\left(\frac{\sqrt{5}+1}{2}\right).$$

Question 12.

∫a−ax+x316−x2 · dx

Solution:

Let I = $\int_{-a}^{a} \frac{x+x^3}{16-x^2} \cdot dx$

Let f(x) = $\frac{x+x^3}{16-x^2}$

∴ f(−x) = $\frac{(-x)+(-x)^3}{16-(-x)^2}$

= $\frac{-(x+x^3)}{16-x^2}$

= $-f(x)$

∴ f is an odd function.

∴ $\int_{-a}^{a} f(x) \cdot dx = 0$, i.e. $\int_{a}^{a} \frac{x+x^3}{16-x^2} \cdot dx$ = 0.

Question 13.
∫10t21−t−−−−√⋅ dt
Solution:

We use the property,

$$\int_0^a f(t)\,dt = \int_0^a f(a-t)\,dt$$

$$\therefore \int_0^1 t^2\sqrt{1-t}\,dt = \int_0^1 (1-t)^2\sqrt{1-1+t}\,dt$$

$$= \int_0^1 (1-2t+t^2)\sqrt{t}\,dt$$

$$= \int_0^1 (t^{\frac{1}{2}} - 2t^{\frac{3}{2}} + t^{\frac{5}{2}})\,dt$$

$$= \left[\frac{t^{\frac{3}{2}}}{3/2} - 2\cdot\frac{t^{\frac{5}{2}}}{5/2} + \frac{t^{\frac{7}{2}}}{7/2}\right]_0^1$$

$$= \frac{2}{3}(1)^{\frac{3}{2}} - \frac{4}{5}(1)^{\frac{5}{2}} + \frac{2}{7}(1)^{\frac{7}{2}} - 0$$

$$= \frac{2}{3} - \frac{4}{5} + \frac{2}{7} = \frac{70-84+30}{105} = \frac{16}{105}.$$

Question 14.
∫π0x⋅ sinx⋅ cos2x⋅ dx
Solution:

$$\text{Let } I = \int_0^{\pi} x \sin x \cos^2 x\,dx$$

$$= \frac{1}{2}\int_0^{\pi} x(2\sin x\cos x)\cos x\,dx$$

$$= \frac{1}{2}\int_0^{\pi} x(\sin 2x\cos x)\,dx$$

$$=\frac{1}{4}\int_0^{\pi} x(2\sin 2x\cos x)\,dx$$

$$=\frac{1}{4}\int_0^{\pi} x\,[\sin(2x+x)+\sin(2x-x)]\,dx$$

$$=\frac{1}{4}\left[\int_0^{\pi} x\sin 3x\,dx+\int_0^{\pi} x\sin x\,dx\right]$$

$$=\frac{1}{4}[I_1+I_2] \qquad \ldots (1)$$

$$I_1=\int_0^{\pi} x\sin 3x\,dx$$

$$=[x\int \sin 3x\,dx]_0^{\pi}-\int\left[\left\{\frac{d}{dx}(x)\int \sin 3x\,dx\right\}\right]dx$$

$$=\left[x\left(\frac{-\cos 3x}{3}\right)\right]_0^{\pi}-\int_0^{\pi} 1\left(\frac{-\cos 3x}{3}\right)dx$$

$$=\left[-\frac{\pi\cos 3\pi}{3}+0\right]+\frac{1}{3}\int_0^{\pi}\cos 3x\,dx$$

$$=-\frac{\pi}{3}(-1)+\frac{1}{3}\left[\frac{\sin 3x}{3}\right]_0^{\pi}$$

$$=\frac{\pi}{3}+\frac{1}{3}\left[\frac{\sin 3\pi}{3}-\frac{\sin 0}{3}\right]$$

$$=\frac{\pi}{3}+\frac{1}{3}[0-0]=\frac{\pi}{3} \qquad \ldots (2)$$

$$I_2=\int_0^{\pi} x\sin x\,dx$$

$$=[x\int \sin x\,dx]_0^{\pi}-\int_0^{\pi}\left[\left\{\frac{d}{dx}(x)\int \sin x\,dx\right\}\right]dx$$

$$=[x(-\cos x)]_0^{\pi}-\int_0^{\pi} 1\cdot(-\cos x)\,dx$$

$$=[-\pi\cos\pi+0]+\int_0^{\pi}\cos x\,dx$$

$$=-\pi(-1)+[\sin x]_0^{\pi}$$

$$=\pi+[\sin\pi-\sin 0]=\pi+(0-0)=\pi \qquad \ldots (3)$$

From (1), (2) and (3), we get

$$I=\frac{1}{4}\left[\frac{\pi}{3}+\pi\right]=\frac{1}{4}\left(\frac{4\pi}{3}\right)=\frac{\pi}{3}.$$

Question 15.
∫10logx1−x2√ dx
Solution:

$$\text{Let } I=\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx$$

Put $x=\sin\theta$ $\therefore$ $dx=\cos\theta\,d\theta$ and

$$\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\sqrt{\cos^2\theta}=\cos\theta$$

When $x = 0$, $\sin\theta = 0 \quad \therefore \theta = 0$

When $x = 1$, $\sin\theta = 1 \quad \therefore \theta = \pi/2$

$$\therefore I = \int_0^{\pi/2} \frac{\log \sin\theta}{\cos\theta} \cdot \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \log \sin\theta \, d\theta$$

