Maharashtra State Board 12th Commerce Maths Solutions Chapter 8 Differential Equation and Applications Ex 8.1

Question 1.
Determine the order and degree of each of the following differential equations:

(i) $\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2+8=0$

Solution:

The given D.E. is $\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2+8=0$

This D.E. has highest order derivative $\frac{d^2x}{dt^2}$ with power 1.

∴ the given D.E. is of order 2 and degree 1.

(ii) $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^2=a^x$

Solution:

The given D.E. is $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^2=a^x$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 2.

∴ the given D.E. is of order 2 and degree 2.

(iii) $\frac{d^4y}{dx^4}+\left[1+\left(\frac{dy}{dx}\right)^2\right]^3$

Solution:

The given D.E. is $\frac{d^4y}{dx^4}+\left[1+\left(\frac{dy}{dx}\right)^2\right]^3$

This D.E. has highest order derivative $\frac{d^4y}{dx^4}$ with power 1.

∴ the given D.E. is of order 4 and degree 1.

(iv) $(y''')^2 + 2(y'')^2 + 6y' + 7y = 0$

Solution:

The given D.E. is $(y''')^2 + 2(y'')^2 + 6y' + 7y = 0$

This can be written as $\left(\frac{d^3y}{dx^3}\right)^2+2\left(\frac{d^2y}{dx^2}\right)^2+6\frac{dy}{dx}+7y=0$

This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 2.

∴ the given D.E. is of order 3 and degree 2.

(v) $\sqrt{1+\frac{1}{\left(\frac{dy}{dx}\right)^2}}=\left(\frac{dy}{dx}\right)^{3/2}$

Solution:

The given D.E. is $\sqrt{1+\frac{1}{\left(\frac{dy}{dx}\right)^2}}=\left(\frac{dy}{dx}\right)^{3/2}$

On squaring both sides, we get

$1+\frac{1}{\left(\frac{dy}{dx}\right)^2}=\left(\frac{dy}{dx}\right)^3$

∴ $\left(\frac{dy}{dx}\right)^2+1=\left(\frac{dy}{dx}\right)^5$

This D.E. has highest order derivative $\frac{dy}{dx}$ with power 5.

∴ the given D.E. is of order 1 and degree 5.

(vi) $\frac{dy}{dx}=7\frac{d^2y}{dx^2}$

Solution:

The given D.E. is $\frac{dy}{dx}=7\frac{d^2y}{dx^2}$

This D.E. has highest order derivative $\frac{d^2y}{dx^2}$ with power 1.

∴ the given D.E. is of order 2 and degree 1.

(vii) $\left(\frac{d^3y}{dx^3}\right)^{1/6}=9$

Solution:

The given D.E. is $\left(\frac{d^3y}{dx^3}\right)^{1/6}=9$

i.e., $\frac{d^3y}{dx^3}=9^6$

This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 1.

∴ the given D.E. is of order 3 and degree 1.

Question 2.
In each of the following examples, verify that the given function is a solution of the corresponding differential equation:

	Solution	D.E.
(i)	$xy = \log y + k$	$y'(1-xy) = y^2$
(ii)	$y = x^n$	$x^2\frac{d^2y}{dx^2} - nx\frac{dy}{dx} + ny = 0$
(iii)	$y = e^x$	$\frac{dy}{dx} = y$
(iv)	$y = 1 - \log x$	$x^2\frac{d^2y}{dx^2} = 1$
(v)	$y = ae^x + be^{-x}$	$\frac{d^2y}{dx^2} = y$
(vi)	$ax^2 + by^2 = 5$	$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = y\cdot\frac{dy}{dx}$

Solution:
(i) xy = log y + k
Differentiating w.r.t. x, we get

$$x\cdot\frac{dy}{dx} + y \times 1 = \frac{1}{y}\cdot\frac{dy}{dx} + 0$$

$$\therefore\ x\frac{dy}{dx} + y = \frac{1}{y}\cdot\frac{dy}{dx}$$

$$\left(x - \frac{1}{y}\right)\frac{dy}{dx} = -y$$

$$\therefore\ \left(\frac{xy-1}{y}\right)\frac{dy}{dx} = -y$$

$$\therefore\ \frac{dy}{dx} = \frac{-y^2}{xy-1} = \frac{y^2}{1-xy},\ \text{if } xy \neq 1$$

$$\therefore\ y' = \frac{y^2}{1-xy},\ \text{if } xy \neq 1.$$

$$\therefore\ y'(1-xy) = y^2$$

Hence, xy = log y + k is a solution of the D.E. y'(1 – xy) = y^2.

(ii) y = x^n
Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx}(nx^{n-1}) = n\frac{d}{dx}(x^{n-1}) = n(n-1)x^{n-2}$$

$$\therefore\ x^2\frac{d^2y}{dx^2} - nx\frac{dy}{dx} + ny$$
$$= x^2\cdot n(n-1)x^{n-2} - nx\cdot nx^{n-1} + n\cdot x^n$$
$$= n(n-1)x^n - n^2x^n + nx^n$$
$$= (n^2 - n - n^2 + n)x^n = 0$$

This shows that y = x^n is a solution of the D.E.
$x^2\frac{d^2y}{dx^2} - nx\frac{dy}{dx} + ny = 0$

(iii) y = e^x
Differentiating w.r.t. x, we get
$\frac{dy}{dx}$ = e^x = y
Hence, y = ex is a solution of the D.E. $\frac{dy}{dx}$ = y.

(iv) $y = 1 - \log x$
Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(1) - \frac{d}{dx}(\log x)$$

$$= 0 - \frac{1}{x} = -\frac{1}{x}$$

Differentiating again w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}(x^{-1}) = -(-1)x^{-2} = \frac{1}{x^2}$$

$$\therefore\ x^2\frac{d^2y}{dx^2} = 1$$

Hence, $y = 1 - \log x$ is a solution of the D.E.
$x^2\frac{d^2y}{dx^2}=1$

(v) $y = ae^x + be^{-x}$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = a(e^x) + b(-e^{-x}) = ae^x - be^{-x}$
Differentiating again w.r.t. x, we get
$\frac{d^2y}{dx^2} = a(e^x) - b(-e^{-x})$
$= ae^x + be^{-x}$
$= y$
Hence, $y = ae^x + be^{-x}$ is a solution of the D.E. $\frac{d^2y}{dx^2} = y$.

(vi) $ax^2 + by^2 = 5$
Differentiating w.r.t. x, we get

$$a(2x) + b\left(2y\frac{dy}{dx}\right) = 0$$

$$\therefore\ ax + by\frac{dy}{dx} = 0$$

$$\therefore\ ax = -by\frac{dy}{dx} \quad \text{........ (1)}$$

Differentiating again w.r.t. x, we get

$$a\cdot 1 = -b\left[y\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{dy}{dx}\right]$$

$$\therefore\ a = -b\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] \quad \text{....... (2)}$$

Dividing (1) by (2), we get

$$x = \frac{y\frac{dy}{dx}}{y\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2}$$

$$\therefore\ xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = y\frac{dy}{dx}$$

Hence, $ax^2 + by^2 = 5$ is a solution of the D.E.
$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2=y\left(\frac{dy}{dx}\right)$

Maharashtra State Board 12th Commerce Maths Solutions Chapter 8 Differential Equation and Applications Ex 8.2

Question 1.
Obtain the differential equation by eliminating arbitrary constants from the following equations:
(i) $y = Ae^{3x} + Be^{-3x}$
Solution:
$y = Ae^{3x} + Be^{-3x}$ ......(1)
Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = Ae^{3x} \times 3 + Be^{-3x} \times (-3)$$

$$\therefore \frac{dy}{dx} = 3Ae^{3x} - 3Be^{-3x}$$

$$\text{and } \frac{d^2y}{dx^2} = 3Ae^{3x} \times 3 - 3Be^{-3x} \times (-3)$$

$$= 9Ae^{3x} + 9Be^{-3x}$$

$$= 9(Ae^{3x} + Be^{-3x}) = 9y \quad \text{....... [By (1)]}$$

$$\therefore \frac{d^2y}{dx^2} = 9y$$

This is the required D.E.

(ii) $y = c_2 + c_1 x$
Solution:
$y = c_2 + c_1 x$
$\therefore xy = c_2x + c_1$
Differentiating w.r.t. x, we get

$$x\frac{dy}{dx} + y \cdot 1 = c_2 \cdot 1 + 0 = c_2$$

$$\therefore x\frac{dy}{dx} + y = c_2$$

Differentiating again w.r.t. x, we get

$$x \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x) + \frac{dy}{dx} = 0$$

$$\therefore x\frac{d^2y}{dx^2} + \frac{dy}{dx} \times 1 + \frac{dy}{dx} = 0$$

$$\therefore x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0 \text{ is the required D.E.}$$

(iii) $y = (c_1 + c_2x)\, e^x$
Solution:
$y = (c_1 + c_2x)\, e^x$

$$\therefore e^{-x}y = c_1 + c_2x$$

Differentiating w.r.t. x, we get

$$e^{-x} \cdot \frac{dy}{dx} + y \cdot e^{-x}(-1) = 0 + c_2 \times 1$$

$\therefore\ e^{-x}\left(\frac{dy}{dx}-y\right)=c_2$

Differentiating again w.r.t. x, we get

$$e^{-x}\left(\frac{d^2y}{dx^2}-\frac{dy}{dx}\right)+\left(\frac{dy}{dx}-y\right)\cdot e^{-x}(-1)=0$$

$$\therefore\ e^{-x}\left(\frac{d^2y}{dx^2}-\frac{dy}{dx}-\frac{dy}{dx}+y\right)=0$$

$$\therefore\ \frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$$

This is the required D.E.

(iv) $y = c_1 e^{3x} + c_2 e^{2x}$
Solution:

$$y=c_1e^{3x}+c_2e^{2x} \quad \ldots (1)$$

Differentiating twice w.r.t. x, we get

$$\frac{dy}{dx}=3c_1e^{3x}+2c_2e^{2x} \quad \ldots (2)$$

$$\frac{d^2y}{dx^2}=9c_1e^{3x}+4c_2e^{2x} \quad \ldots (3)$$

These three equations in c_1e^{3x} and c_2e^{2x} are consistent.

