Maharashtra State Board 11th Maths Solutions Chapter 7 Conic Sections Ex 7.1

Question 1.
Find co-ordinates of focus, equation of directrix, length of latus rectum and the co-ordinates of end points of latus rectum of the parabola:
(i) $5y^2 = 24x$
(ii) $y^2 = -20x$
(iii) $3x^2 = 8y$
(iv) $x^2 = -8y$
(v) $3y^2 = -16x$
Solution:
(i) Given equation of the parabola is $5y^2 = 24x$.
⇒ $y^2 = 245x$
Comparing this equation with $y^2 = 4ax$, we get
⇒ $4a = 245$
⇒ $a = 65$
Co-ordinates of focus are S(a, 0), i.e., S(65, 0)
Equation of the directrix is x + a = 0.
⇒ $x + 65 = 0$
⇒ $5x + 6 = 0$
Length of latus rectum = 4a
= 4(65)
= 245
Co-ordinates of end points of latus rectum are (a, 2a) and (a, -2a),
⇒ $(65,125)$ and $(65,-125)$

(ii) Given equation of the parabola is $y^2 = -20x$.
Comparing this equation with $y^2 = -4ax$, we get
⇒ 4a = 20
⇒ a = 5
Co-ordinates of focus are S(-a, 0), i.e., S(-5, 0)
Equation of the directrix is x – a = 0
⇒ x – 5 = 0
Length of latus rectum = 4a = 4(5) = 20
Co-ordinates of end points of latus rectum are (-a, 2a) and (-a, -2a),
⇒ (-5, 10) and (-5, -10).

(iii) Given equation of the parabola is $3x^2 = 8y$
⇒ $x^2 = 83 y$
Comparing this equation with $x^2 = 4by$, we get
⇒ $4b = 83$
⇒ $b = 23$
Co-ordinates of focus are S(0, b), i.e., S(0, 23)
Equation of the directrix is y + b = 0,
⇒ $y + 23 = 0$
⇒ $3y + 2 = 0$
Length of latus rectum = 4b = 4(23) = 83
Co-ordinates of end points of latus rectum are (2b, b) and (-2b, b),
⇒ $(43,23)$ and $(-43,23)$.

(iv) Given equation of the parabola is $x^2 = -8y$.
Comparing this equation with $x^2 = -4by$, we get
⇒ 4b = 8
⇒ b = 2
Co-ordinates of focus are S(0, -b), i.e., S(0, – 2)
Equation of the directrix is y – b = 0, i.e., y – 2 = 0
Length of latus rectum = 4b = 4(2) = 8
∴ Co-ordinates of end points of latus rectum are (2b, -b) and (-2b, -b), i.e., (4, -2) and (-4, -2).

(v) Given equation of the parabola is $3y^2 = -16x$.
⇒ $y^2 = -163x$
Comparing this equation withy = -4ax, we get
⇒ $4a = 163$

⇒ $a = \frac{4}{3}$

Co-ordinates of focus are S(-a, 0), i.e., $(-\frac{4}{3}, 0)$

Equation of the directrix is x – a = 0,

⇒ $x - -\frac{4}{3} = 0$

⇒ 3x – 4 = 0

Length of latus rectum = 4a = $4(\frac{4}{3}) = \frac{16}{3}$

Co-ordinates of end points of latus rectum are (-a, 2a) and (-a, -2a),

i.e., $\left(-\frac{4}{3}, \frac{8}{3}\right)$ and $\left(-\frac{4}{3}, -\frac{8}{3}\right)$

Question 2.
Find the equation of the parabola with vertex at the origin, the axis along the Y-axis, and passing through the point (-10, -5).
Solution:
Vertex of the parabola is at origin (0, 0) and its axis is along Y-axis.
Equation of the parabola can be either $x^2 = 4by$ or $x^2 = -4by$
Since the parabola passes through (-10, -5), it lies in 3rd quadrant.
Required parabola is $x^2 = -4by$.
Substituting x = -10 and y = -5 in $x^2 = -4by$, we get
⇒ $(-10)^2 = -4b(-5)$
⇒ $b = \frac{100}{20} = 5$
∴ The required equation of the parabola is $x^2 = -4(5)y$, i.e., $x^2 = -20y$.

Question 3.
Find the equation of the parabola with vertex at the origin, the axis along the X-axis, and passing through the point (3, 4).
Solution:
Vertex of the parabola is at the origin (0, 0) and its axis is along X-axis.
Equation of the parabola can be either $y^2 = 4ax$ or $y^2 = -4ax$.
Since the parabola passes through (3, 4), it lies in the 1st quadrant.
Required parabola is $y^2 = 4ax$.
Substituting x = 3 and y = 4 in $y^2 = 4ax$, we get
⇒ $(4)^2 = 4a(3)$
⇒ $a = \frac{16}{12} = \frac{4}{3}$
The required equation of the parabola is
$y^2 = 4(\frac{4}{3})x$
⇒ $3y^2 = 16x$

Question 4.
Find the equation of the parabola whose vertex is O(0, 0) and focus at (-7, 0).
Solution:
Focus of the parabola is S(-7, 0) and vertex is O(0, 0).
Since focus lies on X-axis, it is the axis of the parabola.
Focus S(-7, 0) lies on the left-hand side of the origin.
It is a left-handed parabola.
Required parabola is y = -4ax.
Focus is S(-a, 0).
a = 7
∴ The required equation of the parabola is $y^2 = -4(7)x$, i.e., $y^2 = -28x$.

Question 5.
Find the equation of the parabola with vertex at the origin, the axis along X-axis, and passing through the point
(i) (1, -6)
(ii) (2, 3)
Solution:
(i) Vertex of the parabola is at origin (0, 0) and its axis is along X-axis.
Equation of the parabola can be either $y^2 = 4ax$ or $y^2 = -4ax$.
Since the parabola passes through (1, -6), it lies in the 4th quadrant.
Required parabola is $y^2 = 4ax$.
Substituting x = 1 and y = -6 in $y^2 = 4ax$, we get
⇒ $(-6)^2 = 4a(1)$
⇒ 36 = 4a
⇒ a = 9
∴ The required equation of the parabola is $y^2 = 4(9)x$, i.e., $y^2 = 36x$.

(ii) Vertex of the parabola is at origin (0, 0) and its axis is along X-axis.
Equation of the parabola can be either $y^2 = 4ax$ or $y^2 = -4ax$.

Since the parabola passes through (2, 3), it lies in 1st quadrant.
∴ Required parabola is $y^2 = 4ax$.
Substituting x = 2 and y = 3 in $y^2 = 4ax$, we get
⇒ $(3)^2 = 4a(2)$
⇒ 9 = 8a
⇒ a = $\frac{9}{8}$
The required equation of the parabola is
$y^2 = 4(\frac{9}{8})x$
⇒ $y^2 = \frac{9}{2} x$
⇒ $2y^2 = 9x$.

Question 6.
For the parabola $3y^2 = 16x$, find the parameter of the point:
(i) (3, -4)
(ii) (27, -12)
Solution:
Given the equation of the parabola is $3y^2 = 16x$.
⇒ $y^2 = \frac{16}{3}x$
Comparing this equation with $y^2 = 4ax$, we get
⇒ $4a = \frac{16}{3}$
⇒ $a = \frac{4}{3}$
If t is the parameter of the point P on the parabola, then
$P(t) = (at^2, 2at)$
i.e., $x = at^2$ and y = 2at ....(i)
(i) Given point is (3, -4)
Substituting x = 3, y = -4 and $a = \frac{4}{3}$ in (i), we get
$3 = \frac{4}{3} t^2$ and $-4 = 2(\frac{4}{3}) t$

$$t^2 = \frac{9}{4} \text{ and } t = \frac{-3}{2}$$
$$t = \pm\frac{3}{2} \text{ and } t = \frac{-3}{2}$$
$$t = -\frac{3}{2}$$

∴ The parameter of the given point is $\frac{-3}{2}$

(ii) Given point is (27, -12)
Substituting x = 27, y = -12 and $a = \frac{4}{3}$ in (i), we get

$$27 = \frac{4}{3}t^2 \text{ and } -12 = 2\left(\frac{4}{3}\right)t$$
$$t^2 = \frac{81}{4} \text{ and } t = \frac{-9}{2}$$
$$t = \pm\frac{9}{2} \text{ and } t = \frac{-9}{2}$$
$$t = \frac{-9}{2}$$

∴ The parameter of the given point is $\frac{-9}{2}$

Question 7.
Find the focal distance of a point on the parabola $y^2 = 16x$ whose ordinate is 2 times the abscissa.
Solution:
Given the equation of the parabola is $y^2 = 16x$.
Comparing this equation with $y^2 = 4ax$, we get
⇒ 4a = 16
⇒ a = 4
Since ordinate is 2 times the abscissa,
y = 2x
Substituting y = 2x in $y^2 = 16x$, we get
⇒ $(2x)^2 = 16x$
⇒ $4x^2 = 16x$
⇒ $4x^2 – 16x = 0$
⇒ 4x(x – 4) = 0
⇒ x = 0 or x = 4
When x = 4,
focal distance = x + a = 4 + 4 = 8

When $x = 0$,
focal distance = $a = 4$
∴ Focal distance is 4 or 8.

Question 8.
Find coordinates of the point on the parabola. Also, find focal distance.
(i) $y^2 = 12x$ whose parameter is $\frac{1}{3}$
(ii) $2y^2 = 7x$ whose parameter is -2
Solution:
(i) Given equation of the parabola is $y^2 = 12x$.
Comparing this equation with $y^2 = 4ax$, we get
⇒ $4a = 12$
⇒ $a = 3$
If t is the parameter of the point P on the parabola, then
$P(t) = (at^2, 2at)$
i.e., $x = at^2$ and $y = 2at$ ……..(i)
Given, $t = \frac{1}{3}$
Substituting $a = 3$ and $t = \frac{1}{3}$ in (i), we get
$x = 3(\frac{1}{3})^2$ and $y = 2(3)(\frac{1}{3})$
$x = \frac{1}{3}$ and $y = 2$
The co-ordinates of the point on the parabola are $(\frac{1}{3}, 2)$
∴ Focal distance = $x + a$
$= \frac{1}{3} + 3$
$= \frac{10}{3}$

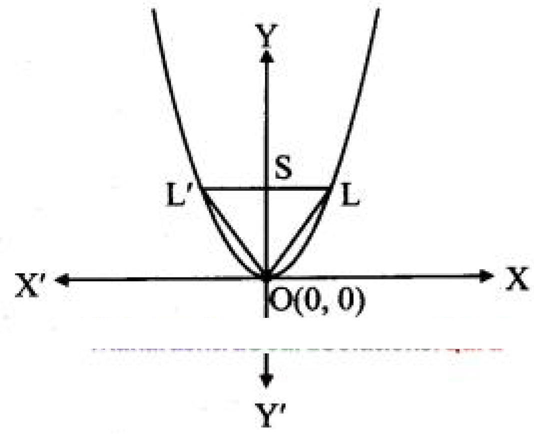
(ii) Given equation of the parabola is $2y^2 = 7x$.
⇒ $y^2 = \frac{7}{2}x$
Comparing this equation with $y^2 = 4ax$, we get
⇒ $4a = \frac{7}{2}$
⇒ $a = \frac{7}{8}$
If t is the parameter of the point P on the parabola, then
$P(t) = (at^2, 2at)$
i.e., $x = at^2$ and $y = 2at$ …..(i)
Given, $t = -2$
Substituting $a = \frac{7}{8}$ and $t = -2$ in (i), we get
$x = \frac{7}{8}(-2)^2$ and $y = 2(\frac{7}{8})(-2)$
$x = \frac{7}{2}$ and $y = \frac{-7}{2}$
The co-ordinates of the point on the parabola are $(\frac{7}{2}, \frac{-7}{2})$
∴ Focal distance = $x + a$
$= \frac{7}{2} + \frac{7}{8}$
$= \frac{35}{8}$

Question 9.
For the parabola $y^2 = 4x$, find the coordinates of the point whose focal distance is 17.
Solution:
Given the equation of the parabola is $y^2 = 4x$.
Comparing this equation with $y^2 = 4ax$, we get
⇒ $4a = 4$
⇒ $a = 1$
Focal distance of a point = $x + a$
Given, focal distance = 17
⇒ $x + 1 = 17$
⇒ $x = 16$
Substituting $x = 16$ in $y^2 = 4x$, we get
⇒ $y^2 = 4(16)$
⇒ $y^2 = 64$
⇒ $y = \pm 8$
∴ The co-ordinates of the point on the parabola are (16, 8) or (16, -8).

Question 10.
Find the length of the latus rectum of the parabola $y^2 = 4ax$ passing through the point (2, -6).
Solution:
Given equation of the parabola is $y^2 = 4ax$ and it passes through point (2, -6).
Substituting $x = 2$ and $y = -6$ in $y^2 = 4ax$, we get
⇒ $(-6)^2 = 4a(2)$

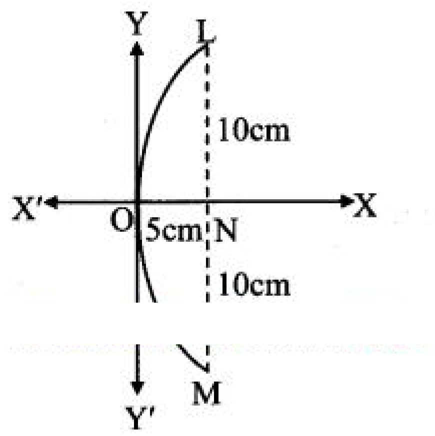
⇒ 4a = 18
∴ Length of latus rectum = 4a = 18 units

Question 11.
Find the area of the triangle formed by the line joining the vertex of the parabola $x^2 = 12y$ to the endpoints of the latus rectum.
Solution:



Given the equation of the parabola is $x^2 = 12y$.
Comparing this equation with $x^2 = 4by$, we get
⇒ 4b = 12
⇒ b = 3
The co-ordinates of focus are S(0, b), i.e., S(0, 3)
End points of the latus-rectum are L(2b, b) and L'(-2b, b),
i.e., L(6, 3) and L'(-6, 3)
Also l(LL′) = length of latus-rectum = 4b = 12
l(OS) = b = 3
Area of ΔOLL′ = $\frac{1}{2}$ × l(LL′) × l(OS)
= $\frac{1}{2}$ × 12 × 3
Area of ΔOLL′ = 18 sq. units

Question 12.
If a parabolic reflector is 20 cm in diameter and 5 cm deep, find its focus.
Solution:



Let LOM be the parabolic reflector such that LM is the diameter and ON is its depth.
It is given that ON = 5 cm and LM = 20 cm.
LN = 10 cm
Taking O as the origin, ON along X-axis and a line through O ⊥ ON as Y-axis.
Let the equation of the reflector be $y^2 = 4ax$ ……(i)
The point L has the co-ordinates (5, 10) and lies on parabola given by (i).
Substituting x = 5 and y = 10 in (i), we get
⇒ $10^2 = 4a(5)$
⇒ 100 = 20a
⇒ a = 5
Focus is at (a, 0), i.e., (5, 0)

Question 13.
Find co-ordinates of focus, vertex, and equation of directrix and the axis of the parabola $y = x^2 – 2x + 3$.
Solution:
Given equation of the parabola is $y = x^2 – 2x + 3$
⇒ $y = x^2 – 2x + 1 + 2$
⇒ $y – 2 = (x – 1)^2$
⇒ $(x – 1)^2 = y – 2$
Comparing this equation with $X^2 = 4bY$, we get
X = x – 1, Y = y – 2
⇒ 4b = 1
⇒ b = $\frac{1}{4}$

The co-ordinates of vertex are (X = 0, Y = 0)
⇒ x – 1 = 0 and y – 2 = 0
⇒ x = 1 and y = 2
The co-ordinates of vertex are (1, 2).
The co-ordinates of focus are S(X = 0, Y = b)
⇒ x – 1 = 0 and y – 2 = 14
⇒ x = 1 and y = 94
The co-ordinates of focus are (1, 94)
Equation of the axis is X = 0
x – 1 = 0, i.e., x = 1
Equation of directrix is Y + b = 0
⇒ y – 2 + 14 = 0
⇒ y – 74 = 0
⇒ 4y – 7 = 0

Question 14.
Find the equation of tangent to the parabola
(i) y^2 = 12x from the point (2, 5)
(ii) y^2 = 36x from the point (2, 9)
Solution:
(i) Given equation of the parabola is y^2 = 12x.
Comparing this equation with y^2 = 4ax, we get
⇒ 4a = 12
⇒ a = 3
Equation of tangent to the parabola y^2 = 4ax having slope m is
y = mx + am
Since the tangent passes through the point (2, 5)
⇒ 5 = 2m + 3m
⇒ 5m = $2m^2$ + 3
⇒ $2m^2$ – 5m + 3 = 0
⇒ $2m^2$ – 2m – 3m + 3 = 0
⇒ 2m(m – 1) – 3(m – 1) = 0
⇒ (m- 1)(2m – 3) = 0
⇒ m = 1 or m = 32
These are the slopes of the required tangents.
By slope point form, y – y_1 = m(x – x_1), the equations of the tangents are
⇒ y – 5 = 1(x – 2) and y – 5 = 32 (x – 2)
⇒ y – 5 = x – 2 and 2y – 10 = 3x – 6
⇒ x – y + 3 = 0 and 3x – 2y + 4 = 0

(ii) Given equation of the parabola is y^2 = 36x.
Comparing this equation with y^2 = 4ax, we get
⇒ 4a = 36
⇒ a = 9
Equation of tangent to the parabola y^2 = 4ax having slope m is
y = mx + am
Since the tangent passes through the point (2, 9),
⇒ 9 = 2m + 9m
⇒ 9m = $2m^2$ + 9
⇒ $2m^2$ – 9m + 9 = 0
⇒ $2m^2$ – 6m – 3m + 9 = 0
⇒ 2m(m – 3) – 3(m – 3) = 0
⇒ (m – 3)(2m – 3) = 0
⇒ m = 3 or m = 32
These are the slopes of the required tangents.
By slope point form, y – y_1 = m(x – x_1), the equations of the tangents are
⇒ y – 9 = 3(x – 2) and y – 9 = 32 (x – 2)
⇒ y – 9 = 3x – 6 and 2y – 18 = 3x – 6
⇒ 3x – y + 3 = 0 and 3x – 2y + 12 = 0

Question 15.
If the tangents drawn from the point (-6, 9) to the parabola y^2 = kx are perpendicular to each other, find k.
Solution:
Given equation of the parabola is y^2 = kx
Comparing this equation with y^2 = 4ax, we get
⇒ 4a = k

⇒ a = $k4$

Equation of tangent to the parabola $y^2 = 4ax$ having slope m is

y = mx + am

Since the tangent passes through the point (-6, 9),

⇒ 9 = -6m + $k4m$

⇒ 36m = -24m2 + k

⇒ $24m^2 + 36m - k = 0$

The roots m_1 and m_2 of this quadratic equation are the slopes of the tangents.

m_1m_2 = $-k24$

Since the tangents are perpendicular to each other,

$m_1m_2 = -1$

⇒ $-k24$ = -1

⇒ k = 24

Alternate method:

We know that, tangents drawn from a point on directrix are perpendicular.

(-6, 9) lies on the directrix x = -a.

⇒ -6 = -a

⇒ a = 6

Since 4a = k

⇒ k = 4(6) = 24

Question 16.

Two tangents to the parabola $y^2 = 8x$ meet the tangents at the vertex in the points P and Q. If PQ = 4, prove that the equation of the locus of the point of intersection of two tangents is $y^2 = 8(x + 2)$.