Using the property, $\int_0^{2a} f(x)\,dx = \int_0^{a} [f(x) + f(2a - x)]\,dx$,

we get,

$$I = \int_0^{\pi/4} \left[\log \sin\theta + \log \sin\left(\frac{\pi}{2} - \theta\right)\right] d\theta$$

$$= \int_0^{\pi/4} (\log \sin\theta + \log \cos\theta)\, d\theta$$

$$= \int_0^{\pi/4} \log \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2 \sin\theta \cos\theta}{2}\right) d\theta$$

$$= \int_0^{\pi/4} (\log \sin 2\theta - \log 2)\, d\theta$$

$$= \int_0^{\pi/4} \log \sin 2\theta \, d\theta - \int_0^{\pi/4} \log 2 \, d\theta$$

$$= I_1 - I_2 \quad \text{........ (Say)}$$

$$I_2 = \int_0^{\pi/4} \log 2 \, d\theta = \log 2 \int_0^{\pi/4} 1 \, d\theta$$

$$= \log 2 \, [\theta]_0^{\pi/4} = (\log 2)\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{4} \log 2$$

$$I_1 = \int_0^{\pi/4} \log \sin 2\theta \, d\theta$$

Put $2\theta = t$. Then $d\theta = \dfrac{dt}{2}$

When $\theta = 0$, $t = 0$

When $\theta = \pi/4$, $t = 2\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{2}$

$$\therefore I_1 = \int_0^{\pi/2} \log \sin t \times \frac{dt}{2}$$

$$= \frac{1}{2}\int_0^{\pi/2} \log \sin\theta \, d\theta = \frac{1}{2} I \quad \ldots \left[\because \int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right]$$

$$\therefore I = \frac{1}{2} I - \frac{\pi}{4} \log 2$$

$$\therefore \frac{1}{2} I = -\frac{\pi}{4} \log 2$$

$$\therefore I = -\frac{\pi}{2} \log 2 = \frac{\pi}{2} \log\left(\frac{1}{2}\right).$$

Maharashtra State Board 12th Maths Solutions Chapter 4 Definite Integration Miscellaneous Exercise 4

I. Choose the correct option from the given alternatives:

Question 1.
∫32dxx(x3−1)=
(a) 13log(208189)
(b) 13log(189208)
(c) log(208189)
(d) log(189208)
Answer:
(a) 13log(208189)

Question 2.
∫π/20sin2x⋅ dx(1+cosx)2=
(a) 4−π2
(b) π−42
(c) 4 – π2
(d) 4+π2
Answer:
(a) 4−π2

Question 3.
∫log50exex−1√ex+3⋅ dx=
(a) 3 + 2π
(b) 4 – π
(c) 2 + π
(d) 4 + π
Answer:
(b) 4 – π

Question 4.
∫π/20sin6xcos2x⋅ dx=
(a) 7π256
(b) 3π256
(c) 5π256
(d) −5π256
Answer:
(c) 5π256

Question 5.
If ∫10dx1+x√−x√=k3, then k is equal to
(a) √2(2√2 – 2)
(b) 2√3(2 – 2√2)
(c) 22√−23
(d) 4√2
Answer:
(d) 4√2

Question 6.
∫211x2e1x⋅ dx=
(a) √e + 1
(b) √e – 1
(c) √e(√e – 1)
(d) e√−1e

Answer:
(c) √e(√e − 1)

Question 7.
If $\int_e^{e^2}\left[\frac{1}{\log x}-\frac{1}{(\log x)^2}\right] \cdot dx = a + b\log 2$, then
(a) a = e, b = -2
(b) a = e, b = 2
(c) a = -e, b = 2
(d) a = -e, b = -2
Answer:
(a) a = e, b = -2

Question 8.
Let $I_1=\int_e^{e^2}\frac{dx}{\log x}$ and $I_2=\int_1^2\frac{e^x}{x} \cdot dx$, then
(a) $I_1 = \frac{1}{3} I_2$
(b) $I_1 + I_2 = 0$
(c) $I_1 = 2I_2$
(d) $I_1 = I_2$
Answer:
(d) $I_1 = I_2$

Question 9.
$\int_0^9\frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} \cdot dx=$
(a) 9
(b) $\frac{9}{2}$
(c) 0
(d) 1
Answer:
(b) $\frac{9}{2}$

Question 10.
The value of $\int_{-\pi/4}^{\pi/4}\log\left(\frac{2+\sin\theta}{2-\sin\theta}\right) \cdot d\theta$ is
(a) 0
(b) 1
(c) 2
(d) π
Answer:
(a) 0

II. Evaluate the following:

Question 1.
$\int_0^{\pi/2}\frac{\cos x}{3\cos x+\sin x}dx$
Solution:
Let I = $\int_0^{\pi/2}\frac{\cos x}{3\cos x+\sin x}dx$
Put Numerator = A(Denominator) + B[$\frac{d}{dx}$(Denominator)]
∴ cos x = A(3 cos x + sin x) + B[$\frac{d}{dx}$(3 cos x + sin x)]
= A(3 cos x + sin x) + B(-3 sin x + cos x)
∴ cos x + 0 . sin x = (3A + B) cos x + (A – 3B) sin x
Comparing the coefficients of sinx and cos x on both the sides, we get
3A + B = 1 ………. (1)
A – 3B = 0 ………. (2)
Multiplying equation (1) by 3, we get
9A + 3B = 3 ………(3)
Adding (2) and (3), we get
10A = 3
∴ A = $\frac{3}{10}$

