Maharashtra State Board 12th Maths Solutions Chapter 3 Trigonometric Functions Ex 3.1

Question 1.
Find the principal solutions of the following equations :
(i) cos θ= 12
Solution:
We know that, cosπ3 = 12 and cos (2π – θ) = cos θ
∴ cosπ3 = cos(2π – π3) = cos5π3
∴ cosπ3 = cos5π3 = 12, where
0 < π3 < 2π and 0 < 5π3 < 2π
∴ cos θ = 12 gives cos θ = cosπ3 = cos5π3
∴ θ = π3 and θ = 5π3
Hence, the required principal solutions are
θ = π3 and θ = 5π3

(ii) sec θ = 23√
Solution:

(iii) cot θ = 3–√
Solution:
The given equation is cot θ = 3–√ which is same as tan θ = 13√.
We know that,

$$\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}} \text{ and } \tan(\pi+\theta)=\tan\theta$$

$$\therefore \tan\frac{\pi}{6}=\tan\left(\pi+\frac{\pi}{6}\right)=\tan\frac{7\pi}{6}$$

$$\therefore \tan\frac{\pi}{6}=\tan\frac{7\pi}{6}=\frac{1}{\sqrt{3}}, \text{ where}$$

$$0<\frac{\pi}{6}<2\pi \text{ and } 0<\frac{7\pi}{6}<2\pi$$

$$\therefore \cot\theta=\sqrt{3}, \text{ i.e. } \tan\theta=\frac{1}{\sqrt{3}} \text{ gives}$$

$$\tan\theta=\tan\frac{\pi}{6}=\tan\frac{7\pi}{6}$$

$$\therefore \theta=\frac{\pi}{6} \text{ and } \theta=\frac{7\pi}{6}$$

Hence, the required principal solution are
θ = π6 and θ = 7π6.

(iv) cot θ = 0.
Solution:

Question 2.
Find the principal solutions of the following equations:
(i) sinθ = −12
Solution:
We know that,
sinπ6 = 12 and sin (π + θ) = -sinθ,
sin(2π – θ) = -sinθ

$$\therefore \sin\left(\pi+\frac{\pi}{6}\right)=-\sin\frac{\pi}{6}=-\frac{1}{2}$$

$$\text{and } \sin\left(2\pi-\frac{\pi}{6}\right)=-\sin\frac{\pi}{6}=-\frac{1}{2}$$

$$\therefore \sin\frac{7\pi}{6}=\sin\frac{11\pi}{6}=-\frac{1}{2}, \text{ where}$$

$$0<\frac{7\pi}{6}<2\pi \text{ and } 0<\frac{11\pi}{6}<2\pi$$

$$\therefore \sin\theta=-\frac{1}{2} \text{ gives,}$$

$$\sin\theta=\sin\frac{7\pi}{6}=\sin\frac{11\pi}{6}$$

$$\therefore \theta=\frac{7\pi}{6} \text{ and } \theta=\frac{11\pi}{6}$$

Hence, the required principal solutions are
θ = 7π6 and θ = 11π6.

(ii) tanθ = -1
Solution:
We know that,
tanπ4 = 1 and tan(π – θ) = -tanθ,
tan(2π – θ) = -tanθ

$$\therefore \tan\left(\pi-\frac{\pi}{4}\right)=-\tan\frac{\pi}{4}=-1$$

$$\text{and } \tan\left(2\pi-\frac{\pi}{4}\right)=-\tan\frac{\pi}{4}=-1$$

$$\therefore \tan\frac{3\pi}{4}=\tan\frac{7\pi}{4}=-1, \text{ where}$$

$$0<\frac{3\pi}{4}<2\pi \text{ and } 0<\frac{7\pi}{4}<2\pi$$

$$\therefore \tan\theta=-1 \text{ gives,}$$

$$\tan\theta=\tan\frac{3\pi}{4}=\tan\frac{7\pi}{4}$$

$$\therefore \theta=\frac{3\pi}{4} \text{ and } \theta=\frac{7\pi}{4}$$

Hence, the required principal solutions are
θ = 3π4 and θ = 7π4.

(iii) 3–√ cosecθ + 2 = 0.
Solution:

Question 3.
Find the general solutions of the following equations :
(i) sinθ = 12
Solution:
(i) The general solution of sin θ = sin ∝ is
θ = nπ + (-1)n∝, n ∈ Z
Now, sinθ = 12 = sinπ6 …[∵ sinπ6 = 12]
∴ the required general solution is
θ = nπ + (-1)nπ6, n ∈ Z.

(ii) cosθ = 3√2
Solution:
The general solution of cos θ = cos ∝ is
θ = 2nπ ± ∝, n ∈ Z
Now, cosθ = 3√2 = cosπ6 …[∵ cosπ6 = 3√2]
∴ the required general solution is
θ = 2nπ ± π6, n ∈ Z.

(iii) tanθ = 13√
Solution:
The general solution of tan θ = tan ∝ is
θ = nπ + ∝, n ∈ Z
Now, tan θ = 13√ = tanπ6 ...[tanπ6 = 13√]
∴ the required general solution is
θ = nπ + π6 , n ∈ Z.

(iv) cotθ = 0.
Solution:
The general solution of tan θ = tan ∝ is
θ = nπ + ∝, n ∈ Z
Now, cot θ = 0 ∴ tan θ does not exist
∴ tanθ = tanπ2 [∵ tanπ2 does not exist]
∴ the required general solution is
θ = nπ + π2, n ∈ Z.

Question 4.
Find the general solutions of the following equations:
(i) secθ = 2–√
Solution:
The general solution of cos θ = cos ∝ is
θ = nπ ± ∝, n ∈ Z.
Now, secθ = 2–√ ∴ cosθ = 12√
∴ cosθ = cosπ4 ....[cosπ4 = 12√]
∴ the required general solution is
θ = 2nπ ± π4, n ∈ Z.

(ii) cosecθ = –2–√
Solution:
The general solution of sinθ = sin∝ is

$\theta = n\pi + (-1)^n\alpha,\ n \in Z$

Now, $\operatorname{cosec}\theta = -\sqrt{2}$

$\therefore\ \sin\theta = -\frac{1}{\sqrt{2}}$

$\therefore\ \sin\theta = -\sin\frac{\pi}{4}$ $\quad\ldots\left[\because\ \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$

$\therefore\ \sin\theta = \sin\left(\pi + \frac{\pi}{4}\right)$ $\quad\ldots[\because\ \sin(\pi+\theta) = -\sin\theta]$

$\therefore\ \sin\theta = \sin\frac{5\pi}{4}$

∴ the required general solution is

$\theta = n\pi + (-1)^n\left(\frac{5\pi}{4}\right),\ n \in Z.$

(iii) tanθ = -1
Solution:

The general solution of tanθ = tan∝ is

$$\theta = n\pi + \alpha,\ n \in Z$$

Now, $\tan\theta = -1$

$$\therefore\ \tan\theta = -\tan\frac{\pi}{4} \qquad \ldots \left[\because\ \tan\frac{\pi}{4} = 1\right]$$

$$\therefore\ \tan\theta = \tan\left(\pi - \frac{\pi}{4}\right) \ \ldots\ [\because\ \tan(\pi - \theta) = -\tan\theta]$$

$$\therefore\ \tan\theta = \tan\frac{3\pi}{4}$$

$\therefore$ the required general solution is

$$\theta = n\pi + \frac{3\pi}{4},\ n \in Z.$$

Question 5.
Find the general solutions of the following equations :
(i) sin 2θ = 12
Solution:
The general solution of sin θ = sin ∝ is
θ = nπ + $(-1)^n$∝, n ∈ Z

Now, $\sin 2\theta = \frac{1}{2}$

$$\therefore\ \sin 2\theta = \sin\frac{\pi}{6} \qquad \ldots \left[\because\ \sin\frac{\pi}{6} = \frac{1}{2}\right]$$

$\therefore$ the required general solution is given by

$$2\theta = n\pi + (-1)^n\left(\frac{\pi}{6}\right),\ n \in Z$$

$$\text{i.e. } \theta = \frac{n\pi}{2} + (-1)^n\left(\frac{\pi}{12}\right),\ n \in Z.$$

(ii) tan 2θ3 = 3–√
Solution:
The general solution of tan θ = tan ∝ is

$$\theta = n\pi + \alpha,\ n \in Z$$

Now, $\tan\frac{2\theta}{3} = \sqrt{3}$

$$\therefore\ \tan\frac{2\theta}{3} = \tan\frac{\pi}{3} \qquad \ldots \left[\because\ \tan\frac{\pi}{3} = \sqrt{3}\right]$$

$\therefore$ the required general solution is given by

$$\frac{2\theta}{3} = n\pi + \frac{\pi}{3},\ n \in Z$$

$$\text{i.e. } \theta = \frac{3n\pi}{2} + \frac{\pi}{2},\ n \in Z.$$

(iii) cot 4θ = -1
Solution:

The general solution of tan θ = tan ∝ is

$$\theta = n\pi + \alpha,\ n \in Z$$

Now, $\cot 4\theta = -1$

$\therefore\ \tan 4\theta = -1$

$\therefore\ \tan 4\theta = -\tan\frac{\pi}{4}$ ... $\left[\because\ \tan\frac{\pi}{4} = 1\right]$

$\therefore\ \tan 4\theta = \tan\left(\pi - \frac{\pi}{4}\right)$ ... $[\because\ \tan(\pi - \theta) = -\tan\theta]$

$\therefore\ \tan 4\theta = \tan\frac{3\pi}{4}$

$\therefore$ the required general solution is given by

$$4\theta = n\pi + \frac{3\pi}{4},\ n \in Z$$

i.e. $\theta = \frac{n\pi}{4} + \frac{3\pi}{16},\ n \in Z.$

Question 6.
Find the general solutions of the following equations :
(i) 4 $\cos^2\theta$ = 3
Solution:
The general solution of $\cos^2\theta = \cos^2 \alpha$ is
θ = nπ ± ∝, n ∈ Z
Now, 4 $\cos^2\theta$ = 3

$\therefore\ \cos^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$

$\therefore\ \cos^2\theta = \left(\cos\frac{\pi}{6}\right)^2$ ... $\left[\because\ \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}\right]$

$\therefore\ \cos^2\theta = \cos^2\frac{\pi}{6}$

$\therefore$ the required general solution is given by

$$\theta = n\pi \pm \frac{\pi}{6},\ n \in Z.$$

(ii) 4 $\sin^2\theta$ = 1
Solution:
The general solution of $\sin^2\theta = \sin^2 \alpha$ is
θ = nπ ± ∝, n ∈ Z
Now, 4 $\sin^2\theta$ = 3

$\therefore\ \sin^2\theta = \frac{1}{4} = \left(\frac{1}{2}\right)^2$

$\therefore\ \sin^2\theta = \left(\sin\frac{\pi}{6}\right)^2$ ... $\left[\because\ \sin\frac{\pi}{6} = \frac{1}{2}\right]$

$\therefore\ \sin^2\theta = \sin^2\frac{\pi}{6}$

$\therefore$ the required general solution is $\theta = n\pi \pm \frac{\pi}{6},\ n \in Z.$

(iii) cos 4θ = cos 2θ
Solution:
The general solution of cos θ = cos ∝ is
θ = 2nπ ± ∝, n ∈ Z
∴ the general solution of cos 4θ = cos 2θ is given by
4θ = 2nπ ± 2θ, n ∈ Z
Taking positive sign, we get
4θ = 2nπ + 2θ, n ∈ Z
∴ 2θ = 2nπ, n ∈ Z
∴ θ = nπ, n ∈ Z
Taking negative sign, we get

$4\theta = 2n\pi - 2\theta$, $n \in Z$
$\therefore 6\theta = 2n\pi$, $n \in Z$
$\therefore \theta = \frac{n\pi}{3}$, $n \in Z$
Hence, the required general solution is
$\theta = \frac{n\pi}{3}$, $n \in Z$ or $\therefore \theta = n\pi$, $n \in Z$.
Alternative Method:
$\cos 4\theta = \cos 2\theta$
$\therefore \cos 4\theta - \cos 20 = 0$
$\therefore -2\sin\left(\frac{4\theta+2\theta}{2}\right)\cdot\sin\left(\frac{4\theta-2\theta}{2}\right) = 0$
$\therefore \sin 3\theta \cdot \sin\theta = 0$
$\therefore$ either $\sin 3\theta = 0$ or $\sin\theta = 0$
The general solution of $\sin\theta = 0$ is
$\theta = n\pi$, $n \in Z$.
$\therefore$ the required general solution is given by
$3\theta = n\pi$, $n \in Z$ or $\theta = n\pi$, $n \in Z$
i.e. $\theta = \frac{n\pi}{3}$, $n \in Z$ or $\theta = n\pi$, $n \in Z$.