Solution:

Given equation of the parabola is $y^2 = 8x$

Comparing this equation with $y^2 = 4ax$, we get

⇒ 4a = 8

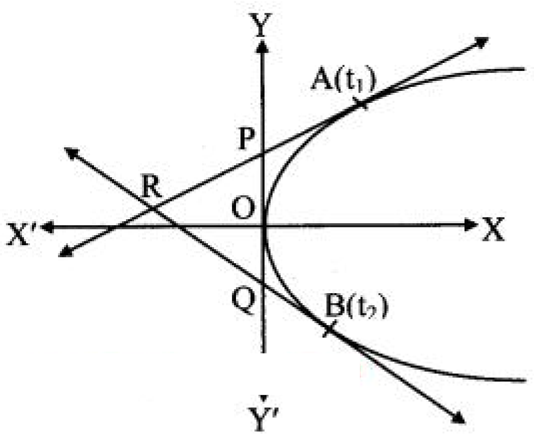
⇒ a = 2

Equation of tangent to given parabola at A(t_1) is y

t_1 = x + 2$t21$ ....(i)

Equation of tangent to given parabola at B(t_2) is y

t_2 = x + 2$t22$ .....(ii)



A tangent at the vertex is Y-axis whose equation is x = 0.

x-coordinate of points P and Q is 0.

Let P be(0, k_1) and Q be (0, k_2).

Then, from (i) and (ii), we get

$k_1t_1 = 0 + 2t_1^2$ and $k_2t_2 = 0 + 2t_2^2$
$k_1 = 2t_1$ and $k_2 = 2t_2$
P is $(0, 2t_1)$ and Q is $(0, 2t_2)$
$PQ = 2|t_1 - t_2|$
But PQ is given to be 4.
$2|t_1 - t_2| = 4$
$(t_1 - t_2)^2 = 4$ …(iii)
Let $R(x_1, y_1)$ be point of intersection of (i) and (ii).
$y_1t_1 = x_1 + 2t_1^2$ …(iv)
and $y_1t_2 = x_1 + 2t_2^2$
Subtracting, we get
$y_1t_1 - y_1t_2 = 2t_1^2 - 2t_2^2$
$y_1(t_1 - t_2) = 2(t_1 + t_2)(t_1 - t_2)$
$y_1 = 2(t_1 + t_2)$ …(v) $[\because t_1 \neq t_2]$
$y_1t_1 = 2(t_1 + t_2)t_1$
$x_1 + 2t_1^2 = 2t_1^2 + 2t_1t_2$ …[From (iv)]
$x_1 = 2t_1t_2$ …(vi)
To find the equation of locus of $R(x_1, y_1)$, eliminate t_1 and t_2 from the equations (iii), (v) and (vi).
Squaring (v), we get
$y_1^2 = 4(t_1 + t_2)^2$
$= 4[(t_1 - t_2)^2 + 4t_1t_2]$
$= 4[4 + 2x_1]$ …[From (iii) and (vi)]
$y_1^2 = 8(x_1 + 2)$
∴ Equation of locus of R is $y^2 = 8(x + 2)$.

Question 17.
Find the equation of common tangent to the parabolas $y^2 = 4x$ and $x^2 = 32y$.
Solution:
Given equation of the parabola is $y^2 = 4x$
Comparing this equation with $y^2 = 4ax$, we get
⇒ $4a = 4$
⇒ $a = 1$
Let the equation of common tangent be
$y = mx + \frac{1}{m}$ …..(i)
Substituting $y = mx + \frac{1}{m}$ in $x^2 = 32y$, we get
⇒ $x^2 = 32(mx + \frac{1}{m}) = 32mx + \frac{32}{m}$
⇒ $mx^2 = 32m^2x + 32$
⇒ $mx^2 - 32m^2x - 32 = 0$ ……..(ii)
Line (i) touches the parabola $x^2 = 32y$.
The quadratic equation (ii) in x has equal roots.
Discriminant = 0
⇒ $(-32m^2)^2 - 4(m)(-32) = 0$
⇒ $1024m^4 + 128m = 0$
⇒ $128m(8m^3 + 1) = 0$
⇒ $8m^3 + 1 = 0$ …..$[\because m \neq 0]$
⇒ $m^3 = -\frac{1}{8}$
⇒ $m = -\frac{1}{2}$
Substituting $m = -\frac{1}{2}$ in (i), we get
⇒ $y = -\frac{1}{2}x + \frac{1}{\left(-\frac{1}{2}\right)}$
⇒ $y = -\frac{1}{2}x - 2$
⇒ $x + 2y + 4 = 0$, which is the equation of the common tangent.

Question 18.
Find the equation of the locus of a point, the tangents from which to the parabola $y^2 = 18x$ are such that sum of their slopes is -3.
Solution:
Given equation of the parabola is $y^2 = 18x$
Comparing this equation with $y^2 = 4ax$, we get
⇒ $4a = 18$
⇒ $a = \frac{9}{2}$
Equation of tangent to the parabola $y^2 = 4ax$ having slope m is
⇒ $y = mx + \frac{a}{m}$
⇒ $y = mx + \frac{9}{2m}$

⇒ $2ym = 2xm^2 + 9$
⇒ $2xm^2 – 2ym + 9 = 0$
The roots m_1 and m_2 of this quadratic equation are the slopes of the tangents.
$m_1 + m_2 = -\frac{(-2y)}{2x} = \frac{y}{x}$
But, $m_1 + m_2 = -3$
$\frac{y}{x} = -3$
y = -3x, which is the required equation of locus.

Question 19.
The towers of a bridge, hung in the form of a parabola, have their tops 30 metres above the roadway and are 200 metres apart. If the cable is 5 metres above the roadway at the centre of the bridge, find the length of the vertical supporting cable 30 metres from the centre.
Solution:
Let CAB be the cable of the bridge and X'OX be the roadway.
Let A be the centre of the bridge.
From the figure, vertex of parabola is at A(0, 5).
Let the equation of parabola be
$x^2 = 4b(y – 5)$ .....(i)
Since the parabola passes through (100, 30).
Substituting x = 100 and y = 30 in (i), we get
⇒ $100^2 = 4b (30 – 5)$
⇒ $100^2 = 4b(25)$
⇒ $100^2 = 100b$
⇒ b = 100
Substituting the value of b in (i), we get
$x^2 = 400(y – 5)$ .....(ii)
Let l metres be the length of vertical supporting cable.
Then P(30, l) lies on (ii).
⇒ $30^2 = 400(l – 5)$
⇒ 900 = 400(l – 5)
⇒ $\frac{9}{4} = l – 5$
⇒ $l = \frac{9}{4} + 5$
⇒ $l = \frac{29}{4}$ m = 7.25 m
The length of the vertical supporting cable is 7.25 m.

Question 20.
A circle whose centre is (4, -1) passes through the focus of the parabola $x^2 + 16y = 0$. Show that the circle touches the directrix of the parabola.
Solution:
Given equation of the parabola is $x^2 + 16y = 0$.
⇒ $x^2 = -16y$
Comparing this equation with $x^2 = -4by$, we get
⇒ 4b = 16
⇒ b = 4
Focus = S(0, -b) = (0, -4)
Centre of the circle is C(4, -1) and it passes through focus S of the parabola.
Radius = CS
$= \sqrt{(4-0)^2+(-1+4)^2}$
$= \sqrt{16+9}$
= 5
Equation of the directrix is y – b = 0, i.e.,y – 4 = 0
Length of the perpendicular from centre C(4, -1) to the directrix
$= \left|\frac{0(4)+1(-1)-4}{\sqrt{(0)^2+(1)^2}}\right|$
$= \left|\frac{-1-4}{1}\right|$
= 5
= radius
∴ The circle touches the directrix of the parabola.

Maharashtra State Board 11th Maths Solutions Chapter 7 Conic Sections Ex 7.2

Question 1.
Find the
(i) lengths of the principal axes
(ii) co-ordinates of the foci
(iii) equations of directrices
(iv) length of the latus rectum
(v) distance between foci
(vi) distance between directrices of the ellipse:
(a) $\frac{x^2}{25}+\frac{y^2}{9}=1$
(b) $3x^2 + 4y^2 = 12$
(c) $2x^2 + 6y^2 = 6$
(d) $3x^2 + 4y^2 = 1$
Solution:
(a) Given equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$
Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get
$a^2 = 25$ and $b^2 = 9$
a = 5 and b = 3
Since a > b,
X-axis is the major axis and Y-axis is the minor axis.

(i) Length of major axis = 2a = 2(5) = 10
Length of minor axis = 2b = 2(3) = 6
Lengths of the principal axes are 10 and 6.

(ii) We know that e = $\frac{\sqrt{a^2-b^2}}{a}$
= $\frac{\sqrt{25-9}}{5}$
= $\frac{\sqrt{16}}{5}$
= $\frac{4}{5}$
Co-ordinates of the foci are S(ae, 0) and S'(-ae, 0),
i.e., S(5($\frac{4}{5}$), 0) and S'(-5($\frac{4}{5}$), 0)
i.e., S(4, 0) and S'(-4, 0)

(iii) Equations of the directrices are x = $\pm\frac{a}{e}$
= $\pm\frac{5}{\frac{4}{5}}$
= $\pm\frac{25}{4}$

(iv) Length of latus rectum = $\frac{2b^2}{a}=\frac{2(3)^2}{5}=\frac{18}{5}$

(v) Distance between foci = 2ae
= 2(5)($\frac{4}{5}$)
= 8

(vi) Distance between directrices = $\frac{2a}{e}$
= $\frac{2(5)}{\frac{4}{5}}$
= $\frac{25}{2}$

(b) Given equation of the ellipse is $3x^2 + 4y^2 = 12$
$\frac{x^2}{4}+\frac{y^2}{3}=1$
Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get
$a^2 = 4$ and $b^2 = 3$
a = 2 and b = √3
Since a > b,
X-axis is the major axis and Y-axis is the minor axis.

(i) Length of major axis = 2a = 2(2) = 4
Length of minor axis = 2b = 2√3
Lengths of the principal axes are 4 and 2√3.

(ii) We know that e = $a_2-b_2\sqrt{}a$
= 4−3√2
= 12
Co-ordinates of the foci are S(ae, 0) and S'(-ae, 0),
i.e., S(2(12), 0) and S'(-2(12), 0)
i.e., S(1, 0) and S'(-1, 0)

(iii) Equations of the directrices are x = ±ae
= $\pm 2_{12}$
= ±4

(iv) Length of latus rectum = $2b_2a=2(3\sqrt{})_22=3$

(v) Distance between foci = 2ae = 2(2)(12) = 2

(vi) Distance between directrices = 2ae
= $2(2)_{12}$
= 8

(c) Given equation of the ellipse is $2x_2 + 6y_2 = 6$
$x_23+y_21=1$
Comparing this equation with $x_2a_2+y_2\ b_2=1$, we get
a_2 = 3 and b_2 = 1
a = √3 and b = 1
Since a > b,
X-axis is the major axis and Y-axis is the minor axis.

(i) Length of major axis = 2a = 2√3
Length of minor axis = 2b = 2(1) = 2
Lengths of the principal axes are 2√3 and 2.

(ii) We know that e = $a_2-b_2\sqrt{}a$
= 3−1√3√
= 2√3√
Co-ordinates of the foci are S(ae, 0) and S'(-ae, 0),
i.e., S(√3(2√3√), o) and S'(-√3(2√3√), 0)
i.e., S(√2, 0) and S'(-√2, 0)

(iii) Equations of the directrices are x = ±ae,
= $\pm 3\sqrt{}_{2\sqrt{}3\sqrt{}}$
= ±32√

(iv) Length of latus rectum = $2b_2a=2(1)_23\sqrt{}=23\sqrt{}$

(v) Distance between foci = 2ae
= 2(3−√)(2√3√)
= 2√2

(vi) Distance between directrices = 2ae
= $23\sqrt{}_{2\sqrt{}3\sqrt{}}$
= 2×32√
= 3√2

(d) Given equation of the ellipse is $3x_2 + 4y = 1$.
$x_{2}13+y_{2}14=1$
Comparing this equation with $x_2a_2+y_2\ b_2=1$, we get
a_2 = 13 and b_2 = 14
a = 13√ and b = 12

Since a > b,
X-axis is the major axis and Y-axis is the minor axis.

(i) Length of major axis = 2a = 2(13√) = 23√
Length of minor axis = 2b = 2(12) = 1
Lengths of the principal axes are 23√ and 1.

(ii) We know that e = a2−b2√a

e = 13−14√13√=112√13√=312−−√=14−−√=12

Co-ordinates of the foci are S(ae, 0) and S'(-ae, 0),
i.e., S(13√(12),0) and S′(−13√(12),0)
i.e., S(123√, 0) and S'(-123√, 0)

(iii) Equations of the directrices are x = ±ae,
= ±13√12
= ±23√

(iv) Length of latus rectum = 2b2a
= 2(12)213√
= 3√2

(v) Distance between foci = 2ae
= 2(13√)(12)
= 13√

(vi) Distance between directrices = 2ae
= 2(13√)12
= 43√

Question 2.
Find the equation of the ellipse in standard form if
(i) eccentricity = 38 and distance between its foci = 6.
(ii) the length of the major axis is 10 and the distance between foci is 8.
(iii) distance between directrices is 18 and eccentricity is 13.
(iv) minor axis is 16 and eccentricity is 13.
(v) the distance between foci is 6 and the distance between directrices is 503.
(vi) the latus rectum has length 6 and foci are (±2, 0).
(vii) passing through the points (-3, 1) and (2, -2).
(viii) the distance between its directrices is 10 and which passes through (-√5, 2).
(ix) eccentricity is 23 and passes through (2, −53).
Solution:
(i) Let the required equation of ellipse be x2a2+y2b2=1, where a > b.
Given, eccentricity (e) = 38
Distance between foci = 2ae
Given, distance between foci = 6
2ae = 6

$$2a\left(\frac{3}{8}\right) = 6$$

$$\frac{3a}{4} = 6$$

$$a = \frac{6 \times 4}{3} = 8$$

$$a^2 = 64$$

$$\text{Now, } b^2 = a^2(1 - e^2)$$

$$= 64\left[1 - \left(\frac{3}{8}\right)^2\right]$$

$$= 64\left(1 - \frac{9}{64}\right) = 64\left(\frac{55}{64}\right) = 55$$

The required equation of ellipse is x264+y255=1.

(ii) Let the required equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > b.
Length of major axis = 2a
Given, length of major axis = 10
2a = 10
a = 5
$a^2 = 25$
Distance between foci = 2ae
Given, distance between foci = 8
2ae = 8
2(5)e = 8

$$e = \frac{8}{10} = \frac{4}{5}$$

$$\text{Now, } b^2 = a^2(1-e^2)$$

$$= 25\left[1-\left(\frac{4}{5}\right)^2\right]$$

$$= 25\left(1-\frac{16}{25}\right) = 25\left(\frac{9}{25}\right) = 9$$

The required equation of ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$.

(iii) Let the required equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > b.
Given, eccentricity (e) = $\frac{1}{3}$
Distance between directrices = $\frac{2a}{e}$
Given, distance between directrices = 18

$$\frac{2a}{e} = 18$$

$$\frac{2a}{\frac{1}{3}} = 18$$

$$6a = 18$$

$$a = \frac{18}{6} = 3$$

$$a^2 = 9$$

$$\text{Now, } b^2 = a^2(1-e^2)$$

$$= 9\left[1-\left(\frac{1}{3}\right)^2\right]$$

$$= 9\left(1-\frac{1}{9}\right) = 9\left(\frac{8}{9}\right) = 8$$

The required equation of ellipse is $\frac{x^2}{9}+\frac{y^2}{8}=1$

(iv) Let the required equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > b.
Length of r
Given, length of minor axis = 16
2b = 16
b = 8
$b^2 = 64$
Given, eccentricity (e) = $\frac{1}{3}$
Now, $b^2 = a^2(1 - e^2)$

$$64 = a^2\left[1-\left(\frac{1}{3}\right)^2\right]$$

$$64 = a^2\left(1-\frac{1}{9}\right)$$

$$64 = a^2\left(\frac{8}{9}\right)$$

$$a^2 = \frac{64\times 9}{8} = 72$$

The required equation of ellipse is $\frac{x^2}{72}+\frac{y^2}{64}=1$.

(v) Let the required equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > b.
Distance between foci = 2ae
Given, distance between foci = 6
2ae = 6

$ae = 3$

$a = \frac{3}{e}$ ....(i)

Distance between directrices = $\frac{2a}{e}$

Given, distance between directrices = $\frac{50}{3}$

$$\frac{2a}{e} = \frac{50}{3}$$

$$\frac{a}{e} = \frac{25}{3}$$

$$\frac{\frac{3}{e}}{e} = \frac{25}{3} \quad \text{...[From (i)]}$$

$$\frac{3}{e^2} = \frac{25}{3}$$

$$e^2 = \frac{9}{25}$$

$$e = \frac{3}{5} \quad \text{...[}\because 0 < e < 1\text{]}$$

Substituting $e = \frac{3}{5}$ in (i), we get

$$a = \frac{3}{\frac{3}{5}}$$

$$a = 5$$

$$a^2 = 25$$

Now, $b^2 = a^2(1 - e^2)$

$$= 25\left[1 - \left(\frac{3}{5}\right)^2\right]$$

$$= 25\left(1 - \frac{9}{25}\right) = 25\left(\frac{16}{25}\right) = 16$$

The required equation of ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1$.

(vi) Given, the length of the latus rectum is 6, and co-ordinates of foci are (±2, 0).
The foci of the ellipse are on the X-axis.

Let the required equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > b.

Length of latus rectum = $\frac{2b^2}{a}$

$\frac{2b^2}{a} = 6$

$b^2 = 3a$ .....(i)

Co-ordinates of foci are (±ae, 0)

$ae = 2$

$a^2e^2 = 4$ .....(ii)

Now, $b^2 = a^2(1 - e^2)$

$b^2 = a^2 - a^2e^2$

$3a = a^2 - 4$ .....[From (i) and (ii)]

$a^2 - 3a - 4 = 0$

$a^2 - 4a + a - 4 = 0$

$a(a - 4) + 1(a - 4) = 0$

$(a - 4)(a + 1) = 0$

$a - 4 = 0$ or $a + 1 = 0$

$a = 4$ or $a = -1$

Since $a = -1$ is not possible,

$a = 4$

$a^2 = 16$

Substituting $a = 4$ in (i), we get

$b^2 = 3(4) = 12$

The required equation of ellipse is $\frac{x^2}{16}+\frac{y^2}{12}=1$.

(vii) Let the required equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > b.

The ellipse passes through the points (-3, 1) and (2, -2).