$\therefore$ from (1), $3\left(\frac{3}{10}\right)+B=1$

$\therefore\ B=1-\frac{9}{10}=\frac{1}{10}$

$\therefore\ \cos x=\frac{3}{10}(3\cos x+\sin x)+\frac{1}{10}(-3\sin x+\cos x)$

$$\therefore\ I=\int_0^{\pi/2}\left[\frac{\frac{3}{10}(3\cos x+\sin x)+\frac{1}{10}(-3\sin x+\cos x)}{3\cos x+\sin x}\right]dx$$

$$=\int_0^{\pi/2}\left[\frac{3}{10}+\frac{\frac{1}{10}(-3\sin x+\cos x)}{3\cos x+\sin x}\right]dx$$

$$=\frac{3}{10}\int_0^{\pi/2}1\,dx+\frac{1}{10}\int_0^{\pi/2}\frac{-3\sin x+\cos x}{3\cos x+\sin x}\,dx$$

$$=\frac{3}{10}[x]_0^{\pi/2}+\frac{1}{10}[\log|3\cos x+\sin x|]_0^{\pi/2}$$

$$\ldots\left[\because\ \int\frac{f'(x)}{f(x)}\,dx=\log|f(x)|+c\right]$$

$$=\frac{3}{10}\left[\frac{\pi}{2}-0\right]+$$

$$\frac{1}{10}\left[\log\left|3\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\right|-\log|3\cos 0+\sin 0|\right]$$

$$=\frac{3\pi}{20}+\frac{1}{10}[\log|3\times 0+1|-\log|3\times 1+0|]$$

$$=\frac{3\pi}{20}+\frac{1}{10}[\log 1-\log 3]$$

$$=\frac{3\pi}{20}-\frac{1}{10}\log 3. \qquad \ldots[\because\ \log 1=0]$$

Question 2.
$\int \pi/2\pi/4 \cos\theta[\cos\theta 2+\sin\theta 2]3 d\theta$
Solution:

$$\text{Let } I=\int_{\pi/4}^{\pi/2}\frac{\cos\theta}{\left[\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right]^3}d\theta$$

$$=\int_{\pi/4}^{\pi/2}\frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\left[\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right]^3}d\theta$$

$$=\int_{\pi/4}^{\pi/2}\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}{\left[\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right]^3}d\theta$$

$$=\int_{\pi/4}^{\pi/2}\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\left[\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right]^2}d\theta$$

Put $\cos\frac{\theta}{2}+\sin\frac{\theta}{2}=t$

$$\therefore\ \left(-\frac{1}{2}\sin\frac{\theta}{2}+\frac{1}{2}\cos\frac{\theta}{2}\right)d\theta=dt$$

$$\therefore\ \left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)d\theta=2\cdot dt$$

When $\theta=\frac{\pi}{4}$, $t=\cos\frac{\pi}{8}+\sin\frac{\pi}{8}$

When $\theta=\frac{\pi}{2}$, $t=\cos\frac{\pi}{4}+\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$

$$\therefore\ I=\int_{\cos\frac{\pi}{8}+\sin\frac{\pi}{8}}^{\sqrt{2}}\frac{1}{t^2}\cdot 2dt=2\int_{\cos\frac{\pi}{8}+\sin\frac{\pi}{8}}^{\sqrt{2}}t^{-2}\,dt$$

$$=2\left[\frac{t^{-1}}{-1}\right]_{\cos\frac{\pi}{8}+\sin\frac{\pi}{8}}^{\sqrt{2}}=\left[\frac{-2}{t}\right]_{\cos\frac{\pi}{8}+\sin\frac{\pi}{8}}^{\sqrt{2}}$$

$$=-\frac{2}{\sqrt{2}}+\frac{2}{\cos\frac{\pi}{8}+\sin\frac{\pi}{8}}$$

$$=\frac{2}{\cos\frac{\pi}{8}+\sin\frac{\pi}{8}}-\sqrt{2}.$$

Question 3.
∫1011+x√dx
Solution:
Let I = ∫1011+x√dx
Put √x = t
∴ $x = t^2$ and dx = 2t . dt
When x = 0, t = 0

When x = 1, t = 1

$$\text{Let } I = \int_0^1 \frac{1}{1+\sqrt{x}} dx$$

Put $\sqrt{x} = t$ $\therefore$ $x = t^2$ and $dx = 2t \cdot dt$

When $x = 0$, $t = 0$

When $x = 1$, $t = 1$

$$\therefore I = \int_0^1 \frac{1}{1+t} . 2t\, dt$$

$$= 2\int_0^1 \frac{t}{1+t} dt = 2\int_0^1 \frac{(1+t)-1}{1+t} dt$$

$$= 2\int_0^1 \left(1 - \frac{1}{1+t}\right) dt$$

$$= 2[t - \log|1+t|]_0^1$$

$$= 2[1 - \log 2 - 0 + \log 1]$$

$$= 2(1 - \log 2) \quad \ldots [\because \log 1 = 0]$$

$$= 2 - 2\log 2 = 2 - \log 4.$$

Question 4.
∫π/40tan3x1+cos2xdx
Solution:

$$\text{Let } I = \int_0^{\pi/4} \frac{\tan^3 x}{1+\cos 2x} dx$$

$$= \int_0^{\pi/4} \frac{\tan^3 x}{2\cos^2 x} dx$$

$$= \frac{1}{2}\int_0^{\pi/4} \tan^3 x \cdot \sec^2 x\, dx$$

Put $\tan x = t$ $\therefore$ $\sec^2 x\, dx = dt$

When $x = 0$, $t = \tan 0 = 0$

When $x = \frac{\pi}{4}$, $t = \tan\frac{\pi}{4} = 1$

$$\therefore I = \frac{1}{2}\int_0^1 t^3\, dt = \frac{1}{2} \cdot \left[\frac{t^4}{4}\right]_0^1$$

$$= \frac{1}{8}[t^4]_0^1 = \frac{1}{8}[1-0] = \frac{1}{8}.$$

Question 5.
∫10ts1−t2−−−−−√dt

Solution:

$$\text{Let } I = \int_0^1 t^5 \sqrt{1-t^2}\, dt$$

Put $t = \sin\theta \quad \therefore \; dt = \cos\theta\, d\theta$

When $t = 1$, $\theta = \sin^{-1}1 = \frac{\pi}{2}$

When $t = 0$, $\theta = \sin^{-1}0 = 0$

$$\therefore \; I = \int_0^{\pi/2} \sin^5\theta \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta$$

$$I = \int_0^{\pi/2} \sin^5\theta \cdot \cos\theta \cdot \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^5\theta\,(1-\sin^2\theta)\, d\theta$$

$$= \int_0^{\pi/2} (\sin^5\theta - \sin^7\theta)\, d\theta$$

$$= \int_0^{\pi/2} \sin^5\theta\, d\theta - \int_0^{\pi/2} \sin^7\theta\, d\theta.$$

Using Reduction formula, we get

$$I = \frac{4}{5}\cdot\frac{2}{3} - \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}$$

$$= \frac{8}{15}\left[1-\frac{6}{7}\right] = \frac{8}{15}\times\frac{1}{7} = \frac{8}{105}.$$

Question 6.