Question 7.
Find the general solutions of the following equations :
(i) $\sin\theta = \tan\theta$
Solution:
$\sin\theta = \tan\theta$
$\therefore \sin\theta = \frac{\sin\theta}{\cos\theta}$
$\therefore \sin\theta \cos\theta = \sin\theta$
$\therefore \sin\theta \cos\theta - \sin\theta = 0$
$\therefore \sin\theta (\cos\theta - 1) = \theta$
$\therefore$ either $\sin\theta = 0$ or $\cos\theta - 1 = 0$
$\therefore$ either $\sin\theta = 0$ or $\cos\theta = 1$
$\therefore$ either $\sin\theta = 0$ or $\cos\theta = \cos\theta$ ...[$\because \cos 0 = 1$]
The general solution of $\sin\theta = 0$ is $\theta = n\pi$, $n \in Z$ and $\cos\theta = \cos\propto$ is $\theta = 2n\pi \pm \propto$, where $n \in Z$.
$\therefore$ the required general solution is given by
$\theta = n\pi$, $n \in Z$ or $\theta = 2n\pi \pm 0$, $n \in Z$
$\therefore \theta = n\pi$, $n \in Z$ or $\theta = 2n\pi$, $n \in Z$.

(ii) $\tan^3\theta = 3\tan\theta$
Solution:
$\tan^3\theta = 3\tan\theta$
$\therefore \tan^3\theta - 3\tan\theta = 0$
$\therefore \tan\theta (\tan^2\theta - 3) = 0$
$\therefore$ either $\tan\theta = 0$ or $\tan^2\theta - 3 = 0$
$\therefore$ either $\tan\theta = 0$ or $\tan^2\theta = 3$
$\therefore$ either $\tan\theta = 0$ or $\tan^2\theta = (\sqrt{3})^3$
$\therefore$ either $\tan\theta = 0$ or $\tan^2\theta = (\tan\frac{\pi}{3})^3$ ...[$\tan\frac{\pi}{3} = \sqrt{3}$]
$\therefore$ either $\tan\theta = 0$ or $\tan^2\theta = \tan^2\frac{\pi}{3}$
The general solution of
$\tan\theta = 0$ is $\theta = n\pi$, $n \in Z$ and
$\tan^2\theta = \tan^2\propto$ is $\theta = n\pi \pm \propto$, $n \in Z$.
$\therefore$ the required general solution is given by
$\theta = n\pi$, $n \in Z$ or $\theta = n\pi \pm \frac{\pi}{3}$, $n \in Z$.

(iii) $\cos\theta + \sin\theta = 1$.
Solution:
$\cos\theta + \sin\theta = 1$

Dividing both sides by $\sqrt{(1)^2+(1)^2}=\sqrt{2}$, we get

$$\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta=\frac{1}{\sqrt{2}}$$

$$\therefore\ \cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}\sin\theta=\cos\frac{\pi}{4}$$

$$\therefore\ \cos\left(\theta-\frac{\pi}{4}\right)=\cos\frac{\pi}{4} \quad \ldots (1)$$

The general solution of

$\cos\theta=\cos\alpha$ is $\theta=2n\pi\pm\alpha,\ n\in Z$.

$\therefore$ the general solution of (1) is given by

$$\theta-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{4},\ n\in Z$$

Taking positive sign, we get

$$\theta-\frac{\pi}{4}=2n\pi+\frac{\pi}{4},\ n\in Z$$

$$\therefore\ \theta=2n\pi+\frac{\pi}{2},\ n\in Z$$

Taking negative sign, we get

$$\theta-\frac{\pi}{4}=2n\pi-\frac{\pi}{4},\ n\in Z$$

$$\therefore\ \theta=2n\pi,\ n\in Z$$

$\therefore$ the required general solution is

$$\theta=2n\pi+\frac{\pi}{2},\ n\in Z \text{ or } \theta=2n\pi,\ n\in Z.$$

Alternative Method :

$$\cos\theta+\sin\theta=1$$

$$\therefore\ \sin\theta=1-\cos\theta$$

$$\therefore\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2\sin^2\frac{\theta}{2}$$

$$\therefore\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}-2\sin^2\frac{\theta}{2}=0$$

$$\therefore\ 2\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)=0$$

$$\therefore\ 2\sin\frac{\theta}{2}=0 \text{ or } \cos\frac{\theta}{2}-\sin\frac{\theta}{2}=0$$

$$\therefore\ \sin\frac{\theta}{2}=0 \text{ or } \sin\frac{\theta}{2}=\cos\frac{\theta}{2}$$

$$\therefore\ \sin\frac{\theta}{2}=0 \text{ or } \tan\frac{\theta}{2}=1 \quad \ldots \left[\because\ \cos\frac{\theta}{2}\neq 0\right]$$

$$\therefore\ \sin\frac{\theta}{2}=0 \text{ or } \tan\frac{\theta}{2}=\tan\frac{\pi}{4} \quad \ldots \left[\because\ \tan\frac{\pi}{4}=1\right]$$

The general solution of $\sin\theta=0$ is $\theta=n\pi,\ n\in Z$ and

$\tan\theta=\tan\alpha$ is $\theta=n\pi+\alpha,\ n\in Z$

$\therefore$ the required general solution is

$$\frac{\theta}{2}=n\pi,\ n\in Z \text{ or } \frac{\theta}{2}=n\pi+\frac{\pi}{4},\ n\in Z$$

i.e. $\theta=2n\pi,\ n\in Z$ or $\theta=2n\pi+\frac{\pi}{2},\ n\in Z$.

Question 8.
Which of the following equations have solutions ?
(i) cos 2θ = -1
Solution:

$\cos 2\theta = -1$
Since $-1 \le \cos\theta \le 1$ for any θ,
$\cos 2\theta = -1$ has solution.

(ii) $\cos^2\theta = -1$
Solution:
$\cos^2\theta = -1$
This is not possible because $\cos^2\theta \ge 0$ for any θ.
$\therefore$ $\cos^2\theta = -1$ does not have any solution.

(iii) $2\sin\theta = 3$
Solution:
$2\sin\theta = 3 \therefore \sin\theta = \frac{3}{2}$
This is not possible because $-1 \le \sin\theta \le 1$ for any θ.
$\therefore$ $2\sin\theta = 3$ does not have any solution.

(iv) $3\tan\theta = 5$
Solution:
$3\tan\theta = 5 \therefore \tan\theta = \frac{5}{3}$
This is possible because $\tan\theta$ is any real number.
$\therefore$ $3\tan\theta = 5$ has solution.

Maharashtra State Board 12th Maths Solutions Chapter 3 Trigonometric Functions Ex 3.2

Question 1.
Find the Cartesian co-ordinates of the point whose polar co-ordinates are:
(i) $\left(\sqrt{2}, \frac{\pi}{4}\right)$
Solution:
Here, $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$
Let the cartesian coordinates be (x, y)
Then, $x = r\cos\theta = \sqrt{2}\cos\frac{\pi}{4} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1$
$y = r\sin\theta = \sqrt{2}\sin\frac{\pi}{4} = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1$
$\therefore$ the cartesian coordinates of the given point are (1, 1).

(ii) $\left(4, \frac{\pi}{2}\right)$
Solution:

(iii) $\left(\frac{3}{4}, \frac{3\pi}{4}\right)$
Solution:
Here, $r = \frac{3}{4}$ and $\theta = \frac{3\pi}{4}$

Let the cartesian coordinates be (x, y)

$$\text{Then, } x = r\cos\theta = \frac{3}{4}\cos\frac{3\pi}{4} = \frac{3}{4}\cos\left(\pi - \frac{\pi}{4}\right)$$

$$= -\frac{3}{4}\cos\frac{\pi}{4} = -\frac{3}{4}\times\frac{1}{\sqrt{2}} = -\frac{3}{4\sqrt{2}}$$

$$y = r\sin\theta = \frac{3}{4}\sin\frac{3\pi}{4} = \frac{3}{4}\sin\left(\pi - \frac{\pi}{4}\right)$$

$$= \frac{3}{4}\sin\frac{\pi}{4} = \frac{3}{4}\times\frac{1}{\sqrt{2}} = \frac{3}{4\sqrt{2}}$$

∴ **the cartesian coordinates of the given point are**

$$\left(-\frac{3}{4\sqrt{2}}, \frac{3}{4\sqrt{2}}\right).$$

(iv) (12,7π3)
Solution:
Here, r = 12 and θ = 7π4
Let the cartesian coordinates be (x, y)

$$\text{Then, } x = r\cos\theta = \frac{1}{2}\cos\frac{7\pi}{3} = \frac{1}{2}\cos\left(2\pi + \frac{\pi}{3}\right)$$

$$= \frac{1}{2}\cos\frac{\pi}{3} = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$$

$$y = r\sin\theta = \frac{1}{2}\sin\frac{7\pi}{3} = \frac{1}{2}\sin\left(2\pi + \frac{\pi}{3}\right)$$

$$= \frac{1}{2}\sin\frac{\pi}{3} = \frac{1}{2}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$$

∴ the cartesian coordinates of the given point are (14,3√4)

Question 2.
Find the of the polar co-ordinates point whose Cartesian co-ordinates are.
(i) (2–√,2–√)
Solution:
Here x = 2–√ and y = 2–√
∴ the point lies in the first quadrant.
Let the polar coordinates be (r, θ)
Then, r2 = x2 + y2 = (2–√)2 + (2–√)2 = 2 + 2 = 4
∴ r = 2 … [∵ r > 0]
cos θ = xr=2√2=12√
and sin θ = yr=2√2=12√
∴ tan θ = 1
Since the point lies in the first quadrant and
0 ≤ θ ≤ 2π, tan θ = 1 = tanπ4
∴ θ = π4
∴ the polar coordinates of the given point are (2,π4).

(ii) (0,12)
Solution:
Here x = 0 and y = 12
the point lies on the positive side of Y-axis. Let the polar coordinates be (r, θ)
Then, r2 = x2 + y2 = (0)2 + (12)2=0+14=14
∴ r = 12 …[∵ r > 0]
cosθ = xr=0(1/2) = 0
and sin θ = yr=(1/2)(1/2) = 1

Since, the point lies on the positive side of Y-axis and $0 \leq \theta \leq 2\pi$
$\cos\theta = 0 = \cos\pi 2$ and $\sin\theta = 1 = \sin\pi 2$
$\therefore \theta = \pi 2$

$\therefore$ the polar coordinates of the given point are $(12, \pi 2)$.

(iii) $(1, -3-\sqrt{})$
Solution:
Here x = 1 and y = $-3-\sqrt{}$
$\therefore$ the point lies in the fourth quadrant.
Let the polar coordinates be (r, θ).
Then, $r2 = x2 + y2 = (1)2 + (-3-\sqrt{}\ )2 = 1 + 3 = 4$
$\therefore r = 2$ ... [$\because r > 0$]

$$\cos\theta = \frac{x}{r} = \frac{1}{2}$$

$$\text{and } \sin\theta = \frac{y}{r} = -\frac{\sqrt{3}}{2}$$

$$\therefore \tan\theta = -\sqrt{3}$$

Since, the point lies in the fourth quadrant and $0 \leqslant \theta < 2\pi$.

$$\tan\theta = -\sqrt{3} = -\tan\frac{\pi}{3}$$

$$= \tan\left(2\pi - \frac{\pi}{3}\right) \quad \ldots [\because \tan(2\pi - \theta) = -\tan\theta]$$

$$= \tan\frac{5\pi}{3}$$

$$\therefore \theta = \frac{5\pi}{3}$$

$\therefore$ the polar coordinates of the given point are $(2, 5\pi 3)$.

(iv) $(32, 33\sqrt{2})$
Solution:

Question 3.
In ΔABC, if $\angle A = 45^\circ$, $\angle B = 60^\circ$ then find the ratio of its sides.
Solution:
By the sine rule,
$a\sin A = b\sin B = c\sin C$

$\therefore ab = \sin A\sin B$ and $bc = \sin B\sin C$

$\therefore a : b : c = \sin A : \sin B : \sin C$
Given $\angle A = 45^\circ$ and $\angle B = 60^\circ$
$\because \angle A + \angle B + \angle C = 180^\circ$
$\therefore 45^\circ + 60^\circ + \angle C = 180^\circ$

$\therefore \angle C = 180° - 105° = 75°$

$$\text{Now, } \sin A = \sin 45° = \frac{1}{\sqrt{2}}$$

$$\sin B = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\text{and } \sin C = \sin 75° = \sin(45° + 30°)$$
$$= \sin 45° \cos 30° + \cos 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$\therefore$ the ratio of the sides of $\triangle ABC$

$$= a : b : c = \sin A : \sin B : \sin C$$
$$= \frac{1}{\sqrt{2}} : \frac{\sqrt{3}}{2} : \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\therefore\ a : b : c = 2 : \sqrt{6} : (\sqrt{3}+1)$$

Question 4.
In ΔABC, prove that $\sin\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{a}\right)\cos\frac{A}{2}$.
Solution:
By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\therefore\ a = k\sin A,\ b = k\sin B,\ c = k\sin C$$

$$\text{RHS} = \left(\frac{b-c}{a}\right)\cos\frac{A}{2}$$
$$= \left(\frac{k\sin B - k\sin C}{k\sin A}\right)\cos\frac{A}{2}$$
$$= \left(\frac{\sin B - \sin C}{\sin A}\right)\cos\frac{A}{2}$$
$$= \frac{2\cos\left(\frac{B+C}{2}\right)\cdot\sin\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cdot\cos\frac{A}{2}}\cdot\cos\frac{A}{2}$$
$$= \frac{\cos\left(\frac{B+C}{2}\right)\cdot\sin\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}}$$
$$= \frac{\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)\cdot\sin\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}} \quad \ldots [\because A+B+C=\pi]$$
$$= \frac{\sin\frac{A}{2}\cdot\sin\left(\frac{B-C}{2}\right)}{\sin\frac{A}{2}}$$
$$= \sin\left(\frac{B-C}{2}\right) = \text{LHS.}$$

Question 5.
With usual notations prove that $2\left\{a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right\} = a - b + c$.