$\therefore$ determinant of their consistency condition is zero.

$$\therefore\ \begin{vmatrix} y & 1 & 1 \\ \frac{dy}{dx} & 3 & 2 \\ \frac{d^2y}{dx^2} & 9 & 4 \end{vmatrix}=0$$

$$\therefore\ y(12-18)-1\left(4\frac{dy}{dx}-2\frac{d^2y}{dx^2}\right)+1\left(9\frac{dy}{dx}-3\frac{d^2y}{dx^2}\right)=0$$

$$\therefore\ -6y-4\frac{dy}{dx}+2\frac{d^2y}{dx^2}+9\frac{dy}{dx}-3\frac{d^2y}{dx^2}=0$$

$$\therefore\ \frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

This is the required D.E.

Alternative Method :

$y=c_1e^{3x}+c_2e^{2x}$

Dividing both sides by e^{2x}, we get

$e^{-2x}y=c_1e^{x}+c_2$

Differentiating w.r.t. x, we get

$$e^{-2x}\frac{dy}{dx}+y\cdot e^{-2x}(-2)=c_1e^{x}+0$$

$$\therefore\ e^{-2x}\left(\frac{dy}{dx}-2y\right)=c_1e^{x}$$

Dividing both sides by e^{x}, we get

$$e^{-3x}\left(\frac{dy}{dx}-2y\right)=c_1$$

Differentiating w.r.t. x, we get

$$e^{-3x}\left(\frac{d^2y}{dx^2}-2\frac{dy}{dx}\right)+\left(\frac{dy}{dx}-2y\right)\cdot e^{-3x}(-3)=0$$

$$\therefore\ e^{-3x}\left(\frac{d^2y}{dx^2}-2\frac{dy}{dx}-3\frac{dy}{dx}+6y\right)=0$$

$$\therefore\ \frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

This is the required D.E.

(v) $y^2 = (x + c)^3$
Solution:
$y^2 = (x + c)^3$
Differentiating w.r.t. x, we get

$$2y\frac{dy}{dx} = 3(x+c)^2 \cdot (1) = 3(x+c)^2$$

$$\therefore\ (x+c)^2 = \frac{2y}{3}\cdot\frac{dy}{dx}$$

$$\therefore\ (x+c)^6 = \left(\frac{2y}{3}\cdot\frac{dy}{dx}\right)^3$$

$$\therefore\ (y^2)^2 = \frac{8y^3}{27}\cdot\left(\frac{dy}{dx}\right)^3 \quad \text{....... [By (1)]}$$

$$\therefore\ 27y^4 = 8y^3\left(\frac{dy}{dx}\right)^3$$

$$\therefore\ 27y = 8\left(\frac{dy}{dx}\right)^3$$

$$\therefore\ \left(\frac{dy}{dx}\right)^3 = \frac{27y}{8}$$

$$\therefore\ \frac{dy}{dx} = \frac{3}{2}(\sqrt[3]{y})$$

This is the required D.E.

Question 2.
Find the differential equation by eliminating arbitrary constant from the relation $x^2 + y^2 = 2ax$.
Solution:
$x^2 + y^2 = 2ax$
Differentiating both sides w.r.t. x, we get
$2x + 2y\frac{dy}{dx} = 2a$
Substituting value of 2a in equation (1), we get
$x^2 + y^2 = [2x + 2y\frac{dy}{dx}]x = 2x^2 + 2xy\frac{dy}{dx}$
$\therefore 2xy\frac{dy}{dx} = y^2 - x^2$ is the required D.E.

Question 3.
Form the differential equation by eliminating arbitrary constants from the relation bx + ay = ab.
Solution:
bx + ay = ab
∴ ay = -bx + ab
∴ y = $-\frac{b}{a}x + b$
Differentiating w.r.t. x, we get
$\frac{dy}{dx} = -\frac{b}{a} \times 1 + 0 = -\frac{b}{a}$
Differentiating again w.r.t. x, we get
$\frac{d^2y}{dx^2} = 0$ is the required D.E.

Question 4.
Find the differential equation whose general solution is $x^3 + y^3 = 35ax$.

Solution:

$x^3 + y^3 = 35ax$ ...(i)

Differentiating w.r.t. x, we get

$3x^2 + 3y^2\frac{dy}{dx} = 35a$ ...(ii)

Substituting (ii) in (i), we get

$$x^3 + y^3 = \left(3x^2 + 3y^2\frac{dy}{dx}\right)x$$

$$\therefore x^3 + y^3 = 3x^3 + 3x \cdot y^2\frac{dy}{dx}$$

$\therefore 2x^3 - y^3 + 3xy^2\frac{dy}{dx} = 0$, which is the required differential equation.

Question 5.
Form the differential equation from the relation $x^2 + 4y^2 = 4b^2$.
Sol ution:
$x^2 + 4y^2 = 4b^2$
Differentiating w.r.t. x, we get
$2x + 4(2y\frac{dy}{dx}) = 0$
i.e. $x + 4y\frac{dy}{dx} = 0$ is the required D.E.

Maharashtra State Board 12th Commerce Maths Solutions Chapter 8 Differential Equation and Applications Ex 8.3

Question 1.
Solve the following differential equations:
(i) $\frac{dy}{dx} = x^2y + y$
Solution:
$\frac{dy}{dx} = x^2y + y$
$\therefore \frac{dy}{dx} = y(x^2 + 1)$
$\therefore \frac{1}{y}$ dy = $(x^2 + 1)$ dx
Integrating, we get
$\int\frac{1}{y}$ dy = $\int(x^2 + 1)$ dx
$\therefore \log|y| = \frac{x^3}{3} + x + c$
This is the general solution.

(ii) $\frac{d\theta}{dt} = -k(\theta - \theta_0)$
Solution:
$\frac{d\theta}{dt} = -k(\theta - \theta_0)$

$$\therefore \frac{1}{\theta-\theta_0} d\theta = -k\,dt$$

Integrating, we get

$$\int \frac{1}{\theta-\theta_0} d\theta = -k \int dt$$

$$\therefore \log|\theta-\theta_0| = -kt + \log c_1$$

$$\therefore \log|\theta-\theta_0| - \log c_1 = -kt$$

$$\therefore \log\left|\frac{\theta-\theta_0}{c_1}\right| = -kt$$

$$\therefore \frac{\theta-\theta_0}{c_1} = e^{-kt}$$

$$\therefore \theta-\theta_0 = c_1 e^{-kt}$$

$$\therefore \theta-\theta_0 = e^c \cdot e^{-kt}, \text{ where } c_1 = e^c$$

$$\therefore \theta-\theta_0 = e^{-kt+c}$$

This is the general solution.

(iii) $(x^2 - yx^2)\,dy + (y^2 + xy^2)\,dx = 0$

Solution:

$(x^2 - yx^2)\,dy + (y^2 + xy^2)\,dx = 0$

$\therefore x^2(1 - y)\,dy + y^2(1 + x)\,dx = 0$

$\therefore \frac{1-y}{y^2}dy + \frac{1+x}{x^2}dx = 0$

Integrating, we get

$$\int \frac{1-y}{y^2} dy + \int \frac{1+x}{x^2} dx = c$$

$$\therefore \int \left(\frac{1}{y^2} - \frac{1}{y}\right) dy + \int \left(\frac{1}{x^2} + \frac{1}{x}\right) dx = c$$

$$\therefore \int y^{-2}\,dy - \int \frac{1}{y} dy + \int x^{-2}\,dx + \int \frac{1}{x} dx = c$$

$$\therefore \frac{y^{-1}}{-1} - \log|y| + \frac{x^{-1}}{-1} + \log|x| = c$$

$$\therefore -\frac{1}{y} - \log|y| - \frac{1}{x} + \log|x| = c$$

$$\therefore \log|x| - \log|y| = \frac{1}{x} + \frac{1}{y} + c$$

This is the general solution.

(iv) $y^3 - \frac{dy}{dx} = x\frac{dy}{dx}$

Solution:

$y^3 - \frac{dy}{dx} = x\frac{dy}{dx}$

$$\therefore y^3 = x\frac{dy}{dx} + \frac{dy}{dx} = (x+1)\frac{dy}{dx}$$

$$\therefore \frac{dx}{x+1} = \frac{dy}{y^3}$$

Integrating, we get

$$\int \frac{dx}{x+1} = \int y^{-3}\,dy$$

$$\therefore \log|x+1| = \frac{y^{-2}}{(-2)} + c = \frac{-1}{2y^2} + c$$

$\therefore 2y^2 \log|x + 1| = 2cy^2 - 1$ is the required solution.

Question 2.
For each of the following differential equations find the particular solution:
(i) $(x - y^2x)\,dx - (y + x^2y)\,dy = 0$, when $x = 2$, $y = 0$.
Solution:
$(x - y^2x)\,dx - (y + x^2y)\,dy = 0$
$\therefore x(1 - y^2)\,dx - y(1 + x^2)\,dy = 0$

$$\therefore \frac{x}{1+x^2}dx - \frac{y}{1-y^2}dy = 0$$

$$\therefore \frac{2x}{1+x^2} - \frac{2y}{1-y^2}dy = 0$$

Integrating, we get

$$\int \frac{2x}{1+x^2}dx + \int \frac{-2y}{1-y^2}dy = c_1$$

Each of these integrals is of the type

$$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$$

$\therefore$ the general solution is
$\log|1 + x^2| + \log|1 - y^2| = \log c$, where $c_1 = \log c$
$\therefore \log|(1 + x^2)(1 - y^2)| = \log c$
$\therefore (1 + x^2)(1 - y^2) = c$
When $x = 2$, $y = 0$, we have
$(1 + 4)(1 - 0) = c$
$\therefore c = 5$
$\therefore$ the particular solution is $(1 + x^2)(1 - y^2) = 5$.