∫10(cos-1x)2dx

Solution:

$$\text{Let } I=\int_0^1 (\cos^{-1}x)^2 dx$$

Put $\cos^{-1}x=t \qquad \therefore\ x=\cos t$

$\therefore\ dx=-\sin t\cdot dt$

When $x=0,\ t=\cos^{-1}0=\frac{\pi}{2}$

When $x=1,\ t=\cos^{-1}1=0$

$$\therefore\ I=\int_{\pi/2}^{0} t^2\cdot(-\sin t)\,dt$$

$$=-\int_{\pi/2}^{0} t^2\sin t\,dt=\int_0^{\pi/2} t^2\sin t\,dt$$

$$\ldots\left[\because\ \int_a^b f(x)\,dx=-\int_b^a f(x)\,dx\right]$$

$$=\left[t^2\int\sin t\,dt\right]_0^{\pi/2}-\int_0^{\pi/2}\left[\frac{d}{dt}(t^2)\int\sin t\,dt\right]dt$$

$$=[t^2(-\cos t)]_0^{\pi/2}-\int_0^{\pi/2} 2t\cdot(-\cos t)\,dt$$

$$=[-t^2\cos t]_0^{\pi/2}+2\int_0^{\pi/2} t\cdot\cos t\,dt$$

$$=\left[-\frac{\pi^2}{4}\cos\frac{\pi}{2}+0\right]+2\left\{\left[t\int\cos t\,dt\right]_0^{\pi/2}-\int_0^{\pi/2}\left[\frac{d}{dt}(t)\int\cos t\,dt\right]dt\right\}$$

$$=0+2\left\{[t\sin t]_0^{\pi/2}-\int_0^{\pi/2}1\cdot\sin t\,dt\right\}\qquad\ldots\left[\because\ \cos\frac{\pi}{2}=0\right]$$

$$=2[t\sin t]_0^{\pi/2}-2[(-\cos t)]_0^{\pi/2}$$

$$=2\left[\frac{\pi}{2}\sin\frac{\pi}{2}-0\right]-2\left[-\cos\frac{\pi}{2}+\cos 0\right]$$

$$=2\left[\frac{\pi}{2}\times 1\right]-2[-0+1]$$

$$=\pi-2.$$

Question 7.

$\int_{-1}^{1}\frac{1+x^3}{9-x^2}dx$

Solution:

$$\text{Let } I = \int_{-1}^{1} \frac{1+x^3}{9-x^2}\,dx$$

$$= \int_{-1}^{1} \left[\frac{1}{9-x^2} + \frac{x^3}{9-x^2}\right] dx$$

$$= \int_{-1}^{1} \frac{1}{9-x^2}\,dx + \int_{-1}^{1} \frac{x^3}{9-x^2}\,dx$$

$$\therefore\ I = I_1 + I_2 \qquad \dots (1)$$

$$I_1 = \int_{-1}^{1} \frac{1}{3^2-x^2}\,dx$$

$$= \frac{1}{2\times 3}\left[\log\left|\frac{3+x}{3-x}\right|\right]_{-1}^{1}$$

$$= \frac{1}{6}\left[\log\left(\frac{4}{2}\right) - \log\left(\frac{2}{4}\right)\right]$$

$$= \frac{1}{6}\left[\log\left(\frac{2}{1/2}\right)\right] = \frac{1}{6}\log 4$$

$$= \frac{1}{6}\log 2^2 = \frac{1}{6}\times 2\log 2 = \frac{1}{3}\log 2 \qquad \dots (2)$$

$$I_2 = \int_{-1}^{1} \frac{x^3}{9-x^2}\,dx$$

$$\text{Let } f(x) = \frac{x^3}{9-x^2}$$

$$\therefore\ f(-x) = \frac{(-x)^3}{9-(-x)^2} = \frac{-x^3}{9-x^2} = -f(x)$$

$\therefore$ f is an odd function.

$$\therefore\ \int_{-1}^{1} f(x)\,dx = 0$$

$$\therefore\ I_2 = \int_{-1}^{1} \frac{x^3}{9-x^2}\,dx = 0 \qquad \dots (3)$$

From (1), (2) and (3), we get

$$I = \frac{1}{3}\log 2 + 0 = \frac{1}{3}\log 2.$$

Question 8.
$\int_0^\pi x \cdot \sin x \cdot \cos^4 x\,dx$
Solution:

$$\text{Let } I = \int_0^{\pi} x \cdot \sin x \cdot \cos^4 x \, dx \quad \ldots (1)$$