Solution:

$$\text{LHS} = 2\left\{a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right\}$$

$$= a\left(2\sin^2\frac{C}{2}\right) + c\left(2\sin^2\frac{A}{2}\right)$$

$$= a(1 - \cos C) + c(1 - \cos A)$$

$$= a\left[1 - \frac{a^2 + b^2 - c^2}{2ab}\right] + c\left[1 - \frac{b^2 + c^2 - a^2}{2bc}\right]$$

... [By cosine rule]

$$= a\left[\frac{2ab - a^2 - b^2 + c^2}{2ab}\right] + c\left[\frac{2bc - b^2 - c^2 + a^2}{2bc}\right]$$

$$= \frac{2ab - a^2 - b^2 + c^2}{2b} + \frac{2bc - b^2 - c^2 + a^2}{2b}$$

$$= \frac{2ab - a^2 - b^2 + c^2 + 2bc - b^2 - c^2 + a^2}{2b}$$

$$= \frac{2ab - 2b^2 + 2bc}{2b}$$

$$= a - b + c = \text{RHS.}$$

Question 6.
In ΔABC, prove that $a^3\sin(B – C) + b^3\sin(C – A) + c^3\sin(A – B) = 0$
Solution:
By the sine rule,
asinA = bsinB = csinC = k
∴ a = k sin A, b = k sin B, c = k sin C
LHS = $a^3\sin(B – C) + b^3\sin(C – A) + c^3\sin(A – B)$
= a^3(sin B cos C – cos B sin C) + b^3(sinCcos A – cos C sin A) + c^3(sinAcosB – cos A sin B)

$$= a^3\left(\frac{b}{k}\cos C - \frac{c}{k}\cos B\right) + b^3\left(\frac{c}{k}\cos A - \frac{a}{k}\cos C\right) + c^3\left(\frac{a}{k}\cos B - \frac{b}{k}\cos A\right)$$

$$= \frac{1}{k}[a^3b\cos C - a^3c\cos B + b^3c\cos A - b^3a\cos C + c^3a\cos B - c^3b\cos A]$$

$$= \frac{1}{k}\left[a^3b\left(\frac{a^2 + b^2 - c^2}{2ab}\right) - a^3c\left(\frac{c^2 + a^2 - b^2}{2ca}\right) + b^3c\left(\frac{b^2 + c^2 - a^2}{2bc}\right) - ab^3\left(\frac{a^2 + b^2 - c^2}{2ab}\right) + ac^3\left(\frac{c^2 + a^2 - b^2}{2ca}\right) - bc^3\left(\frac{b^2 + c^2 - a^2}{2bc}\right)\right]$$

... [By cosine rule]

= *12k* [$a^2(a^2 + b^2 – c^2) – a^2(a^2 + c^2 – b^2) + b^2(b^2 + c^2 – a^2) – b^2(a^2 + b^2 – c^2) + c^2(c^2 + a^2 – b^2) – c^2(b^2 + c^2 – a^2)$]
= *12k* [$a^4 + a^2b^2 – a^2c^2 – a^4 – a^2c^2 + a^2b^2 + b^4 + b^2c^2 – a^2b^2 – a^2b^2 – b^4 + b^2c^2 + c^4 + a^2c^2 – b^2c^2 – b^2c^2 – c^4 + a^2c^2$]
= *12k*(0) = 0 = RHS.

Question 7.
In ΔABC, if cot A, cot B, cot C are in A.P. then show that a^2, b^2, c^2 are also in A.P
Solution:
By the sine rule,
sinAa = sinBb =sinCc = k
∴ sin A = ka, sin B = kb, sin C = kc ...(1)
Now, cot A, cotB, cotC are in A.P.
∴ cotC – cotB = cotB – cot A
∴ cotA + cotC = 2cotB

$$\therefore \frac{\cos A}{\sin A}+\frac{\cos C}{\sin C}=2\cot B$$

$$\therefore \frac{\sin C\cos A+\sin A\cos C}{\sin A\cdot\sin C}=2\cot B$$

$$\therefore \frac{\sin(A+C)}{\sin A\cdot\sin C}=2\cot B$$

$$\therefore \frac{\sin(\pi-B)}{\sin A\cdot\sin C}=2\cot B \qquad \ldots [\because A+B+C=\pi]$$

$$\therefore \frac{\sin B}{\sin A\cdot\sin C}=\frac{2\cos B}{\sin B}$$

$$\therefore \frac{\sin^2 B}{\sin A\cdot\sin C}=2\cos B$$

$$\therefore \frac{k^2b^2}{(ka)(kc)}=2\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$$\therefore \frac{b^2}{ac}=\frac{a^2+c^2-b^2}{ac}$$

$$\therefore b^2=a^2+c^2-b^2 \qquad \therefore 2b^2=a^2+c^2$$

Hence, a^2, b^2, c^2 are in A.P.

Question 8.
In ΔABC, if a cos A = b cos B then prove that the triangle is right angled or an isosceles traingle.
Solution:
By the sine rule,
$\frac{a}{\sin A}=\frac{b}{\sin B}=k$
a = k sin A and b = k sin B
∴ a cos A = b cos B gives
k sin A cos A = k sin B cos B
∴ 2 sin A cos A = 2 sin B cos B
∴ sin 2A = sin 2B ∴ sin 2A – sin 2B = 0
∴ 2 cos (A + B)·sin (A -B) = 0
∴ 2cos (π – C)·sin(A – B) = 0 … [∵ A + B + C = π]
∴ -2 cos C·sin (A – B) = 0
∴ cos C = 0 OR sin(A -B) = 0
∴ C = 90° OR A – B = 0
∴ C = 90° OR A = B
∴ the triangle is either rightangled or an isosceles triangle.

Question 9.
With usual notations prove that 2(bc cos A + ac cos B + ab cos C) = $a^2+b^2+c^2$.
Solution:
LHS = 2 (bc cos A + ac cos B + ab cos C)
= 2bc cos A + 2ac cos B + 2ab cos C
= 2bc $\left(\frac{b^2+c^2-a^2}{2bc}\right)$ + 2ac$\left(\frac{c^2+a^2-b^2}{2ca}\right)$ + 2ab$\left(\frac{a^2+b^2-c^2}{2ab}\right)$ …(By cosine rule]
= $b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2=a^2+b^2+c^2$ = RHS.

Question 10.
In △ABC, if a = 18, b = 24, c = 30 then find the values of
(i) cos A
Solution:
Given : a = 18, b = 24 and c = 30
∴ 2s = a + b + c = 18 + 24 + 30 = 72 ∴ s = 36

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{(24)^2+(30)^2-(18)^2}{2(24)(30)}$$

$$=\frac{576+900-324}{1440}=\frac{1152}{1440}=\frac{4}{5}.$$

(ii) sinA2
Solution:

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}=\sqrt{\frac{(36-24)(36-30)}{(24)(30)}}$$
$$=\sqrt{\frac{12\times 6}{24\times 30}}=\sqrt{\frac{1}{10}}=\frac{1}{\sqrt{10}}.$$

(iii) cosA2
Solution:

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}=\sqrt{\frac{36(36-18)}{(24)(30)}}$$
$$=\sqrt{\frac{36\times 18}{24\times 30}}=\sqrt{\frac{9}{10}}=\frac{3}{\sqrt{10}}.$$

(iv) tanA2
Solution:

$$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{1/\sqrt{10}}{3/\sqrt{10}}=\frac{1}{3}.$$

(v) A(△ABC)
Solution:

$$\begin{aligned}A(\triangle ABC)&=\sqrt{s(s-a)(s-b)(s-c)}\\&=\sqrt{36(36-18)(36-24)(36-30)}\\&=\sqrt{36\times 18\times 12\times 6}\\&=\sqrt{36\times 18\times 4\times 18}\\&=6\times 18\times 2=216\text{ sq units.}\end{aligned}$$

(iv) sin A.
Solution:

$$A(\triangle ABC)=\tfrac{1}{2}bc\sin A$$
$$\therefore\ 216=\tfrac{1}{2}(24)(30)\sin A$$
$$\therefore\ \sin A=\frac{216}{12\times 30}=\frac{216}{360}=\frac{3}{5}.$$

Question 11.
In △ABC prove that (b + c – a) tan A2 = (c + a – b) tanB2 = (a + b – c) tanC2.
Solution:
(b + c – a) tan A2

$$= (a+b+c-2a)\cdot\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= (2s-2a)\cdot\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= 2\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \qquad \text{... (1)}$$

$$(c+a-b)\tan\frac{B}{2} = (a+b+c-2b)\cdot\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$= (2s-2b)\cdot\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$= 2\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \qquad \text{... (2)}$$

$$(a+b-c)\tan\frac{C}{2} = (a+b+c-2c)\cdot\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= (2s-2c)\cdot\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$= 2\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \qquad \text{... (3)}$$

From (1), (2) and (3), we get

$$(b+c-a)\tan\frac{A}{2} = (c+a-b)\tan\frac{B}{2} = (a+b-c)\tan\frac{C}{2}.$$

Question 12.
In △ABC prove that $\sin\frac{A}{2}\cdot\sin\frac{A}{2}\cdot\sin\frac{A}{2} = \frac{[A(\triangle ABC)]^2}{abcs}$
Solution:
LHS = $\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}$

$$= \sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{(s-a)(s-c)}{ac}}\cdot\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= \sqrt{\frac{(s-a)^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$$

$$= \frac{(s-a)(s-b)(s-c)}{abc}$$

$$= \frac{s(s-a)(s-b)(s-c)}{abcs}$$

$$= \frac{[A(\triangle ABC)]^2}{abcs} \text{ ... } \left[\because A(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}\right]$$

$$= \text{RHS.}$$

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Maharashtra State Board 12th Maths Solutions Chapter 3 Trigonometric Functions Ex 3.3

Question 1.
Find the principal values of the following :
(i) $\sin^{-1}(12)$
Solution:
The principal value branch of $\sin^{-1}x$ is $[-\pi2, \pi2]$.
Let $\sin^{-1}(12) = \propto$, where $-\pi2 \le \propto \le \pi2$
$\therefore \sin\propto = 12 = \sin\pi6$
$\therefore \propto = \pi6$ …[$\because -\pi2 \le \pi6 \le \pi2$]
$\therefore$ the principal value of $\sin^{-1}(12)$ is $\pi6$.

(ii) $\text{cosec}^{-1}(2)$
Solution:
The principal value branch of $\text{cosec}^{-1}x$ is $[-\pi2, \pi2] - \{0\}$.
Let $\text{cosec}^{-1}(2) = \propto$, where $-\pi2 \le \propto \le \pi2$, $\propto \ne 0$
$\therefore \text{cosec}^{-1} \propto = 2 = \text{cosec}\pi6$
$\therefore \propto = \pi6$ …[$\because -\pi2 \le \pi6 \le \pi2$]
$\therefore$ the principal value of $\text{cosec}^{-1}(2)$ is $\pi6$.

(iii) $\tan^{-1}(-1)$
Solution:
The principal value branch of $\tan^{-1}x$ is $(-\pi2, \pi2)$
Let $\tan^{-1}(-1) = \propto$, where $-\pi2 < \propto < \pi2$
$\therefore \tan\propto = -1 = -\tan\pi4$
$\therefore \tan\propto = \tan(-\pi4)$ …[$\because \tan(-\theta) = -\tan\theta$]
$\therefore \propto = -\pi4$ …[$\because -\pi2 < -\pi4 < \pi2$]
$\therefore$ the principal value of $\tan^{-1}(-1)$ is $-\pi4$.

(iv) $\tan^{-1}(-3-\surd)$
Solution:
The principal value branch of $\tan^{-1}x$ is $(-\pi2, \pi2)$.
Let $\tan^{-1}(-3-\surd) = \propto$, where $-\pi2 < \propto < \pi2$
$\therefore \tan\propto = -3-\surd = -\tan\pi3$
$\therefore \tan\propto = \tan(-\pi3)$ …[$\because \tan(-\theta) = -\tan\theta$]
$\therefore \propto = -\pi3$ …[$\because -\pi2 < -\pi3 < \pi2$]
$\therefore$ the principal value of $\tan^{-1}(-3-\surd)$ is $-\pi3$.

(v) $\sin^{-1}(12\surd)$
Solution:
The principal value branch of $\sin^{-1}x$ is $[-\pi2, \pi2]$.
Let $\sin^{-1}(12\surd) = \propto$, where $-\pi2 < \propto < \pi2$
$\therefore \sin\propto = (12\surd) = \sin\pi4$
$\therefore \propto = \pi4$ …[$\because -\pi2 \le \pi4 \le \pi2$]
$\therefore$ the principal value of $\sin^{-1}(12\surd)$ is $\pi4$.