(ii) $(x + 1)\frac{dy}{dx} - 1 = 2e^{-y}$, when $y = 0$, $x = 1$.
Solution:

$$(x+1)\frac{dy}{dx} - 1 = 2e^{-y}$$

$$\therefore (x+1)\frac{dy}{dx} = \frac{2}{e^y} + 1 = \frac{2+e^y}{e^y}$$

$$\therefore \frac{e^y}{2+e^y}dy = \frac{1}{x+1}dx$$

Integrating, we get

$$\int \frac{e^y}{2+e^y}dy = \int \frac{1}{x+1}dx$$

$$\therefore \log|2+e^y| = \log|x+1| + \log c$$

$$\ldots \left[\because \frac{d}{dy}(2+e^y) = e^y \text{ and } \int \frac{f'(y)}{f(y)}dy = \log|f(y)| + c\right]$$

$\therefore \log|2 + e^y| = \log|c(x + 1)|$
$\therefore 2 + e^y = c(x + 1)$
This is the general solution.
Now, $y = 0$, when $x = 1$
$\therefore 2 + e^0 = c(1 + 1)$
$\therefore 3 = 2c$
$\therefore c = \frac{3}{2}$
$\therefore$ the particular solution is
$2 + e^y = \frac{3}{2}(x + 1)$
$\therefore 4 + 2e^y = 3x + 3$
$\therefore 3x - 2e^y - 1 = 0$

(iii) $y(1 + \log x)\frac{dx}{dy} - x\log x = 0$, when $x = e$, $y = e^2$.
Solution:

$$y(1+\log x)\frac{dx}{dy}-x\log x=0$$

$$\therefore\ \frac{1+\log x}{x\log x}\,dx-\frac{dy}{y}=0$$

$$\therefore\ \int\frac{1+\log x}{x\ \log\ x}\,dx-\int\frac{dy}{y}=c_1 \qquad \text{...... (1)}$$

Put $x\log x=t$.

$$\text{Then}\left[x\cdot\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(x)\right]dx=dt$$

$$\therefore\ \left[\frac{x}{x}+(\log x)(1)\right]dx=dt \quad \therefore\ (1+\log x)\,dx=dt$$

$$\therefore\ \int\frac{1+\log x}{x\ \log\ x}\,dx=\int\frac{dt}{t}=\log|t|=\log|x\ \log\ x|$$

∴ from (1), the general solution is
log |x log x| – log |y| = log c, where c_1 = log c
∴ log |$\frac{x\log x}{y}$| = log c
∴ $\frac{x\log x}{y}$ = c
∴ x log x = cy
This is the general solution.
Now, y = e^2, when x = e
e log e = ce^2
1 = ce ……[∵ log e = 1]
c = $\frac{1}{e}$
∴ the particular solution is x log x = ($\frac{1}{e}$) y
∴ y = ex log x

(iv) $\frac{dy}{dx}$ = 4x + y + 1, when y = 1, x = 0.
Solution:
$\frac{dy}{dx}$ = 4x + y + 1
Put 4x + y + 1 = v

$$\therefore\ 4+\frac{dy}{dx}=\frac{dv}{dx}$$

$$\therefore\ \frac{dy}{dx}=\frac{dv}{dx}-4$$

∴ the given D.E. becomes

$$\frac{dv}{dx}-4=v \qquad \therefore\ \frac{dv}{dx}=4+v$$

$$\therefore\ \frac{dv}{v+4}=dx$$

Integrating we get

$$\int\frac{dv}{v+4}=\int dx$$

∴ log |v + 4| = x + c
∴ log |4x + y + 1 + 4| = x + c
i.e. log |4x + y + 5| = x + c
This is the general solution.
Now, y = 1 when x = 0
∴ log|0 + 1 + 5| = 0 + c,
i.e. c = log 6
∴ the particular solution is
log |4x + y + 5| = x + log 6
∴ $\log\left|\frac{4x+y+5}{6}\right| = x$

Maharashtra State Board 12th Commerce Maths Solutions Chapter 8 Differential Equation and Applications Ex 8.4

Solve the following differential equations:

Question 1.
x dx + 2y dy = 0
Solution:
x dx + 2y dy = 0
Integrating, we get
$\int x\, dx + 2 \int y\, dy = c_1$
$\therefore \frac{x^2}{2} + 2\left(\frac{y^2}{2}\right) = c_1$
$\therefore x^2 + 2y^2 = c$, where $c = 2c_1$
This is the general solution.

Question 2.
$y^2\, dx + (xy + x^2)\, dy = 0$
Solution:
$y^2\, dx + (xy + x^2)\, dy = 0$
$\therefore (xy + x^2)\, dy = -y^2\, dx$
$\therefore \frac{dy}{dx} = \frac{-y^2}{xy + x^2}$ ....(1)
Put y = vx
$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$
Substituting these values in (1), we get

$$\therefore \int\left(\frac{1}{v}-\frac{1}{2v+1}\right)dv=-\int\frac{1}{x}dx$$

$$\therefore \int\frac{1}{v}dv-\int\frac{1}{2v+1}dv=-\int\frac{1}{x}dx$$

$$\therefore \log|v|-\frac{1}{2}\log|2v+1|=-\log|x|+\log c$$

$$\therefore 2\log|v|-\log|2v+1|=-2\log|x|+2\log c$$

$$\therefore \log|v^2|-\log|2v+1|=-\log|x^2|+\log c^2$$

$$\therefore \log\left|\frac{v^2}{2v+1}\right|=\log\left|\frac{c^2}{x^2}\right|$$

$$\therefore \frac{v^2}{2v+1}=\frac{c^2}{x^2}$$

$$\therefore \frac{\left(\frac{y^2}{x^2}\right)}{2\left(\frac{y}{x}\right)+1}=\frac{c^2}{x^2} \qquad \therefore \frac{y^2}{x(2y+x)}=\frac{c^2}{x^2}$$

$$\therefore xy^2=c^2(x+2y)$$

This is the general solution.

Question 3.
$x^2y\,dx - (x^3 + y^3)\,dy = 0$
Solution:
$x^2y\,dx - (x^3 + y^3)\,dy = 0$
$\therefore (x^3 + y^3)\,dy = x^2y\,dx$
$\therefore \frac{dy}{dx}=\frac{x^2y}{x^3+y^3}$ ......(1)
Put y = vx
$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\therefore \text{(1) becomes, } v+x\frac{dv}{dx}=\frac{x^2\cdot vx}{x^3+v^3x^3}=\frac{v}{1+v^3}$$

$$\therefore x\frac{dv}{dx}=\frac{v}{1+v^3}-v=\frac{v-v-v^4}{1+v^3}$$

$$\therefore x\frac{dv}{dx}=\frac{-v^4}{1+v^3}$$

$$\therefore \frac{1+v^3}{v^4}dv=-\frac{1}{x}dx$$

Integrating, we get

$$\int\frac{1+v^3}{v^4}dv=-\int\frac{1}{x}dx$$

$$\therefore \int\left(\frac{1}{v^4}+\frac{1}{v}\right)dv=-\int\frac{1}{x}dx$$

$$\therefore \int v^{-4}dv+\int\frac{1}{v}dv=-\int\frac{1}{x}dx$$

$$\therefore \frac{v^{-3}}{-3} + \log|v| = -\log|x| + c_1$$

$$\therefore -\frac{1}{3v^3} + \log|v| = -\log|x| + c_1$$

$$\therefore -\frac{1}{3} \cdot \frac{1}{\left(\frac{y}{x}\right)^3} + \log\left|\frac{y}{x}\right| = -\log|x| + c_1$$

$$\therefore -\frac{x^3}{3y^3} + \log|y| - \log|x| = -\log|x| - \log c,$$

where $c_1 = -\log c$

$$\therefore \frac{x^3}{3y^3} = \log c + \log y$$

$$\therefore \frac{x^3}{3y^3} = \log$$

This is the general solution.

Question 4.
dydx+x−2y2x−y=0
Solution:

$$\frac{dy}{dx} + \frac{x-2y}{2x-y} = 0$$

$$\therefore \frac{dy}{dx} = -\left(\frac{x-2y}{2x-y}\right) \quad \text{........ (1)}$$

Put $y = vx$. $\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \text{(1) becomes, } v + x\frac{dv}{dx} = -\left(\frac{x-2vx}{2x-vx}\right)$$

$$\therefore v + x\frac{dv}{dx} = -\left(\frac{1-2v}{2-v}\right)$$

$$\therefore x\frac{dv}{dx} = -\left(\frac{1-2v}{2-v}\right) - v$$

$$\therefore x\frac{dv}{dx} = \frac{-1+2v-2v+v^2}{2-v}$$

$$\therefore x\frac{dv}{dx} = \frac{v^2-1}{2-v}$$

$$\therefore \frac{2-v}{v^2-1}dv = \frac{1}{x}dx$$

Integrating, we get

$$\int \frac{2-v}{v^2-1}dv = \int \frac{1}{x}dx$$

$$\therefore 2\int \frac{1}{v^2-1}dv - \frac{1}{2}\int \frac{2v}{v^2-1}dv = \int \frac{1}{x}dx$$

$$\therefore 2 \times \frac{1}{2}\log\left|\frac{v-1}{v+1}\right| - \frac{1}{2}\log|v^2-1| = \log|x| + \log c_1$$

$$\ldots \left[\because \frac{d}{dv}(v^2-1) = 2v \text{ and } \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$$

$\therefore \log\left|\frac{v-1}{v+1}\right| - \log|(v^2-1)^{\frac{1}{2}}| = \log|c_1x|$

$\therefore \log\left|\frac{v-1}{v+1} \cdot \frac{1}{\sqrt{v^2-1}}\right| = \log|c_1x|$

$\therefore \frac{v-1}{v+1} \cdot \frac{1}{\sqrt{v^2-1}} = c_1x$

$\therefore \frac{\frac{y}{x}-1}{\frac{y}{x}+1} \cdot \frac{1}{\sqrt{\frac{y^2}{x^2}-1}} = c_1x$

$\therefore \frac{y-x}{y+x} \cdot \frac{x}{\sqrt{y^2-x^2}} = c_1x$

$\therefore \frac{y-x}{y+x} = c_1\sqrt{y^2-x^2}$

$\therefore \frac{y-x}{y+x} = c_1\sqrt{y-x} \cdot \sqrt{y+x}$

$\therefore \sqrt{y-x} = c_1(y+x)^{\frac{3}{2}}$

$\therefore y-x = c_1{}^2(x+y)^3$

$\therefore y-x = c(x+y)^3$, where $c = c_1{}^2$

This is the general solution.