We use the property, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Here $a = \pi$. Hence changing x by $\pi - x$, we get

$$I = \int_0^{\pi} (\pi - x) \cdot \sin(\pi - x) \cdot [\cos(\pi - x)]^4 \, dx$$

$$= \int_0^{\pi} (\pi - x) \cdot \sin x \cdot \cos^4 x \, dx \quad \ldots (2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} x \cdot \sin x \cdot \cos^4 x \, dx + \int_0^{\pi} (\pi - x) \cdot \sin x \cdot \cos^4 x \, dx$$

$$= \int_0^{\pi} (x + \pi - x) \cdot \sin x \cdot \cos^4 x \, dx$$

$$= \pi \int_0^{\pi} \sin x \cdot \cos^4 x \, dx$$

$$\therefore\ I = \frac{\pi}{2} \int_0^{\pi} \cos^4 x \cdot \sin x \, dx$$

Put $\cos x = t \qquad \therefore\ -\sin x\,dx = dt$

$\therefore\ \sin x\,dx = -dt$

When $x = 0$, $t = \cos 0 = 1$

When $x = \pi$, $t = \cos \pi = -1$

$$\therefore\ I = \frac{\pi}{2} \int_1^{-1} t^4(-dt) = -\frac{\pi}{2} \int_1^{-1} t^4 \, dt$$

$$= -\frac{\pi}{2}\left[\frac{t^5}{5}\right]_1^{-1} = -\frac{\pi}{10}[t^5]_1^{-1}$$

$$= -\frac{\pi}{10}[(-1)^5 - (1)^5]$$

$$= -\frac{\pi}{10}(-1-1) = \frac{2\pi}{10} = \frac{\pi}{5}.$$

Question 9.

$\int_0^{\pi} \frac{x}{1+\sin^2 x} dx$

Solution:

$$\text{Let } I = \int_0^{\pi} \frac{x}{1+\sin^2 x}\,dx \qquad \ldots (1)$$

We use the property, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Here $a = \pi$. Hence in I, changing x to $\pi - x$, we get

$$I = \int_0^{\pi} \frac{\pi - x}{1+\sin^2(\pi - x)}\,dx = \int_0^{\pi} \frac{\pi - x}{1+\sin^2 x}\,dx$$

$$= \int_0^{\pi} \frac{\pi}{1+\sin^2 x}\,dx - \int_0^{\pi} \frac{x}{1+\sin^2 x}\,dx$$

$$= \int_0^{\pi} \frac{\pi}{1+\sin^2 x}\,dx - I \qquad \text{[By (1)]}$$

$$\therefore\ 2I = \pi \int_0^{\pi} \frac{1}{1+\sin^2 x}\,dx$$

Dividing numerator and denominator by $\cos^2 x$, we get

$$2I = \pi \int_0^{\pi} \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\,dx$$

$$= \pi \int_0^{\pi} \frac{\sec^2 x}{1+2\tan^2 x}\,dx$$

Put $\tan x = t$ $\quad\therefore\ \sec^2 x\,dx = dt$

When $x = \pi$, $t = \tan \pi = 0$

When $x = 0$, $t = \tan 0 = 0$

$$\therefore\ 2I = \pi \int_0^{0} \frac{dt}{1+2t^2} = 0$$

$$\therefore\ I = 0. \qquad \ldots \left[\because \int_a^a f(x)\,dx = 0\right]$$

Question 10.
∫∞11x√(1+x)dx
Solution:

$$\text{Let } I=\int_1^{\infty}\frac{1}{\sqrt{x}\ (1+x)}\cdot dx$$

Put $x=\tan^2 t$

$$\therefore\ dx=\left[2\tan t\frac{d}{dt}(\tan t)\right]dt=2\tan t\sec^2 t\,dt$$

When $x=\infty$, $\tan^2 t=\infty \quad \therefore\ t=\frac{\pi}{2}$

When $x=1$, $\tan^2 t=1 \quad \therefore\ t=\frac{\pi}{4}$

$$\therefore\ I=\int_{\pi/4}^{\pi/2}\frac{2\tan t\sec^2 t}{\sqrt{\tan^2 t}\ (1+\tan^2 t)}dt$$

$$=\int_{\pi/4}^{\pi/2}\frac{2\sec^2 t}{\sec^2 t}dt$$

$$=2\int_{\pi/4}^{\pi/2}1dt=2[t]_{\pi/4}^{\pi/2}$$

$$=2\left[\frac{\pi}{2}-\frac{\pi}{4}\right]=2\left[\frac{\pi}{4}\right]=\frac{\pi}{2}.$$

III. Evaluate the following:

Question 1.
$\int 10(11+x2)\sin-1(2x1+x2)dx$
Solution:

$$\text{Let } I=\int_0^1\left(\frac{1}{1+x^2}\right)\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Put $x=\tan t$, i.e. $t=\tan^{-1}x$

$\therefore\ dx=\sec^2 t\,dt$

When $x=1$, $t=\tan^{-1}1=\frac{\pi}{4}$

When $x=0$, $t=\tan^{-1}0=0$

$$\therefore\ I=\int_0^{\pi/4}\left(\frac{1}{1+\tan^2 t}\right)\sin^{-1}\left(\frac{2\tan t}{1+\tan^2 t}\right)\sec^2 t\,dt$$

$$=\int_0^{\pi/4}\frac{1}{\sec^2 t}\sin^{-1}(\sin 2t)\sec^2 t\,dt$$

$$=\int_0^{\pi/4}2t\,dt=2\int_0^{\pi/4}t\,dt=2\left[\frac{t^2}{2}\right]_0^{\pi/4}$$

$$=2\left[\frac{\pi^2}{32}-0\right]=\frac{\pi^2}{16}.$$

Question 2.
$\int \pi/2016-\cos x dx$
Solution:

Let I = $\int_0^{\frac{\pi}{2}} \frac{1}{6-\cos x} \cdot dx$

Put $\tan\left(\frac{x}{2}\right)$ = t

∴ x = 2 $\tan^{-1}$ t

∴ dx = $\frac{2dt}{1+t}$

and

cos x = $\frac{1-t^2}{1+t^2}$

When x = $\frac{\pi}{2}, t = \tan\left(\frac{\pi}{2}\right)$ = 1

When x = 0, t = tan 0 = 0

∴ I = $\frac{\frac{2dt}{1+t^2}}{6-\cos\left(\frac{1-t^2}{1+t^2}\right)}$

= $\int_0^1 \frac{2dt}{6(1+t^2)+1(1-t^2)}$

= $2\int_0^1 \frac{1}{t^2+7} \cdot dt$

= $2\left[\frac{1}{35}\tan^{-1}\frac{t}{5}\right]_0^1$

= $2\left[\frac{1}{35}\tan^{-1}\frac{1}{3} - \frac{1}{5}\tan^{-1}0\right]$

= $\frac{2}{35}\tan^{-1}\frac{1}{3} - \frac{7}{5} \times 0$

= $\frac{2}{\sqrt{35}}\tan^{-1}\sqrt{\frac{7}{5}}$.