(vi) $\cos^{-1}(-12)$
Solution:
The principal value branch of $\cos^{-1}x$ is $(0, \pi)$.
Let $\cos^{-1}(-12) = \propto$, where $0 \le \propto \le \pi$
$\therefore \cos\propto = -12 = -\cos\pi3$
$\therefore \cos\propto = \cos(\pi-\pi3)$ …[$\because \cos(\pi - \theta) = -\cos\theta$]
$\therefore \cos\propto = \cos2\pi3$

∴ ∝ = 2π3 ...[∵ 0 ≤ 2π3 ≤ π]

∴ the principal value of cos-1(−12) is 2π3.

Question 2.
Evaluate the following :

(i) tan-1(1) + cos-1(12) + sin-1(12)

Solution:

Let $\tan^{-1}(1) = \alpha$, where $\frac{-\pi}{2} < \alpha < \frac{\pi}{2}$

$\therefore \tan\alpha = 1 = \tan\frac{\pi}{4}$

$\therefore \alpha = \frac{\pi}{4}$ ... $\left[\because \frac{-\pi}{2} < \frac{\pi}{4} < \frac{\pi}{2}\right]$

$\therefore \tan^{-1}(1) = \frac{\pi}{4}$ ... (1)

Let $\cos^{-1}\left(\frac{1}{2}\right) = \beta$, where $0 \leqslant \beta \leqslant \pi$

$\therefore \cos\beta = \frac{1}{2} = \cos\frac{\pi}{3}$

$\therefore \beta = \frac{\pi}{3}$ ... $\left[\because 0 < \frac{\pi}{3} < \pi\right]$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ ... (2)

Let $\sin^{-1}\left(\frac{1}{2}\right) = \gamma$, where $\frac{-\pi}{2} \leqslant \gamma \leqslant \frac{\pi}{2}$

$\therefore \sin\gamma = \frac{1}{2} = \sin\frac{\pi}{6}$

$\therefore \gamma = \frac{\pi}{6}$ ... $\left[\because \frac{-\pi}{2} \leqslant \frac{\pi}{6} \leqslant \frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ ... (3)

$\therefore \tan^{-1}(1) + \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{4} + \frac{\pi}{3} + \frac{\pi}{6}$ ... [By (1), (2) and (3)]

$= \frac{3\pi + 4\pi + 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}.$

(ii) cos-1(12) + 2 sin-1(12)

Solution:

Let $\cos^{-1}\left(\frac{1}{2}\right)$ = α, where 0 ≤ α ≤ π

$\therefore$ cos α = $\frac{1}{2} = \cos\ \frac{\pi}{3}$

$\therefore$ α = $\frac{\pi}{3}$ $\quad\ldots\left[\because 0 < \frac{\pi}{3} < \pi\right]$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ ...(1)

Let $\sin^{-1}\left(\frac{1}{2}\right) = \beta$, where $\frac{-\pi}{2} \leq \beta \leq \frac{\pi}{2}$

$\therefore$ sin β = $\frac{1}{2} = \sin\ \frac{\pi}{6}$

$\therefore$ β = $\frac{\pi}{6}$ $\quad\ldots\left[\because \frac{-\pi}{2} \leq \frac{\pi}{6} \leq \frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ ...(2)

$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ and $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3} + 2\left(\frac{\pi}{6}\right)$

$= \frac{\pi}{3} + \frac{\pi}{3}$

$= \frac{2\pi}{3}$.

(iii) tan-13−−√ – sec-1(-2)

Solution:

Let $\tan^{-1}(\sqrt{3}) = \alpha$, where $\frac{-\pi}{2} < \alpha < \frac{\pi}{2}$

$\therefore\ \tan\alpha = \sqrt{3} = \tan\frac{\pi}{3}$

$\therefore\ \alpha = \frac{\pi}{3}$ $\quad\ldots\left[\because\ \frac{-\pi}{2} < \frac{\pi}{3} < \frac{\pi}{2}\right]$

$\therefore\ \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ ... (1)

Let $\sec^{-1}(-2) = \beta$, where $0 \leqslant \beta \leqslant \pi$, $\beta \neq \frac{\pi}{2}$

$\therefore\ \sec\beta = -2 = -\sec\frac{\pi}{3}$

$\therefore\ \sec\beta = \sec\left(\pi - \frac{\pi}{3}\right)$ ... $[\because\ \sec(\pi - \theta) = -\sec\theta]$

$\therefore\ \sec\beta = \sec\frac{2\pi}{3}$

$\therefore\ \beta = \frac{2\pi}{3}$ $\quad\ldots\ [\because\ 0 \leqslant \frac{2\pi}{3} \leqslant \pi]$

$\therefore\ \sec^{-1}(-2) = \frac{2\pi}{3}$ ... (2)

∴ tan-13−−√ – sec-1(-2)

= π3−2π3 ...[By (1) and (2)]

= −π3.

(iv) cosec-1(−2−√) + cot-1(3−√)

Solution:

Let $\operatorname{cosec}^{-1}(-\sqrt{2}) = \alpha$, where $\frac{-\pi}{2} \leqslant y \leqslant \frac{\pi}{2}, y \neq 0$

$\therefore \operatorname{cosec} \alpha = -\sqrt{2} = -\operatorname{cosec}\frac{\pi}{4}$

$\therefore \operatorname{cosec} \alpha = \operatorname{cosec}\left(-\frac{\pi}{4}\right)$

... $[\because \operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta]$

$\therefore \alpha = -\frac{\pi}{4}$ ... $\left[\frac{-\pi}{2} \leqslant \frac{-\pi}{4} \leqslant \frac{\pi}{2}\right]$

$\therefore \operatorname{cosec}^{-1}(-\sqrt{2}) = -\frac{\pi}{4}$ ... (1)

Let $\cot^{-1}(\sqrt{3}) = \beta$, where $0 < \beta < \pi$

$\therefore \cot\beta = \sqrt{3} = \cot\frac{\pi}{6}$

$\therefore \beta = \frac{\pi}{6}$ ... $\left[\because 0 < \frac{\pi}{6} < \pi\right]$

$\therefore \cot^{-1}(\sqrt{3}) = \frac{\pi}{6}$ ... (2)

$\therefore \operatorname{cosec}^{-1}(-\sqrt{2}) + \cot^{-1}(\sqrt{3})$

$= -\frac{\pi}{4} + \frac{\pi}{6}$ ... [By (1) and (2)]

$= \frac{-3\pi + 2\pi}{12} = -\frac{\pi}{12}.$

Question 3.
Prove the following :

(i) sin-1(12√) – 3sin-1(3√2) = −−3π4

Question is modified.

sin-1(12√) – 3sin-1(3√2) = −3π4

Solution:

$$\text{Let } \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \alpha, \text{ where } -\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2}$$

$$\therefore \ \sin\alpha = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

$$\therefore \ \alpha = \frac{\pi}{4} \qquad \dots \left[\because \ -\frac{\pi}{2} \leqslant \frac{\pi}{4} \leqslant \frac{\pi}{2}\right]$$

$$\therefore \ \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \qquad \dots (1)$$

$$\text{Let } \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \beta, \text{ where } -\frac{\pi}{2} \leqslant \beta \leqslant \frac{\pi}{2}$$

$$\therefore \ \sin\beta = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$$

$$\therefore \ \beta = \frac{\pi}{3} \qquad \dots \left[\because \ -\frac{\pi}{2} \leqslant \frac{\pi}{3} \leqslant \frac{\pi}{2}\right]$$

$$\therefore \ \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \qquad \dots (2)$$

$$\text{LHS} = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 3\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{4} - 3\left(\frac{\pi}{3}\right) \qquad \dots [\text{By (1) and (2)}]$$

$$= \frac{\pi}{4} - \pi = -\frac{3\pi}{4} = \text{RHS.}$$

(ii) sin-1(−12) + cos-1(−3√2) = cos-1(−12)
Solution:

$$\text{Let } \sin^{-1}\left(-\frac{1}{2}\right) = \alpha, \text{ where } -\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2}$$

$$\therefore \ \sin\alpha = -\frac{1}{2} = -\sin\frac{\pi}{6}$$

$$\therefore \ \sin\alpha = \sin\left(-\frac{\pi}{6}\right) \qquad \dots [\because \ \sin(-\theta) = -\sin\theta]$$

$$\therefore \ \alpha = -\frac{\pi}{6} \qquad \dots \left[\because \ -\frac{\pi}{2} \leqslant -\frac{\pi}{6} \leqslant \frac{\pi}{2}\right]$$

$$\therefore \ \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \qquad \dots (1)$$

$$\text{Let } \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \beta, \text{ where } 0 \leqslant \beta \leqslant \pi$$

$$\therefore \ \cos\beta = -\frac{\sqrt{3}}{2} = -\cos\frac{\pi}{6}$$

$$\therefore \ \cos\beta = \cos\left(\pi - \frac{\pi}{6}\right) \qquad \dots [\because \ \cos(\pi - \theta) = -\cos\theta]$$

$$\therefore \ \cos\beta = \cos\frac{5\pi}{6}$$

$\therefore\ \beta = \frac{5\pi}{6}$ ... $\left[\because\ 0 \leqslant \frac{5\pi}{6} \leqslant \pi\right]$

$\therefore\ \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ ... (2)

Let $\cos^{-1}\left(-\frac{1}{2}\right) = \gamma$, where $0 \leqslant \gamma \leqslant \pi$

$\therefore\ \cos\gamma = -\frac{1}{2} = -\cos\frac{\pi}{3}$

$\therefore\ \cos\gamma = \cos\left(\pi - \frac{\pi}{3}\right)$ ... $[\because\ \cos(\pi - \theta) = -\cos\theta]$

$\therefore\ \cos\gamma = \cos\frac{2\pi}{3}$

$\therefore\ \gamma = \frac{2\pi}{3}$ ... $\left[\because\ 0 \leqslant \frac{2\pi}{3} \leqslant \pi\right]$

$\therefore\ \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ ... (3)

$\text{LHS} = \sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= -\frac{\pi}{6} + \frac{5\pi}{6}$ ... [By (1) and (2)]

$= \frac{4\pi}{6} = \frac{2\pi}{3}$

$= \cos^{-1}\left(-\frac{1}{2}\right)$ ... [By (3)]

$= \text{RHS}.$

(iii) sin-1(35) + cos-1(1213) = sin-1(5665)
Solution:

Let $\sin^{-1}\left(\frac{3}{5}\right) = x$, $\cos^{-1}\left(\frac{12}{13}\right) = y$ and $\sin^{-1}\left(\frac{56}{65}\right) = z$.

Then $\sin x = \frac{3}{5}$, where $0 < x < \frac{\pi}{2}$

$\cos y = \frac{12}{13}$, where $0 < y < \frac{\pi}{2}$

and $\sin z = \frac{56}{65}$, where $0 < z < \frac{\pi}{2}$

$\therefore\ \cos x > 0,\ \sin y > 0$

Now, $\cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

and $\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - \frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$

We have to prove that, $x + y = z$

Now, $\sin(x + y) = \sin x \cos y + \cos x \sin y$

$$= \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{5}{13}\right)$$
$$= \frac{36}{65} + \frac{20}{65} = \frac{56}{65}$$

$\therefore\ \sin(x + y) = \sin z \qquad \therefore\ x + y = z$

Hence, $\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\left(\frac{56}{65}\right)$.

(iv) $\cos^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$

Solution:

Let $\cos^{-1}\left(\frac{3}{5}\right) = x$

$\therefore \cos x = \left(\frac{3}{5}\right)$, where $0 < x < \frac{\pi}{2}$ $\therefore \sin x > 0$

Now, $\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\therefore \; x = \sin^{-1}\left(\frac{4}{5}\right)$

$\therefore \; \cos^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{4}{5}\right)$ ... (1)

$$\text{LHS} = \cos^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right)$$

$$= \sin^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right) \quad \text{... [By (1)]}$$

$$= \frac{\pi}{2} \quad \text{...} \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

$= \text{RHS.}$

(v) $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

Solution:

LHS = $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}\right)$

$= \tan^{-1}\left(\frac{3+2}{6-1}\right) = \tan^{-1}(1)$

$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$

= RHS.