Substituting x = -3 and y = 1 in equation of ellipse, we get

$$\frac{(-3)^2}{a^2}+\frac{1^2}{b^2}=1$$

$$\frac{9}{a^2}+\frac{1}{b^2}=1 \quad \ldots(i)$$

Substituting $x = 2$ and $y = -2$ in equation of ellipse, we get

$$\frac{2^2}{a^2}+\frac{(-2)^2}{b^2}=1$$

$$\frac{4}{a^2}+\frac{4}{b^2}=1 \quad \ldots(ii)$$

Let $\frac{1}{a^2}$ = A and $\frac{1}{b^2}$ = B

Equations (i) and (ii) become
9A + B = 1 .....(iii)
4A + 4B = 1 .....(iv)
Multiplying (iii) by 4, we get
36A + 4B = 4 .....(v)
Subtracting (iv) from (v), we get
32A = 3
A = $\frac{3}{32}$
Substituting A = $\frac{3}{32}$ in (iv), we get
4($\frac{3}{32}$) + 4B = 1
$\frac{3}{8}$ + 4B = 1
4B = 1 – $\frac{3}{8}$
4B = $\frac{5}{8}$
B = $\frac{5}{32}$

Since $\frac{1}{a^2}$ = A and $\frac{1}{b^2}$ = B,

$$\frac{1}{a^2}=\frac{3}{32} \text{ and } \frac{1}{b^2}=\frac{5}{32}$$

$$a^2=\frac{32}{3} \text{ and } b^2=\frac{32}{5}$$

The required equation of ellipse is

$$\frac{x^2}{\left(\frac{32}{3}\right)}+\frac{y^2}{\left(\frac{32}{5}\right)}, \text{ i.e., } 3x^2+5y^2=32.$$

(viii) Let the required equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > b.
Distance between directrices = $\frac{2a}{e}$
Given, distance between directrices = 10
$\frac{2a}{e}$ = 10
a = 5e .....(i)
The ellipse passes through (-√5, 2).
Substituting x = -√5 and y = 2 in equation of ellipse, we get

$$\frac{(-\sqrt{5})^2}{a^2}+\frac{2^2}{b^2}=1$$

$$\frac{5}{a^2}+\frac{4}{b^2}=1$$

$$\frac{5}{a^2}+\frac{4}{a^2(1-e^2)}=1 \qquad \ldots[\because b^2=a^2(1-e^2)]$$

Multiplying throughout by a^2, we get

$$5+\frac{4}{1-e^2}=a^2$$

$$5+\frac{4}{1-e^2}=(5e)^2 \qquad \ldots[\text{From (i)}]$$

$5(1-e^2)+4=25e^2(1-e^2)$

$5-5e^2+4=25e^2-25e^4$

$25e^4-30e^2+9=0$

$(5e^2-3)^2=0$

$5e^2-3=0$

$e^2=\frac{3}{5}$

From (i), a = 5e

$a^2=25e^2=25\times\frac{3}{5}$

$a^2=15$

We know that,

$b^2=a^2(1-e^2)$

$b^2=15\left(1-\frac{3}{5}\right)$

$b^2 = 15\left(\frac{2}{5}\right)$

$b^2 = 6$

The required equation of ellipse is $\frac{x^2}{15}+\frac{y^2}{6}=1$.

(ix) Let the required equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > b.

Given, eccentricity (e) = $\frac{2}{3}$

The ellipse passes through (2, $-\frac{5}{3}$).

Substituting x = 2 and y = $-\frac{5}{3}$ in equation of ellipse, we get

$$\frac{2^2}{a^2}+\frac{\left(-\frac{5}{3}\right)^2}{b^2}=1$$

$$\frac{4}{a^2}+\frac{25}{9b^2}=1$$

$$\frac{4}{a^2}+\frac{25}{9a^2(1-e^2)}=1 \qquad \ldots[\because b^2=a^2(1-e^2)]$$

Multiplying throughout by a^2, we get

$$4+\frac{25}{9(1-e^2)}=a^2$$

$$4+\frac{25}{9\left[1-\left(\frac{2}{3}\right)^2\right]}=a^2$$

$$4+\frac{25}{9\left(1-\frac{4}{9}\right)}=a^2$$

$$4+\frac{25}{5}=a^2$$

$$4+5=a^2$$

$$a^2=9$$

Now, $b^2 = a^2(1-e^2)$

$$= 9\left[1-\left(\frac{2}{3}\right)^2\right]$$

$$= 9\left(1-\frac{4}{9}\right) = 9\left(\frac{5}{9}\right) = 5$$

The required equation of ellipse is $\frac{x^2}{9}+\frac{y^2}{5}=1$.

Question 3.
Find the eccentricity of an ellipse, if the length of its latus rectum is one-third of its minor axis.
Solution:

Let the equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > b.

Length of latus rectum = $\frac{2b^2}{a}$
Length of minor axis = 2b
According to the given condition,
Length of latus rectum = $\frac{1}{3}$ (Minor axis)

$$\frac{2b^2}{a}=\frac{1}{3}(2b)$$

$$\frac{b}{a}=\frac{1}{3}$$

$$b=\frac{1}{3}a$$

$$b^2=\frac{1}{9}a^2 \qquad \ldots(i)$$

Now, $b^2=a^2(1-e^2)$

$$\frac{1}{9}a^2=a^2(1-e^2) \qquad \ldots[\text{From (i)}]$$

$$\frac{1}{9}=1-e^2$$

$$e^2=1-\frac{1}{9}$$

$$e^2=\frac{8}{9}$$

$$e=\frac{2\sqrt{2}}{3} \qquad \ldots[\because 0<e<1]$$

Question 4.
Find the eccentricity of an ellipse, if the distance between its directrices is three times the distance between its foci.
Solution:

Let the required equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > b.

Distance between directrices = $\frac{2a}{e}$

Distance between foci = 2ae

According to the given condition,

distance between directrices = 3(distance between foci)

$\frac{2a}{e}$ = 3(2ae)

$\frac{1}{e}$ = 3e

$\frac{1}{3}$ = e^2

e = $\frac{1}{\sqrt{3}}$ ......[∵ 0 < e < 1]

Eccentricity of the ellipse is $\frac{1}{\sqrt{3}}$

Question 5.

Show that the product of the lengths of the perpendicular segments drawn from the foci to any tangent line to the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ is equal to 16.

Solution:

Given equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1$.

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

a^2 = 25, b^2 = 16

a = 5, b = 4

We know that $e = \frac{\sqrt{a^2 - b^2}}{a}$

$$e = \frac{\sqrt{25-16}}{5} = \frac{\sqrt{9}}{5} = \frac{3}{5}$$

$$ae = 5\left(\frac{3}{5}\right) = 3$$

Co-ordinates of foci are S(ae, 0) and S′(– ae, 0),

i.e., S(3, 0) and S′(–3, 0)

Equations of tangents to the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ having slope m are

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

Equation of one of the tangents to the ellipse is

$$y = mx + \sqrt{25m^2 + 16}$$

$$mx - y + \sqrt{25m^2+16} = 0 \qquad \ldots(i)$$

p_1 = length of perpendicular segment from S(3, 0) to the tangent (i)

$$= \left|\frac{m(3) - 0 + \sqrt{25m^2+16}}{\sqrt{m^2+1}}\right|$$

$$p_1 = \left|\frac{3m + \sqrt{25m^2+16}}{\sqrt{m^2+1}}\right|$$

p_2 = length of perpendicular segment from S′(–3, 0) to the tangent (i)

$$= \left|\frac{m(-3) - 0 + \sqrt{25m^2+16}}{\sqrt{m^2+1}}\right|$$

$$p_2 = \left|\frac{-3m + \sqrt{25m^2+16}}{\sqrt{m^2+1}}\right|$$

$$p_1 p_2 = \left|\frac{3m + \sqrt{25m^2+16}}{\sqrt{m^2+1}}\right| \left|\frac{-3m + \sqrt{25m^2+16}}{\sqrt{m^2+1}}\right|$$

$$= \frac{(25m^2+16) - 9m^2}{m^2+1} = \frac{16(m^2+1)}{m^2+1} = 16$$

Question 6.

A tangent having slope $\left(-\frac{1}{2}\right)$ the ellipse $3x^2 + 4y^2 = 12$ intersects the X and Y axes in the points A and B respectively. If O is the origin, find the area of the triangle AOB.

Solution:

Given equation of the ellipse is $3x^2 + 4y^2 = 12$.

$\frac{x^2}{4}+\frac{y^2}{3}=1$

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2 = 4$, $b^2 = 3$

Equations of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having slope m are

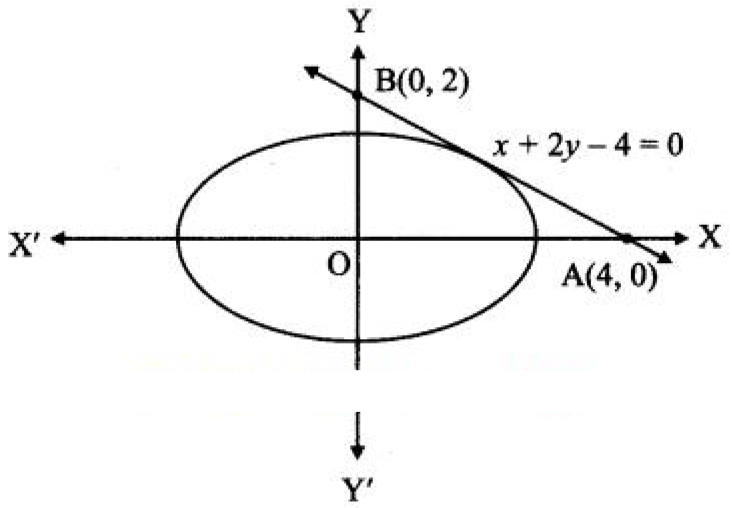
$y = mx \pm \sqrt{a^2m^2+b^2}$

Here, $m = -\frac{1}{2}$

Equations of the tangents are

$y = -\frac{1}{2}x \pm \sqrt{4\left(-\frac{1}{2}\right)^2+3} = -\frac{1}{2}x \pm 2$

$2y = -x \pm 4$
$x + 2y \pm 4 = 0$
Consider the tangent $x + 2y - 4 = 0$
Let this tangent intersect the X-axis at $A(x_1, 0)$ and Y-axis at $B(0, y_1)$.
$x_1 + 0 - 4 = 0$ and $0 + 2y_1 - 4 = 0$
$x_1 = 4$ and $y_1 = 2$
A = (4, 0) and B = (0, 2)
l(OA) = 4 and l(OB) = 2



Area of ΔAOB = $\frac{1}{2}$ × l(OA) × l(OB)
= $\frac{1}{2}$ × 4 × 2
= 4 sq.units

Question 7.
Show that the line $x - y = 5$ is a tangent to the ellipse $9x^2 + 16y^2 = 144$. Find the point of contact.
Solution:
Given equation of the ellipse is $9x^2 + 16y^2 = 144$

$\frac{x^2}{16}+\frac{y^2}{9}=1$

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2 = 16$ and $b^2 = 9$
Given equation of line is $x - y = 5$, i.e., $y = x - 5$
$c^2 = a^2 m^2 + b^2$
Comparing this equation with $y = mx + c$, we get
$m = 1$ and $c = -5$

For the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we must have

$c^2 = a^2 m^2 + b^2$
$c^2 = (-5)^2 = 25$
$a^2 m^2 + b^2 = 16(1)^2 + 9 = 16 + 9 = 25 = c^2$
The given line is a tangent to the given ellipse and point of contact

$= \left(\frac{-a^2 m}{c}, \frac{b^2}{c}\right)$

$= \left(\frac{(-16)(1)}{-5}, \frac{9}{-5}\right)$

$= \left(\frac{16}{5}, \frac{-9}{5}\right)$

Question 8.
Show that the line $8y + x = 17$ touches the ellipse $x^2 + 4y^2 = 17$. Find the point of contact.
Solution:
Given equation of the ellipse is $x^2 + 4y^2 = 17$.

$\frac{x^2}{17}+\frac{y^2}{\frac{17}{4}}=1$

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2 = 17$ and $b^2 = \frac{17}{4}$

Given equation of line is $8y + x = 17$,

$y = -\frac{1}{8}x+\frac{17}{8}$

Comparing this equation with y = mx + c, we get
m = $-\frac{1}{8}$ and c = $\frac{17}{8}$

For the line y = mx + c to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we must have

$$c^2 = a^2 m^2 + b^2$$

$$c^2 = \left(\frac{17}{8}\right)^2 = \frac{289}{64}$$

$$a^2 m^2 + b^2 = 17\left(\frac{-1}{8}\right)^2 + \frac{17}{4} = \frac{17}{64} + \frac{17}{4} = \frac{289}{64} = c^2$$

The given line touches the given ellipse and point of contact is

$$\left(\frac{-a^2 m}{c}, \frac{b^2}{c}\right) = \left(\frac{-17\left(\frac{-1}{8}\right)}{\frac{17}{8}}, \frac{\frac{17}{4}}{\frac{17}{8}}\right)$$

$$= (1, 2)$$

Question 9.

Determine whether the line x + 3y√2 = 9 is a tangent to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$. If so, find the co-ordinates of the point of contact.

Solution:

Given equation of the ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

a^2 = 9 and b^2 = 4

Given equation of line is x + 3y√2 = 9,

i.e., y = $-\frac{1}{3\sqrt{2}}x+\frac{3}{\sqrt{2}}$

Comparing this equation with y = mx + c, we get
m = $-\frac{1}{3\sqrt{2}}$ and c = $\frac{3}{\sqrt{2}}$

For the line y = mx + c to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we must have

$$c^2 = a^2 m^2 + b^2$$

$$c^2 = \left(\frac{3}{\sqrt{2}}\right)^2 = \frac{9}{2}$$

$$a^2m^2 + b^2 = 9\left(\frac{-1}{3\sqrt{2}}\right)^2 + 4 = \frac{1}{2} + 4 = \frac{9}{2} = c^2$$

The given line is a tangent to the given ellipse and point of contact is

$$\left(\frac{-a^2 m}{c}, \frac{b^2}{c}\right) = \left(\frac{(-9)\left(\frac{-1}{3\sqrt{2}}\right)}{\frac{3}{\sqrt{2}}}, \frac{4}{\frac{3}{\sqrt{2}}}\right) = \left(1, \frac{4\sqrt{2}}{3}\right)$$

Question 10.

Find k, if the line 3x + 4y + k = 0 touches $9x^2 + 16y^2 = 144$.

Solution:

Given equation of the ellipse is $9x^2 + 16y^2 = 144$.

$\frac{x^2}{16}+\frac{y^2}{9}=1$

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

a^2 = 16 and b^2 = 9

Given equation of line is 3x + 4y + k = 0,

i.e., y = $-\frac{3}{4}x-\frac{k}{4}$

Comparing this equation with y = mx + c, we get
m = $-\frac{3}{4}$ and c = $-\frac{k}{4}$

For the line y = mx + c to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we must have

$c^2 = a^2 m^2 + b^2$

$\left(-\frac{k}{4}\right)^2=16\left(-\frac{3}{4}\right)^2+9$

$\frac{k^2}{16}$ = 9 + 9

$\frac{k^2}{16}$ = 18

k^2 = 288

k = ±12√2

Question 11.
Find the equations of the tangents to the ellipse:
(i) $\frac{x^2}{5}+\frac{y^2}{4}=1$ passing through the point (2, -2).
(ii) $4x^2 + 7y^2 = 28$ from the point (3, -2).
(iii) $2x^2 + y^2 = 6$ from the point (2, 1).
(iv) $x^2 + 4y^2 = 9$ which are parallel to the line 2x + 3y – 5 = 0.
(v) $\frac{x^2}{25}+\frac{y^2}{4}=1$ which are parallel to the line x + y + 1 = 0.
(vi) $5x^2 + 9y^2 = 45$ which are ⊥ to the line 3x + 2y + 1 = 0.
(vii) $x^2 + 4y^2 = 20$ which are ⊥ to the line 4x + 3y = 7.
Solution:
(i) Given equation of the ellipse is $\frac{x^2}{5}+\frac{y^2}{4}=1$.
Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get
$a^2 = 5$ and $b^2 = 4$
Equations of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having slope m are
$y = mx \pm \sqrt{a^2m^2+b^2}$
Since (2, -2) lies on both the tangents,
$-2 = 2m \pm\sqrt{5m^2+4}$
$-2 - 2m = \pm\sqrt{5m^2+4}$
Squaring both the sides, we get
$4m^2 + 8m + 4 = 5m^2 + 4$
$m^2 - 8m = 0$
$m(m - 8) = 0$
m = 0 or m = 8
These are the slopes of the required tangents.
By slope point form $y - y_1 = m(x - x_1)$,
the equations of the tangents are
y + 2 = 0(x – 2) and y + 2 = 8(x – 2)
y + 2 = 0 and y + 2 = 8x – 16
y + 2 = 0 and 8x – y – 18 = 0

(ii) Given equation of the ellipse is $4x^2 + 7y^2 = 28$.
$\frac{x^2}{7}+\frac{y^2}{4}=1$
Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get
$a^2 = 7$ and $b^2 = 4$
Equations of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having slope m are
$y = mx \pm \sqrt{a^2m^2+b^2}$
Since (3, -2) lies on both the tangents,
$-2 = 3m \pm \sqrt{7m^2+4}$
$-2 - 3m = \pm\sqrt{7m^2+4}$
Squaring both the sides, we get
$9m^2 + 12m + 4 = 7m^2 + 4$
$2m^2 + 12m = 0$
2m(m + 6) = 0
m = 0 or m = -6
These are the slopes of the required tangents.
By slope point form $y - y_1 = m(x - x_1)$,
the equations of the tangents are
y + 2 = 0(x – 3) and y + 2 = -6(x – 3)
y + 2 = 0 and y + 2 = -6x + 18
y + 2 = 0 and 6x + y – 16 = 0

(iii) Given equation of the ellipse is $2x^2 + y^2 = 6$.
$\frac{x^2}{3}+\frac{y^2}{6}=1$
Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get
$a^2 = 3$ and $b^2 = 6$
Equations of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having slope m are
$y = mx \pm \sqrt{a^2m^2+b^2}$
Since (2, 1) lies on both the tangents,
$1 = 2m \pm \sqrt{3m^2+6}$

$1 - 2m = \pm\sqrt{3m^2+6}$

Squaring both the sides, we get
$1 - 4m + 4m^2 = 3m^2 + 6$
$m^2 - 4m - 5 = 0$
$(m - 5)(m + 1) = 0$
$m = 5$ or $m = -1$
These are the slopes of the required tangents.
By slope point form $y - y_1 = m(x - x_1)$,
the equations of the tangents are
$y - 1 = 5(x - 2)$ and $y - 1 = -1(x - 2)$
$y - 1 = 5x - 10$ and $y - 1 = -x + 2$
$5x - y - 9 = 0$ and $x + y - 3 = 0$

(iv) Given equation of the ellipse is $x^2 + 4y^2 = 9$.

$\frac{x^2}{9}+\frac{y^2}{\frac{9}{4}}=1$

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2 = 9$ and $b^2 = \frac{9}{4}$
Slope of the line $2x + 3y - 5 = 0$ is $\frac{-2}{3}$.
Since the given line is parallel to the required tangents, slope of the required tangents is
$m = \frac{-2}{3}$

Equations of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having slope m are

$$y = mx \pm \sqrt{a^2m^2+b^2}$$
$$y = -\frac{2x}{3} \pm \sqrt{9\left(\frac{-2}{3}\right)^2+\frac{9}{4}}$$
$$y = -\frac{2x}{3} \pm \sqrt{4+\frac{9}{4}}$$
$$y = -\frac{2x}{3} \pm \sqrt{\frac{25}{4}}$$
$$y = -\frac{2x}{3} \pm \frac{5}{2}$$
$$4x + 6y = \pm 15$$

(v) Given equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{4}=1$.

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get
$a^2 = 25$ and $b^2 = 4$
Slope of the given line $x + y + 1 = 0$ is -1.
Since the given line is parallel to the required tangents,
the slope of the required tangents is $m = -1$.
Equations of tangents to the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ having slope m are

$$y = mx \pm \sqrt{a^2m^2+b^2}$$
$$y = -x \pm \sqrt{25(-1)^2+4}$$
$$y = -x \pm \sqrt{29}$$
$$x + y = \pm\sqrt{29}$$

(vi) Given equation of the ellipse is $5x^2 + 9y^2 = 45$.

$\frac{x^2}{9}+\frac{y^2}{5}=1$

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2 = 9$ and $b^2 = 5$
Slope of the given line $3x + 2y + 1 = 0$ is $\frac{-3}{2}$

Since the given line is perpendicular to the required tangents, slope of the required tangents is
$m = \frac{2}{3}$

Equations of tangents to the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ having slope m are

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

$$y = \frac{2}{3}x \pm \sqrt{9\left(\frac{2}{3}\right)^2 + 5} = \frac{2}{3}x \pm \sqrt{9\left(\frac{4}{9}\right) + 5}$$

$$y = \frac{2}{3}x \pm 3$$

$$2x - 3y = \pm 9$$

(vii) Given equation of the ellipse is $x^2 + 4y^2 = 20$.

$$\frac{x^2}{20} + \frac{y^2}{5} = 1$$

Comparing this equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 20$ and $b^2 = 5$

Slope of the given line $4x + 3y = 7$ is $-\frac{4}{3}$.