Question 5.
$(x^2 - y^2)\,dx + 2xy\,dy = 0$
Solution:
$(x^2 - y^2)\,dx + 2xy\,dy = 0$
$\therefore 2xy\,dy = -(x^2 - y^2)\,dx = (y^2 - x^2)\,dx$
$\therefore \frac{dy}{dx} = \frac{y^2-x^2}{2xy}$ ....(1)

Put $y = vx$ $\quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x \cdot vx}$

$$\therefore v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$\therefore x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\therefore x\frac{dv}{dx} = \frac{-1 - v^2}{2v} = -\left(\frac{1 + v^2}{2v}\right)$$

$$\therefore \frac{2v}{1 + v^2}dv = -\frac{1}{x}dx$$

Integrating, we get

$$\int \frac{2v}{1 + v^2}dv = -\int \frac{1}{x}dx$$

$$\therefore \log|1 + v^2| = -\log x + \log c$$

$$\ldots \left[\because \frac{d}{dv}(1 + v^2) = 2v \text{ and } \int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c\right]$$

$$\therefore \log\left|1 + \frac{y^2}{x^2}\right| = -\log x + \log c$$

$$\therefore \log\left|\frac{x^2 + y^2}{x^2}\right| = \log\left|\frac{c}{x}\right|$$

$$\therefore \frac{x^2 + y^2}{x^2} = \frac{c}{x}$$

$$\therefore x^2 + y^2 = cx$$

This is the general solution.

Question 6.
$xy\frac{dy}{dx} = x^2 + 2y^2$
Solution:

$$xy\frac{dy}{dx} = x^2 + 2y^2$$

$$\therefore \frac{dy}{dx} = \frac{x^2 + 2y^2}{xy} \quad \text{......... (1)}$$

Put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (1) becomes, $v + x\frac{dv}{dx} = \frac{x^2 + 2v^2x^2}{x \cdot vx} = \frac{1 + 2v^2}{v}$

$$\therefore x\frac{dv}{dx} = \frac{1 + 2v^2}{v} - v = \frac{1 + 2v^2 - v^2}{v}$$

$$\therefore x\frac{dv}{dx} = \frac{1 + v^2}{v}$$

$$\therefore \frac{v}{1 + v^2}dv = \frac{1}{x}dx$$

Integrating, we get

$$\int \frac{v}{1 + v^2}dv = \int \frac{1}{x}dx$$

$$\therefore \frac{1}{2}\int \frac{2v}{1 + v^2}dv = \int \frac{1}{x}dx + \log c_1$$

$$\therefore \frac{1}{2}\log|1 + v^2| = \log|x| + \log c_1$$

$$\therefore \log|1 + v^2| = 2\log|x| + 2\log c_1$$

$$\therefore \log|1 + v^2| = \log|x^2| + \log {c_1}^2$$

$\therefore \log|1 + v^2| = \log|cx^2|$, where $c = {c_1}^2$

$$\therefore 1 + v^2 = cx^2$$

$$\therefore 1 + \frac{y^2}{x^2} = cx^2$$

$$\therefore \frac{x^2 + y^2}{x^2} = cx^2$$

$$\therefore x^2 + y^2 = cx^4$$

This is the general solution.

Question 7.
$x^2\frac{dy}{dx} = x^2 + xy - y^2$
Solution:

$$x^2\frac{dy}{dx}=x^2+xy-y^2$$

$$\therefore\ \frac{dy}{dx}=1+\frac{y}{x}-\frac{y^2}{x^2} \quad \text{......... (1)}$$

Put $y=vx$ i.e. $\frac{y}{x}=v$

$$\therefore\ \frac{ay}{dx}=v+x\frac{av}{dx}$$

Maharashtra State Board 12th Commerce Maths Solutions Chapter 8 Differential Equation and Applications Ex 8.5

Solve the following differential equations.

Question 1.
$\frac{dy}{dx}+y=e^{-x}$
Solution:
$\frac{dy}{dx}+y=e^{-x}$ ....(1)
This is the linear differential equation of the form

$\frac{dy}{dx}+P\cdot y=Q$, where $P=1$ and $Q=e^{-x}$

$\therefore$ I.F. $=e^{\int P\,dx}=e^{\int 1\,dx}=e^x$

$\therefore$ the solution of (1) is given by

$y\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$

$\therefore\ y\cdot e^x=\int e^{-x}\cdot e^x\,dx+c$

$\therefore\ e^x\cdot y=\int 1\,dx+c$

$\therefore\ e^x\cdot y=x+c \qquad \therefore\ ye^x=x+c$

This is the general solution.

Question 2.
$\frac{dy}{dx}$ + y = 3
Solution:
$\frac{dy}{dx}$ + y = 3
This is the linear differential equation of the form

$\frac{dy}{dx} + P \cdot y = Q$, where $P = 1$, $Q = 3$

$\therefore$ I.F. $= e^{\int P\,dx} = e^{\int 1\,dx} = e^x$

$\therefore$ the solution of (1) is given by

$y \cdot (\text{I.F.}) = \int Q(\text{I.F.})\,dx + c$

$\therefore\ ye^x = \int 3e^x dx + c = 3e^x + c$

$\therefore\ ye^x = 3e^x + c$

This is the general solution.

Question 3.
$x\frac{dy}{dx} + 2y = x^2 \cdot \log x$.
Solution:
$x\frac{dy}{dx} + 2y = x^2 \cdot \log x$
$\therefore \frac{dy}{dx} + \left(\frac{2}{x}\right) \cdot y = x \cdot \log x$ ....(1)
This is the linear differential equation of the form

$\frac{dy}{dx} + P \cdot y = Q$, where $P = \frac{2}{x}$ and $Q = x \cdot \log x$

$$\therefore \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2}{x}dx} = e^{2\int \frac{1}{x}dx}$$
$$= e^{2\log x} = e^{\log x^2} = x^2$$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$$
$$\therefore\ y \cdot x^2 = \int (x\log x) \cdot x^2\,dx + c$$
$$\therefore\ x^2 \cdot y = \int x^3 \cdot \log x\,dx + c$$
$$= (\log x)\int x^3\,dx - \int\left[\frac{d}{dx}(\log x)\int x^3\,dx\right]dx + c$$
$$= (\log x) \cdot \frac{x^4}{4} - \int \frac{1}{x} \cdot \frac{x^4}{4}dx + c$$
$$= \frac{1}{4}x^4\log x - \frac{1}{4}\int x^3 dx + c$$
$$\therefore\ x^2 \cdot y = \frac{1}{4}x^4\log x - \frac{1}{4} \cdot \frac{x^4}{4} + c$$
$$\therefore\ y \cdot x^2 = \frac{x^4\log x}{4} - \frac{x^4}{16} + c$$

This is the general solution.

Question 4.
$(x + y)\frac{dy}{dx} = 1$
Solution:
$(x + y)\frac{dy}{dx} = 1$
$\therefore \frac{dx}{dy} = x + y$
$\therefore \frac{dx}{dy} - x = y$
$\therefore \frac{dx}{dy} + (-1)x = y$ ......(1)
This is the linear differential equation of the form

$\frac{dx}{dy} + P \cdot x = Q$, where $P = -1$ and $Q = y$

$\therefore$ I.F. $= e^{\int P\,dy} = e^{\int -1\,dy} = e^{-y}$

$\therefore$ the solution of (1) is given by

$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dy + c$

$\therefore x \cdot e^{-y} = \int y \cdot e^{-y}\,dy + c$

$\therefore e^{-y} \cdot x = y \int e^{-y}\,dy - \int \left[\frac{d}{dy}(y) \int e^{-y}\,dy\right] dy + c$

$= y \cdot \frac{e^{-y}}{-1} - \int 1 \cdot \frac{e^{-y}}{-1}\,dy + c$

$= -ye^{-y} + \int e^{-y}\,dy + c$

$\therefore e^{-y} \cdot x = -ye^{-y} + \frac{e^{-y}}{-1} + c$

$\therefore e^{-y} \cdot x + ye^{-y} + e^{-y} = c$

$\therefore e^{-y}(x + y + 1) = c$

$\therefore x + y + 1 = ce^{y}$

This is the general solution.

Question 5.
y dx + (x – y^2) dy = 0
Solution:
y dx + (x – y^2) dy = 0
∴ y dx = -(x – y^2) dy
∴ $\frac{dx}{dy} = \frac{-(x - y^2)}{y} = -\frac{x}{y} + y$
∴ $\frac{dx}{dy} + \left(\frac{1}{y}\right) \cdot x = y$ ……(1)
This is the linear differential equation of the form

$\frac{dx}{dy} + P \cdot x = Q$, where $P = \frac{1}{y}$ and $Q = y$

$\therefore$ I.F. $= e^{\int P\,dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y_1$

$\therefore$ the solution of (1) is given by

$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dy + c_1$

$\therefore xy = \int y \cdot y\,dy + c_1$

$\therefore xy = \int y^2\,dy + c_1$

$\therefore xy = \frac{y^3}{3} + c_1$

$\therefore 3xy = y^3 + 3c_1$

$\therefore 3xy = y^2 + c$, where $c = 3c_1$

This is the general solution.

Question 6.
$\frac{dy}{dx}$ + 2xy = x
Solution:
$\frac{dy}{dx}$ + 2xy = x ………(1)
This is the linear differential equation of the form

$$\frac{dy}{dx} + P \cdot y = Q, \text{ where } P = 2x, Q = x$$

$$\therefore \text{ I.F.} = e^{\int P\,dx} = e^{\int 2x\,dx}$$

$$= e^{2\int x\,dx} = e^{2\left(\frac{x^2}{2}\right)} = e^{x^2}$$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$$

$$\therefore ye^{x^2} = \int xe^{x^2}\,dx + c$$

Put $x^2 = t$ $\quad \therefore 2x\,dx = dt$

$$\therefore x\,dx = \frac{1}{2}dt$$

$\therefore$ (1) becomes

$$ye^{x^2} = \frac{1}{2}\int e^t\,dt + c$$

$$\therefore ye^{x^2} = \frac{1}{2}e^{x^2} + c$$

This is the general solution.

Question 7.
(x + a) $\frac{dy}{dx}$ = -y + a
Solution:
(x + a) $\frac{dy}{dx}$ + y = a
$\therefore \frac{dy}{dx} + \left(\frac{1}{x+a}\right)y = \frac{a}{x+a}$ ....(1)
This is the linear differential equation of the form

$$\frac{dy}{dx} + P \cdot y = Q, \text{ where } P = \frac{1}{x+a}, Q = \frac{a}{x+a}.$$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{x+a}dx}$$

$$= e^{\log(x+a)} = x + a$$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$$

$$\therefore y(x+a) = \int \left(\frac{a}{x+a}\right)(x+a)\,dx + c$$

$$= a\int dx + c$$

$$\therefore y(x+a) = ax + c$$

This is the general solution.