(vi) $2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

Solution:

$$\text{LHS} = 2\tan^{-1}\left(\frac{1}{3}\right)$$

$$= \tan^{-1}\left[\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}\right]$$

$$\ldots \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left[\frac{\left(\frac{2}{3}\right)}{1-\frac{1}{9}}\right] = \tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right)$$

$$= \text{RHS.}$$

Alternative Method :

$$\text{LHS} = 2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{3}\right)$$

$$= \tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{3}}{1-\frac{1}{3}\times\frac{1}{3}}\right]$$

$$= \tan^{-1}\left(\frac{3+3}{9-1}\right) = \tan^{-1}\left(\frac{6}{8}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right)$$

$$= \text{RHS.}$$

(vii) tan-1$\left[\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right]$ = $\frac{\pi}{4}$ + θ if θ ∈ $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$

Solution:

$$\text{LHS} = \tan^{-1}\left[\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}\right]$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\,\tan\theta}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$= \frac{\pi}{4}+\theta \qquad \ldots [\because \tan^{-1}(\tan\theta) = \theta]$$

$$= \text{RHS.}$$

(viii) $\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\theta}{2}$, if θ ∈ (0, π)

Solution:

$$\frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^2(\theta/2)}{2\cos^2(\theta/2)}$$

$$= \tan^2\left(\frac{\theta}{2}\right)$$

$$\therefore \sqrt{\frac{1-\cos\theta}{1+\text{co}\,\theta}} = \sqrt{\tan^2\left(\frac{\theta}{2}\right)} = \tan\left(\frac{\theta}{2}\right)$$

$$\therefore \text{ LHS} = \tan^{-1}\left[\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right]$$

= $\frac{\theta}{2}$ ...[∵ $\tan^{-1}(\tan\theta) = \theta$]
= RHS.

Maharashtra State Board 12th Maths Solutions Chapter 3 Trigonometric Functions Miscellaneous Exercise 3

I) Select the correct option from the given alternatives.
Question 1.
The principal of solutions equation sinθ = $-\frac{1}{2}$ are _______.

(a) $\frac{5\pi}{6}, \frac{\pi}{6}$
(b) $\frac{7\pi}{6}, \frac{11\pi}{6}$
(c) $\frac{\pi}{6}, \frac{7\pi}{6}$
(d) $\frac{7\pi}{6}, \frac{\pi}{3}$.

Solution:
(b) $\frac{7\pi}{6}, \frac{11\pi}{6}$

Question 2.
The principal solution of equation cot θ = $\sqrt{3}$ _________

(a) $\frac{\pi}{6}, \frac{7\pi}{6}$
(b) $\frac{\pi}{6}, \frac{5\pi}{6}$
(c) $\frac{\pi}{6}, \frac{8\pi}{6}$
(d) $\frac{7\pi}{6}, \frac{\pi}{3}$.

Solution:
(a) $\frac{\pi}{6}, \frac{7\pi}{6}$

Question 3.
The general solution of sec x = $\sqrt{2}$ is ________.
(a) $2n\pi \pm \frac{\pi}{4}$, n ∈ Z
(b) $2n\pi \pm \frac{\pi}{2}$, n ∈ Z
(c) $n\pi \pm \frac{\pi}{2}$, n ∈ Z
(d) $2n\pi \pm \frac{\pi}{3}$, n ∈ Z

Solution:
(a) 2nπ ± $\frac{\pi}{4}$, n ∈ Z

Question 4.
If cos pθ = cosqθ, p ≠ q rhen _______.
(a) θ = $\frac{2n\pi}{p \pm q}$
(b) θ = 2nπ
(c) θ = 2nπ ± p
(d) nπ ± q
Solution:
(a) θ = $\frac{2n\pi}{p \pm q}$

Question 5.
If polar co-ordinates of a point are $\left(2, \frac{\pi}{4}\right)$ then its cartesian co-ordinates are _____.
(a) (2, $\sqrt{2}$)
(b) ($\sqrt{2}$, 2)
(c) (2, 2)
(d) ($\sqrt{2}$, $\sqrt{2}$)
Solution:
(d) ($\sqrt{2}$, $\sqrt{2}$)

Question 6.
If $\sqrt{3}$ cosx – sin x = 1, then general value of x is ________.
(a) 2nπ ± $\frac{\pi}{3}$
(b) 2nπ ± $\frac{\pi}{6}$
(c) 2nπ ± $\frac{\pi}{3} - \frac{\pi}{6}$
(d) nπ + $(-1)^n \frac{\pi}{3}$
Solution:
(c) 2nπ ± $\frac{\pi}{3} - \frac{\pi}{6}$

Question 7.
In ΔABC if ∠A = 45°, ∠B = 60° then the ratio of its sides are ________.
(a) 2 : $\sqrt{2}$: $\sqrt{3}$ + 1
(b) $\sqrt{2}$: 2 : $\sqrt{3}$ + 1
(c) 2 $\sqrt{2}$: $\sqrt{2}$: $\sqrt{3}$
(d) 2 : 2 $\sqrt{2}$: $\sqrt{3}$ + 1
Solution:
(a) 2 : $\sqrt{2}$: $\sqrt{3}$ + 1

Question 8.
In ΔABC, if $c^2 + a^2 - b^2 = ac$, then ∠B = _________.
(a) $\frac{\pi}{4}$
(b) $\frac{\pi}{3}$
(c) $\frac{\pi}{2}$
(d) $\frac{\pi}{6}$
Solution:
(b) $\frac{\pi}{3}$

Question 9.
In ABC, ac cos B – bc cos A = ___________.
(a) $a^2 - b^2$
(b) $b^2 - c^2$
(c) $c^2 - a^2$
(d) $a^2 - b^2 - c^2$
Solution:
(a) $a^2 - b^2$

Question 10.
If in a triangle, the are in A.P. and b : c = $\sqrt{3}$: $\sqrt{2}$ then A is equal to _________.
(a) 30°
(b) 60°
(c) 75°

(d) 45°
Solution:
(c) 75°

Question 11.
$\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ = _______.

a) $\frac{7\pi}{6}$
b) $\frac{5\pi}{6}$
c) $\frac{\pi}{6}$
d) $\frac{3\pi}{2}$

Question 12.
The value of cot ($\tan^{-1} 2x + \cot^{-1} 2x$) is _________.
(a) 0
(b) 2x
(c) π + 2x
(d) π – 2x
Solution:
(a) 0

Question 13.
The principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is __________.

a) $\left(-\frac{2\pi}{3}\right)$
b) $\frac{4\pi}{3}$
c) $\frac{5\pi}{3}$
d) $-\frac{\pi}{3}$

Solution:
(d) $-\frac{\pi}{3}$

Question 14.
If $\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \sin^{-1} \propto$, then ∝ = ___________.
(a) $\frac{63}{65}$
(b) $\frac{62}{65}$
(c) $\frac{61}{65}$
(d) $\frac{60}{65}$
Solution:
(a) $\frac{63}{65}$

Question 15.
If $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$, then x = _______.
(a) -1
(b) $\frac{1}{6}$
(c) $\frac{2}{6}$
(d) $\frac{3}{2}$
Solution:
(b) $\frac{1}{6}$

Question 16.
$2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$ = _____.
(a) $\tan^{-1}\frac{4}{5}$
(b) $\frac{\pi}{2}$
(c) 1
(d) $\frac{\pi}{4}$
Solution:
(d) $\frac{\pi}{4}$

Question 17.
$\tan\left(2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right)$ = _____.

(a) 177

(b) −177

(c) 717

(d) −717

Solution:

(d) −717

$$\left[\textbf{Hint : } 2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left[\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}\right] = \tan^{-1}\left(\frac{5}{12}\right)\right.$$

$$\therefore\ 2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4} = \tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}(1)$$

$$= \tan^{-1}\left[\frac{\frac{5}{12}-1}{1+\frac{5}{12}\times 1}\right] = \tan^{-1}\left(\frac{-7}{17}\right)$$

$$\therefore\ \tan\left[2\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}\right] = \tan\left[\tan^{-1}\left(-\frac{7}{17}\right)\right]$$

$$\left. = -\frac{7}{17}.\right]$$

Question 18.
The principal value branch of sec-1 x is _________.

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ (b) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

(c) $(0, \pi)$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution:
(b) [0, π] – {π2}

Question 19.
cos[tan-113 + tan-112] = ________.
(a) 12√
(b) 3√2
(c) 12
(d) π4
Solution:
(a) 12√

Question 20.
If tan θ + tan 2θ + tan 3θ = tan θ·tan 2θ·tan 3θ, then the general value of the θ is _______.
(a) nπ
(b) nπ6
(c) nπ ± nπ4
(d) nπ2
Solution:
(b) nπ6
[Hint: tan(A + B + C) = tanA+tanB+tanC−tanA⋅ tanB⋅ tanC1−tanA⋅ tanB−tanB⋅ tanC−tanC⋅ tanA
Since , tan θ + tan 2θ + tan 3θ = tan θ · tan 2θ · tan 3θ,
we get, tan (θ + 2θ + 3θ) = θ
∴ tan6θ = 0
∴ 6θ = nπ, θ = nπ6.]

Question 21.
If any ΔABC, if a cos B = b cos A, then the triangle is ________.
(a) Equilateral triangle
(b) Isosceles triangle
(c) Scalene
(d) Right angled
Solution:
(b) Isosceles triangle

II: Solve the following
Question 1.
Find the principal solutions of the following equations :

(i) sin2θ = $-\frac{1}{2}$

Solution:

sin2θ = $-\frac{1}{2}$

Since, θ ∈ (0, 2π), 2∈ ∈ (0, 4π)

$$\therefore \sin 2\theta = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(\pi+\frac{\pi}{6}\right) = \sin\left(2\pi-\frac{\pi}{6}\right)$$

$$= \sin\left(3\pi+\frac{\pi}{6}\right) = \sin\left(4\pi-\frac{\pi}{6}\right)$$

$$\ldots \left[\because \sin(\pi+\theta) = \sin(2\pi-\theta) = \sin(3\pi+\theta) = \sin(4\pi-\theta) = -\sin\theta\right]$$

$$\therefore \sin 2\theta = \sin\frac{7\pi}{6} = \sin\frac{11\pi}{6} = \sin\frac{19\pi}{6} = \sin\frac{23\pi}{6}$$

$$\therefore 2\theta = \frac{7\pi}{6} \text{ or } 2\theta = \frac{11\pi}{6} \text{ or } 2\theta = \frac{19\pi}{6} \text{ or } 2\theta = \frac{23\pi}{6}$$

$$\therefore \theta = \frac{7\pi}{12} \text{ or } \theta = \frac{11\pi}{12} \text{ or } \theta = \frac{19\pi}{12} \text{ or } \theta = \frac{23\pi}{12}$$

Hence, the required principal solutions are

$$\left\{\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}.$$

(ii) tan3θ = -1
Solution:
Since, θ ∈ (0, 2π), 3∈ ∈ (0, 6π)

$$\therefore \tan 3\theta = -1 = -\tan\frac{\pi}{4} = \tan\left(\pi-\frac{\pi}{4}\right)$$

$$= \tan\left(2\pi-\frac{\pi}{4}\right) = \tan\left(3\pi-\frac{\pi}{4}\right)$$

$$= \tan\left(4\pi-\frac{\pi}{4}\right) = \tan\left(5\pi-\frac{\pi}{4}\right)$$

$$= \tan\left(6\pi-\frac{\pi}{4}\right)$$

… [∵ tan(π – θ) = tan(2π – θ) = tan(3π – θ)
= tan (4π – θ) = tan (5π – θ) = tan (6π – θ) = -tan θ]
∴ tan3θ = tan$\frac{3\pi}{4}$ = tan$\frac{7\pi}{4}$ = tan$\frac{11\pi}{4}$ = tan$\frac{15\pi}{4}$

$$= \tan\frac{19\pi}{4} = \tan\frac{23\pi}{4}$$

$$\therefore 3\theta = \frac{3\pi}{4} \text{ or } 3\theta = \frac{7\pi}{4} \text{ or } 3\theta = \frac{11\pi}{4} \text{ or } 3\theta = \frac{15\pi}{4}$$

$$\text{or } 3\theta = \frac{19\pi}{4} \text{ or } 3\theta = \frac{23\pi}{4}$$

$$\therefore \theta = \frac{\pi}{4} \text{ or } \theta = \frac{7\pi}{12} \text{ or } \theta = \frac{11\pi}{12} \text{ or } \theta = \frac{5\pi}{4}$$

$$\text{or } \theta = \frac{19\pi}{12} \text{ or } \theta = \frac{23\pi}{12}$$

Hence, the required principal solutions are,

$$\left\{\frac{\pi}{4}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}.$$

(iii) cotθ = 0
Solution:
cotθ = 0
Since θ ∈ (0, 2π),
cotθ = 0 = cot $\frac{\pi}{2}$ = cot (π + $\frac{\pi}{2}$ …[∵ cos(π + θ) = cotθ]

∴ cotθ = cotπ2 = cot3π2
∴ θ = π2 or θ = 3π2

Hence, the required principal solutions are {π2,3π2}

Question 2.
Find the principal solutions of the following equations :

(i) sin2θ = −12√
Solution:

(ii) tan5θ = -1
Solution:

(iii) cot2θ = 0
Solution:

Question 3.
Which of the following equations have no solutions ?
(i) cos 2θ = 13
Solution:
cos 2θ = 13
Since 13 ≤ cosθ ≤ 1 for any θ
cos2θ = 13 has solution

(ii) cos2 θ = -1
Solution:
cos2θ = -1
This is not possible because cos2θ ≥ 0 for any θ.
∴ cos2θ = -1 does not have any solution.

(iii) 2 sinθ = 3
Solution:
2 sin θ = 3 ∴ sin θ = 32
This is not possible because -1 ≤ sin θ ≤ 1 for any θ.
∴ 2 sin θ = 3 does not have any solution.