Since the given line is perpendicular to the required tangents,

slope of the required tangents is $m = \frac{3}{4}$.

Equations of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ having slope m are

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

$$y = \frac{3}{4}x \pm \sqrt{20\left(\frac{3}{4}\right)^2 + 5}$$

$$y = \frac{3}{4}x \pm \sqrt{\frac{45}{4} + 5}$$

$$y = \frac{3}{4}x \pm \frac{\sqrt{65}}{2}$$

$$4y = 3x \pm 2\sqrt{65}$$

$$3x - 4y = \pm 2\sqrt{65}$$

Question 12.

Find the equation of the locus of a point, the tangents from which to the ellipse $3x^2 + 5y^2 = 15$ are at right angles.

Solution:

Given equation of the ellipse is $3x^2 + 5y^2 = 15$.

$$\frac{x^2}{5} + \frac{y^2}{3} = 1$$

Comparing this equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 5$ and $b^2 = 3$

Equations of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ having slope m are

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

$$y = mx \pm \sqrt{5m^2 + 3}$$

$$y - mx = \pm \sqrt{5m^2 + 3}$$

Squaring both the sides, we get

$y^2 - 2mxy + m^2x^2 = 5m^2 + 3$

$(x^2 - 5)m^2 - 2xym + (y^2 - 3) = 0$

The roots m_1 and m_2 of this quadratic equation are the slopes of the tangents.

$m_1m_2 = \frac{y^2 - 3}{x^2 - 5}$

Since the tangents are at right angles,

$m_1m_2 = -1$

$$\frac{y^2 - 3}{x^2 - 5} = -1$$

$y^2 - 3 = -x^2 + 5$

$x^2 + y^2 = 8$, which is the required equation of the locus.

Alternate method:

The locus of the point of intersection of perpendicular tangents is the director circle of an ellipse.

The equation of the director circle of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 + b^2$

Here, $a^2 = 5$ and $b^2 = 3$

$x^2 + y^2 = 5 + 3$

$x^2 + y^2 = 8$, which is the required equation of the locus.

Question 13.

Tangents are drawn through a point P to the ellipse $4x^2 + 5y^2 = 20$ having inclinations θ_1 and θ_2 such that $\tan\theta_1 + \tan\theta_2 = 2$. Find the equation of the locus of P.

Solution:

Given equation of the ellipse is $4x^2 + 5y^2 = 20$.

$\frac{x^2}{5}+\frac{y^2}{4}=1$

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2 = 5$ and $b^2 = 4$

Since inclinations of tangents are θ_1 and θ_2,

$m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$

Equation of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having slope m are

$y = mx \pm \sqrt{a^2m^2+b^2}$

$y = mx \pm \sqrt{5m^2+4}$

$y - mx = \pm \sqrt{5m^2+4}$

Squaring both the sides, we get

$y^2 - 2mxy + m^2x^2 = 5m^2 + 4$

$(x^2 - 5)m^2 - 2xym + (y^2 - 4) = 0$

The roots m_1 and m_2 of this quadratic equation are the slopes of the tangents.

$m_1 + m_2 = \frac{-(-2xy)}{x^2-5}=\frac{2xy}{x^2-5}$

Given, $\tan \theta_1 + \tan \theta_2 = 2$

$m_1 + m_2 = 2$

$\frac{2xy}{x^2-5} = 2$

$xy = x^2 - 5$

$x^2 - xy - 5 = 0$, which is the required equation of the locus of P.

Question 14.

Show that the locus of the point of intersection of tangents at two points on an ellipse, whose eccentric angles differ by a constant, is an ellipse.

Solution:

Let $P(\theta_1)$ and $Q(\theta_2)$ be any two points on the given ellipse such that $\theta_1 - \theta_2 = k$, where k is a constant.

The equation of the tangent at point $P(\theta_1)$ is

$\frac{x\cos\theta_1}{a}+\frac{y\sin\theta_1}{b}=1$ ......(i)

The equation of the tangent at point $Q(\theta_2)$ is

$\frac{x\cos\theta_2}{a}+\frac{y\sin\theta_2}{b}=1$ ......(ii)

Multiplying equation (i) by $\cos \theta_2$ and equation (ii) by $\cos \theta_1$ and subtracting, we get

$\frac{y}{b}(\sin \theta_1 \cos \theta_2 - \sin \theta_2 \cos \theta_1) = \cos \theta_2 - \cos \theta_1$

$\frac{y}{b}[\sin(\theta_1 - \theta_2)] = \cos\theta_2 - \cos\theta_1$

$\frac{y}{b}\sin k = \cos\theta_2 - \cos\theta_1$

$\frac{y}{b} = \frac{\cos\theta_2 - \cos\theta_1}{\sin k}$ …(iii)

Similarly,
Multiplying equation (i) by $\sin\theta_2$ and equation (ii) by $\sin\theta_1$ and subtracting, we get

$\frac{x}{a}(\cos\theta_1 \sin\theta_2 - \cos\theta_2 \sin\theta_1) = \sin\theta_2 - \sin\theta_1$

$-\frac{x}{a}[\sin(\theta_1 - \theta_2)] = \sin\theta_2 - \sin\theta_1$

$-\frac{x}{a}\sin k = \sin\theta_2 - \sin\theta_1$

$-\frac{x}{a} = \frac{\sin\theta_2 - \sin\theta_1}{\sin k}$ …(iv)

Squaring (iii) and (iv) and adding, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(\cos\theta_2 - \cos\theta_1)^2 + (\sin\theta_2 - \sin\theta_1)^2}{\sin^2 k}$$

$$= \frac{1}{\sin^2 k}[\cos^2\theta_2 + \sin^2\theta_2 + \cos^2\theta_1 + \sin^2\theta_1 - 2(\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2)]$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2 - 2\cos(\theta_1 - \theta_2)}{\sin^2 k}$$

$$= \frac{2 - 2\cos k}{\sin^2 k}$$

$$= \frac{2(1 - \cos k)}{\sin^2 k} = \frac{4\sin^2\frac{k}{2}}{4\sin^2\frac{k}{2}\cos^2\frac{k}{2}}$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2\frac{k}{2}$, which is an ellipse.

Question 15.
P and Q are two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with eccentric angles θ_1 and θ_2. Find the equation of the locus of the point of intersection of the tangents at P and Q if $\theta_1 + \theta_2 = \frac{\pi}{2}$.
Solution:
Given equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
θ_1 and θ_2 are the eccentric angles of a tangent.
Equation of tangent at point P is
$\frac{x}{a}\cos\theta_1+\frac{y}{b}\sin\theta_1=1$ ……(i)
Equation of tangent at point Q is
$\frac{x}{a}\cos\theta_2+\frac{y}{b}\sin\theta_2=1$ ………(ii)
$\theta_1 + \theta_2 = \frac{\pi}{2}$ …..[Given]

$\theta_2 = \frac{\pi}{2} - \theta_1$

$\frac{x}{a}\cos\left(\frac{\pi}{2} - \theta_1\right) + \frac{y}{b}\sin\left(\frac{\pi}{2} - \theta_1\right) = 1$

$\frac{x}{a}\sin\theta_1 + \frac{y}{b}\cos\theta_1 = 1 \quad \ldots\text{(ii)}$

From (i) and (ii), we get

$\frac{x}{a}\cos\theta_1 + \frac{y}{b}\sin\theta_1 = \frac{x}{a}\sin\theta_1 + \frac{y}{b}\cos\theta_1$

Let $M(x_1, y_1)$ be the point of intersection of the tangents.

$\frac{x_1}{a}\cos\theta_1 + \frac{y_1}{b}\sin\theta_1 = \frac{x_1}{a}\sin\theta_1 + \frac{y_1}{b}\cos\theta_1$

$\frac{x_1}{a}(\cos\theta_1 - \sin\theta_1) = \frac{y_1}{b}(\cos\theta_1 - \sin\theta_1) \ldots\text{(iii)}$

If $\cos\theta_1 - \sin\theta_1 = 0$,

$\cos\theta_1 = \sin\theta_1$

$\tan\theta_1 = 1$

$\theta_1 = \frac{\pi}{4}$

Since $\theta_1 + \theta_2 = \frac{\pi}{2}$, $\theta_2 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

i.e., points P and Q coincide, which is not possible, as P and Q are two different points.

$\cos\theta_1 - \sin\theta_1 \neq 0$

Dividing equation (iii) by $(\cos\theta_1 - \sin\theta_1)$, we get

$\frac{x_1}{a} = \frac{y_1}{b}$

$bx_1 - ay_1 = 0$

$bx - ay = 0$, which is the required equation of locus of point M.

Question 16.

The eccentric angles of two points P and Q of the ellipse $4x^2 + y^2 = 4$ differ by $\frac{2\pi}{3}$. Show that the locus of the point of intersection of the tangents at P and Q is the ellipse $4x^2 + y^2 = 16$.

Solution:

Given equation of the ellipse is $4x^2 + y^2 = 4$.

$\frac{x^2}{1} + \frac{y^2}{4} = 1$

Let $P(\theta_1)$ and $Q(\theta_2)$ be any two points on the given ellipse such that

$\theta_1 - \theta_2 = \frac{2\pi}{3}$

The equation of the tangent at point $P(\theta_1)$ is

$\frac{x\cos\theta_1}{1} + \frac{y\sin\theta_1}{2} = 1$ ......(i)

The equation of the tangent at point $Q(\theta_2)$ is

$\frac{x\cos\theta_2}{1} + \frac{y\sin\theta_2}{2} = 1$

Multiplying equation (i) by $\cos\theta_2$ and equation (ii) by $\cos\theta_1$ and subtracting, we get

$$\frac{y}{2}(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1) = \cos\theta_2 - \cos\theta_1$$

$$\frac{y}{2}[\sin(\theta_1 - \theta_2)] = \cos\theta_2 - \cos\theta_1$$

$$\frac{y}{2}\left[\sin\left(\frac{2\pi}{3}\right)\right] = \cos\theta_2 - \cos\theta_1$$

$$\frac{y}{2}\sin\left(\pi - \frac{\pi}{3}\right) = \cos\theta_2 - \cos\theta_1$$

$$\frac{y}{2}\sin\left(\frac{\pi}{3}\right) = \cos\theta_2 - \cos\theta_1$$

$$\frac{y}{2}\left(\frac{\sqrt{3}}{2}\right) = \cos\theta_2 - \cos\theta_1$$

$$\frac{\sqrt{3}\,y}{4} = \cos\theta_2 - \cos\theta_1 \qquad \ldots\text{(iii)}$$

Multiplying equation (i) by $\sin\theta_2$ and equation (ii) by $\sin\theta_1$ and subtracting, we get

$$x(\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1) = \sin\theta_2 - \sin\theta_1$$

$$-x\sin(\theta_1 - \theta_2) = \sin\theta_2 - \sin\theta_1$$

$$-x\sin\left(\frac{2\pi}{3}\right) = \sin\theta_2 - \sin\theta_1$$

$$-x\sin\left(\pi - \frac{\pi}{3}\right) = \sin\theta_2 - \sin\theta_1$$

$$-x\sin\frac{\pi}{3} = \sin\theta_2 - \sin\theta_1$$

$$-\frac{\sqrt{3}}{2}x = \sin\theta_2 - \sin\theta_1 \qquad \ldots\text{(iv)}$$

Squaring (iii) and (iv) and adding, we get

$$\frac{3x^2}{4} + \frac{3y^2}{16} = \sin^2\theta_2 - 2\sin\theta_2\sin\theta_1 + \sin^2\theta_1 + \cos^2\theta_2 - 2\cos\theta_2\cos\theta_1 + \cos^2\theta_1$$

$$= (\cos^2\theta_2 + \sin^2\theta_2) + (\cos^2\theta_1 + \sin^2\theta_1) - 2\cos\theta_2\cos\theta_1 - 2\sin\theta_2\sin\theta_1$$

$$= 1 + 1 - 2(\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1)$$

$$= 2 - 2[\cos(\theta_1 - \theta_2)]$$

$$= 2 - 2\cos\left(\frac{2\pi}{3}\right)$$

$$= 2 - 2\left(\frac{-1}{2}\right)$$

$$= 2 + 1$$

$$\frac{3x^2}{4} + \frac{3y^2}{16} = 3$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

$4x^2 + y^2 = 16$, which is the required equation of locus.

Question 17.

Find the equations of the tangents to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$, making equal intercepts on co-ordinate axes.

Solution:

Given equation of the ellipse is $\frac{x^2}{16}+\frac{y^2}{9}=1$

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2 = 16$ and $b^2 = 9$

Since the tangents make equal intercepts on the co-ordinate axes,

$m = -1$.

Equations of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having slope m are

$y = mx \pm \sqrt{a^2m^2+b^2}$

$y = -x \pm \sqrt{16(-1)^2+9}$

$y = -x \pm \sqrt{25}$
x + y = ±5

Question 18.
A tangent having slope $\left(-\frac{1}{2}\right)$ to the ellipse $3x^2 + 4y^2 = 12$ intersects the X and Y axes in the points A and B respectively. If O is the origin, find the area of the triangle AOB.
Solution:
The equation of the ellipse is $3x^2 + 4y^2 = 12$

$\frac{x^2}{4}+\frac{y^2}{3}=1$

Comparing with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2 = 4$, $b^2 = 3$

The equation of tangent with slope m is

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

$$\text{i.e., } y = mx \pm \sqrt{4m^2 + 3} \quad \ldots[\because a^2 = 4, b^2 = 3]$$

$$\therefore y = -\frac{1}{2}x \pm \sqrt{4\left(\frac{1}{4}\right) + 3} \quad \ldots\left[\because m = -\frac{1}{2}\right]$$

$$\therefore y = -\frac{x}{2} \pm 2$$

$$\therefore x + 2y \pm 4 = 0 \quad \ldots(1)$$

It meets X axis at A
∴ for A, put y = 0 in equation (1), we get,
x = ±4
∴ A = (±4, 0)
Similarly, B = (0, ±2)
∴ OA = 4, OB = 2
∴ Area of ΔOAB = $\frac{1}{2}$ × OA × OB
= $\frac{1}{2}$ × 4 × 2
= 4 sq. units

Maharashtra State Board 11th Maths Solutions Chapter 7 Conic Sections Ex 7.3

Question 1.
Find the length of the transverse axis, length of the conjugate axis, the eccentricity, the co-ordinates of foci, equations of directrices, and the length of the latus rectum of the hyperbolae.

(i) $\frac{x^2}{25}-\frac{y^2}{16}=1$

(ii) $\frac{x^2}{25}-\frac{y^2}{16}=-1$

(iii) $16x^2 - 9y^2 = 144$
(iv) $21x^2 - 4y^2 = 84$
(v) $3x^2 - y^2 = 4$
(vi) $x^2 - y^2 = 16$

(vii) $\frac{y^2}{25}-\frac{x^2}{9}=1$

(viii) $\frac{y^2}{25}-\frac{x^2}{144}=1$

(ix) $\frac{x^2}{100}-\frac{y^2}{25}=1$

(x) x = 2 sec θ, y = 2√3 tan θ

Solution:

(i) Given equation of the hyperbola is $\frac{x^2}{25}-\frac{y^2}{16}=1$

Comparing this equation with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$a^2 = 25$ and $b^2 = 16$

⇒ a = 5 and b = 4

Length of transverse axis = 2a = 2(5) = 10

Length of conjugate axis = 2b = 2(4) = 8

We know that

$$e = \frac{\sqrt{a^2+b^2}}{a} = \frac{\sqrt{25+16}}{5} = \frac{\sqrt{41}}{5}$$

Co-ordinates of foci are S(ae, 0) and S′(– ae, 0),

i.e., $S\left(5\left(\frac{\sqrt{41}}{5}\right), 0\right)$ and $S'\left(-5\left(\frac{\sqrt{41}}{5}\right), 0\right)$,

i.e., $S(\sqrt{41}, 0)$ and $S'(-\sqrt{41}, 0)$

Equations of the directrices are $x = \pm \frac{a}{e}$.

$$x = \pm \frac{5}{\left(\frac{\sqrt{41}}{5}\right)}$$

$$x = \pm \frac{25}{\sqrt{41}}$$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2(16)}{5} = \frac{32}{5}$

(ii) Given equation of the hyperbola is $\frac{x^2}{25}-\frac{y^2}{16}=-1$

$\frac{y^2}{16}-\frac{x^2}{25}=1$

Comparing this equation with $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$, we get

$b^2 = 16$ and $a^2 = 25$

⇒ b = 4 and a = 5

Length of transverse axis = 2b = 2(4) = 8

Length of conjugate axis = 2a = 2(5) = 10

Co-ordinates of vertices are B(0, b) and B′ (0, -b)

i.e., B(0, 4) and B'(0, -4)

We know that

$$e = \frac{\sqrt{b^2+a^2}}{b} = \frac{\sqrt{16+25}}{4} = \frac{\sqrt{41}}{4}$$

Co-ordinates of foci are S(0, be) and S′(0, –be),

i.e., $S\left(0, 4\left(\frac{\sqrt{41}}{4}\right)\right)$ and $S'\left(0, -4\left(\frac{\sqrt{41}}{4}\right)\right)$,

i.e., $S(0, \sqrt{41})$ and $S'(0, -\sqrt{41})$

Equations of the directrices are $y = \pm \frac{b}{e}$.

$$y = \pm \frac{4}{\frac{\sqrt{41}}{4}}$$

$$y = \pm \frac{16}{\sqrt{41}}$$

Length of latus-rectum = $\frac{2a^2}{b} = \frac{2(25)}{4} = \frac{25}{2}$

(iii) Given equation of the hyperbola is $16x^2 – 9y^2 = 144$.

$\frac{x^2}{9}-\frac{y^2}{16}=1$

Comparing this equation with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$a^2 = 9$ and $b^2 = 16$

⇒ a = 3 and b = 4

Length of transverse axis = 2a = 2(3) = 6
Length of conjugate axis = 2b = 2(4) = 8
We know that

e = $\frac{\sqrt{a^2+b^2}}{a}=\frac{\sqrt{9+16}}{3}=\frac{\sqrt{25}}{3}=\frac{5}{3}$

Co-ordinates of foci are S(ae, 0) and S'(-ae, 0),
i.e., S(3($\frac{5}{3}$), 0) and S'(-3($\frac{5}{3}$), 0)
i.e., S(5, 0) and S'(-5, 0)
Equations of the directrices are x = $\pm\frac{a}{e}$

$= \pm\frac{3}{\left(\frac{5}{3}\right)}$

$= \pm\frac{9}{5}$

Length of latus rectum = $\frac{2b^2}{a}=\frac{2(16)}{3}=\frac{32}{3}$

(iv) Given equation of the hyperbola is $21x^2 - 4y^2 = 84$.

$\frac{x^2}{4}-\frac{y^2}{21}=1$

Comparing this equation with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get
$a^2 = 4$ and $b^2 = 21$
⇒ a = 2 and b = √21
Length of transverse axis = 2a = 2(2) = 4
Length of conjugate axis = 2b = 2√21
We know that

$$e = \frac{\sqrt{a^2+b^2}}{a} = \frac{\sqrt{4+21}}{2} = \frac{\sqrt{25}}{2} = \frac{5}{2}$$

Co-ordinates of foci are S(ae, 0) and S′(– ae, 0),

i.e., $S\left(2\left(\frac{5}{2}\right), 0\right)$ and $S'\left(-2\left(\frac{5}{2}\right), 0\right)$,

i.e., S(5, 0) and S′(– 5, 0)

Equations of the directrices are $x = \pm\frac{a}{e}$.

$$x = \pm\frac{2}{\left(\frac{5}{2}\right)}$$

$$x = \pm\frac{4}{5}$$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2(21)}{2} = 21$

(v) Given equation of the hyperbola is $3x^2 - y^2 = 4$.