Question 3.

∫a01a2+ax−x2dx

Solution:

$$\text{Let } I=\int_0^a \frac{1}{a^2+ax-x^2}dx$$

$$a^2+ax-x^2=a^2-\left(x^2-ax+\frac{a^2}{4}\right)+\frac{a^2}{4}$$

$$=\frac{5a^2}{4}-\left(x-\frac{a}{2}\right)^2$$

$$=\left(\frac{\sqrt{5}a}{2}\right)^2-\left(x-\frac{a}{2}\right)^2$$

$$\therefore\ I=\int_0^a \frac{dx}{\left(\frac{\sqrt{5}a}{2}\right)^2-\left(x-\frac{a}{2}\right)^2}$$

$$=\frac{1}{2\times\frac{\sqrt{5}a}{2}}\cdot\left[\log\left|\frac{\frac{\sqrt{5}a}{2}+x-\frac{a}{2}}{\frac{\sqrt{5}a}{2}-x+\frac{a}{2}}\right|\right]_0^a$$

$$=\frac{1}{\sqrt{5}a}\left[\log\left|\frac{\frac{\sqrt{5}a}{2}+a-\frac{a}{2}}{\frac{\sqrt{5}a}{2}-a+\frac{a}{2}}\right|-\log\left|\frac{\frac{\sqrt{5}a}{2}-\frac{a}{2}}{\frac{\sqrt{5}a}{2}+\frac{a}{2}}\right|\right]$$

$$=\frac{1}{\sqrt{5}a}\left[\log\left|\frac{\frac{\sqrt{5}}{2}+\frac{1}{2}}{\frac{\sqrt{5}}{2}-\frac{1}{2}}\right|-\log\left|\frac{\frac{\sqrt{5}}{2}-\frac{1}{2}}{\frac{\sqrt{5}}{2}+\frac{1}{2}}\right|\right]$$

$$=\frac{1}{\sqrt{5}a}\left[\log\left|\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)\right|-\log\left|\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)\right|\right]$$

$$=\frac{1}{\sqrt{5}a}\log\left|\frac{\sqrt{5}+1}{\sqrt{5}-1}\times\frac{\sqrt{5}+1}{\sqrt{5}-1}\right|$$

$$=\frac{1}{\sqrt{5}a}\log\left[\left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)^2\right]$$

$$=\frac{1}{\sqrt{5}a}\log\left|\frac{5+1+2\sqrt{5}}{5+1-2\sqrt{5}}\right|$$

$$=\frac{1}{\sqrt{5}a}\log\left|\frac{6+2\sqrt{5}}{6-2\sqrt{5}}\right|$$

$$=\frac{1}{\sqrt{5}a}\log\left|\frac{6+2\sqrt{5}}{6-2\sqrt{5}}\times\frac{6+2\sqrt{5}}{6+2\sqrt{5}}\right|$$

$$=\frac{1}{\sqrt{5}a}\log\left|\frac{36+20+24\sqrt{5}}{36-20}\right|$$

$$=\frac{1}{\sqrt{5}a}\log\left|\frac{56+24\sqrt{5}}{16}\right|=\frac{1}{\sqrt{5}a}\log\left|\frac{7+3\sqrt{5}}{2}\right|.$$

Question 4.
$\int_{\pi/5}^{3\pi/10}\frac{\sin x}{\sin x+\cos x}dx$
Solution:

$$\text{Let } I = \int_{\pi/5}^{3\pi/10} \frac{\sin x}{\sin x + \cos x}\, dx \qquad \ldots (1)$$

We use the property, $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$.

Here $a = \frac{\pi}{5}$, $b = \frac{3\pi}{10}$. Hence changing x by $\frac{\pi}{5} + \frac{3\pi}{10} - x$,

we get,

$$I = \int_{\pi/5}^{3\pi/10} \frac{\sin\left(\frac{\pi}{5} + \frac{3\pi}{10} - x\right)}{\sin\left(\frac{\pi}{5} + \frac{3\pi}{10} - x\right) + \cos\left(\frac{\pi}{5} + \frac{3\pi}{10} - x\right)}\, dx$$

$$= \int_{\pi/5}^{3\pi/10} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\, dx$$

$$= \int_{\pi/5}^{3\pi/10} \frac{\cos x}{\cos x + \sin x}\, dx \qquad \ldots (2)$$

Adding (1) and (2), we get,

$$2I = \int_{\pi/5}^{3\pi/10} \frac{\sin x}{\sin x + \cos x}\, dx + \int_{\pi/5}^{3\pi/10} \frac{\cos x}{\cos x + \sin x}\, dx$$

$$= \int_{\pi/5}^{3\pi/10} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx$$

$$= \int_{\pi/5}^{3\pi/10} 1\, dx = [x]_{\pi/5}^{3\pi/10}$$

$$= \frac{3\pi}{10} - \frac{\pi}{5} = \frac{\pi}{10}$$

$$\therefore\ I = \frac{\pi}{20}.$$

Question 5.