Question 8.
dy + (2y) dx = 8 dx
Solution:
dy + (2y) dx = 8 dx
$\therefore \frac{dy}{dx}$ + 2y = 8 ....(1)
This is the linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = 2$, $Q = 8$

$\therefore$ I.F. $= e^{\int P\,dx} = e^{2\int dx} = e^{2x}$

$\therefore$ the solution $f(1)$ is given by

$y \cdot (\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$

$\therefore ye^{2x} = \int 8e^{2x}\,dx + c$

$= 8\left(\frac{e^{2x}}{2}\right) + c$

$\therefore ye^{2x} = 4e^{2x} + c$

This is the general solution.

Maharashtra State Board 12th Commerce Maths Solutions Chapter 8 Differential Equation and Applications Ex 8.6

Question 1.
In a certain culture of bacteria, the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, find the number of times the bacteria are increased in 12 hours.
Solution:
Let x be the number of bacteria in the culture at time t.
Then the rate of increase is $\frac{dx}{dt}$ which is proportional to x.
$\therefore \frac{dx}{dt} \propto x$
$\therefore \frac{dx}{dt} = kx$, where k is a constant
$\therefore \frac{dx}{x} = k\,dt$
On integrating, we get
$\int \frac{dx}{x} = k \int dt$
$\therefore \log x = kt + c$
Initially, i.e. when t = 0, let $x = x_0$
$\therefore \log x_0 = k \times 0 + c$
$\therefore c = \log x_0$
$\therefore \log x = kt + \log x_0$
$\therefore \log x - \log x_0 = kt$
$\therefore \log\left(\frac{x}{x_0}\right) = kt$ ......(1)
Since the number doubles in 4 hours, i.e. when t = 4, $x = 2x_0$

$\therefore \log\left(\frac{2x_0}{x_0}\right) = 4k \qquad \therefore k = \frac{1}{4}\log 2$

$\therefore$ (1) becomes, $\log\left(\frac{x}{x_0}\right) = \frac{t}{4}\log 2$

When $t = 12$, we get

$\log\left(\frac{x}{x_0}\right) = \frac{12}{4}\log 2 = 3\log 2$

$\therefore \log\left(\frac{x}{x_0}\right) = \log 8$

$\therefore \frac{x}{x_0} = 8 \qquad \therefore x = 8x_0$

$\therefore$ the number of bacteria will be 8 times the original number in 12 hours.

Question 2.
If the population of a town increases at a rate proportional to the population at that time. If the population increases from 40 thousand to 60 thousand in 40 years, what will be the population in another 20 years? (Given: $\sqrt{\frac{3}{2}}$ = 1.2247)
Solution:

Let P be the population of the city at time t.
Then $\frac{dP}{dt}$, the rate of increase of population, is proportional to P.

$\therefore \frac{dP}{dt} \propto P$

$\therefore \frac{dP}{dt} = kP$, k is a constant

$\therefore \frac{dP}{P} = k\,dt$

Integrating, we get

$\int \frac{dP}{P} = k\int dt$

$\therefore \log P = kt + c$

Initially, i.e. when t = 0, P = 40000

$\therefore \log 40000 = 0 + c$

$\therefore c = \log 40000$

$\therefore \log P = kt + \log 40000$

$\therefore \log P - \log 40000 = kt$

$\therefore \log\left(\frac{P}{40000}\right) = kt$ ....(1)

When t = 40, P = 60000

$$\therefore \log\left(\frac{60000}{40000}\right) = 40k$$

$$\therefore k = \frac{1}{40}\log\left(\frac{3}{2}\right)$$

$\therefore$ (1) becomes

$$\log\left(\frac{P}{40000}\right) = \frac{t}{40}\log\left(\frac{3}{2}\right) = \log\left(\frac{3}{2}\right)^{\frac{t}{40}}$$

$$\therefore \frac{P}{40000} = \left(\frac{3}{2}\right)^{\frac{t}{40}}$$

We have to find P in another 20 years

i.e. at $t = 40 + 20 = 60$

If $t = 60$, then

$$\frac{P}{40000} = \left(\frac{3}{2}\right)^{\frac{60}{40}} = \left(\frac{3}{2}\right)^{\frac{3}{2}} = \frac{3}{2}\sqrt{\frac{3}{2}}$$

$$\therefore P = \frac{40000 \times 3}{2} \times 1.2247 \qquad \text{... [By data]}$$

$$= 73482$$

$\therefore$ population after 60 years will be 73482.

Question 3.
The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number doubles in 1 hour, find the number of bacteria after $\frac{5}{2}$ hours. [Given: $\sqrt{2} = 1.414$]
Solution:
Let x be the number of bacteria at time t.
Then the rate of increase is $\frac{dx}{dt}$ which is proportional to x.

$\therefore \frac{dx}{dt} \propto x$

$\therefore \frac{dx}{dt} = kx$, where k is a constant

$\therefore \frac{dx}{x} = k\,dt$

On integrating, we get

$\int \frac{dx}{x} = k\int dt$

$\therefore \log x = kt + c$

Initially, i.e. when t = 0, x = 1000

$\therefore \log 1000 = k \times 0 + c$

$\therefore c = \log 1000$

$\therefore \log x = kt + \log 1000$

$\therefore \log x - \log 1000 = kt$

$\therefore \log\left(\frac{x}{1000}\right) = kt$ ....(1)

Now, when t = 1, x = 2 × 1000 = 2000

$$\therefore \log\left(\frac{2000}{1000}\right) = k \qquad \therefore k = \log 2$$

$$\therefore \text{(1) becomes, } \log\left(\frac{x}{1000}\right) = t\log 2$$

If $t = \frac{5}{2}$, then

$$\log\left(\frac{x}{1000}\right) = \frac{5}{2}\log 2 = \log(2)^{\frac{5}{2}}$$

$$\therefore \left(\frac{x}{1000}\right) = (2)^{\frac{5}{2}} = 4\sqrt{2} = 4 \times 1.414 = 5.656$$

$$\therefore x = 5.656 \times 1000 = 5656$$

∴ number of bacteria after $\frac{5}{2}$ hours = 5656.

Question 4.
Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years, the population increased from 30,000 to 40,000.
Solution:
Let P be the population of the city at time t.
Then $\frac{dP}{dt}$, the rate of increase of population, is proportional to P.
∴ $\frac{dP}{dt} \propto P$
∴ $\frac{dP}{dt} = kP$, where k is a constant.
∴ $\frac{dP}{P} = k\,dt$
On integrating, we get
$\int \frac{1}{P} dP = k\int dt$
∴ log P = kt + c
Initially, i.e. when t = 0, P = 30000
∴ log 30000 = k x 0 + c
∴ c = log 30000
∴ log P = kt + log 30000
∴ log P – log 30000 = kt

$$\therefore \log\left(\frac{P}{30000}\right) = kt \qquad \dots (1)$$

Now, when $t = 40$, $P = 40000$

$$\therefore \log\left(\frac{40000}{30000}\right) = k \times 40$$

$$\therefore k = \frac{1}{40}\log\left(\frac{4}{3}\right)$$

$$\therefore \text{(1) becomes, } \log\left(\frac{P}{30000}\right) = \frac{t}{40}\log\left(\frac{4}{3}\right) = \log\left(\frac{4}{3}\right)^{\frac{t}{40}}$$

$$\therefore \frac{P}{30000} = \left(\frac{4}{3}\right)^{\frac{t}{40}}$$

$$\therefore P = 30000\left(\frac{4}{3}\right)^{\frac{t}{40}}$$

∴ the population of the city at time t = $30000\left(\frac{4}{3}\right)^{\frac{t}{40}}$.

Question 5.
The rate of depreciation $\frac{dV}{dt}$ of a machine is inversely proportional to the square of t + 1, where V is the value of the machine t years after it was purchased. The initial value of the machine was ₹ 8,00,000 and its value decreased ₹ 1,00,000 in the first year. Find the value after 6 years.
Solution:
Let V be the value of the machine at the end of t years.
Then $\frac{dV}{dt}$, the rate of depreciation, is inversly proportional to $(t + 1)^2$.

$\therefore \frac{dV}{dt} \propto \frac{1}{(t+1)^2}$

$\therefore \frac{dV}{dt} = -\frac{k}{(t+1)^2}$, $k > 0$ is a constant

$\therefore dV = \frac{-kdt}{(t+1)^2}$

On integrating, we get

$\int dV = -k \int \frac{dt}{(t+1)^2}$

$\therefore V = -k\left[\frac{-1}{t+1}\right] + c$

$\therefore V = \frac{k}{t+1} + c$

Initially, i.e. when t = 0, V = 800000
∴ 800000 = $\frac{k}{1}$ + c = k + c ....(1)
Now, when t = 1, V = 800000 – 100000 = 700000
∴ 700000 = $\frac{k}{1+1}$ + c = $\frac{k}{2}$ + c ......(2)
Subtracting (2) from (1), we get
100000 = $\frac{1k}{2}$
∴ k = 200000
∴ from (1), 800000 = 200000 + c
∴ c = 600000 200000
∴ V = $\frac{200000}{t+1}$ + 600000
When t = 6,
V = $\frac{200000}{7}$ + 600000
= 28571.43 + 600000
= 628571.43 ~ 628571
Hence, the value of the machine after 6 years will be ₹ 6,28,571.