$\frac{x^2}{\left(\frac{4}{3}\right)}-\frac{y^2}{4}=1$

Comparing this equation with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$a^2 = \frac{4}{3}$ and $b^2 = 4$

$$a = \frac{2}{\sqrt{3}} \text{ and } b = 2$$

Length of transverse axis $= 2a = 2\left(\frac{2}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}}$

Length of conjugate axis $= 2b = 2(2) = 4$

We know that

$$e = \frac{\sqrt{a^2+b^2}}{a} = \frac{\sqrt{\frac{4}{3}+4}}{\left(\frac{2}{\sqrt{3}}\right)} = \frac{\sqrt{\frac{16}{3}}}{\left(\frac{2}{\sqrt{3}}\right)} = \frac{4}{2} = 2$$

Co-ordinates of foci are $S(ae, 0)$ and $S'(-ae, 0)$,

i.e., $S\left(\frac{2}{\sqrt{3}}(2), 0\right)$ and $S'\left(-\frac{2}{\sqrt{3}}(2), 0\right)$,

i.e., $S\left(\frac{4}{\sqrt{3}}, 0\right)$ and $S'\left(-\frac{4}{\sqrt{3}}, 0\right)$

Equations of the directrices are $x = \pm \frac{a}{e}$.

$$x = \pm \frac{\left(\frac{2}{\sqrt{3}}\right)}{2}$$

$$x = \pm \frac{1}{\sqrt{3}}$$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2(4)}{\left(\frac{2}{\sqrt{3}}\right)} = 4\sqrt{3}$

(vi) Given equation of the hyperbola is $x^2 - y^2 = 16$.

$\frac{x^2}{16} - \frac{y^2}{16} = 1$

Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a^2 = 16$ and $b^2 = 16$

⇒ $a = 4$ and $b = 4$

Length of transverse axis $= 2a = 2(4) = 8$

Length of conjugate axis $= 2b = 2(4) = 8$

We know that

$e = \frac{\sqrt{a^2+b^2}}{a} = \frac{\sqrt{16+16}}{4} = \frac{\sqrt{32}}{4} = \frac{4\sqrt{2}}{4} = \sqrt{2}$

Co-ordinates of foci are S(ae, 0) and S'(-ae, 0),

i.e., S $(4\sqrt{2}, 0)$ and S′ $(-4\sqrt{2}, 0)$

Equations of the directrices are $x = \pm\frac{a}{e}$

⇒ $x = \pm\frac{4}{\sqrt{2}}$

⇒ $x = \pm 2\sqrt{2}$

Length of latus rectum $= \frac{2b^2}{a} = \frac{2(16)}{4} = 8$

(vii) Given equation of the hyperbola is $\frac{y^2}{25} - \frac{x^2}{9} = 1$.

Comparing this equation with $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, we get

$b^2 = 25$ and $a^2 = 9$

⇒ $b = 5$ and $a = 3$

Length of transverse axis $= 2b = 2(5) = 10$

Length of conjugate axis $= 2a = 2(3) = 6$

Co-ordinates of vertices are B(0, b) and B′ (0, -b),

i.e., B(0, 5) and B′ (0, -5)

We know that

$$e = \frac{\sqrt{b^2 + a^2}}{b} = \frac{\sqrt{25+9}}{5} = \frac{\sqrt{34}}{5}$$

Co-ordinates of foci are S(0, be) and S′(0, –be),

i.e., $S\left(0, 5\left(\frac{\sqrt{34}}{5}\right)\right)$ and $S'\left(0, -5\left(\frac{\sqrt{34}}{5}\right)\right)$,

i.e., S $(0, \sqrt{34})$ and S′ $(0, -\sqrt{34})$

Equations of the directrices are $y = \pm \frac{b}{e}$.

$$y = \pm \frac{5}{\left(\frac{\sqrt{34}}{5}\right)}$$

$$y = \pm \frac{25}{\sqrt{34}}$$

Length of latus-rectum $= \frac{2a^2}{b} = \frac{2(9)}{5} = \frac{18}{5}$

(viii) Given equation of the hyperbola is $\frac{y^2}{25} - \frac{x^2}{144} = 1$.

Comparing this equation with $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, we get

$b^2 = 25$ and $a^2 = 144$

⇒ b = 5 and a = 12

Length of transverse axis = 2b = 2(5) = 10

Length of conjugate axis = 2a = 2(12) = 24

Co-ordinates of vertices are B(0, b) and B′ (0, -b),

i.e., B(0, 5) and B′ (0, -5)

We know that

$$e = \frac{\sqrt{b^2 + a^2}}{b} = \frac{\sqrt{25+144}}{5} = \frac{\sqrt{169}}{5} = \frac{13}{5}$$

Co-ordinates of foci are S(0, be) and S′(0, –be),

i.e., $S\left(0, 5\left(\frac{13}{5}\right)\right)$ and $S'\left(0, -5\left(\frac{13}{5}\right)\right)$,

i.e., S (0, 13) and S′ (0, – 13)

Equations of the directrices are $y = \pm \frac{b}{e}$.

$$y = \pm \frac{5}{\left(\frac{13}{5}\right)}$$

$$y = \pm \frac{25}{13}$$

Length of latus-rectum $= \frac{2a^2}{b} = \frac{2(144)}{5} = \frac{288}{5}$

(ix) Given equation of the hyperbola is $\frac{x^2}{100} - \frac{y^2}{25} = 1$

Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a^2 = 100$ and $b^2 = 25$

⇒ a = 10 and b = 5

Length of transverse axis = 2a = 2(10) = 20

Length of conjugate axis = 2b = 2(5) = 10

We know that

$$e = \frac{\sqrt{a^2+b^2}}{a} = \frac{\sqrt{100+25}}{10}$$
$$= \frac{\sqrt{125}}{10}$$
$$= \frac{5\sqrt{5}}{10} = \frac{\sqrt{5}}{2}$$

Co-ordinates of foci are S(ae, 0) and S′(– ae, 0),

i.e., $S\left(10\left(\frac{\sqrt{5}}{2}\right), 0\right)$ and $S'\left(-10\left(\frac{\sqrt{5}}{2}\right), 0\right)$,

i.e., $S\left(5\sqrt{5}, 0\right)$ and $S'\left(-5\sqrt{5}, 0\right)$

Equations of the directrices are $x = \pm \frac{a}{e}$.

$$x = \pm \frac{10}{\left(\frac{\sqrt{5}}{2}\right)}$$

$$x = \pm \frac{20}{\sqrt{5}}$$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2(25)}{10} = 5$

(x) Given equation of the hyperbola is x = 2 sec θ, y = 2√3 tan θ.
Since $\sec^2 \theta - \tan^2 \theta = 1$,

$$\left(\frac{x}{2}\right)^2 - \left(\frac{y}{2\sqrt{3}}\right)^2 = 1$$

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get
$a^2 = 4$ and $b^2 = 12$
⇒ a = 2 and b = 2√3
Length of transverse axis = 2a = 2(2) = 4
Length of conjugate axis = 2b = 2(2√3) = 4√3
We know that
e = $\frac{\sqrt{a^2+b^2}}{a} = \frac{\sqrt{4+12}}{2}$ = 2
Co-ordinates of foci are S(ae, 0) and S'(-ae, 0),
i.e., S(2(2), 0) and S'(-2(2), 0),
i.e., S(4, 0) and S'(-4, 0)
Equations of the directrices are x = $\pm\frac{a}{e}$.
⇒ x = $\pm\frac{2}{2}$
⇒ x = ±1
Length of latus rectum = $\frac{2b^2}{a} = \frac{2(12)}{2}$ = 12

Question 2.
Find the equation of the hyperbola with centre at the origin, length of the conjugate axis as 10, and one of the foci as (-7, 0).
Solution:
Given, one of the foci of the hyperbola is (-7, 0).
Since this focus lies on the X-axis, it is a standard hyperbola.
Let the required equation of hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Length of conjugate axis = 2b
Given, length of conjugate axis = 10
⇒ 2b = 10
⇒ b = 5
⇒ b^2 = 25
Co-ordinates of focus are (-ae, 0)
ae = 7
⇒ a^2e^2 = 49
Now, $b^2 = a^2(e^2 - 1)$
⇒ 25 = 49 – a^2
⇒ a^2 = 49 – 25 = 24
The required equation of hyperbola is $\frac{x^2}{24} - \frac{y^2}{25} = 1$

Question 3.
Find the eccentricity of the hyperbola, which is conjugate to the hyperbola $x^2 - 3y^2 = 3$
Solution:
Given, equation of hyperbola is $x^2 - 3y^2 = 3$.

$\frac{x^2}{3}-\frac{y^2}{1}=1$

Equation of the hyperbola conjugate to the above hyperbola is $\frac{y^2}{1}-\frac{x^2}{3}=1$

Comparing this equation with $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$, we get
$b^2 = 1$ and $a^2 = 3$
Now, $a^2 = b^2(e^2 - 1)$
⇒ $3 = 1(e^2 - 1)$
⇒ $3 = e - 1$
⇒ $e^2 = 4$
⇒ $e = 2$ .....[∵ $e > 1$]

Question 4.
If e and e′ are the eccentricities of a hyperbola and its conjugate hyperbola respectively, prove that $\frac{1}{e^2}+\frac{1}{(e')^2}=1$.
Solution:
Let e be the eccentricity of a hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

$$e=\frac{\sqrt{a^2+b^2}}{a}$$

$$e^2=\frac{a^2+b^2}{a^2}$$

$$\frac{1}{e^2}=\frac{a^2}{a^2+b^2} \quad \ldots(i)$$

Also, e′ is the eccentricity of conjugate hyperbola $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

$$e'=\frac{\sqrt{b^2+a^2}}{b}$$

$$(e')^2=\frac{b^2+a^2}{b^2}$$

$$\frac{1}{(e')^2}=\frac{b^2}{b^2+a^2} \quad \ldots(ii)$$

Adding (i) and (ii), we get

$$\frac{1}{e^2}+\frac{1}{(e')^2}=\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}=\frac{a^2+b^2}{a^2+b^2}$$

$$\frac{1}{e^2}+\frac{1}{(e')^2}=1$$

Question 5.
Find the equation of the hyperbola referred to its principal axes:
(i) whose distance between foci is 10 and eccentricity is $\frac{5}{2}$
(ii) whose distance between foci is 10 and length of the conjugate axis is 6.
(iii) whose distance between directrices is $\frac{8}{3}$ and eccentricity is $\frac{3}{2}$.
(iv) whose length of conjugate axis = 12 and passing through (1, -2).
(v) which passes through the points (6, 9) and (3, 0).
(vi) whose vertices are (±7, 0) and endpoints of the conjugate axis are (0, ±3).
(vii) whose foci are at (±2, 0) and eccentricity is $\frac{3}{2}$.
(viii) whose lengths of transverse and conjugate axes are 6 and 9 respectively.
(ix) whose length of transverse axis is 8 and distance between foci is 10.
Solution:
(i) Let the required equation of hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Given, eccentricity (e) = $\frac{5}{2}$
Distance between foci = 2ae
Given, distance between foci = 10
⇒ 2ae = 10
⇒ ae = 5
⇒ $a(\frac{5}{2}) = 5$
⇒ a = 2

⇒ $a^2 = 4$

Now, $b^2 = a^2(e^2 - 1)$

$$b^2 = 4\left[\left(\frac{5}{2}\right)^2 - 1\right]$$

$$= 4\left(\frac{25}{4} - 1\right)$$

$$= 4\left(\frac{21}{4}\right)$$

$b^2 = 21$

The required equation of hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{21} = 1.$$

(ii) Let the required equation of hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Length of conjugate axis = 2b
Given, length of conjugate axis = 6
⇒ 2b = 6
⇒ b = 3
⇒ $b^2 = 9$
Distance between foci = 2ae
Given, distance between foci = 10
⇒ 2ae = 10
⇒ ae = 5
⇒ $a^2e^2 = 25$
Now, $b^2 = a^2 (e^2 – 1)$
⇒ $b^2 = a^2 e^2 – a^2$
⇒ $9 = 25 – a^2$
⇒ $a^2 = 25 – 9$
⇒ $a^2 = 16$

The required equation of hyperbola is $\frac{x^2}{16}-\frac{y^2}{9}=1$

(iii) Let the required equation of hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Given, eccentricity (e) = $\frac{3}{2}$

Distance between directrices = $\frac{2a}{e}$

Given, distance between directrices = $\frac{8}{3}$

$$\frac{2a}{e} = \frac{8}{3}$$

$$\frac{2a}{\frac{3}{2}} = \frac{8}{3}$$

$$\frac{4a}{3} = \frac{8}{3}$$

a = 2

$a^2 = 4$

Now, $b^2 = a^2 (e^2 - 1)$

$$b^2 = 4\left[\left(\frac{3}{2}\right)^2 - 1\right]$$

$$= 4\left(\frac{9}{4} - 1\right)$$

$$= 4\left(\frac{5}{4}\right)$$

$b^2 = 5$

The required equation of hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{5} = 1.$$

(iv) Let the required equation of hyperbola be

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ......(i)

Length of conjugate axis = 2b

Given, length of conjugate axis = 12
⇒ 2b = 12
⇒ b = 6 .....(ii)
⇒ b^2 = 36
The hyperbola passes through (1, -2)
Substituting x = 1 and y = -2 in (i), we get

$$\frac{1^2}{a^2} - \frac{(-2)^2}{b^2} = 1$$

$$\frac{1}{a^2} - \frac{4}{b^2} = 1$$

$$\frac{1}{a^2} - \frac{4}{6^2} = 1 \quad \ldots[\text{From(ii)}]$$

$$\frac{1}{a^2} - \frac{4}{36} = 1$$

$$\frac{1}{a^2} = 1 + \frac{1}{9}$$

$$\frac{1}{a^2} = \frac{10}{9}$$

$$a^2 = \frac{9}{10}$$

The required equation of hyperbola is

$$\frac{x^2}{\frac{9}{10}} - \frac{y^2}{36} = 1, \text{ i.e., } \frac{10x^2}{9} - \frac{y^2}{36} = 1.$$

(v) Let the required equation of hyperbola be

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ......(i)

The hyperbola passes through the points (6, 9) and (3, 0).

Substituting $x = 6$ and $y = 9$ in (i), we get

$$\frac{6^2}{a^2} - \frac{9^2}{b^2} = 1$$

$$\frac{36}{a^2} - \frac{81}{b^2} = 1 \quad \ldots(\text{ii})$$

Substituting $x = 3$ and $y = 0$ in (i), we get

$$\frac{3^2}{a^2} - \frac{0^2}{b^2} = 1$$

$$\frac{9}{a^2} - 0 = 1$$

$$a^2 = 9$$

Substituting $a^2 = 9$ in (ii), we get

$$\frac{36}{9} - \frac{81}{b^2} = 1$$

$$\frac{81}{b^2} = \frac{36}{9} - 1$$

$$\frac{81}{b^2} = 4 - 1 = 3$$

$$b^2 = \frac{81}{3} = 27$$

The required equation of hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{27} = 1.$$

(vi) Let the required equation of hyperbola be

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Co-ordinates of vertices are (±a, 0).
Given that, co-ordinates of vertices are (±7, 0)
∴ a = 7
Endpoints of the conjugate axis are (0, b) and (0, -b).
Given, the endpoints of the conjugate axis are (0, ±3).
∴ b = 3

The required equation of hyperbola is $\frac{x^2}{7^2}-\frac{y^2}{3^2}=1$

i.e., $\frac{x^2}{49}-\frac{y^2}{9}=1$

(vii) Let the required equation of hyperbola be

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ……(i)

Given, eccentricity (e) = $\frac{3}{2}$

Co-ordinates of foci are (±ae, 0).

Given co-ordinates of foci are (±2, 0)

ae = 2

⇒ a($\frac{3}{2}$) = 2

⇒ a = $\frac{4}{3}$

⇒ a^2 = $\frac{16}{9}$

(viii) Let the required equation of hyperbola be

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Length of transverse axis = 2a

Given, length of transverse axis = 6

⇒ 2a = 6

⇒ a = 3

⇒ a^2 = 9

Length of conjugate axis = 2b

Given, length of conjugate axis = 9

⇒ 2b = 9

⇒ b = $\frac{9}{2}$

⇒ b^2 = $\frac{81}{4}$

The required equation of hyperbola is

$\frac{x^2}{9}-\frac{y^2}{\left(\frac{81}{4}\right)}=1$

i.e., $\frac{x^2}{9}-\frac{4y^2}{81}=1$

(ix) Let the required equation of hyperbola be

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Length of transverse axis = 2a

Given, length of transverse axis = 8

⇒ 2a = 8

⇒ a = 4

⇒ a^2 = 16

Distance between foci = 2ae

Given, distance between foci = 10

⇒ 2ae = 10

⇒ ae = 5

⇒ a^2e^2 = 25

Now, $b^2 = a^2 (e^2 – 1)$

⇒ $b^2 = a^2 e^2 – a^2$

⇒ b^2 = 25 – 16 = 9

The required equation of hyperbola is $\frac{x^2}{16}-\frac{y^2}{9}=1$

Question 6.

Find the equation of the tangent to the hyperbola.

(i) $3x^2 – y^2 = 4$ at the point (2, 2√2).

(ii) $3x^2 – y^2 = 12$ at the point (4, 6)

(iii) $\frac{x^2}{144}-\frac{y^2}{25}=1$ at the point whose eccentric angle is $\frac{\pi}{3}$.

(iv) $\frac{x^2}{16}-\frac{y^2}{9}=1$ at the point in a first quadrant whose ordinate is 3.

(v) $9x^2 – 16y^2 = 144$ at the point L of the latus rectum in the first quadrant.

Solution:

i. Given equation of the hyperbola is $3x^2 - y^2 = 4$.

$\therefore \quad \frac{x^2}{\left(\frac{4}{3}\right)} - \frac{y^2}{4} = 1$

Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

we get

$a^2 = \frac{4}{3}$ and $b^2 = 4$

Equation of the tangent to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

$\therefore$ Equation of the tangent at $\left(2, 2\sqrt{2}\right)$ is

$\frac{2x}{\left(\frac{4}{3}\right)} - \frac{2\sqrt{2}y}{4} = 1$

$\therefore \quad \frac{3x}{2} - \frac{\sqrt{2}y}{2} = 1$

$\therefore \quad 3x - \sqrt{2}y = 2$

ii. Given equation of the hyperbola is $3x^2 - y^2 = 12$.

$\therefore \quad \frac{x^2}{4} - \frac{y^2}{12} = 1$

Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

we get

$a^2 = 4$ and $b^2 = 12$

Equation of the tangent to the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

Equation of the tangent at (4, 6) is

$\frac{4x}{4} - \frac{6y}{12} = 1$

$x - \frac{y}{2} = 1$

$2x - y = 2$

iii. Given equation of the hyperbola is $\frac{x^2}{144} - \frac{y^2}{25} = 1$.

Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

we get

$a^2 = 144$ and $b^2 = 25$

$\therefore$ $a = 12$ and $b = 5$

Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P(\theta)$ is

$$\frac{x \sec\theta}{a} - \frac{y \tan\theta}{b} = 1$$

Eccentric angle $(\theta) = \frac{\pi}{3}$

$\therefore$ Equation of the tangent at $P\left(\frac{\pi}{3}\right)$ is

$$\frac{x \sec\frac{\pi}{3}}{12} - \frac{y \tan\frac{\pi}{3}}{5} = 1$$

$\therefore$ $\frac{2x}{12} - \frac{\sqrt{3}y}{5} = 1$

$\therefore$ $\frac{x}{6} - \frac{\sqrt{3}y}{5} = 1$

$\therefore$ $5x - 6\sqrt{3}\,y = 30$

iv. Given equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

we get

$a^2 = 16$ and $b^2 = 9$

Let $P(x_1, 3)$ be the point on the hyperbola in the first quadrant at which the tangent is drawn.