(iv) 3 sin θ = 5
Solution:
3 sin θ = 5
∴ sin θ = 53
This is not possible because -1 ≤ sin θ ≤ 1 for any θ.
∴ 3 sin θ = 5 does not have any solution.

Question 4.
Find the general solutions of the following equations :
(i) tanθ = −X−−√
Solution:
The general solution of tan θ = tan ∝ is
θ = nπ + ∝, n ∈ Z.
Now, tanθ = −X−−√
∴ tanθ = tanπ3 ...[∵ tanπ3 = 3–√]
∴ tanθ = tan(π−π3) ...[∵ tan(π – θ) = -tanθ]
∴ tanθ = tan2π3
∴ the required general solution is
θ = nπ + 2π3, n ∈ Z.

(ii) tan2θ = 3
Solution:
The general solution of tan2θ = tan2∝ is
θ = nπ ± ∝, n ∈ Z.
Now, tan2θ = 3 = (X−−√)2
∴ tan2θ = (tanπ3)2 ...[∵ tanπ3 = 3–√]
∴ tan2θ = tan2π3
∴ the required general solution is
θ = nπ ± π3, n ∈ Z.

(iii) sin θ – cosθ = 1
Solution:
∴ cosθ – sin θ = -1

Dividing both sides by $\sqrt{(1)^2+(-1)^2}=\sqrt{2}$, we get

$$\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta=-\frac{1}{\sqrt{2}}$$

$$\therefore\ \cos\frac{\pi}{4}\cos\theta-\sin\frac{\pi}{4}\sin\theta=-\cos\frac{\pi}{4}$$

$$\therefore\ \cos\left(\theta-\frac{\pi}{4}\right)=\cos\left(\pi-\frac{\pi}{4}\right)$$

$$\ldots\ [\because\ \cos(\pi-\theta)=-\cos\theta]$$

$$\therefore\ \cos\left(\theta-\frac{\pi}{4}\right)=\cos\frac{3\pi}{4} \qquad \ldots (1)$$

The general solution of $\cos\theta=\cos\alpha$ is

$\theta=2n\pi\pm\alpha,\ n\in Z$

∴ the general solution of (1) is given by

$$\theta-\frac{\pi}{4}=2n\pi\pm\frac{3\pi}{4},\ n\in Z$$

Taking positive sign, we get

$$\theta-\frac{\pi}{4}=2n\pi+\frac{3\pi}{4},\ n\in Z.$$

$$\therefore\ \theta=2n\pi+\pi=(2n+1)\pi,\ n\in Z$$

Taking negative sign, we get

$$\theta-\frac{\pi}{4}=2n\pi-\frac{3\pi}{4},\ n\in Z.$$

$$\therefore\ \theta=2n\pi-\frac{\pi}{2},\ n\in Z$$

∴ the required general solution is

$$\theta=(2n+1)\pi,\ n\in Z\ \text{ or }\ \theta=2n\pi-\frac{\pi}{2},\ n\in Z.$$

(iv) $\sin^2\theta - \cos^2\theta = 1$
Solution:
$\sin^2\theta - \cos^2\theta = 1$
∴ $\cos^2\theta - \sin^2\theta = -1$
∴ cos2θ = cosπ ...(1)
The general solution of cos θ = cos ∝ is
θ = 2nπ ± ∝, n ∈ Z
∴ the general solution of (1) is given by
2θ = 2nπ ± π, n ∈ Z
∴ θ = nπ ± $\frac{\pi}{2}$, n ∈ Z

Question 5.
In ΔABC prove that $\cos\left(\frac{A-B}{2}\right)=\left(\frac{a+b}{c}\right)\sin\frac{C}{2}$
Solution:
By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\therefore\ a = k\sin A,\ b = k\sin B,\ c = k\sin C$$

$$\text{RHS} = \left(\frac{a+b}{c}\right)\sin\frac{C}{2}$$

$$= \left(\frac{k\sin A + k\sin B}{k\sin C}\right)\sin\frac{C}{2}$$

$$= \left(\frac{\sin A + \sin B}{\sin C}\right)\sin\frac{C}{2}$$

$$= k \times 2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right) \times 2\cos\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)$$

$$= k \times 2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A+B}{2}\right) \times 2\sin\left(\frac{A-B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$$

Question 6.
With usual notations prove that $\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2-b^2}{c^2}$.
Solution:
By the sine rule,
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$
∴ a = ksinA, b = ksinB, c = ksinC

$$\text{RHS} = \frac{a^2-b^2}{c^2} = \frac{k^2\sin^2 A - k^2\sin^2 B}{k^2\sin^2 C}$$

$$= \frac{\sin^2 A - \sin^2 B}{\sin^2 C}$$

$$= \frac{(\sin A + \sin B)(\sin A - \sin B)}{[\sin\{\pi - (A+B)\}]^2} \quad \ldots [\because A + B + C = \pi]$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right) \times 2\cos\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)}{\sin^2(A+B)}$$

$$= \frac{2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A+B}{2}\right) \times 2\sin\left(\frac{A-B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)}{\sin^2(A+B)}$$

$$= \frac{\sin(A+B)\cdot\sin(A-B)}{\sin^2(A+B)}$$

$$= \frac{\sin(A-B)}{\sin(A+B)} = \text{LHS.}$$

Question 7.
In ΔABC prove that $(a - b)^2\ 2\cos^2\frac{C}{2} + (a + b)^2 \sin^2\frac{C}{2} = c^2$.
Solution:
LHS $(a - b)^2\ 2\cos^2\frac{C}{2} + (a + b)^2 \sin^2\frac{C}{2}$
$= (a^2 + b^2 - 2ab) \cos^2\frac{C}{2} + (a^2 + b^2 + 2ab) \sin^2\frac{C}{2}$
$= (a^2 + b^2) \cos^2\frac{C}{2} - 2ab \cos^2\frac{C}{2} + (a^2 + b^2) \sin^2\frac{C}{2} + 2ab \sin^2\frac{C}{2}$
$= (a^2 + b^2) (\cos^2\frac{C}{2} + \sin^2\frac{C}{2}) - 2ab(\cos^2\frac{C}{2} - \sin^2\frac{C}{2})$
$= a^2 + b^2 - 2ab \cos C$
$= c^2 =$ RHS.

Question 8.
In ΔABC if cosA = sin B – cos C then show that it is a right angled triangle.
Solution:
cos A= sin B – cos C
∴ cos A + cos C = sin B

$$\therefore\ 2\cos\left(\frac{A+C}{2}\right)\cdot\cos\left(\frac{A-C}{2}\right)=\sin B$$

$$\therefore\ 2\cos\left(\frac{\pi}{2}-\frac{B}{2}\right)\cdot\cos\left(\frac{A-C}{2}\right)=\sin B$$

$$\ldots\ [\because\ A+B+C=\pi]$$

$$\therefore\ 2\sin\frac{B}{2}\cdot\cos\left(\frac{A-C}{2}\right)=2\sin\frac{B}{2}\cdot\cos\frac{B}{2}$$

$$\therefore\ \cos\left(\frac{A-C}{2}\right)=\cos\frac{B}{2}$$

$$\therefore\ \frac{A-C}{2}=\frac{B}{2}$$

∴ A – C = B
∴ A = B + C
∴ A + B + C = 180° gives
A + A = 180°
∴ 2A = 180 ∴ A = 90°
∴ Δ ABC is a rightangled triangle.

Question 9.
If $\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)}$ then show that a^2, b^2, c^2, are in A.P.
Solution:
By sine rule,
$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k$
∴ sin A = ka, sin B = kb,sin C = kc
Now, $\frac{\sin A}{\sin C}=\frac{\sin(A-B)}{\sin(B-C)}$
∴ sinA·sin(B – C) = sinC·sin(A -B)
∴ sin [π – (B + C)] · sin (B – C)
= sin [π – (A + B)]·sin (A – B) … [∵ A + B + C = π]
∴ sin(B + C) · sin(B – C) = sin (A + B) · sin (A – B)
∴ $\sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$
∴ $2\sin^2 B = \sin^2 A + \sin^2 C$
∴ $2k^2b^2 = k^2a^2 + k^2c^2$
∴ $2b^2 = a^2 + c^2$
Hence, a^2, b^2, c^2 are in A.P.

Question 10.
Solve the triangle in which a = $(\sqrt{3} + 1)$, b = $(\sqrt{3} - 1)$ and ∠C = 60°.
Solution:
Given : a = $\sqrt{3} + 1$, b = $\sqrt{3} - 1$ and ∠C = 60°.
By cosine rule,
$c^2 = a^2 + b^2 - 2ab \cos C$
$= (\sqrt{3} + 1)^2 + (\sqrt{3} - 1)^2 - 2(\sqrt{3} + 1)(\sqrt{3} - 1)\cos 60°$
$= 3 + 1 + 2\sqrt{3} + 3 + 1 - 2\sqrt{3} - 2(3 - 1)\left(\frac{1}{2}\right)$
= 8 – 2 = 6
∴ c = $\sqrt{6}$ …[∵ c > 0)
By sine rule,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\therefore\ \frac{\sqrt{3}+1}{\sin A}=\frac{\sqrt{3}-1}{\sin B}=\frac{\sqrt{6}}{\sin 60^\circ}$$

$$\therefore\ \frac{\sqrt{3}+1}{\sin A}=\frac{\sqrt{3}-1}{\sin B}=\frac{\sqrt{6}}{(\sqrt{3}/2)}=2\sqrt{2}$$

$$\therefore\ \sin A=\frac{\sqrt{3}+1}{2\sqrt{2}}\ \text{ and }\ \sin B=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\therefore\ \sin A=\frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\ \text{ and }\ \sin B=\frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}$$

$$\therefore\ \sin A=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}$$

$$\text{and } \sin B=\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}-\frac{1}{2}\times\frac{1}{\sqrt{2}}$$

∴ sin A = sin 60° cos 45° + cos 60° sin 45°
and sin B = sin 60° cos 45° – cos 60° sin 45°
∴ sin A = sin (60° + 45°) – sin 105°
and sin B = sin (60° – 45°) = sin 15°
∴ A = 105° and B = 15°

Hence, A = 105°, B 15° and C = 6–√ units.

Question 11.
In ΔABC prove the following :
(i) a sin A – b sin B = c sin (A – B)
Solution:
By sine rule,
asinA = *bsinB* = *csinC* = k
∴ a = ksinA, b = ksinB, c = ksinC,
LHS = a sin A – b sinB
= ksinA·sinA – ksinB·sinB
= k (sin2A – sin2B)
= k (sin A + sin B)(sin A – sin B)

$$=k\times 2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)\times 2\cos\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)$$

$$=k\times 2\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A+B}{2}\right)\times 2\sin\left(\frac{A-B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$$

= k × sin (A + B) × sin (A – B)
= ksin(π – C)·sin(A – B) … [∵ A + B + C = π]
= k sinC·sin (A – B)
= c sin (A – B) = RHS.

(ii) $\frac{c-b\cos A}{b-c\cos A}=\frac{\cos B}{\cos C}$.

Solution:

$$\text{LHS}=\frac{c-b\cos A}{b-c\cos A}=\frac{c-b\left(\frac{b^2+c^2-a^2}{2bc}\right)}{b-c\left(\frac{b^2+c^2-a^2}{2bc}\right)}$$

$$=\frac{c-\left(\frac{b^2+c^2-a^2}{2c}\right)}{b-\left(\frac{b^2+c^2-a^2}{2b}\right)}=\frac{\left(\frac{2c^2-b^2-c^2+a^2}{2c}\right)}{\left(\frac{2b^2-b^2-c^2+a^2}{2b}\right)}$$

$$=\frac{\left(\frac{c^2+a^2-b^2}{2c}\right)}{\left(\frac{a^2+b^2-c^2}{2b}\right)}=\frac{\left(\frac{c^2+a^2-b^2}{2ca}\right)}{\left(\frac{a^2+b^2-c^2}{2ab}\right)}$$

$$=\frac{\cos B}{\cos C}=\text{RHS}.$$

(iii) $a^2 \sin(B - C) = (b^2 - c^2) \sin A$

Solution:

By sine rule,

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$

$\therefore$ a = ksinA, b = ksinB, c = ksinC

RHS = $(b^2 - c^2) \sin A$

= $(k^2\sin^2 B - k^2\sin^2 C)\sin A$

= $k^2(\sin^2 B - \sin^2 C) \sin A$

= $k^2(\sin B + \sin C)(\sin B - \sin C) \sin A$

$$=k^2\times 2\sin\left(\frac{B+C}{2}\right)\cdot\cos\left(\frac{B-C}{2}\right)\times 2\cos\left(\frac{B+C}{2}\right)\cdot\sin\left(\frac{B-C}{2}\right)\times\sin A$$

$$=k^2\times 2\sin\left(\frac{B+C}{2}\right)\cdot\cos\left(\frac{B+C}{2}\right)\times 2\sin\left(\frac{B-C}{2}\right)\cdot\cos\left(\frac{B-C}{2}\right)\times\sin A$$

= $k^2 \times \sin(B + C) \times \sin(B - C) \times \sin A$

= $k^2\cdot\sin(\pi - A)\cdot\sin(B - C)\cdot\sin A$ ... [$\because$ A + B + C = π]

= $k^2\sin A\cdot\sin(B - C)\cdot\sin A$

= $(k \sin A)^2\cdot\sin(B - C)$

= $a^2\sin(B - C)$ = LHS.