$\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$

Solution:

Let I = $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Put $x = \tan t$, i.e. $t = \tan^{-1}x$

$\therefore\ dx = \sec^2 t\ dt$

When $x = 0$, $t = \tan^{-1} 0 = 0$

When $x = 1$, $t = \tan^{-1} 1 = \frac{\pi}{4}$

$$\therefore\ I = \int_0^{\pi/4} \sin^{-1}\left(\frac{2\tan t}{1+\tan^2 t}\right)\sec^2 t\ dt$$

$$= \int_0^{\pi/4} \sin^{-1}(\sin 2t)\sec^2 t\ dt$$

$$= \int_0^{\pi/4} 2t\sec^2 t\ dt$$

$$= \left[2t\int \sec^2 t\ dt\right]_0^{\pi/4} - \int_0^{\pi/4}\left[\frac{d}{dt}(2t)\int \sec^2 t\ dt\right]$$

$$= [2t\tan t]_0^{\pi/4} - \int_0^{\pi/4} 2\tan t\ dt$$

$$= \left[2\cdot\frac{\pi}{4}\tan\frac{\pi}{4} - 0\right] - 2[\log(\sec t)]_0^{\pi/4}$$

$$= \frac{\pi}{2} - 2\left[\log\left(\sec\frac{\pi}{4}\right) - \log(\sec 0)\right]$$

$$= \frac{\pi}{2} - 2[\log\sqrt{2} - \log 1] = \frac{\pi}{2} - 2\left[\frac{1}{2}\log 2 - 0\right]$$

$$= \frac{\pi}{2} - \log 2.$$

Question 6.

$\int_0^{\pi/4}\frac{\cos 2x}{1+\cos 2x+\sin 2x}dx$

Solution:

$$\text{Let } I=\int_0^{\pi/4} \frac{\cos 2x}{1+\cos 2x+\sin 2x}\,dx$$

$$=\int_0^{\pi/4} \frac{\cos^2 x-\sin^2 x}{2\cos^2 x+2\sin x\cos x}\,dx$$

$$=\int_0^{\pi/4} \frac{(\cos x-\sin x)(\cos x+\sin x)}{2\cos x(\cos x+\sin x)}\,dx$$

$$=\int_0^{\pi/4} \frac{\cos x-\sin x}{2\cos x}\,dx$$

$$=\frac{1}{2}\int_0^{\pi/4} \left[\frac{\cos x}{\cos x}-\frac{\sin x}{\cos x}\right]dx$$

$$=\frac{1}{2}\left[\int_0^{\pi/4} 1dx-\int_0^{\pi/4} \tan x\,dx\right]$$

$$=\frac{1}{2}\{[x]_0^{\pi/4}-[\log(\sec x)]_0^{\pi/4}\}$$

$$=\frac{1}{2}\left[\left(\frac{\pi}{4}-0\right)-\left(\log\sec\frac{\pi}{4}-\log\sec 0\right)\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{4}-\log\sqrt{2}+\log 1\right]$$

$$=\frac{1}{2}\left[\frac{\pi}{4}-\log\sqrt{2}\right] \qquad \ldots [\because \log 1=0]$$

Question 7.
∫π/20[2log(sinx)–log(sin2x)]dx

Solution:

$$\text{Let } I = \int_0^{\pi/2} (2\log\sin x - \log\sin 2x)\,dx$$

$$= \int_0^{\pi/2} [2\log\sin x - \log(2\sin x\cos x)]\,dx$$

$$= \int_0^{\pi/2} [2\log\sin x - (\log 2 + \log\sin x + \log\cos x)]\,dx$$

$$= \int_0^{\pi/2} (2\log\sin x - \log 2 - \log\sin x - \log\cos x)\,dx$$

$$= \int_0^{\pi/2} (\log\sin x - \log\cos x - \log 2)\,dx$$

$$= \int_0^{\pi/2} \log\sin x\,dx - \int_0^{\pi/2} \log\cos x\,dx - \log 2\int_0^{\pi/2} 1\,dx$$

$$= \int_0^{\pi/2} \log\left[\sin\left(\frac{\pi}{2} - x\right)\right]dx - \int_0^{\pi/2} \log\cos x\,dx - \log 2\,[x]_0^{\pi/2}$$

$$\ldots\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \log\cos x\,dx - \int_0^{\pi/2} \log\cos x\,dx - \log 2\left[\frac{\pi}{2} - 0\right]$$

$$= -\frac{\pi}{2}\log 2.$$

Question 8.
$\int_0^\pi (\sin^{-1}x+\cos^{-1}x)^3\sin^3x\,dx$
Solution:

$$\text{Let } I = \int_0^{\pi} (\sin^{-1}x + \cos^{-1}x)^3 \sin^3 x \, dx$$

We know that, $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ and

$\sin 3x = 3 \sin x - 4 \sin^3 x$

$\therefore\ 4 \sin^3 x = 3 \sin x - \sin 3x$

$\therefore\ \sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$

$$\therefore\ I = \int_0^{\pi} \left(\frac{\pi}{2}\right)^3 \left[\frac{3}{4} \sin x - \frac{1}{4} \sin 3x\right] dx$$

$$= \frac{\pi^3}{8} \times \frac{3}{4} \int_0^{\pi} \sin x \, dx - \frac{\pi^3}{8} \times \frac{1}{4} \int_0^{\pi} \sin 3x$$

$$= \frac{3\pi^3}{32} [-\cos x]_0^{\pi} - \frac{\pi^3}{32} \left[-\frac{\cos 3x}{3}\right]$$

$$= \frac{3\pi^3}{32} [-\cos \pi - (-\cos 0)] - \frac{\pi^3}{32} \left[-\frac{\cos 3\pi}{3} - \left(\frac{-\cos 0}{3}\right)\right]$$

$$= \frac{3\pi^3}{32} [1+1] - \frac{\pi^3}{32} \left[\frac{1}{3} + \frac{1}{3}\right]$$