Substituting $x = x_1$ and $y = 3$ in equation of hyperbola, we get

$$\frac{x_1^2}{16} - \frac{3^2}{9} = 1$$

$\therefore$ $\frac{x_1^2}{16} - 1 = 1$

$\therefore$ $\frac{x_1^2}{16} = 2$

$\therefore$ $x_1^2 = 32$

$\therefore$ $x_1 = \pm 4\sqrt{2}$

Since P lies in the first quadrant,

$P \equiv \left(4\sqrt{2}, 3\right)$

Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Equation of the tangent at $P\left(4\sqrt{2}, 3\right)$ is

$$\frac{4\sqrt{2}x}{16} - \frac{3y}{9} = 1$$

$\therefore$ $\frac{\sqrt{2}}{4}x - \frac{y}{3} = 1$

$\therefore$ $3\sqrt{2}\,x - 4y = 12$

v. Given equation of the hyperbola is
$9x^2 - 16y^2 = 144.$

$\therefore \quad \frac{x^2}{16} - \frac{y^2}{9} = 1$

Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,
we get
$a^2 = 16$ and $b^2 = 9$

$\therefore \quad a = 4$ and $b = 3$

Since the point L lies in the first quadrant,
Latus rectum of the hyperbola is
$L\left(ae, \frac{b^2}{a}\right)$

Now, $b^2 = a^2(e^2 - 1)$

$\therefore \quad 9 = 16(e^2 - 1)$

$\therefore \quad \frac{9}{16} = e^2 - 1$

$\therefore \quad e^2 = \frac{9}{16} + 1 = \frac{25}{16}$

$\therefore \quad e = \frac{5}{4} \qquad \ldots[\because e > 1]$

$\therefore \quad ae = 4\left(\frac{5}{4}\right) = 5$

$\therefore \quad L\left(ae, \frac{b^2}{a}\right) = L\left(5, \frac{9}{4}\right)$

Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

$\therefore$ Equation of the tangent at $L\left(5, \frac{9}{4}\right)$ is

$$\frac{5x}{16} - \frac{\frac{9}{4}y}{9} = 1$$

$\therefore \quad \frac{5x}{16} - \frac{y}{4} = 1$

$\therefore \quad 5x - 4y = 16$

Question 7.
Show that the line $3x - 4y + 10 = 0$ is a tangent to the hyperbola $x^2 - 4y^2 = 20$. Also, find the point of contact.
Solution:
Given equation of the hyperbola is $x^2 - 4y^2 = 20$

$\frac{x^2}{20} - \frac{y^2}{5} = 1$

Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get
$a^2 = 20$ and $b^2 = 5$
Given equation of line is $3x - 4y + 10 = 0$.

$y = \frac{3x}{4} + \frac{5}{2}$

Comparing this equation with $y = mx + c$, we get
$m = \frac{3}{4}$ and $c = \frac{5}{2}$

For the line y = mx + c to be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we must have

$$c^2 = a^2m^2 - b^2$$

$$c^2 = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$$

$$a^2m^2 - b^2 = 20\left(\frac{3}{4}\right)^2 - 5$$

$$= 20\left(\frac{9}{16}\right) - 5 = \frac{45}{4} - 5 = \frac{25}{4} = c^2$$

The given line is a tangent to the given hyperbola and point of contact

$$= \left(\frac{-a^2m}{c}, \frac{-b^2}{c}\right)$$

$$= \left(\frac{-20\left(\frac{3}{4}\right)}{\left(\frac{5}{2}\right)}, \frac{-5}{\left(\frac{5}{2}\right)}\right) = (-6, -2)$$

Question 8.
If the line 3x – 4y = k touches the hyperbola $\frac{x^2}{5}-\frac{4y^2}{5}=1$, then find the value of k.
Solution:
Given equation of the hyperbola is
$\frac{x^2}{5}-\frac{4y^2}{5}=1$
$\frac{x^2}{5}-\frac{y^2}{\frac{5}{4}}=1$
Comparing this equation with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get
$a^2 = 5$, $b^2 = \frac{5}{4}$
Given equation of line is 3x – 4y = k
y = $\frac{3}{4}x-\frac{k}{4}$
Comparing this equation with y = mx + c, we get
m = $\frac{3}{4}$, c = $-\frac{k}{4}$
For the line y = mx + c to be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we must have
$c^2 = a^2 m^2 - b^2$
⇒ $\left(-\frac{k}{4}\right)^2=5\left(\frac{3}{4}\right)^2-\frac{5}{4}$
⇒ $\frac{k^2}{16}=\frac{5}{16}(9-4)$
⇒ $\frac{k^2}{16}=\frac{5}{16}(5)$
⇒ $k^2 = 25$
⇒ k = ±5

Alternate method:
Given equation of the hyperbola is
$\frac{x^2}{5}-\frac{4y^2}{5}=1$ …….(i)
Given equation of the line is 3x – 4y = k
y = $\frac{3x-k}{4}$
Substituting this value ofy in (i), we get
$\frac{x^2}{5}-\frac{4}{5}\left(\frac{3x-k}{4}\right)^2=1$
⇒ $\frac{x^2}{5}-\frac{4}{5}\left(\frac{9x^2-6kx+k^2}{16}\right)=1$
⇒ $4x^2 – (9x^2 – 6kx + k^2) = 20$
⇒ $4x^2 – 9x^2 + 6kx – k^2 = 20$
⇒ $-5x^2 + 6kx – k^2 = 20$
⇒ $5x^2 – 6kx + (k^2 + 20) = 0$ …..(ii)
Since, the given line touches the given hyperbola.
The quadratic equation (ii) in x has equal roots.
$(-6k)^2 – 4(5)(k^2 + 20) = 0$
⇒ $36k^2 – 20k^2 – 400 = 0$
⇒ $16k^2 = 400$
⇒ $k^2 = 25$
⇒ k = ±5

Question 9.

Find the equations of the tangents to the hyperbola $\frac{x^2}{25}-\frac{y^2}{9}=1$ making equal intercepts on the co-ordinate axes.

Solution:

Given equation of the hyperbola is $\frac{x^2}{25}-\frac{y^2}{9}=1$.

Comparing this equation with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$a^2 = 25$ and $b^2 = 9$

Since the tangents make equal intercepts on the co-ordinate axes,

∴ m = -1

Equations of tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ having slope m are

y = mx ± $\sqrt{a^2m^2-b^2}$

⇒ y = -x ± $\sqrt{25(-1)^2-9}$

⇒ y = -x ± √16

⇒ x + y = ±4

Question 10.

Find the equations of the tangents to the hyperbola $5x^2 - 4y^2 = 20$ which are parallel to the line 3x + 2y + 12 = 0.

Solution:

Given equation of the hyperbola is $5x^2 - 4y^2 = 20$

$\frac{x^2}{4}-\frac{y^2}{5}=1$

Comparing this equation with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$a^2 = 4$ and $b^2 = 5$

Slope of the line 3x + 2y + 12 = 0 is $-\frac{3}{2}$

Since the given line is parallel to the tangents,

Slope of the required tangents (m) = $-\frac{3}{2}$

Equations of tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ having slope m are

y = mx ± $\sqrt{a^2m^2-b^2}$

$$y = -\frac{3}{2}x \pm \sqrt{4\left(-\frac{3}{2}\right)^2 - 5}$$

$$y = -\frac{3}{2}x \pm \sqrt{4\left(\frac{9}{4}\right) - 5}$$

$$y = -\frac{3}{2}x \pm \sqrt{4}$$

$$y = -\frac{3}{2}x \pm 2$$

$$3x + 2y = \pm 4$$

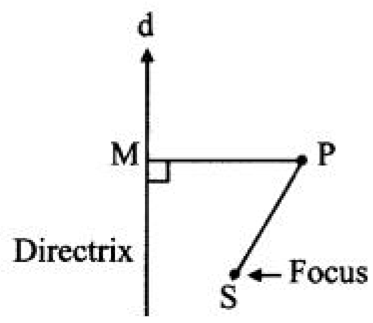
Maharashtra State Board 11th Maths Solutions Chapter 7 Conic Sections Miscellaneous Exercise 7

(I) Select the correct option from the given alternatives.

Question 1.
The line y = mx + 1 is a tangent to the parabola $y^2 = 4x$, if m is ________
(A) 1
(B) 2
(C) 3
(D) 4
Answer:
(A) 1
Hint:
$y^2 = 4x$
Compare with $y^2 = 4ax$
∴ a = 1
Equation of tangent is y = mx + 1
Compare with $y = mx + \frac{a}{m}$
$\frac{a}{m} = 1$
∴ a = m = 1

Question 2.
The length of latus rectum of the parabola $x^2 - 4x - 8y + 12 = 0$ is ________
(A) 4
(B) 6
(C) 8
(D) 10
Answer:
(C) 8
Hint:
Given equation of parabola is
$x^2 - 4x - 8y + 12 = 0$
$\Rightarrow x^2 - 4x = 8y - 12$
$\Rightarrow x^2 - 4x + 4 = 8y - 12 + 4$
$\Rightarrow (x - 2)^2 = 8(y - 1)$
Comparing this equation with $(x - h)^2 = 4b(y - k)$, we get
4b = 8
∴ Length of latus rectum = 4b = 8

Question 3.
If the focus of the parabola is (0, -3), its directrix is y = 3, then its equation is ________
(A) $x^2 = -12y$
(B) $x^2 = 12y$
(C) $y^2 = 12x$
(D) $y^2 = -12x$
Answer:
(A) $x^2 = -12y$
Hint:



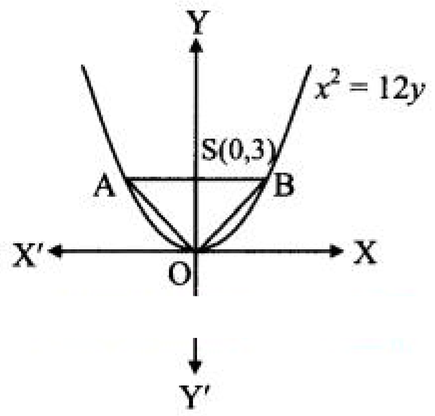
$SP^2 = PM^2$
$\Rightarrow (x - 0)^2 + (y + 3)^2 = \left|\frac{y-3}{\sqrt{1}}\right|^2$
$\Rightarrow x^2 + y^2 + 6y + 9 = y^2 - 6y + 9$
$\Rightarrow x^2 = -12y$

Question 4.
The co-ordinates of a point on the parabola $y^2 = 8x$ whose focal distance is 4 are ________
(A) $(\frac{1}{2}, \pm 2)$
(B) $(1, \pm 2\sqrt{2})$
(C) (2, ±4)
(D) none of these
Answer:
(C) (2, ±4)

Question 5.
The end points of latus rectum of the parabola $y^2 = 24x$ are ________
(A) (6, ±12)
(B) (12, ±6)
(C) (6, ±6)
(D) none of these
Answer:
(A) (6, ±12)

Question 6.
Equation of the parabola with vertex at the origin and directrix with equation x + 8 = 0 is ________
(A) $y^2 = 8x$
(B) $y^2 = 32x$
(C) $y^2 = 16x$
(D) $x^2 = 32y$
Answer:
(B) $y^2 = 32x$
Hint:
Since directrix is parallel to Y-axis,
The X-axis is the axis of the parabola.
Let the equation of parabola be $y^2 = 4ax$.
Equation of directrix is x + 8 = 0
∴ a = 8
∴ required equation of parabola is $y^2 = 32x$

Question 7.
The area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the endpoints of its latus rectum is ________
(A) 22 sq. units
(B) 20 sq. units
(C) 18 sq. units
(D) 14 sq. units
Answer:
(C) 18 sq. units
Hint:
$x^2 = 12y$
4b = 12
b = 3



Area of triangle = $\frac{1}{2}$ × AB × OS
= $\frac{1}{2}$ × 4a × a
= $\frac{1}{2}$ × 12 × 3
= 18 sq. units

Question 8.
If $P(\frac{\pi}{4})$ is any point on the ellipse $9x^2 + 25y^2 = 225$, S and S′ are its foci, then SP . S′P = ________
(A) 13
(B) 14
(C) 17
(D) 19
Answer:
(C) 17
Hint:
$9x^2 + 25y^2 = 225$
$\frac{x^2}{25}+\frac{y^2}{9}=1$
Here, a = 5, b = 3
Eccentricity (e) = $\frac{4}{5}$
$\therefore ae=5(\frac{4}{5})=254$
Coordinates of foci are S(4, 0) and S'(-4, 0)

P(θ) = (a cos θ, b sin θ)

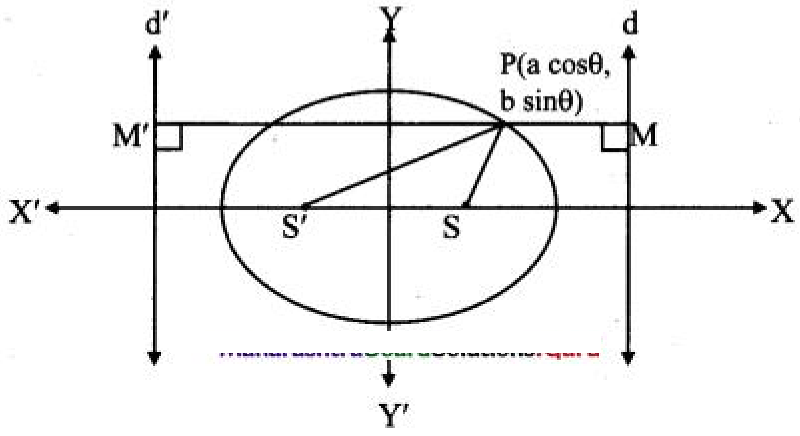
$\therefore \quad P\left(\frac{\pi}{4}\right) = \left(5\cos\frac{\pi}{4}, 3\sin\frac{\pi}{4}\right) = \left(\frac{5}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right).$

$PM = \frac{25}{4} - \frac{5}{\sqrt{2}} = \frac{25-10\sqrt{2}}{4}$

$SP = ePM = \frac{4}{5}\left(\frac{25-10\sqrt{2}}{4}\right) = 5 - 2\sqrt{2}$

Similarly, $S'P = 5 + 2\sqrt{2}$

$\therefore \quad SP.S'P = (5 - 2\sqrt{2})(5 + 2\sqrt{2}) = 25 - 8 = 17$



Question 9.
The equation of the parabola having (2, 4) and (2, -4) as end points of its latus rectum is _______
(A) $y^2 = 4x$
(B) $y^2 = 8x$
(C) $y^2 = -16x$
(D) $x^2 = 8y$
Answer:
(B) $y^2 = 8x$
Hint:
The given points lie in the 1st and 4th quadrants.
∴ Equation of the parabola is $y^2 = 4ax$
End points of latus rectum are (a, 2a) and (a, -2a)
∴ a = 2
∴ required equation of parabola is y = 8x

Question 10.
If the parabola $y^2 = 4ax$ passes through (3, 2), then the length of its latus rectum is _______
(A) 23
(B) 43
(C) 13
(D) 4
Answer:
(B) 43
Hint:
Length of latus rectum = 4a
The given parabola passes through (3, 2)
∴ $(2)^2 = 4a(3)$
∴ 4a = 43

Question 11.
The eccentricity of rectangular hyperbola is
(A) 12
(B) 12_{12}
(C) 2_{12}
(D) 13_{12}
Answer:
(C) 2_{12}

Question 12.
The equation of the ellipse having one of the foci at (4, 0) and eccentricity 13 is
(A) $9x^2 + 16y^2 = 144$
(B) $144x^2 + 9y^2 = 1296$

(C) $128x^2 + 144y^2 = 18432$
(D) $144x^2 + 128y^2 = 18432$
Answer:
(C) $128x^2 + 144y^2 = 18432$

Question 13.
The equation of the ellipse having eccentricity $\frac{\sqrt{3}}{2}$ and passing through (-8, 3) is
(A) $4x^2 + y^2 = 4$
(B) $x^2 + 4y^2 = 100$
(C) $4x^2 + y^2 = 100$
(D) $x^2 + 4y^2 = 4$
Answer:
(B) $x^2 + 4y^2 = 100$

Question 14.
If the line $4x – 3y + k = 0$ touches the ellipse $5x^2 + 9y^2 = 45$, then the value of k is
(A) 21
(B) $\pm 3\sqrt{21}$
(C) 3
(D) 3(21)
Answer:
(B) $\pm 3\sqrt{21}$

Question 15.
The equation of the ellipse is $16x^2 + 25y^2 = 400$. The equations of the tangents making an angle of 180° with the major axis are
(A) $x = 4$
(B) $y = \pm 4$
(C) $x = -4$
(D) $x = \pm 5$
Answer:
(B) $y = \pm 4$

Question 16.
The equation of the tangent to the ellipse $4x^2 + 9y^2 = 36$ which is perpendicular to $3x + 4y = 17$ is
(A) $y = 4x + 6$
(B) $3y + 4x = 6$
(C) $3y = 4x + 6\sqrt{5}$
(D) $3y = x + 25$
Answer:
(C) $3y = 4x + 6\sqrt{5}$

Question 17.
Eccentricity of the hyperbola $16x^2 – 3y^2 – 32x – 12y – 44 = 0$ is
(A) $\sqrt{\frac{17}{3}}$
(B) $\sqrt{\frac{19}{3}}$
(C) $\frac{19}{\sqrt{3}}$
(D) $\frac{17}{\sqrt{3}}$
Answer:
(B) $\sqrt{\frac{19}{3}}$
Hint:
$16x^2 – 3y^2 – 32x – 12y – 44 = 0$
$\Rightarrow 16(x – 1)^2 – 3(y + 2)^2 = 48$
$\Rightarrow \frac{(x-1)^2}{3} - \frac{(y+2)^2}{16} = 1$
Here, $a^2 = 3$ and $b^2 = 16$
$e = \frac{\sqrt{a^2+b^2}}{a} = \frac{\sqrt{3+16}}{\sqrt{3}} = \sqrt{\frac{19}{3}}$

Question 18.
Centre of the ellipse $9x^2 + 5y^2 – 36x – 50y – 164 = 0$ is at
(A) (2, 5)
(B) (1, -2)
(C) (-2, 1)
(D) (0, 0)
Answer:
(A) (2, 5)
Hint:
$9x^2 + 5y^2 – 36x – 50y – 164 = 0$

⇒ $9(x-2)^2 + 5(y-5)^2 = 325$

⇒ $\frac{(x-2)^2}{\frac{325}{9}} + \frac{(y-5)^2}{65} = 1$

⇒ centre of the ellipse = (2, 5)

Question 19.
If the line $2x - y = 4$ touches the hyperbola $4x^2 - 3y^2 = 24$, the point of contact is
(A) (1, 2)
(B) (2, 3)
(C) (3, 2)
(D) (-2, -3)
Answer:
(C) (3, 2)

Question 20.
The foci of hyperbola $4x^2 - 9y^2 - 36 = 0$ are
(A) $(\pm\sqrt{13}, 0)$
(B) $(\pm\sqrt{11}, 0)$
(C) $(\pm\sqrt{12}, 0)$
(D) $(0, \pm\sqrt{12})$
Answer:
(A) $(\pm\sqrt{13}, 0)$

II. Answer the following.

Question 1.
For each of the following parabolas, find focus, equation of file directrix, length of the latus rectum and ends of the latus rectum.
(i) If $2y^2 = 17x$
(ii) $5x^2 = 24y$
Solution:
(i) Given equation of the parabola is $2y^2 = 17x$
$y^2 = \frac{17}{2}x$
Comparing this equation with $y^2 = 4ax$, we get
$4a = \frac{17}{2}$
$a = \frac{17}{8}$
Co-ordinates of focus are S(a, 0), i.e., S$(\frac{17}{8}, 0)$
Equation of the directrix is $x + a = 0$
$x + \frac{17}{8} = 0$
$8x + 17 = 0$
Length of latus rectum = $4a = 4(\frac{17}{8}) = \frac{17}{2}$
Co-ordinates of end points of latus rectum are (a, 2a) and (a, -2a)
i.e., $\left(\frac{17}{8}, \frac{17}{4}\right)$ and $\left(\frac{17}{8}, -\frac{17}{4}\right)$

(ii) Given equation of the parabola is $5x^2 = 24y$
$x^2 = \frac{24y}{5}$
Comparing this equation with $x^2 = 4by$, we get
$4b = \frac{24}{5}$
$b = \frac{6}{5}$
Co-ordinates of focus are S(0, b), i.e., S$(0, \frac{6}{5})$
Equation of the directrix is $y + b = 0$
$y + \frac{6}{5} = 0$
$5y + 6 = 0$
Length of latus rectum = $4b = 4(\frac{6}{5}) = \frac{24}{5}$
Co-ordinates of end points of latus rectum are (2b, b) and (-2b, b), i.e., $\left(\frac{12}{5}, \frac{6}{5}\right)$ and $\left(-\frac{12}{5}, \frac{6}{5}\right)$

Question 2.
Find the cartesian co-ordinates of the points on the parabola $y^2 = 12x$ whose parameters are
(i) 2
(ii) -3
Solution:
Given equation of the parabola is $y^2 = 12x$
Comparing this equation with $y^2 = 4ax$, we get
$4a = 12$
∴ $a = 3$

If t is the parameter of the point P on the parabola, then
$P(t) = (at^2, 2at)$
i.e., $x = at^2$ and $y = 2at$ .....(i)
(i) Given, t = 2
Substituting a = 3 and t = 2 in (i), we get
$x = 3(2)^2$ and $y = 2(3)(2)$
x = 12 and y = 12
∴ The cartesian co-ordinates of the point on the parabola are (12, 12).