Maharashtra State Board 12th Commerce Maths Solutions Chapter 8 Differential Equation and Applications Miscellaneous Exercise 8

(I) Choose the correct option from the given alternatives:

Question 1.
The order and degree of $\left(\frac{dy}{dx}\right)^3 - \frac{d^3y}{dx^3} + ye^x = 0$ are respectively.
(a) 3, 1
(b) 1, 3
(c) 3, 3
(d) 1, 1
Answer:
(a) 3, 1

Question 2.
The order and degree of $\left[1+\left(\frac{dy}{dx}\right)^3\right]^{\frac{2}{3}} = 8\frac{d^3y}{dx^3}$ are respectively
(a) 3, 1
(c) 3, 3
(b) 1, 3
(d) 1, 1
Answer:
(c) 3, 3

Question 3.
The differential equation of $y = k_1 + \frac{k_2}{x}$ is
(a) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$
(b) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$
(c) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$
(d) $x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$
Answer:
(b) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

Question 4.
The differential equation of $y = k_1 e^x + k_2 e^{-x}$ is
(a) $\frac{d^2y}{dx^2} - y = 0$
(b) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$
(c) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$
(d) $\frac{d^2y}{dx^2} + y = 0$
Answer:
(a) $\frac{d^2y}{dx^2} - y = 0$

Question 5.
The solution of $\frac{dy}{dx} = 1$ is
(a) $x + y = c$
(b) $xy = c$
(c) $x^2 + y^2 = c$
(d) $y - x = c$
Answer:
(d) $y - x = c$

Question 6.
The solution of $\frac{dy}{dx} + \frac{x^2}{y^2} = 0$ is
(a) $x^3 + y^3 = 7$
(b) $x^2 + y^2 = c$
(c) $x^3 + y^3 = c$
(d) $x + y = c$
Answer:
(c) $x^3 + y^3 = c$

Question 7.
The solution of $x\frac{dy}{dx} = y \log y$ is
(a) $y = ae^x$
(b) $y = be^{2x}$
(c) $y = be^{-2x}$
(d) $y = e^{ax}$
Answer:
(d) $y = e^{ax}$

Question 8.
Bacterial increases at a rate proportional to the number present. If the original number M doubles in 3 hours, then the number of bacteria will be 4M in
(a) 4 hours
(b) 6 hours
(c) 8 hours
(d) 10 hours
Answer:
(b) 6 hours

Question 9.
The integrating factor of $\frac{dy}{dx} - y = e^x$ is
(a) x
(b) -x
(c) e^x

(d) e^{-x}
Answer:
(c) e^{x}

Question 10.
The integrating factor of $\frac{dy}{dx}$ – y = e^{x} is e^{-x}, then its solution is
(a) $ye^{-x} = x + c$
(b) $ye^{x} = x + c$
(c) $ye^{x} = 2x + c$
(d) $ye^{-x} = 2x + c$
Answer:
(a) $ye^{-x} = x + c$

(II) Fill in the blanks:

Question 1.
The order of highest derivative occurring in the differential equation is called _______ of the differential equation.
Answer:
order

Question 2.
The power of the highest ordered derivative when all the derivatives are made free from negative and/or fractional indices if any is called _______ of the differential equation.
Answer:
degree

Question 3.
A solution of differential equation that can be obtained from the general solution by giving particular values to the arbitrary constants is called ________ solution.
Answer:
particular

Question 4.
Order and degree of a differential equation are always ________ integers.
Answer:
positive

Question 5.
The integrating factor of the differential equation $\frac{dy}{dx}$ – y = x is ________
Answer:
e^{-x}

Question 6.
The differential equation by eliminating arbitrary constants from bx + ay = ab is ________
Answer:
$\frac{d^2y}{dx^2} = 0$

(III) State whether each of the following is True or False:

Question 1.
The integrating factor of the differential equation $\frac{dy}{dx}$ – y = x is e^{-x}.
Answer:
True

Question 2.
The order and degree of a differential equation are always positive integers.
Answer:
True

Question 3.
The degree of a differential equation is the power of the highest ordered derivative when all the derivatives are made free from negative and/or fractional indices if any.
Answer:
True

Question 4.
The order of highest derivative occurring in the differential equation is called the degree of the differential equation.
Answer:
False

Question 5.
The power of the highest ordered derivative when all the derivatives are made free from negative and/or fractional indices if any is called the order of the differential equation.
Answer:
False

Question 6.
The degree of the differential equation $e^{\frac{dy}{dx}}=\frac{dy}{dx}+c$ is not defined.
Answer:
True

(IV) Solve the following:

Question 1.
Find the order and degree of the following differential equations:
(i) $\left[\frac{d^3y}{dx^3}+x\right]^{3/2}=\frac{d^2y}{dx^2}$
Solution:
The given differential equation is $\left[\frac{d^3y}{dx^3}+x\right]^{3/2}=\frac{d^2y}{dx^2}$
∴ $\left[\frac{d^3y}{dx^3}+x\right]^3=\left(\frac{d^2y}{dx^2}\right)^2$
This D.E. has highest order derivative $\frac{d^3y}{dx^3}$ with power 3
∴ order = 3 and degree = 3

(ii) $x+\frac{dy}{dx}=1+\left(\frac{dy}{dx}\right)^2$
Solution:
The given differential equation is $x+\frac{dy}{dx}=1+\left(\frac{dy}{dx}\right)^2$
This D.E. has highest order derivative $\frac{dy}{dx}$ with power 2.
∴ order = 1, degree = 2.

Question 2.
Verify that y = log x + c is a solution of the differential equation $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$.
Solution:
y = log x + c
Differentiating both sides w.r.t. x, we get
$\frac{dy}{dx}=\frac{1}{x}+0=\frac{1}{x}$
∴ $x\frac{dy}{dx}$ = 1
Differentiating again w.r.t. x, we get
$x\frac{d^2y}{dx^2}+\frac{dy}{dx}\times 1=0$
∴ $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$
This shows that y = log x + c is a solution of the D.E.
$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$

Question 3.
Solve the following differential equations:
(i) $\frac{dy}{dx}$ = 1 + x + y + xy
Solution:
$\frac{dy}{dx}$ = 1 + x + y + xy
∴ $\frac{dy}{dx}$ = (1 + x) + y(1 + x) = (1 + x)(1 + y)
∴ $\frac{1}{1+y}$ dy = (1 + x) dx
Integrating, we get
$\int\frac{1}{1+y}$ dy = ∫(1 + x) dx
∴ log|1 + y| = x + $\frac{x^2}{2}$ + c
This is the general solution.

(ii) $e^{dy/dx} = x$

Solution:

$$e^{dy/dx} = x \quad \therefore \frac{dy}{dx} = \log x$$

$$\therefore dy = \log x\, dx \quad \therefore \int 1\, dy = \int \log x\, dx$$

$$\text{Now } \int \log x\, dx = \int (\log x)(1)\, dx \quad \text{......... (1)}$$

$$= (\log x) \int 1\, dx - \int \left[\frac{d}{dx}(\log x) \cdot \int 1\, dx\right] dx$$

$$= (\log x)(x) - \int \frac{1}{x} \cdot x\, dx = x \log x - \int 1\, dx$$

$$= x \log x - x$$

∴ from (1), the general solution is

y = x log x – x + c, i.e. y = x(log x – 1) + c.

(iii) dr = ar dθ – θ dr

Solution:

dr = ar dθ – θ dr

∴ dr + θ dr = ar dθ

∴ (1 + θ) dr = ar dθ

∴ $\frac{dr}{r} = \frac{a\, d\theta}{1+\theta}$

On integrating, we get

$\int \frac{dr}{r} = a \int \frac{d\theta}{1+\theta}$

∴ log |r| = a log |1 + θ| + c

This is the general solution.

(iv) Find the differential equation of the family of curves y = e^x (ax + bx^2), where a and b are arbitrary constants.

Solution:

y = e^x (ax + bx^2)

ax + bx^2 = ye^{-x} ....(1)

Differentiating (1) w.r.t. x twice and writing $\frac{dy}{dx}$ as y_1 and $\frac{d^2y}{dx^2}$ as y_2, we get

$$a + 2bx = y(-e^{-x}) + e^{-x} y_1$$

$$\therefore a + 2bx = e^{-x}(y_1 - y) \quad \dots (2)$$

$$\text{and } a(0) + 2b \times 1 = e^{-x}(y_2 - y_1) + (y_1 - y)(-e^{-x})$$

$$\therefore a \cdot 0 + 2b = e^{-x}(y_2 - 2y_1 + y) \quad \dots (3)$$

Eliminating a and b from (1), (2), (3), we get

$$\begin{vmatrix} x & x^2 & e^{-x} y \\ 1 & 2x & e^{-x}(y_1 - y) \\ 0 & 2 & e^{-x}(y_2 - 2y_1 + y) \end{vmatrix} = 0$$

$$\therefore e^{-x} \begin{vmatrix} x & x^2 & y \\ 1 & 2x & y_1 - y \\ 0 & 2 & y_2 - 2y_1 + y \end{vmatrix} = 0$$

... [Taking e^{-x} common from C_3]

$$\therefore \begin{vmatrix} x & x^2 & y \\ 1 & 2x & y_1 - y \\ 0 & 2 & y_2 - 2y_1 + y \end{vmatrix} = 0 \quad \dots [\because e^{-x} \neq 0]$$

$$\therefore x[2x(y_2 - 2y_1 + y) - 2(y_1 - y)] - x^2[y_2 - 2y_1 + y - 0] + y[2 - 0] = 0$$

$$\therefore 2x^2y_2 - 4x^2y_1 + 2x^2y - 2xy_1 + 2xy - x^2y_2 + 2x^2y_1 - x^2y + 2y = 0$$

$$\therefore x^2y_2 - 2x^2y_1 + x^2y - 2xy_1 + 2xy + 2y = 0$$

$$\text{i.e. } x^2\frac{d^2y}{dx^2} - 2x^2\frac{dy}{dx} - 2x\frac{dy}{dx} + x^2y + 2xy + 2y = 0$$

This is the required differential equation.

Question 4.

Solve $\frac{dy}{dx}=\frac{x+y+1}{x+y-1}$ when $x = \frac{2}{3}$ and $y = \frac{1}{3}$.

Solution:

$$\frac{dy}{dx}=\frac{x+y+1}{x+y-1} \quad \text{....... (1)}$$

Put $x+y=v \qquad \therefore\ 1+\frac{dy}{dx}=\frac{dv}{dx}$

$\therefore\ \frac{dy}{dx}=\frac{dv}{dx}-1$

$\therefore$ (1) becomes, $\frac{dv}{dx}-1=\frac{v+1}{v-1}$

$\therefore\ \frac{dv}{dx}=\frac{v+1}{v-1}+1=\frac{v+1+v-1}{v-1}$

$\therefore\ \frac{dv}{dx}=\frac{2v}{v-1}$

$\therefore\ \frac{v-1}{v}dv=2dx$

$\int\frac{v-1}{v}dv=2\int dx$

$\therefore\ \int\left(1-\frac{1}{v}\right)dv=2\int dx+c$

$\therefore\ v-\log|v|=2x+c$

$\therefore\ x+y-\log|x+y|=2x+c$

$\therefore\ \log|x+y|=y-x-c$

This is the general solution.