$$= \frac{6\pi^3}{32} - \frac{2\pi^3}{96}$$

$$= \frac{18\pi^3 - 2\pi^3}{96} = \frac{16\pi^3}{96} = \frac{\pi^3}{6}.$$

Question 9.
∫40[x2+2x+3−−−−−−−−−√]-1dx
Solution:

$$\text{Let } I = \int_0^4 [\sqrt{x^2 + 2x + 3}]^{-1} \, dx$$

$$= \int_0^4 \frac{1}{\sqrt{x^2 + 2x + 1 + 2}} dx = \int_0^4 \frac{1}{\sqrt{(x+1)^2 + 2}} dx$$

$$= [\log [x + 1 + \sqrt{(x+1)^2 + 2}]_0^4$$

$$= \log [4 + 1 + \sqrt{5^2 + 2}] - \log [0 + 1 + \sqrt{1^2 + 2}]$$

$$= \log (5 + 3\sqrt{3}) - \log (1 + \sqrt{3})$$

$$= \log \left(\frac{5 + 3\sqrt{3}}{1 + \sqrt{3}}\right).$$

Question 10.
∫3-2|x−2|dx
Solution:
|x – 2|= 2 – x, if x < 2

= x – 2, if x ≥ 2

$$\therefore \int_{-2}^{3} |x-2|\,dx = \int_{-2}^{2} |x-2|\,dx + \int_{2}^{3} |x-2|\,dx$$

$$= \int_{-2}^{2} (2-x)\,dx + \int_{2}^{3} (x-2)\,dx$$

$$= \left[\frac{(2-x)^2}{(-2)}\right]_{-2}^{2} + \left[\frac{(x-2)^2}{2}\right]_{2}^{3}$$

$$= \left[0 - \frac{(4)^2}{(-2)}\right] + \left[\frac{1^2}{2} - \frac{0^2}{2}\right]$$

$$= 8 + \frac{1}{2} = \frac{17}{2}.$$

IV. Evaluate the following:

Question 1.
If $\int_a^a \sqrt{x}\,dx = 2a\int_0^{\pi/2} \sin^3 x\,dx$, find the value of $\int_a^{a+1} x\,dx$.
Solution:

It is given that

$$\int_a^a \sqrt{x}\,dx = 2a \int_0^{\pi/2} \sin^3 x\,dx.$$

$$\therefore \left[\frac{x^{3/2}}{3/2}\right]_0^a = 2a \cdot \frac{2}{3} \quad \text{... [Using Reduction Formula]}$$

$$\therefore \left[\frac{2a^{3/2}}{3} - 0\right] = \frac{4a}{3}$$

$$\therefore \frac{2a\sqrt{a}}{3} = \frac{4a}{3}$$

$$\therefore 2a(\sqrt{a} - 2) = 0$$

$$\therefore a = 0 \quad \text{or} \quad \sqrt{a} = 2$$

i.e. $a = 0$ or $a = 4$

When $a = 0$, $\int_a^{a+1} x\,dx = \int_0^1 x\,dx$

$$= \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

When $a = 4$, $\int_a^{a+1} d\,dx = \int_4^5 x\,dx$

$$= \left[\frac{x^2}{2}\right]_4^5 = \frac{25}{2} - \frac{16}{2} = \frac{9}{2}.$$

Question 2.
If $\int_0^k \frac{1}{2+8x^2}\,dx = \frac{\pi}{16}$, find k.
Solution:

$$\text{Let } I=\int_0^k \frac{1}{2+8x^2}\,dx$$

$$=\frac{1}{8}\int_0^k \frac{1}{x^2+(1/2)^2}\,dx$$

$$=\frac{1}{8}\times\frac{1}{(1/2)}\left[\tan^{-1}\left(\frac{x}{(1/2)}\right)\right]_0^k$$

$$=\frac{1}{4}[\tan^{-1}2x]_0^k$$

$$=\frac{1}{4}[\tan^{-1}2k-\tan^{-1}0]=\frac{1}{4}\tan^{-1}2k$$

$$\therefore\ I=\frac{\pi}{16} \text{ gives } \frac{1}{4}\tan^{-1}2k=\frac{\pi}{16}$$

$$\therefore\ \tan^{-1}2k=\frac{\pi}{4}$$

$$\therefore\ 2k=\tan\frac{\pi}{4}=1 \qquad \therefore\ k=\frac{1}{2}.$$

Question 3.
If f(x) = a + bx + cx2, show that ∫10f(x)dx=16[f(0)+4f(12)+f(1)]
Solution:

$$\int_0^1 f(x)\,dx=\int_0^1 (a+bx+cx^2)\,dx$$

$$=a\int_0^1 1\,dx+b\int_0^1 x\,dx+c\int_0^1 x^2\,dx$$

$$=\left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}\right]_0^1$$

$$=a+\frac{b}{2}+\frac{c}{3} \qquad \ldots (1)$$

Now, $f(0)=a+b(0)+c(0)^2=a$

$$f\left(\frac{1}{2}\right)=a+b\left(\frac{1}{2}\right)+c\left(\frac{1}{2}\right)^2=a+\frac{b}{2}+\frac{c}{4}$$

and $f(1)=a+b(1)+c(1)^2=a+b+c$

$$\therefore\ \frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right]$$

$$=\frac{1}{6}\left[a+4\left(a+\frac{b}{2}+\frac{c}{4}\right)+(a+b+c)\right]$$

$$=\frac{1}{6}[a+4a+2b+c+a+b+c]$$

$$=\frac{1}{6}[6a+3b+2c]$$

$$=a+\frac{b}{2}+\frac{c}{3} \qquad \ldots (2)$$

$\therefore$ from (1) and (2),

$$\int_0^1 f(x)\,dx=\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right].$$