(ii) Given, t = -3
Substitùting a = 3 and t = -3 in (i), we get
$x = 3(-3)^2$ and $y = 2(3)(-3)$
∴ x = 27 and y = -18
∴ The cartesian co-ordinates of the point on the parabola are (27, -18).

Question 3.
Find the co-ordinates of a point of the parabola $y^2 = 8x$ having focal distance 10.
Solution:
Given equation of the parabola is $y^2 = 8x$
Comparing this equation with $y^2 = 4ax$, we get
4a = 8
∴ a = 2
Focal distance of a point = x + a
Given, focal distance = 10
x + 2 = 10
∴ x = 8
Substituting x = 8 in $y^2 = 8x$, we get
$y^2 = 8(8)$
∴ y = ±8
∴ The co-ordinates of the points on the parabola are (8, 8) and (8, -8).

Question 4.
Find the equation of the tangent to the parabola $y^2 = 9x$ at the point (4, -6) on it.
Solution:
Given equation of the parabola is $y^2 = 9x$
Comparing this equation with $y^2 = 4ax$, we get
4a = 9
∴ $a = \frac{9}{4}$
Equation of the tangent $y^2 = 4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$
The equation of the tangent at (4, -6) is
$y(-6) = 2(\frac{9}{4})(x + 4)$
⇒ $-6y = \frac{9}{2}(x + 4)$
⇒ -12y = 9x + 36
⇒ 9x + 12y + 36 = 0
⇒ 3x + 4y + 12 = 0

Question 5.
Find the equation of the tangent to the parabola $y^2 = 8x$ at t = 1 on it.
Solution:
Given equation of the parabola is $y^2 = 8x$
Comparing this equation with $y^2 = 4ax$, we get
4a = 8
a = 2
t = 1
Equation of tangent with parameter t is $yt = x + at^2$
∴ The equation of tangent with t = 1 is
$y(1) = x + 2(1)^2$
y = x + 2
∴ x – y + 2 = 0

Question 6.
Find the equations of the tangents to the parabola $y^2 = 9x$ through the point (4, 10).
Solution:
Given equation of the parabola is $y^2 = 9x$
Comparing this equation with $y^2 = 4ax$, we get
4a = 9
∴ $a = \frac{9}{4}$
Equation of tangent to the parabola $y^2 = 4ax$ having slope m is

$y = mx + \frac{a}{m}$
$y = mx + \frac{9}{4m}$
But, (4, 10) lies on the tangent.
$10 = 4m + \frac{9}{4m}$
$\Rightarrow 40m = 16m^2 + 9$
$\Rightarrow 16m^2 - 40m + 9 = 0$
$\Rightarrow 16m^2 - 36m - 4m + 9 = 0$
$\Rightarrow 4m(4m - 9) - 1(4m - 9) = 0$
$\Rightarrow (4m - 9)(4m - 1) = 0$
$\Rightarrow 4m - 9 = 0$ or $4m - 1 = 0$
$\Rightarrow m = \frac{9}{4}$ or $m = \frac{1}{4}$
These are the slopes of the required tangents.
By slope point form, $y - y_1 = m(x - x_1)$,
the equations of the tangents are
$y - 10 = \frac{9}{4}(x - 4)$ or $y - 10 = \frac{1}{4}(x - 4)$
$\Rightarrow 4y - 40 = 9x - 36$ or $4y - 40 = x - 4$
$\Rightarrow 9x - 4y + 4 = 0$ or $x - 4y + 36 = 0$

Question 7.
Show that the two tangents drawn to the parabola $y^2 = 24x$ from the point (-6, 9) are at the right angle.
Solution:
Given the equation of the parabola is $y^2 = 24x$.
Comparing this equation with $y^2 = 4ax$, we get
$4a = 24$
$\Rightarrow a = 6$
Equation of tangent to the parabola $y^2 = 4ax$ having slope m is
$y = mx + \frac{a}{m}$
$\Rightarrow y = mx + \frac{6}{m}$
But, (-6, 9) lies on the tangent
$9 = -6m + \frac{6}{m}$
$\Rightarrow 9m = -6m^2 + 6$
$\Rightarrow 6m^2 + 9m - 6 = 0$
The roots m_1 and m_2 of this quadratic equation are the slopes of the tangents.
$m_1 m_2 = -1$
Tangents drawn to the parabola $y^2 = 24x$ from the point (-6, 9) are at a right angle.

Alternate method:
Comparing the given equation with $y^2 = 4ax$, we get
$4a = 24$
$\Rightarrow a = 6$
Equation of the directrix is $x = -6$.
The given point lies on the directrix.
Since tangents are drawn from a point on the directrix are perpendicular,
Tangents drawn to the parabola $y^2 = 24x$ from the point (-6, 9) are at the right angle.

Question 8.
Find the equation of the tangent to the parabola $y^2 = 8x$ which is parallel to the line $2x + 2y + 5 = 0$. Find its point of contact.
Solution:
Given the equation of the parabola is $y^2 = 8x$.
Comparing this equation with $y^2 = 4ax$, we get
$4a = 8$
$a = 2$
Slope of the line $2x + 2y + 5 = 0$ is -1
Since the tangent is parallel to the given line,
slope of the tangent line is $m = -1$
Equation of tangent to the parabola $y^2 = 4ax$ having slope m is $y = mx + \frac{a}{m}$
Equation of the tangent is
$y = -x + \frac{2}{-1}$
$x + y + 2 = 0$
Point of contact $= \left(\frac{a}{m^2}, \frac{2a}{m}\right)$
$= \left(\frac{2}{(-1)^2}, \frac{2(2)}{-1}\right)$
$= (2, -4)$

Question 9.
A line touches the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$. Show that its equation is $y = \pm(x + 2)$.
Solution:

Given equation of the parabola is $y^2 = 8x$
Comparing this equation with $y^2 = 4ax$, we get
$4a = 8$
$a = 2$
Equation of tangent to given parabola with slope m is
y = mx + 2m
$m^2x – my + 2 = 0$ ….(i)
Equation of the circle is $x^2 + y^2 = 2$
Its centre = (0, 0) and Radius = √2
Line (i) touches the circle.
Length of perpendicular from the centre to the line (i) = radius
⇒ ∣∣ m2(0)−m(0)+2m4+m2√ ∣∣ = √2
⇒ 4m4+m2 = 2
⇒ $m^4 + m^2 – 2 – 0$
⇒ $(m^2 + 2)(m^2 – 1) = 0$
Since $m^2 \neq -2$,
$m^2 – 1 = 0$
⇒ m = ±1
When m = 1, equation of the tangent is
y = (1)x + 2(1)
y = (x + 2) …..(i)
When m = -1, equation of the tangent is
y = (-1)x + 2(−1)
y = -x – 2
y = -(x + 2) …..(ii)
From (i) and (ii),
equation of the common tangents to the given parabola is y = ±(x + 2)

Question 10.
Two tangents to the parabola $y^2 = 8x$ meet the tangents at the vertex in P and Q. If PQ = 4, prove that the locus of the point of intersection of the two tangents is $y^2 = 8(x + 2)$.
Solution:
Given parabola is $y^2 = 8x$
Comparing with $y^2 = 4ax$, we get,
$4a = 8$
⇒ a = 2
Let $M(t_1)$ and $N(t_2)$ be any two points on the parabola.
The equations of tangents at M and N are
yt_1 = x + 2t21 …..(1)
yt_2 = x + 2t22 …(2) ….[∵ a = 2]
Let tangent at M meet the tangent at the vertex in P.
But tangent at the vertex is Y-axis whose equation is x = 0.
⇒ to find P, put x = 0 in (1)
⇒ yt_1 = 2t21
⇒ $y = 2t_1$ …..($t_1 \neq 0$ otherwise tangent at M will be x = 0)
⇒ $P = (0, 2t_1)$
Similarly, $Q = (0, 2t_2)$
It is given that PQ = 4
∴ $|2t_1 – 2t_2| = 4$
∴ $|t_1 – t_2| = 2$ …..(3)
Let $R = (x_1, y_1)$ be any point on the required locus.
Then R is the point of intersection of tangents at M and N.
To find R, we solve (1) and (2).
Subtracting (2) from (1), we get
$y(t_1 – t_2)$ = 2t21−2t22
$y(t_1 – t_2) = 2(t_1 – t_2)(t_1 + t_2)$
∴ $y = 2(t_1 + t_2)$ …..[∵ M, N are distinct ∴ $t_1 \neq t_2$]
i.e., $y_1 = 2(t_1 + t_2)$ …..(4)
∴ from (1), we get
$2t_1(t_1 + t_2)$ = x + 2t21
∴ $2t_1t_2 = x$ i.e. $x_1 = 2t_1t_2$ …..(5)
To find the equation of locus of $R(x_1, y_1)$,
we eliminate t_1 and t_2 from the equations (3), (4) and (5).
We know that,
$(t_1 + t_2)^2 = (t_1 + t_2)^2 + 4t_1t_2$

⇒ $(\frac{y_1}{2})^2 = 4 + 4(\frac{x_1}{2})$ ...[By (3), (4) and (5)]

⇒ $y_1^2 = 16 + 8x_1 = 8(x_1 + 2)$

Replacing x_1 by x and y_1 by y,
the equation of required locus is $y^2 = 8(x + 2)$.

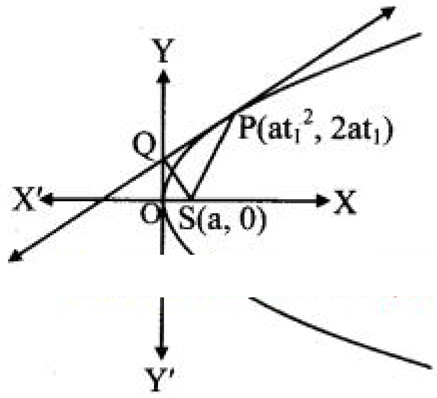
Question 11.
The slopes of the tangents drawn from P to the parabola $y^2 = 4ax$ are m_1 and m_2, showing that
(i) $m_1 – m_2 = k$
(ii) $(\frac{m_1}{m_2}) = k$, where k is a constant.
Solution:
Let $P(x_1, y_1)$ be any point on the parabola $y^2 = 4ax$.
Equation of tangent to the parabola $y^2 = 4ax$ having slope m is $y = mx + \frac{a}{m}$
This tangent passes through $P(x_1, y_1)$.
$y_1 = mx_1 + \frac{a}{m}$
$my_1 = m^2x_1 + a$
$m^2x_1 – my_1 + a = 0$
This is a quadratic equation in 'm'.
The roots m_1 and m_2 of this quadratic equation are the slopes of the tangents drawn from P.
∴ $m_1 + m_2 = \frac{y_1}{x_1}$, $m_1m_2 = \frac{a}{x_1}$

i. $$(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$$
$$= \left(\frac{y_1}{x_1}\right)^2 - \frac{4a}{x_1} = \frac{y_1^2 - 4ax_1}{x_1^2}$$
∴ $$m_1 - m_2 = \sqrt{\frac{y_1^2 - 4ax_1}{x_1^2}}$$

Since (x_1, y_1) and a are constants, $m_1 – m_2$ is a constant.
∴ $m_1 – m_2 = k$, where k is constant.

(ii) Since (x_1, y_1) and a are constants, m_1m_2 is a constant.
$(\frac{m_1}{m_2}) = k$, where k is a constant.

Question 12.
The tangent at point P on the parabola $y^2 = 4ax$ meets the Y-axis in Q. If S is the focus, show that SP subtends a right angle at Q.
Solution:
Let $P(at_1^2, 2at_1)$ be a point on the parabola and
S(a, 0) be the focus of parabola $y^2 = 4ax$



Since the tangent passing through point P meet Y-axis at point Q,
equation of tangent at $P(at_1^2, 2at_1)$ is

$yt_1 = x + at_1^2$ .....(i)
∴ Point Q lie on tangent
∴ put x = 0 in equation (i)
$yt_1 = at_1^2$
$y = at_1$
∴ Co-ordinate of point $Q(0, at_1)$
$S = (a, 0)$, $P(at_1^2, 2at_1)$, $Q(0, at_1)$

Slope of SQ = $\frac{y_2-y_1}{x_2-x_1}=\frac{at_1-0}{0-a}=\frac{at_1}{-a}=-t_1$

Slope of PQ = $\frac{y_2-y_1}{x_2-x_1}$

$=\frac{2at_1-at_1}{at_1^2}=\frac{at_1}{at_1^2}=\frac{1}{t_1}$

Slope of SQ × Slope of PQ

$=-t_1\times\frac{1}{t_1}=-1$

∴ SP subtends a right angle at Q.

Question 13.
Find the
(i) lengths of the principal axes
(ii) co-ordinates of the foci
(iii) equations of directrices
(iv) length of the latus rectum
(v) Distance between foci
(vi) distance between directrices of the curve

(a) $\frac{x^2}{25}+\frac{y^2}{9}=1$

(b) $16x^2 + 25y^2 = 400$

(c) $\frac{x^2}{144}-\frac{y^2}{25}=1$

(d) $x^2 - y^2 = 16$

Solution:

(a) Given equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2 = 25$ and $b^2 = 9$
∴ a = 5 and b = 3
Since a > b,
X-axis is the major axis and Y-axis is the minor axis.
(i) Length of major axis = 2a = 2(5) = 10
Length of minor axis = 2b = 2(3) = 6
∴ Lengths of the principal axes are 10 and 6.

(ii) We know that e = $\frac{\sqrt{a^2-b^2}}{a}$
∴ e = $\frac{\sqrt{25-9}}{5}=\frac{4}{5}$
Co-ordinates of the foci are S(ae, 0) and S'(-ae, 0)
i.e., S(5($\frac{4}{5}$), 0) and S'(-5($\frac{4}{5}$), 0),
i.e., S(4, 0) and S'(-4, 0)

(iii) Equations of the directrices are x = $\pm\frac{a}{e}$
i.e., x = $\pm\frac{5}{\frac{4}{5}}$
i.e., x = $\pm\frac{25}{4}$

(iv) Length of latus rectum = $\frac{2b^2}{a}=\frac{2(3)^2}{5}=\frac{18}{5}$

(v) Distance between foci = 2ae = 2 (5) ($\frac{4}{5}$) = 8

(vi) Distance between directrices = $\frac{2a}{e}=\frac{2(5)}{\frac{4}{5}}=\frac{25}{2}$

(b) Given equation of the ellipse is $16x^2 + 25y^2 = 400$

$\frac{x^2}{25}+\frac{y^2}{16}=1$

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2 = 25$ and $b^2 = 16$
∴ a = 5 and b = 4
Since a > b,
X-axis is the major axis and Y-axis is the minor axis
(i) Length of major axis = 2a = 2(5) = 10
Length of minor axis = 2b = 2(4) = 8
Lengths of the principal axes are 10 and 8.

(ii) $b^2 = a^2 (1 - e^2)$
$16 = 25(1 - e^2)$
$\frac{16}{25} = 1 - e^2$
$e^2 = 1 - \frac{16}{25}$
$e^2 = \frac{9}{25}$
$e = \frac{3}{5}$ ……[$\because$ $0 < e < 1$]
Co-ordinates of the foci are S(ae, 0) and S'(-ae, 0),
i.e., $S(5(\frac{3}{5}), 0)$ and $S'(-5(\frac{3}{5}), 0)$,
i.e., S(3, 0) and S'(-3, 0)

(iii) Equations of the directrices are $x = \pm\frac{a}{e}$
i.e., $x = \pm\frac{5}{(\frac{3}{5})}$
i.e., $x = \pm\frac{25}{3}$

(iv) Length of latus rectum = $\frac{2b^2}{a} = \frac{2(16)}{5} = \frac{32}{5}$

(v) Distance between foci = $2ae = 2(5)(\frac{3}{5}) = 6$

(vi) Distance between directrices = $\frac{2a}{e} = \frac{2(5)}{(\frac{3}{5})} = \frac{50}{3}$

(c) Given equation of the hyperbola $\frac{x^2}{144} - \frac{y^2}{25} = 1$
Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
$a^2 = 144$ and $b^2 = 25$
$\because$ a = 12 and b = 5
(i) Length of transverse axis = 2a = 2(12) = 24
Length of conjugate axis = 2b = 2(5) = 10
lengths of the principal axes are 24 and 10.

(ii) $b^2 = a^2(e^2 - 1)$
$25 = 144 (e^2 - 1)$
$\frac{25}{144} = e^2 - 1$
$e^2 = 1 + \frac{25}{144}$
$e^2 = \frac{169}{144}$
$e = \frac{13}{12}$ …….[$\because$ $e > 1$]
Co-ordinates of foci are S(ae, 0) and S'(-ae, 0)
i.e., $S(12(\frac{13}{12}), 0)$ and $S'(-12(\frac{13}{12}), 0)$
i.e., S(13, 0) and S'(-13, 0)

(iii) Equations of the directrices are $x = \pm\frac{a}{e}$
i.e., $x = \pm\frac{12}{(\frac{13}{12})}$
i.e., $x = \pm\frac{144}{13}$

(iv) Length of latus rectum = $\frac{2b^2}{a} = \frac{2(25)}{12} = \frac{25}{6}$

(v) Distance between foci = $2ae = 2(12)(\frac{13}{12}) = 26$

(vi) Distance between directrices = $\frac{2a}{e} = \frac{2(12)}{(\frac{13}{12})} = \frac{288}{13}$

(d) Given equation of the hyperbola is $x^2 - y^2 = 16$
$\therefore \frac{x^2}{16} - \frac{y^2}{16} = 1$
Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get
$a^2 = 16$ and $b^2 = 16$
$\therefore$ a = 4 and b = 4
(i) Length of transverse axis = 2a = 2(4) = 8
Length of conjugate axis = 2b = 2(4) = 8

(ii) We know that

$$e = \frac{\sqrt{a^2 + b^2}}{a}$$

$$= \frac{\sqrt{16+16}}{4}$$

$$= \frac{\sqrt{32}}{4}$$

$$= \frac{4\sqrt{2}}{4}$$

$$= \sqrt{2}$$

Co-ordinates of foci are S(ae, 0) and S'(-ae, 0),
i.e., S(4√2, 0) and S'(-4√2, 0)

(iii) Equations of the directrices are x = ±ae
∴x = ± 42√
∴ x = ±2√2

(iv) Length of latus rectum = 2b2a = 2(16)4 = 8

(v) Distance between foci = 2ae = 2(4)(√2) = 8√2

(vi) Distance between directrices = 2ae = 2(4)2√ = 4√2.