When $x=\frac{2}{3}$ and $y=\frac{1}{3}$, we get

$\log\left|\frac{2}{3}+\frac{1}{3}\right|=\frac{1}{3}-\frac{2}{3}-c$

$\therefore\ \log 1=-\frac{1}{3}-c$

$\therefore\ 0=-\frac{1}{3}-c \qquad \therefore\ c=-\frac{1}{3}$

$\therefore$ the particular solution is

$\log|x+y|=y-x+\frac{1}{3}$.

Question 5.

Solve y dx – x dy = -log x dx.

Solution:

y dx – x dy = -log x dx

$\therefore$ y dx – x dy + log x dx = 0

$\therefore$ x dy = (y + log x) dx

$\therefore\ \frac{dy}{dx}=\frac{y+\log x}{x}=\frac{y}{x}+\frac{\log x}{x}$

$\therefore\ \frac{dy}{dx}-\frac{1}{x}\cdot y=\frac{\log x}{x}$ ....(1)

This is the linear differential equation of the form

$$\frac{dy}{dx} + P \cdot y = Q, \text{ where } P = -\frac{1}{x} \text{ and } Q = \frac{\log x}{x}$$

$$\therefore \text{ I.F.} = e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x}$$

$$= e^{\log(x)^{-1}} = \frac{1}{x}$$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$$

$$\therefore\ y \cdot \frac{1}{x} = \int \frac{\log x}{x} \cdot \frac{1}{x}\,dx + c$$

$$\therefore\ \frac{y}{x} = \int \frac{\log x}{x^2}\,dx + c$$

$$\therefore\ \frac{y}{x} = (\log x) \int x^{-2}\,dx - \int \left[\frac{d}{dx}(\log x) \int x^{-2}\,dx\right] dx + c$$

$$\therefore\ \frac{y}{x} = (\log x) \cdot \frac{x^{-1}}{-1} - \int \frac{1}{x} \cdot \frac{x^{-1}}{-1}\,dx + c$$

$$\therefore\ \frac{y}{x} = -\frac{\log x}{x} + \int x^{-2}\,dx + c$$

$$\therefore\ \frac{y}{x} = -\frac{\log x}{x} + \frac{x^{-1}}{-1} + c$$

$$\therefore\ \frac{y}{x} = -\frac{\log x}{x} - \frac{1}{x} + c$$

$$\therefore\ y = -\log x - 1 + cx$$

$$\therefore\ y = cx - (1 + \log x)$$

This is the general solution.

Question 6.
Solve y log y $\frac{dx}{dy}$ + x – log y = 0.
Solution:

$$y\log y\cdot\frac{dx}{dy}+x-\log y=0$$

$$\therefore\ y\log y\cdot\frac{dx}{dy}=\log y-x$$

$$\therefore\ \frac{dx}{dy}=\frac{\log y-x}{y\log y}$$

$$\therefore\ \frac{dx}{dy}=\frac{1}{y}-\frac{x}{y\log y}$$

$$\therefore\ \frac{dx}{dy}+\frac{x}{y\log y}=\frac{1}{y} \qquad \ldots (1)$$

This is the linear differential equation of the form

$\frac{dx}{dy}+Px=Q$, where $P=\frac{1}{y\log y}$ and $Q=\frac{1}{y}$

$$\therefore\ \text{I.F.}=e^{\int P\,dy}=e^{\int\frac{1}{y\log y}dy}$$

$$=e^{\int\frac{(1/y)}{\log y}dy}=e^{\log|\log y|}=\log y$$

$\therefore$ the solution of (1) is given by

$$x\cdot(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dy+c_1$$

$$\therefore\ x\cdot\log y=\int\frac{1}{y}\cdot\log y\,dy+c_1$$

$$\therefore\ (\log y)\cdot x=\int\frac{\log y}{y}dy+c_1$$

Put $\log y=t \qquad \therefore\ \frac{1}{y}dy=dt$

$$\therefore\ (\log y)\cdot x=\int t\,dt+c_1$$

$$\therefore\ x\log y=\frac{t^2}{2}+c_1$$

$$\therefore\ x\log y=\frac{1}{2}(\log y)^2+c_1$$

$\therefore\ 2x\log y=(\log y)^2+c$, where $c=2c_1$

This is the general solution.

Question 7.
Solve (x + y) dy = a^2 dx
Solution:

$$(x+y)\,dy=a^2dx$$

$$\therefore\ \frac{dy}{dx}=\frac{a^2}{x+y} \qquad \ldots\ldots (1)$$

Put $x+y=v \qquad \therefore\ 1+\frac{dy}{dx}=\frac{dv}{dx}$

$$\therefore\ \frac{dy}{dx}=\frac{dv}{dx}-1$$

$\therefore$ (1) becomes, $\frac{dv}{dx}-1=\frac{a^2}{v}$

$$\therefore\ \frac{dv}{dx}=\frac{a^2}{v}+1=\frac{a^2+v}{v}$$

$$\therefore\ \frac{v}{a^2+v}dv=dx$$

Integrating, we get

$$\int \frac{v}{a^2+v} dv = \int dx$$

$$\therefore \int \frac{(a^2+v)-a^2}{a^2+v} dv = \int dx$$

$$\therefore \int \left(1 - \frac{a^2}{a^2+v}\right) dv = \int dx$$

$$\therefore \int 1\, dv - a^2 \int \frac{1}{a^2+v} dv = \int dx$$

$$\therefore v - a^2 \log|a^2+v| = x + c_1$$

$$\therefore x + y - a^2\log|a^2+x+y| = x + c_1$$

$$\therefore a^2\log|x+y+a^2| = y - c_1$$

$$\therefore \log|x+y+a^2| = \frac{y-c_1}{a^2} = \frac{y}{a^2} - \frac{c_1}{a^2}$$

$$\therefore x+y+a^2 = e^{\left(\frac{y}{a^2} - \frac{c_1}{a^2}\right)} = e^{\frac{y}{a^2}} \cdot e^{\frac{-c_1}{a^2}}$$

$$\therefore x+y+a^2 = c \cdot e^{y/a^2}, \text{ where } c = e^{-c_1/a^2}$$

This is the general solution.

Question 8.
Solve $\frac{dy}{dx}+\frac{2}{x}y=x^2$
Solution:
$\frac{dy}{dx}+\frac{2}{x}y=x^2$ ……..(1)
This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \frac{2}{x},\ \ Q = x^2$$

$$\therefore \text{I.F.} = e^{\int P\,dx} = e^{2\int \frac{1}{x}dx}$$

$$= e^{2\log x} = e^{\log x^2} = x^2$$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q(\text{I.F.})\,dx + c_1$$

$$\therefore yx^2 = \int x^2 \cdot x^2\,dx + c_1$$

$$= \int x^4\,dx + c_1$$

$$\therefore yx^2 = \frac{x^5}{5} + c_1$$

$$\therefore 5x^2y = x^5 + c, \text{ where } c = 5c_1$$

This is the general solution.

Question 9.
The rate of growth of the population is proportional to the number present. If the population doubled in the last 25 years and the present population is 1 lakh, when will the city have a population of 400000?
Solution:
Let P be the population at time t years.
Then the rate of growth of the population is $\frac{dP}{dt}$ which is proportional to P.
$\therefore \frac{dP}{dt} \propto P$
$\therefore \frac{dP}{dt}$ = kP, where k is a constant
$\therefore \frac{dP}{P}$= k dt
On integrating, we get
$\int\frac{dP}{P}=k\int dt$
$\therefore$ log P = kt + c
The population doubled in 25 years and present population is 1,00,000.
$\therefore$ initial population was 50,000
i.e. when t = 0, P = 50000

$\therefore \log 50000 = k \times 0 + c$
$\therefore c = \log 50000$
$\therefore \log P = kt + \log 50000$
When $t = 25$, $P = 100000$
$\therefore \log 100000 = k \times 25 + \log 50000$
$\therefore 25k = \log 100000 - \log 50000 = \log\left(\frac{100000}{50000}\right)$
$\therefore k = \frac{1}{25} \log 2$
$\therefore \log P = \frac{t}{25} \log 2 + \log 50000$
If $P = 400000$, then
$\log 400000 = \frac{t}{25} \log 2 + \log 50000$
$\therefore \log 400000 - \log 50000 = \frac{t}{25} \log 2$
$\therefore \log\left(\frac{400000}{50000}\right) = \log(2)^{t/25}$
$\therefore \log 8 = \log(2)^{t/25}$
$\therefore 8 = (2)^{t/25}$
$\therefore (2)^{t/25} = (2)^3$
$\therefore \frac{t}{25} = 3$
$\therefore t = 75$
$\therefore$ the population will be 400000 in $(75 - 25) = 50$ years.

Question 10.
The resale value of a machine decreases over a 10 years period at a rate that depends on the age of the machine. When the machine is x years old, the rate at which its value is changing is ₹ $2200(x - 10)$ per year. Express the value of the machine as a function of its age and initial value. If the machine was originally worth ₹ 1,20,000 how much will it be worth when it is 10 years old?
Solution:
Let V be the value of the machine after x years.
Then rate of change of the value is $\frac{dV}{dx}$ which is $2200(x - 10)$
$\therefore \frac{dV}{dx} = 2200(x - 10)$
$\therefore dV = 2200(x - 10)\, dx$
On integrating, we get
$\int dV = 2200\int (x - 10)\, dx$
$\therefore V = 2200\left[\frac{x^2}{2} - 10x\right] + c$
Initially, i.e. at $x = 0$, $V = 120000$
$\therefore 120000 = 2200 \times 0 + c = c$
$\therefore c = 120000$
$\therefore V = 2200\left[\frac{x^2}{2} - 10x\right] + 120000$ ....(1)
This gives value of the machine in terms of initial value and age x.
We have to find V when $x = 10$.
When $x = 10$, from (1)
$V = 2200\left[\frac{100}{2} - 100\right] + 120000$
$= 2200 [-50] + 120000$
$= -110000 + 120000$
$= 10000$
Hence, the value of the machine after 10 years will be ₹ 10000.