(iv) $ac \cos B - bc \cos A = (a^2 - b^2)$.

Solution:

LHS = ac cos B – bc cos A

= $ac\left(\frac{c^2+a^2-b^2}{2ca}\right) - bc\left(\frac{b^2+c^2-a^2}{2bc}\right)$

$=\frac{1}{2}(c^2 + a^2 - b^2) - \frac{1}{2}(b^2 + c^2 - a^2)$

= $\frac{1}{2}(c^2 + a^2 - b^2 - b^2 - c^2 + a^2)$

= $\frac{1}{2}(2a^2 - 2b^2) = a^2 - b^2$ = RHS.

(v) $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$.

Solution:

$$\text{LHS} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \frac{\left(\frac{b^2+c^2-a^2}{2bc}\right)}{a} + \frac{\left(\frac{c^2+a^2-b^2}{2ca}\right)}{b} + \frac{\left(\frac{a^2+b^2-c^2}{2ab}\right)}{c}$$

$$= \frac{b^2+c^2-a^2}{2abc} + \frac{c^2+a^2-b^2}{2abc} + \frac{a^2+b^2-c^2}{2abc}$$

$$= \frac{b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2}{2abc}$$

$$= \frac{a^2+b^2+c^2}{2abc} = \text{RHS.}$$

(vi) $\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$.

Solution:
By sine rule,

$\frac{\sin A}{a} = \frac{\sin B}{b}$

$$\therefore \frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2} \quad \ldots (1)$$

$$\text{LHS} = \frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2}$$

$$= \frac{1-2\sin^2 A}{a^2} - \frac{1-2\sin^2 B}{b^2}$$

$$= \frac{1}{a^2} - \frac{2\sin^2 A}{a^2} - \frac{1}{b^2} + \frac{2\sin^2 B}{b^2}$$

$$= \frac{1}{a^2} - \frac{1}{b^2} - 2\left(\frac{\sin^2 A}{a^2} - \frac{\sin^2 B}{b^2}\right)$$

$$= \frac{1}{a^2} - \frac{1}{b^2} - 2\left(\frac{\sin^2 B}{b^2} - \frac{\sin^2 B}{b^2}\right) \quad \ldots [\text{By (1)}]$$

$$= \frac{1}{a^2} - \frac{1}{b^2} - 2 \times 0$$

$$= \frac{1}{a^2} - \frac{1}{b^2} = \text{RHS.}$$

(vii) $\frac{b-c}{a} = \frac{\tan\frac{B}{2} - \tan\frac{C}{2}}{\tan\frac{B}{2} + \tan\frac{C}{2}}$

Solution:
By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$$

$$\therefore a = k\sin A,\ b = k\sin B,\ c = k\sin C$$

$$\text{LHS} = \frac{b-c}{a} = \frac{k\sin B - k\sin C}{k\sin A}$$

$$= \frac{\sin B - \sin C}{\sin A}$$

$$= \frac{\sin B - \sin C}{\sin\{\pi - (B+C)\}} \quad \ldots [\because A+B+C=\pi]$$

$$= \frac{\sin B - \sin C}{\sin(B+C)}$$

$$= \frac{2\cos\left(\frac{B+C}{2}\right)\cdot\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cdot\cos\left(\frac{B+C}{2}\right)}$$

$$= \frac{\sin\left(\frac{B-C}{2}\right)}{\sin\left(\frac{B+C}{2}\right)} = \frac{\sin\left(\frac{B}{2}-\frac{C}{2}\right)}{\sin\left(\frac{B}{2}+\frac{C}{2}\right)}$$

$$= \frac{\sin\frac{B}{2}\cos\frac{C}{2} - \cos\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{B}{2}\cos\frac{C}{2} + \cos\frac{B}{2}\sin\frac{C}{2}}$$

$$= \frac{\dfrac{\sin\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} - \dfrac{\cos\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}}{\dfrac{\sin\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} + \dfrac{\cos\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}}$$

$$= \frac{\dfrac{\sin\frac{B}{2}}{\cos\frac{B}{2}} - \dfrac{\sin\frac{C}{2}}{\cos\frac{C}{2}}}{\dfrac{\sin\frac{B}{2}}{\cos\frac{C}{2}} + \dfrac{\sin\frac{C}{2}}{\cos\frac{C}{2}}} = \frac{\tan\frac{B}{2} - \tan\frac{C}{2}}{\tan\frac{B}{2} + \tan\frac{C}{2}} = \text{RHS.}$$

Alternative Method :

$$\text{RHS} = \frac{\tan\frac{B}{2} - \tan\frac{C}{2}}{\tan\frac{B}{2} + \tan\frac{C}{2}}$$

$$= \frac{\sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}} - \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}}{\sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}} + \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}}$$

$$= \frac{(s-c)\sqrt{(s-a)} - (s-b)\sqrt{(s-a)}}{(s-c)\sqrt{(s-a)} + (s-b)\sqrt{(s-a)}}$$

$$= \frac{(s-c)-(s-b)}{(s-c)+(s-b)} = \frac{s-c-s+b}{s-c+s-b}$$

$$= \frac{b-c}{2s-(b+c)} = \frac{b-c}{(a+b+c)-(b+c)}$$

$$= \frac{b-c}{a} = \text{LHS.}$$

Question 12.
In ΔABC if a^2, b^2, c^2, are in A.P. then $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are also in A.P.
Question is modified

In ΔABC if a, b, c, are in A.P. then $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are also in A.P.
Solution:
a, b, c, are in A.P.
∴ 2b = a + c ...(1)

$$\text{Now, } \cot\frac{A}{2}+\cot\frac{C}{2}=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$$

$$=\frac{\cos\frac{A}{2}\cdot\sin\frac{C}{2}+\sin\frac{A}{2}\cdot\cos\frac{C}{2}}{\sin\frac{A}{2}\cdot\sin\frac{C}{2}}$$

$$=\frac{\sin\left(\frac{A}{2}+\frac{C}{2}\right)}{\sin\frac{A}{2}\cdot\sin\frac{C}{2}}$$

$$=\frac{\sin\left(\frac{\pi}{2}-\frac{B}{2}\right)}{\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{(s-a)(s-b)}{ab}}} \quad \ldots [\because A+B+C=\pi]$$

$$=\frac{\cos\frac{B}{2}}{\frac{(s-b)}{b}\cdot\sqrt{\frac{(s-c)(s-a)}{ca}}}$$

$$=\frac{b\cos\frac{B}{2}}{(s-b)\cdot\sin\frac{B}{2}}$$

$$=\frac{b}{s-b}\cdot\cot\frac{B}{2}$$

$$=\frac{b}{\left(\frac{a+b+c}{2}-b\right)}\cdot\cot\frac{B}{2} \quad \ldots [\because 2s=a+b+c]$$

$$=\left(\frac{2b}{a+c-b}\right)\cdot\cot\frac{B}{2}$$

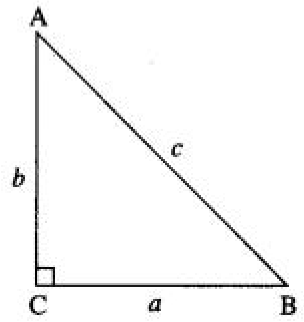
$$=\frac{2b}{(2b-b)}\cdot\cot\frac{B}{2} \quad \ldots \text{[By (1)]}$$

$$=\frac{2b}{b}\cdot\cot\frac{B}{2}$$

$$\therefore \cot\frac{A}{2}+\cot\frac{C}{2}=2\cot\frac{B}{2}$$

Hence, $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are in A.P.

Question 13.
In ΔABC if ∠C = 90° then prove that $\sin(A - B) = \frac{a^2-b^2}{a^2+b^2}$
Solution:
In ΔABC, if ∠C = 90°
∴ $c^2 = a^2 + b^2$ ...(1)
By sine rule,



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin 90°}$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = c \quad \ldots \left[\because \sin 90° = 1\right]$$

$$\therefore \sin A = \frac{a}{c} \text{ and } \sin B = \frac{b}{c} \quad \ldots (2)$$

$$\text{LHS} = \sin(A - B)$$

$$= \sin A \cos B - \cos A \sin B$$

$$= \frac{a}{c}\cos B - \frac{b}{c}\cos A \quad \ldots [\text{By } (2)]$$

$$= \frac{a}{c}\left(\frac{c^2 + a^2 - b^2}{2ca}\right) - \frac{b}{c}\left(\frac{b^2 + c^2 - a^2}{2bc}\right)$$

$$= \frac{c^2 + a^2 - b^2}{2c^2} - \frac{b^2 + c^2 - a^2}{2c^2}$$

$$= \frac{c^2 + a^2 - b^2 - b^2 - c^2 + a^2}{2c^2}$$

$$= \frac{2a^2 - 2b^2}{2c^2} = \frac{a^2 - b^2}{c^2}$$

$$= \frac{a^2 - b^2}{a^2 + b^2} \quad \ldots [\text{By } (1)]$$

$$= \text{RHS.}$$

Question 14.
In ΔABC if $\frac{\cos A}{a} = \frac{\cos B}{b}$, then show that it is an isosceles triangle.
Solution:
Given : $\frac{\cos A}{a} = \frac{\cos B}{b}$ ....(1)
By sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = k$$

$$\therefore a = k\sin A,\ b = k\sin B$$

$$\therefore (1) \text{ gives, } \frac{\cos A}{k\sin A} = \frac{\cos B}{k\sin B}$$

$$\therefore \frac{\cos A}{\sin A} = \frac{\cos B}{\sin B}$$

∴ sin A cos B = cos A sinB
∴ sinA cosB – cosA sinB = 0
∴ sin (A – B) = 0 = sin0
∴ A – B = 0 ∴ A = B
∴ the triangle is an isosceles triangle.

Question 15.
In ΔABC if $\sin^2 A + \sin^2 B = \sin^2 C$ then prove that the triangle is a right angled triangle.
Question is modified
In ΔABC if $\sin^2 A + \sin^2 B = \sin^2 C$ then show that the triangle is a right angled triangle.
Solution:
By sine rule,
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$
∴ sin A = ka, sinB = kb, sin C = kc
∴ $\sin^2 A + \sin^2 B = \sin^2 C$
∴ $k^2a^2 + k^2b^2 = k^2c^2$
∴ $a^2 + b^2 = c^2$
∴ ΔABC is a rightangled triangle, rightangled at C.

Question 16.
In ΔABC prove that $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B) = 0$.
Solution:
By sine rule,
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$
LHS = $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B)$
$= k^2\sin^2 A\,[(1 - \sin^2 B) - (1 - \sin^2 C)] + k^2\sin^2 B\,[(1 - \sin^2 C) - (1 - \sin^2 A)] + k^2\sin^2 C[(1 - \sin^2 A) - (1 - \sin^2 B)]$
$= k^2\sin^2 A\,(\sin^2 C - \sin^2 B) + k^2\sin^2 B(\sin^2 A - \sin^2 C) + k^2\sin^2 C\,(\sin^2 B - \sin^2 A)$
$= k^2(\sin^2 A \sin^2 C - \sin^2 A\sin^2 B + \sin^2 A \sin^2 B - \sin^2 B \sin^2 C + \sin^2 B \sin^2 C - \sin^2 A \sin^2 C)$
$= k^2(0) = 0$ = RHS.

Question 17.
With usual notations show that $(c^2 - a^2 + b^2) \tan A = (a^2 - b^2 + c^2) \tan B = (b^2 - c^2 + a^2) \tan C$.
Solution:
By sine rule,
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$
∴ a = fksinA, b = ksinB, c = ksinC

$$\text{Now, } (c^2 - a^2 + b^2)\tan A = (c^2 - a^2 + b^2)\cdot\frac{\sin A}{\cos A}$$
$$= (c^2 + b^2 - a^2) \times \frac{ka}{\left(\frac{c^2 + b^2 - a^2}{2bc}\right)}$$
$$= (c^2 + b^2 - a^2) \times \frac{2kabc}{c^2 + b^2 - a^2}$$
$$= 2kabc \qquad \ldots (1)$$
$$(a^2 - b^2 + c^2)\tan B = (a^2 - b^2 + c^2)\cdot\frac{\sin B}{\cos B}$$
$$= (a^2 + c^2 - b^2) \times \frac{kb}{\left(\frac{a^2 + c^2 - b^2}{2ac}\right)}$$
$$= (a^2 + c^2 - b^2) \times \frac{2kabc}{a^2 + c^2 - b^2}$$
$$= 2kabc \qquad \ldots (2)$$
$$(b^2 - c^2 + a^2)\tan C = (b^2 - c^2 + a^2)\cdot\frac{\sin C}{\cos C}$$
$$= (a^2 + b^2 - c^2) \times \frac{kc}{\left(\frac{a^2 + b^2 - c^2}{2ab}\right)}$$
$$= (a^2 + b^2 - c^2) \times \frac{2kabc}{a^2 + b^2 - c^2}$$
$$= 2kabc \qquad \ldots (3)$$
From (1), (2) and (3), we get
$(c^2 - a^2 + b^2) \tan A = (a^2 - b^2 + c^2) \tan B$
$= (b^2 - c^2 + a^2) \tan C$.