Question 14.
Find the equation of the ellipse in standard form if
(i) eccentricity = 38 and distance between its foci = 6.
(ii) the length of the major axis is 10 and the distance between foci is 8.
(iii) passing through the points (-3, 1) and (2, -2).
Solution:
(i) Let the required equation of ellipse be x2a2+y2b2=1, where a > b.
Given, eccentricity (e) = 38
Distance between foci = 2ae
Given, distance between foci = 6
∴ 2ae = 6
∴ 2a(38) = 6
∴ 3a4 = 6
∴ a = 8
∴ a2 = 64
Now, b2 = a2 (1 – e2)
= 64[1−(38)2]
= 4(1−964)
= 64(5564)
= 55
∴ The required equation of the ellipse is x264+y255=1

(ii) Let the equation of the ellipse be
x2a2+y2b2=1 ......(1)
Then length of major axis = 2a = 10
∴ a = 5
Also, distance between foci= 2ae = 8
∴ 2 × 5 × e = 8
∴ e = 45
∴ b2 = a2(1 – e2)
= 25(1 – 625)
= 9
∴ from (1), the equation of the required ellipse is x225+y29=1

(iii) Let the required equation of ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a > b.
The ellipse passes through the points (-3, 1) and (2, -2).
∴ Substituting x = -3 and y = 1 in equation of ellipse, we get
$\frac{(-3)^2}{a^2}+\frac{1^2}{b^2}=1$
∴ $\frac{9}{a^2}+\frac{1}{b^2}=1$ …..(i)
Substituting x = 2 and y = -2 in equation of ellipse, we get
$\frac{2^2}{a^2}+\frac{(-2)^2}{b^2}=1$
∴ $\frac{4}{a^2}+\frac{4}{b^2}=1$ ……(ii)
Let $\frac{1}{a^2}$ = A and $\frac{1}{b^2}$ = B
∴ Equations (i) and (ii) become
9A + B = 1 …..(iii)
4A + 4B = 1 …..(iv)
Multiplying (iii) by 4, we get
36A + 4B = 4 …..(v)
Subtracting (iv) from (v), we get
32A = 3
∴ A = $\frac{3}{32}$
Substituting A = $\frac{3}{32}$ in (iv), we get
4($\frac{3}{32}$) + 4B = 1
∴ $\frac{3}{8}$ + 4B = 1
∴ 4B = 1 – $\frac{3}{8}$
∴ 4B = $\frac{5}{8}$
∴ B = $\frac{5}{32}$
Since $\frac{1}{a^2}$ = A and $\frac{1}{b^2}$ = B
$\frac{1}{a^2}=\frac{3}{32}$ and $\frac{1}{b^2}=\frac{5}{32}$
∴ a^2 = $\frac{32}{3}$ and b^2 = $\frac{32}{5}$
∴ The required equation of ellipse is
$\frac{x^2}{\left(\frac{32}{3}\right)}+\frac{y^2}{\left(\frac{32}{5}\right)}$
i.e., $3x^2 + 5y^2 = 32$.

Question 15.
Find the eccentricity of an ellipse if the distance between its directrices is three times the distance between its foci.
Solution:
Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
It is given that,
distance between directrices is three times the distance between the foci.
∴ $\frac{2a}{e}$ = 3(2ae)
∴ 1 = $3e^2$
∴ e^2 = $\frac{1}{3}$
∴ e = $\frac{1}{\sqrt{3}}$ …..[∵ 0 < e < 1]

Question 16.
For the hyperbola $\frac{x^2}{100}-\frac{y^2}{25}=1$, prove that SA . S'A = 25, where S and S' are the foci and A is the vertex.
Solution:
Given equation of the hyperbola is $\frac{x^2}{100}-\frac{y^2}{25}=1$
Comparing this equation with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get
a^2 = 100 and b^2 = 25
∴ a = 10 and b = 5
∴ Co-ordinates of vertex is A(a, 0), i.e., A(10, 0)
Eccentricity, e = $\frac{\sqrt{a^2+b^2}}{a}$
= $\frac{\sqrt{100+25}}{10}$
= $\frac{\sqrt{125}}{10}$
= $\frac{5\sqrt{5}}{10}$
= $\frac{\sqrt{5}}{2}$
Co-ordinates of the foci are S(ae, 0) and S'(-ae, 0)
i.e., S(10($\frac{\sqrt{5}}{2}$), 0) and S'(-10($\frac{\sqrt{5}}{2}$), 0)
i.e., S(5√5, 0) and S'(-5√5, 0)
Since S, A and S' lie on the X-axis,
SA = |5√5 – 10| and S'A = |-5√5 – 10|
= |-(5√5 + 10)|
= |5√5 + 10|

∴ SA . S'A = |5√5 – 10| |5√5 + 10|
= $|(5\sqrt{5})^2 - (10)^2|$
= |125 – 100|
= |25|
SA . S'A = 25

Question 17.
Find the equation of the tangent to the ellipse $\frac{x^2}{5}+\frac{y^2}{4}=1$ passing through the point (2, -2).
Solution:
Given equation of the ellipse is $\frac{x^2}{5}+\frac{y^2}{4}=1$
Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get
$a^2 = 5$ and $b^2 = 4$
Equations of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having slope m are
y = mx ± $\sqrt{a^2m^2+b^2}$
Since (2, -2) lies on both the tangents,
-2 = 2m ± $\sqrt{5m^2+4}$
∴ -2 – 2m = ±$\sqrt{5m^2+4}$
Squaring both the sides, we get
$4m^2 + 8m + 4 = 5m^2 + 4$
∴ $m^2 - 8m = 0$
∴ m(m – 8) = 0
∴ m = 0 or m = 8
These are the slopes of the required tangents.
∴ By slope point form $y - y_1 = m(x - x_1)$,
the equations of the tangents are
y + 2 = 0(x – 2) and y + 2 = 8(x – 2)
∴ y + 2 = 0 and y + 2 = 8x – 16
∴ y + 2 = 0 and 8x – y – 18 = 0.

Question 18.
Find the equation of the tangent to the ellipse $x^2 + 4y^2 = 100$ at (8, 3).
Solution:
Given equation of ellipse is $x^2 + 4y^2 = 100$
∴ $\frac{x^2}{100}+\frac{y^2}{25}=1$
Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get
$a^2 = 100$ and $b^2 = 25$
Equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$
Equation of tangent at (8, 3) is
$\frac{8x}{100}+\frac{3y}{25}=1$
$\frac{2x}{25}+\frac{3y}{25}=1$
2x + 3y = 25

Question 19.
Show that the line 8y + x = 17 touches the ellipse $x^2 + 4y^2 = 17$. Find the point of contact.
Solution:

Given equation of the ellipse is $x^2 + 4y^2 = 17$.

$\therefore \frac{x^2}{17} + \frac{y^2}{\frac{17}{4}} = 1$

Comparing this equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ we get

$a^2 = 17$ and $b^2 = \frac{17}{4}$

Given equation of line is $8y + x = 17$,

i.e., $y = \frac{-1}{8}x + \frac{17}{8}$

Comparing this equation with $y = mx + c$, we get

$m = \frac{-1}{8}$ and $c = \frac{17}{8}$

For the line $y = mx + c$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we must have

$c^2 = a^2 m^2 + b^2$

$c^2 = \left(\frac{17}{8}\right)^7 = \frac{289}{64}$

$a^2m^2 + b^2 = 17\left(\frac{-1}{8}\right)^2 + \frac{17}{4}$

$= \frac{17}{64} + \frac{17}{4}$

$= \frac{289}{64}$

$= c^2$

$\therefore$ The given line touches the given ellipse and point of contact is

$$\left(\frac{-a^2m}{c}, \frac{b^2}{c}\right) = \left(\frac{-17\left(\frac{-1}{8}\right)}{\frac{17}{8}}, \frac{\frac{17}{4}}{\frac{17}{8}}\right)$$

$= (1, 2)$.

Question 20.
Tangents are drawn through a point P to the ellipse $4x^2 + 5y^2 = 20$ having inclinations θ_1 and θ_2 such that $\tan \theta_1 + \tan \theta_2 = 2$. Find the equation of the locus of P.
Solution:
Given equation of the ellipse is $4x^2 + 5y^2 = 20$.

$\therefore \frac{x^2}{5} + \frac{y^2}{4} = 1$

Comparing this equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 5$ and $b^2 = 4$
Since inclinations of tangents are θ_1 and θ_2,
$m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$

Equation of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ having slope m are

$y = mx \pm \sqrt{a^2m^2 + b^2}$

$\therefore y = mx \pm \sqrt{5m^2 + 4}$

$\therefore y - mx = \pm\sqrt{5m^2 + 4}$

Squaring both the sides, we get
$y^2 - 2mxy + m^2x^2 = 5m^2 + 4$
$\therefore (x^2 - 5)m^2 - 2xym + (y^2 - 4) = 0$

The roots m_1 and m_2 of this quadratic equation are the slopes of the tangents.

∴ $m_1 + m_2 = -\frac{(-2xy)}{x^2-5} = \frac{2xy}{x^2-5}$

Given, $\tan\theta_1 + \tan\theta_2 = 2$

∴ $m_1 + m_2 = 2$

∴ $\frac{2xy}{x^2-5}$

∴ $xy = x^2 - 5$

∴ $x^2 - xy - 5 = 0$, which is the required equation of the locus of P.

Question 21.

Show that the product of the lengths of its perpendicular segments drawn from the foci to any tangent line to the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ is equal to 16.

Solution:

Given equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{16}=1$

Comparing this equation with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

∴ $a^2 = 25$, $b^2 = 16$

∴ $a = 5$, $b = 4$

We know that e = $\frac{\sqrt{a^2-b^2}}{a}$

∴ e = $\frac{\sqrt{25-16}}{5}$ = $\frac{3}{5}$

ae = $5\left(\frac{3}{5}\right) = 3$

Co-ordinates of foci are S(ae, 0) and S'(-ae, 0),

i.e., S(3, 0) and S'(-3, 0)

Equations of tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having slope m are

$y = mx \pm \sqrt{a^2m^2+b^2}$

Equation of one of the tangents to the ellipse is

$y = mx + \sqrt{25m^2+16}$

∴ $mx - y + \sqrt{25m^2+16} = 0$ .....(i)

p_1 = length of perpendicular segment from S(3, 0) to the tangent (i)

$$= \left|\frac{m(3) - 0 + \sqrt{25m^2+16}}{\sqrt{m^2+1}}\right|$$

$$\therefore p_1 = \left|\frac{3m + \sqrt{25m^2+16}}{\sqrt{m^2+1}}\right|$$

p_2 = length of perpendicular segment from S'(-3, 0) to the tangent (i)

$$= \left|\frac{m(-3) - 0 + \sqrt{25m^2+16}}{\sqrt{m^2+1}}\right|$$

$$\therefore p_2 = \left|\frac{-3m + \sqrt{25m^2+16}}{\sqrt{m^2+1}}\right|$$

$$\therefore p_1p_2 = \left|\frac{3m + \sqrt{25m^2+16}}{\sqrt{m^2+1}}\right|\left|\frac{-3m + \sqrt{25m^2+16}}{\sqrt{m^2+1}}\right|$$

$$= \frac{\left(25m^2+16\right) - 9m^2}{m^2+1}$$

$$= \frac{16\left(m^2+1\right)}{m^2+1}$$

$$= 16$$

Question 22.
Find the equation of the hyperbola in the standard form if
(i) Length of conjugate axis is 5 and distance between foci is 13.
(ii) eccentricity is $\frac{3}{2}$ and distance between foci is 12.
(iii) length of the conjugate axis is 3 and the distance between the foci is 5.
Solution:
(i) Let the required equation of hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Length of conjugate axis = 2b
Given, length of conjugate axis = 5
2b = 5
b = $\frac{5}{2}$
$b^2 = \frac{25}{4}$
Distance between foci = 2ae
Given, distance between foci = 13
2ae = 13
ae = $\frac{13}{2}$
$a^2e^2 = \frac{169}{4}$
Now, $b^2 = a^2(e^2 - 1)$
$b^2 = a^2e^2 - a^2$
$\frac{25}{4} = \frac{169}{4} - a^2$
$a^2 = \frac{169}{4}-\frac{25}{4} = 36$
∴ The required equation of hyperbola is $\frac{x^2}{36}-\frac{y^2}{\frac{25}{4}}=1$
i.e., $\frac{x^2}{36}-\frac{4y^2}{25}=1$

(ii) Let the required equation of hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Given, eccentricity (e) = $\frac{3}{2}$
Distance between foci = 2ae
Given, distance between foci = 12
∴ 2ae = 12
∴ $2a\left(\frac{3}{2}\right) = 12$
∴ 3a = 12
∴ a = 4
∴ $a^2 = 16$
Now, $b^2 = a^2(e^2 - 1)$
∴ $b^2 = \left[\left(\frac{3}{2}\right)^2-1\right]$
∴ $b^2 = 16\left(\frac{9}{4} - 1\right)$
∴ $b^2 = 16\left(\frac{5}{4}\right)$
∴ $b^2 = 20$
∴ The required equation of hyperbola is $\frac{x^2}{16}-\frac{y^2}{20}=1$

(iii) Let the required equation of hyperbola be $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$
Length of conjugate axis = 2b
Given, length of conjugate axis = 3
∴ 2b = 3
∴ b = $\frac{3}{2}$
∴ $b^2 = \frac{9}{4}$
Distance between foci = 2ae
Given, distance between foci = 5
∴ 2ae = 5
∴ ae = $\frac{5}{2}$
∴ $a^2e^2 = \frac{25}{4}$
Now, $b^2 = a^2(e^2 - 1)$
∴ $b^2 = a^2e^2 - a^2$
∴ $\frac{9}{4} = \frac{25}{4} - a^2$
∴ $a^2 = \frac{25}{4}-\frac{9}{4}$
∴ $a^2 = 4$
∴ The required equation of hyperbola is $\frac{x^2}{4}-\frac{y^2}{\left(\frac{9}{4}\right)}=1$
i.e., $\frac{x^2}{4}-\frac{4y^2}{9}=1$

Question 23.
Find the equation of the tangent to the hyperbola,

(i) $7x^2 - 3y^2 = 51$ at (-3, -2)
(ii) $x = 3 \sec \theta$, $y = 5 \tan \theta$ at $\theta = \pi/3$
(iii) $\frac{x^2}{25}-\frac{y^2}{16}=1$ at P(30°).
Solution:
(i) Given equation of the hyperbola is $7x^2 - 3y^2 = 51$

$$\frac{x^2}{\left(\frac{51}{7}\right)} - \frac{y^2}{17} = 1$$

Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

we get

$a^2 = \frac{51}{7}$ and $b^2 = 17$

Equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = 1$

Equation of the tangent at (–3, –2) is

$$\frac{-3x}{\left(\frac{51}{7}\right)} + \frac{2y}{17} = 1$$

$$\frac{-7x}{17} + \frac{2y}{17} = 1$$

$$7x - 2y + 17 = 0$$

(ii) Given, equation of the hyperbola is
$x = 3 \sec \theta$, $y = 5 \tan \theta$
Since $\sec^2 \theta - \tan^2 \theta = 1$,

$\frac{x^2}{9}-\frac{y^2}{25}=1$

Comparing this equation with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$a^2 = 9$ and $b^2 = 25$
a = 3 and b = 5
Equation of tangent at P(θ) is

$\frac{x \sec\theta}{a}-\frac{y \tan\theta}{b}=1$

∴ Equation of tangent at P(π/3) is

$\frac{x \sec\left(\frac{\pi}{3}\right)}{3}-\frac{y \tan\left(\frac{\pi}{3}\right)}{5}=1$

$\frac{2x}{3}-\frac{\sqrt{3}y}{5}=1$

$10x - 3\sqrt{3}\, y = 15$

(iii) Given equation of hyperbola is $\frac{x^2}{25}-\frac{y^2}{16}=1$

Comparing this equation with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$a^2 = 25$ and $b^2 = 16$
a = 5 and b = 4
Equation of tangent at P(θ) is

$\frac{x \sec\theta}{a}-\frac{y \tan\theta}{b}=1$

The equation of tangent at P(30°) is

$\frac{x \sec 30^\circ}{5}-\frac{y \tan 30^\circ}{4}=1$

$\frac{2x}{5\sqrt{3}}-\frac{y}{4\sqrt{3}}=1$

$8x - 5y = 20\sqrt{3}$

Question 24.
Show that the line $2x - y = 4$ touches the hyperbola $4x^2 - 3y^2 = 24$. Find the point of contact.
Solution:
Given equation of die hyperbola is $4x^2 - 3y^2 = 24$.

∴ $\frac{x^2}{6}-\frac{y^2}{8}=1$

Comparing this equation with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$a^2 = 6$ and $b^2 = 8$
Given equation of line is $2x - y = 4$
∴ $y = 2x - 4$
Comparing this equation with $y = mx + c$, we get
m = 2 and c = -4

For the line $y = mx + c$ to be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we must have

$c^2 = a^2m^2 – b^2$
$c^2 = (-4)^2 = 16$
$a^2m^2 – b^2 = 6(2)^2 – 8 = 24 – 8 = 16$
∴ The given line is a tangent to the given hyperbola and point of contact
$= \left(\frac{-a^2 m}{c}, \frac{-b^2}{c}\right)$
$= \left(\frac{-6(2)}{-4}, \frac{-8}{-4}\right)$
= (3, 2)

Question 25.
Find the equations of the tangents to the hyperbola $3x^2 – y^2 = 48$ which are perpendicular to the line x + 2y – 7 = 0.
Solution:
Given the equation of the hyperbola is $3x^2 – y^2 = 48$.
∴ $\frac{x^2}{16} - \frac{y^2}{48} = 1$
Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get
$a^2 = 16$ and $b^2 = 48$
Slope of the line x + 2y – 7 = 0 is $-\frac{1}{2}$
Since the given line is perpendicular to the tangents,
slope of the required tangent (m) = 2
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Equations of tangents to the ellipse having slope m are
$y = mx \pm \sqrt{a^2m^2 - b^2}$
$y = 2x \pm \sqrt{16(2)^2 - 48}$
$y = 2x \pm \sqrt{16}$
∴ $y = 2x \pm 4$

Question 26.
Two tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ make angles θ_1, θ_2, with the transverse axis. Find the locus of their point of intersection if $\tan \theta_1 + \tan \theta_2 = k$.
Solution:
Given equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
Let θ_1 and θ_2 be the inclinations.
$m_1 = \tan \theta_1$, $m_2 = \tan \theta_2$
Let $P(x_1, y_1)$ be a point on the hyperbola
Equation of a tangent with slope 'm' to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is
$y = mx \pm \sqrt{a^2m^2 - b^2}$
This tangent passes through $P(x_1, y_1)$.
$y_1 = mx_1 \pm \sqrt{a^2m^2 - b^2}$
$(y_1 – mx_1)^2 = a^2m^2 – b^2$
$(x_1^2 - a^2)m^2 - 2x_1y_1 m + (y_1^2 + b^2) = 0$ ……(i)
This is a quadratic equation in 'm'.
It has two roots say m_1 and m_2, which are the slopes of two tangents drawn from P.
∴ $m_1 + m_2 = \frac{2x_1y_1}{x_1^2 - a^2}$
Since $\tan \theta_1 + \tan \theta_2 = k$,
$\frac{2x_1y_1}{x_1^2 - a^2} = k$
∴ $P(x_1, y_1)$ moves on the curve whose equation is $k(x^2 – a^2) = 2xy$.