Question 11.
Solve $y^2\, dx + (xy + x^2)\, dy = 0$
Solution:

$y2\ dx + (xy + x^2)dy = 0$

$\therefore (xy + x^2)\ dy = -y^2\ dx$

$\therefore \frac{dy}{dx} = -\frac{y^2}{xy + x^2}$ ...(i)

Put y = tx ...(ii)

Differentiating w.r.t. x, we get

$\frac{dy}{dx} = t + x\frac{dt}{dx}$ ...(iii)

Substituting (ii) and (iii) in (i), we get

$$\therefore t + x\frac{dt}{dx} = \frac{-t^2x^2}{x.tx + x^2}$$

$$\therefore t + x\frac{dt}{dx} = \frac{-t^2x^2}{x^2(t+1)}$$

$$\therefore x\frac{dt}{dx} = \frac{-t^2}{t+1} - t$$

$$\therefore\ x\frac{dt}{dx} = \frac{-t^2 - t^2 - t}{t+1}$$

$$\therefore\ x\frac{dt}{dx} = \frac{-\left(2t^2 + t\right)}{t+1}$$

$$\therefore\ \frac{t+1}{2t^2+t}dt = -\frac{1}{x}dx$$

Integrating on both sides, we get

$$\int \frac{t+1}{2t^2+t}dt = -\int \frac{1}{x}dx$$

$$\therefore \int \frac{2t+1-t}{t(2t+1)}dt = -\int \frac{1}{x}dx$$

$$\therefore \int \frac{1}{t}dt - \int \frac{1}{2t+1}dt = -\int \frac{1}{x}dx$$

$$\therefore \log|t| - \frac{1}{2}\log|2t+1| = -\log|x| + \log|c|$$

$$\therefore 2\log|t| - \log|2t + 1| = -2\log|x| + 2\log|c|$$

$$\therefore 2\log\left|\frac{y}{x}\right| - \log\left|\frac{2y}{x} + 1\right| = -2\log|x| + 2\log|c|$$

$$\therefore\ 2\log|y| - 2\log|x| - \log|2y + x| + \log|x| = -2\log|x| + 2\log|c|$$

$$\therefore\ \log|y^2| + \log|x| = \log|c^2| + \log|2y + x|$$

$$\therefore\ \log|y^2x| = \log|c^2(x + 2y)|$$

$$\therefore\ xy^2 = c^2(x + 2y)$$

Question 12.
Solve $x^2y\ dx – (x^3 + y^3)\ dy = 0$
Solution:

$x^2y\ dx - (x^3 + y^3)\ dy = 0$

$\therefore\ x^2y\ dx - (x^3 + y^3) = dy$

$$\therefore\ \frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}\quad \text{...(i)}$$

Put y = tx ...(ii)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = t + x\frac{dt}{dx}\quad \text{...(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$t + x\frac{dt}{dx} = \frac{x^2.\,tx}{x^3 + t^3x^3}$$

$$\therefore t + x\frac{dt}{dx} = \frac{x^3.\,t}{x^3(1 + t^3)}$$

$$\therefore x\frac{dt}{dx} = \frac{t}{1 + t^3} - t$$

$$\therefore x\frac{dt}{dx} = \frac{t - t - t^4}{1 + t^3}$$

$$\therefore x\frac{dt}{dx} = \frac{-t^4}{1 + t^3}$$

$$\therefore \frac{1 + t^3}{t^4}dt = -\frac{dx}{x}$$

Integrating on both sides, we get

$$\int \frac{1 + t^3}{t^4}dt = -\int \frac{1}{x}dx$$

$$\therefore \int \left(\frac{1}{t^4} + \frac{1}{t}\right)dt = -\int \frac{1}{x}dx$$

$$\therefore \int t^{-4}dt + \int \frac{1}{t}dt = -\int \frac{1}{x}dx$$

$$\therefore \frac{t^3}{-3} + \log|t| = -\log|x| + c$$

$$\therefore -\frac{1}{3t^3} + \log|t| = -\log|x| + c$$

$$\therefore -\frac{1}{3}.\frac{1}{\left(\frac{y}{x}\right)^3} + \log\left|\frac{y}{x}\right| = -\log|x| + c$$

$$\therefore \frac{x^3}{3y^3} + \log|y| - \log|x| = -\log|x| + c$$

$$\therefore \log|y| - \frac{x^3}{3y^3} = c$$

Question 13.
Solve yx $dydx$ = x2 + 2y2
Solution:

$$xy\,\frac{dy}{dx} = x^2 + 2y^2$$

$$\therefore \frac{dy}{dx} = x^2 + \frac{2y^2}{xy} \quad \text{...(i)}$$

Put y = tx ...(ii)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = t + x\frac{dt}{dx} \quad \text{...(iii)}$$

Substituting (ii) and (iii) in (i), we get

$$t + x\frac{dt}{dx} = \frac{x^2 + 2t^2x^2}{x(tx)}$$

$$\therefore t + x\frac{dt}{dx} = \frac{x^2\left(1 + 2t^2\right)}{x^2t}$$

$$\therefore x\frac{dt}{dx}\ \frac{1 + 2t^2}{t} - t = \frac{1 + t^2}{t}$$

$$\therefore \frac{t}{1 + t^2}dt = \frac{1}{x}dx$$

Integrating on both sides, we get

$$\frac{1}{2}\int \frac{2t}{1 + t^2}\,dt = \int \frac{dx}{x}$$

$$\therefore \frac{1}{2}\log\left|1 + t^2\right| = \log|x| + \log|c|$$

$\therefore$ log $|1 + t^2|$ = 2 log $|x|$ + 2log $|c|$

= log $|x^2|$ + log $|c^2|$

$\therefore$ log $|1 + t^2|$ = log $|c^2 x^2|$

$\therefore 1 + t^2 = c^2x^2$

$$\therefore 1 + \frac{y^2}{x^2} = c^2x^2$$

$\therefore x^2 + y^2 = c^2 x^4$

Question 14.
Solve $(x + 2y^3) \frac{dy}{dx} = y$
Solution:
$(x + 2y^3) \frac{dy}{dx} = y$
$\therefore x + 2y^3 = y \frac{dx}{dy}$

$$\therefore \frac{x}{y} + 2y^2 = \frac{dx}{dy}$$

$$\therefore \frac{dx}{dy} - \frac{x}{y} = 2y^2 \quad \text{........ (1)}$$

This is a linear differential equation of the form

$\frac{dx}{dy} + Px = Q$, where $P = -\frac{1}{y}$, $Q = 2y^2$

$$\therefore \text{I.F.} = e^{\int P\,dy} = e^{\int -\frac{1}{y}dy} = e^{-\int \frac{1}{y}dy}$$

$$= e^{-\log y} = e^{\log\left(\frac{1}{y}\right)} = \frac{1}{y}$$

$\therefore$ the solution of (1) is given by

$$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dy + c$$

$$\therefore \frac{x}{y} = \int 2y^2 \cdot \frac{1}{y}dy + c = 2\int y\,dy + c$$

$$= 2\left(\frac{y^2}{2}\right) + c$$

$$\therefore x = y(y^2 + c)$$

This is the general solution.

Question 15.
Solve y dx – x dy + log x dx = 0
Solution:
y dx – x dy + log x dx = 0
∴ (y + log x) dx = x dy

$$\therefore \frac{y + \log x}{x} = \frac{dy}{dx}$$

$$\therefore \frac{y}{x} + \frac{\log x}{x} = \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} - \frac{1}{x} \cdot y = \frac{\log x}{x} \quad \ldots (1)$$

This is a linear differential equation of the form

$\frac{dy}{dx} + Py = Q$, where $P = -\frac{1}{x}$, $Q = \frac{\log x}{x}$

$$\therefore \text{I.F.} = e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\int \frac{1}{x}dx}$$

$$= e^{-\log x} = e^{\log\left(\frac{1}{x}\right)} = \frac{1}{x}$$

$\therefore$ the solution of (1) is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$$

$$\therefore \frac{y}{x} = \int \frac{\log x}{x} \cdot \frac{1}{x}dx + c$$

$$= \int \log x \cdot x^{-2}\,dx + c$$

$$= (\log x)\int x^{-2}\,dx - \int\left[\left\{\frac{d}{dx}(\log x)\right\}\int x^{-2}\,dx\right]dx$$

$$= (\log x)\left(\frac{x^{-1}}{-1}\right) - \int \frac{1}{x} \cdot \left(\frac{x^{-1}}{-1}\right) dx + c$$

$$= -\frac{\log x}{x} + \int x^{-2}\, dx + c$$

$$\therefore \frac{y}{x} = -\frac{\log x}{x} - \frac{1}{x} + c$$

$$\therefore y = cx - (1 + \log x)$$

This is the general solution.

Question 16.
Solve $\frac{dy}{dx}$ = log x dx
Solution:
$\frac{dy}{dx}$ = log x dx
∴ dy = log x dx
On integrating, we get
∫dy = ∫log x . 1 dx
∴ y = (log x) ∫1 dx − $\int\left[\left\{\frac{d}{dx}(\log x)\right\} \cdot \int 1\,dx\right]dx$
∴ y = (log x) . x − $\int \frac{1}{x} \cdot x\,dx$
∴ y = x log x − ∫1 dx
∴ y = x log x − x + c
This is the general solution.

Question 17.
y log y $\frac{dx}{dy}$ = log y − x
Solution:

$$y \log y \frac{dx}{dy} = \log y - x$$

$$y \log y \frac{dx}{dy} + x = \log y$$

$$\therefore \frac{dx}{dy} + \frac{1}{y \log y} x = \frac{1}{y}$$

The given equation is of the form $\frac{dx}{dy} + px = Q$

where, $P = \frac{1}{y \log y}$ and $Q = \frac{1}{y}$

$$\therefore I.F. = e^{\int pdy} = e^{\int \frac{1}{y \log y} dy} = e^{\log |\log y|} = \log y$$

$\therefore$ Solution of the given equation is

$$x(I.F.) = \int Q(I.F.) dy + c_1$$

$$\therefore x.\log y = 1y \int \log y \, dy + c_1$$

In R. H. S., put log y = t

Differentiating w.r.t. x, we get

$$\frac{1}{y} dy = dt$$

$$\therefore x \log y = tdt \int + c_1 = \frac{t^2}{2} + c_1$$

$$\therefore x \log y = \frac{(\log y)^2}{2} + c_1$$

$\therefore$ 2x log y = $(\log y)^2$ + c ...[$2c_1 = c$]

x log y = $\frac{1}{2}$ $(\log y)^2$ + c