Question 18.
In ΔABC, if $a \cos^2\frac{C}{2} + c \cos^2\frac{A}{2} = \frac{3b}{2}$, then prove that a , b ,c are in A.P.
Solution:
$a \cos^2\frac{C}{2} + c \cos^2\frac{A}{2} = \frac{3b}{2}$
$$\therefore\ a\left(\frac{1 + \cos C}{2}\right) + c\left(\frac{1 + \cos A}{2}\right) = \frac{3b}{2}$$
$$\therefore\ \frac{1}{2}(a + a\cos C + c + c\cos A) = \frac{3b}{2}$$
$$\therefore\ a + c + (a\cos C + c\cos A) = 3b$$
∴ a + c + b = 3b ...[∵ a cos C + c cos A = b]
∴ a + c = 2b
Hence, a, b, c are in A.P.

Question 19.

Show that $2 \sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{24}{7}\right)$.

Solution:

Let $\sin^{2}\left(\frac{3}{5}\right) = x$.

Then $\sin x = \frac{3}{5}$, where $0 < x < \frac{\pi}{2}$

$\therefore \cos x > 0$

Now, $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\therefore \tan x = \frac{\sin x}{\cos x} = \frac{(3/5)}{(4/5)} = \frac{3}{4}$

$\therefore x = \tan^{-1}\left(\frac{3}{4}\right)$

$\therefore \sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

Now, LHS $= 2\sin^{-1}\left(\frac{3}{5}\right) = 2\tan^{-1}\left(\frac{3}{4}\right)$

$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{4}\right)$

$= \tan^{-1}\left[\frac{\frac{3}{4}+\frac{3}{4}}{1-\frac{3}{4}\times\frac{3}{4}}\right] = \tan^{-1}\left[\frac{12+12}{16-9}\right]$

$= \tan^{-1}\left(\frac{24}{7}\right) = \text{RHS}.$

Alternative Method :

$\text{LHS} = 2\sin^{-1}\left(\frac{3}{5}\right) = 2\tan^{-1}\left(\frac{3}{\ }\right)$

$= \tan^{-1}\left[\frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}\right]$

$\ldots \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$

$= \tan^{-1}\left[\frac{\left(\frac{3}{2}\right)}{1-\left(\frac{9}{16}\right)}\right]$

$= \tan^{-1}\left(\frac{3}{2}\times\frac{16}{7}\right)$

$\therefore \tan^{-1}\left(\frac{24}{7}\right) = \text{RHS}$

Question 20.

Show that $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$.

Solution:

$$\text{LHS} = \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left[\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right] + \tan^{-1}\left[\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right]$$

$$= \tan^{-1}\left(\frac{7+5}{35-1}\right) + \tan^{-1}\left(\frac{8+3}{24-1}\right)$$

$$= \tan^{-1}\left(\frac{12}{34}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$= \tan^{-1}\left[\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right]$$

$$= \tan^{-1}\left(\frac{138+187}{391-66}\right) = \tan^{-1}\left(\frac{325}{325}\right)$$

$$= \tan^{-1}(1) = \tan^{-1}\left(\tan\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} = \text{RHS}.$$

Question 21.
Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$, if x ∈ [0, 1].
Solution:
Let $\tan^{-1}\sqrt{x} = y$
∴ $\tan y = \sqrt{x}$ ∴ $x = \tan^2 y$

$$\text{Now, RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2 y}{1+\tan^2 y}\right)$$

$$= \frac{1}{2}\cos^{-1}(\cos 2y) = \frac{1}{2}(2y) = y$$

$$= \tan^{-1}\sqrt{x} = \text{LHS}.$$

Question 22.
Show that $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$.
Question is modified
Show that $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$.
Solution:
We have to show that

$$\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

i.e. to show that,

$$\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)+\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)=\frac{9\pi}{8}$$

Let $\sin^{-1}\left(\frac{1}{3}\right)=x$

$\therefore\ \sin x=\frac{1}{3}$, where $0<x<\frac{\pi}{3}$

$\therefore\ \cos x>0$

Now, $\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\frac{1}{9}}=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}$

$\therefore\ x=\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

$\therefore\ \sin^{-1}\left(\frac{1}{3}\right)=\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ ... (1)

$$\begin{aligned}
\therefore\ \text{LHS} &= \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)+\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\\
&= \frac{9}{4}\left[\sin^{-1}\left(\frac{1}{3}\right)+\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right]\\
&= \frac{9}{4}\left[\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)+\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right] \quad \text{... [By (1)]}\\
&= \frac{9}{4}\left(\frac{\pi}{2}\right) \quad \text{... }\left[\because\ \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]\\
&= \frac{9\pi}{8}=\text{RHS.}
\end{aligned}$$

Question 23.
Show that

$$\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,$$

$$\text{for } -\frac{1}{\sqrt{2}}\leqslant x\leqslant 1.$$

Solution:

$$\text{LHS}=\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

Put $x=\cos\theta$

$\therefore\ \theta=\cos^{-1}x$

$$\therefore \text{ LHS} = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$$

$$= \tan^{-1}\left[\frac{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)}-\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}{\sqrt{2\cos^2\left(\frac{\theta}{2}\right)}+\sqrt{2\sin^2\left(\frac{\theta}{2}\right)}}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{2}\cos\left(\frac{\theta}{2}\right)-\sqrt{2}\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}\cos\left(\frac{\theta}{2}\right)+\sqrt{2}\sin\left(\frac{\theta}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\frac{\sqrt{2}\cos\left(\frac{\theta}{2}\right)}{\sqrt{2}\cos\left(\frac{\theta}{2}\right)}-\frac{\sqrt{2}\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}\cos\left(\frac{\theta}{2}\right)}}{\frac{\sqrt{2}\cos\left(\frac{\theta}{2}\right)}{\sqrt{2}\cos\left(\frac{\theta}{2}\right)}+\frac{\sqrt{2}\sin\left(\frac{\theta}{2}\right)}{\sqrt{2}\cos\left(\frac{\theta}{2}\right)}}\right]$$

$$= \tan^{-1}\left[\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4}-\tan\left(\frac{\theta}{2}\right)}{1+\tan\frac{\pi}{4}.\tan\left(\frac{\theta}{2}\right)}\right] \quad \ldots \left[\because \tan\frac{\pi}{4}=1\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right] = \frac{\pi}{4}-\frac{\theta}{2}$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x \quad \ldots [\because \theta = \cos^{-1}x]$$

$$= \text{RHS.}$$

Question 24.
If sin(sin-115 + cos-1x) = 1, then find the value of x.
Solution:
sin(sin-115 = 1

$$\therefore \sin^{-1}\frac{1}{5}+\cos^{-1}x = \sin^{-1}(1)$$

$$\therefore \sin^{-1}\frac{1}{5}+\cos^{-1}x = \sin^{-1}\left(\sin\frac{\pi}{2}\right)$$

$$\therefore \sin^{-1}\frac{1}{5}+\cos^{-1}x = \frac{\pi}{2}$$

$$\therefore x = \frac{1}{5} \quad \ldots \left[\because \sin^{-1}x+\cos^{-1}x = \frac{\pi}{2}\right]$$

Question 25.
If tan-1(x−1x−2) + tan-1(x+1x+2) = π4 then find the value of x.
Solution:
tan-1(x−1x−2) + tan-1(x+1x+2) = π4

$$\therefore \tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]=\frac{\pi}{4}$$

$$\therefore \frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)}=\tan\frac{\pi}{4}$$

$$\therefore \frac{(x^2+x-2)+(x^2-x-2)}{(x^2-4)-(x^2-1)}=1$$

$$\therefore \frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}=1$$

$$\therefore \frac{2x^2-4}{-3}=1$$

$$\therefore 2x^2-4=-3$$

$$\therefore 2x^2=1 \qquad \therefore x^2=\frac{1}{2}$$

$\therefore x = \pm\frac{1}{\sqrt{2}}$.

Question 26.
If 2 $\tan^{-1}$(cos x) = $\tan^{-1}$(cosec x) then find the value of x.
Solution:
2 $\tan^{-1}$(cos x) = $\tan^{-1}$(cosec x)

$$\therefore \tan^{-1}\left[\frac{2\cos x}{1-\cos^2 x}\right]=\tan^{-1}(2\operatorname{cosec} x)$$

$$\ldots\left[\because\ 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\therefore \frac{2\cos x}{1-\cos^2 x}=2\operatorname{cosec} x$$

$$\therefore \frac{2\cos x}{\sin^2 x}=\frac{2}{\sin x}$$

$$\therefore \cos x=\sin x$$

$$\therefore x=\frac{\pi}{4} \qquad \ldots\left[\because\ \sin\frac{\pi}{4}=\cos\frac{\pi}{4}\right]$$

Question 27.
Solve: $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}(\tan^{-1}x)$, for x > 0.
Solution:
$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}(\tan^{-1}x)$

$$\therefore\ 2\tan^{-1}\left(\frac{1-x}{1+x}\right)=\tan^{-1}x$$

$$\therefore\ \tan^{-1}\left[\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right]=\tan^{-1}x$$

$$\ldots\left[\because\ 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\therefore\ \frac{2\left(\frac{1-x}{1+x}\right)(1+x)^2}{(1+x)^2-(1-x)^2}=x$$

$$\therefore\ \frac{2(1-x)(1+x)}{(1+2x+x^2)-(1-2x+x^2)}=x$$

$$\therefore\ \frac{2(1-x^2)}{1+2x+x^2-1+2x-x^2}=x$$

$$\therefore\ \frac{2-2x^2}{4x}=x$$

$$\therefore\ 2-2x^2=4x^2$$

$$\therefore\ 6x^2=2 \qquad \therefore\ x^2=\frac{1}{3}$$

$$\therefore\ x=\frac{1}{\sqrt{3}} \qquad \ldots[\because\ x>0]$$

Question 28.
If sin-1(1 – x) – 2sin-1x = π2, then find the value of x.
Solution:
sin-1(1 – x) – 2sin-1x = π2

$$\therefore\ \sin^{-1}(1-x)=\frac{\pi}{2}+2\sin^{-1}x$$

$$\therefore\ 1-x=\sin\left(\frac{\pi}{2}+2\sin^{-1}x\right)$$

$$\therefore\ 1-x=\cos(2\sin^{-1}x) \quad \ldots\left[\because\ \sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta\right]$$

$$\therefore\ 1-x=1-2[\sin(\sin^{-1}x)]^2 \ldots[\because\ \cos 2\theta=1-2\sin^2\theta]$$

$$\therefore\ 1-x=1-2x^2$$

$$\therefore\ 2x^2-x=0 \qquad \therefore\ x(2x-1)=0$$

$$\therefore\ x=0 \text{ or } x=\frac{1}{2}$$

When $x=\frac{1}{2}$

$$\text{LHS}=\sin^{-1}\left(1-\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)$$

$$=\sin^{-1}\left(\frac{1}{2}\right)-2\sin^{-1}\left(\frac{1}{2}\right)$$

$$=-\sin^{-1}\left(\frac{1}{2}\right)=-\sin^{-1}\left(\sin\frac{\pi}{6}\right)=-\frac{\pi}{6}\neq\frac{\pi}{2}$$

$$\therefore\ x\neq\frac{1}{2}$$

Hence, $x=0$.

Question 29.
If tan-12x + tan-13x = π4, then find the value of x.
Question is modified
If tan-12x + tan-13x = π2, then find the value of x.
Solution:

$\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

$\therefore \tan^{-1}\left(\frac{2x+3x}{1-2x \times 3x}\right) = \tan\frac{\pi}{4}$, where $2x > 0$, $3x > 0$

$\therefore \frac{5x}{1-6x^2} = \tan\frac{\pi}{4} = 1$

$\therefore 5x = 1 - 6x^2$

$\therefore 6x^2 + 5x - 1 = 0$

$\therefore 6x^2 + 6x - x - 1 = 0$

$\therefore 6x(x + 1) - 1(x + 1) = 0$

$\therefore (x + 1)(6x - 1) = 0$

$\therefore x = -1$ or $x = \frac{1}{6}$

But $x > 0 \therefore x \neq -1$

Hence, $x = \frac{1}{6}$

Question 30.
Show that $\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{2}{9}$.
Solution:
LHS = $\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{4}$

$$= \tan^{-1}\left[\frac{\frac{1}{2}-\frac{1}{4}}{1+\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)}\right]$$

$$= \tan^{-1}\left(\frac{4-2}{8+1}\right)$$

$$= \tan^{-1}\left(\frac{2}{9}\right) = \text{RHS.}$$

Question 31.
Show that $\cot^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{3} = \cot^{-1}\frac{3}{4}$.
Solution:
LHS = $\cot^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{3}$

$$= \tan^{-1} 3 - \tan^{-1}\frac{1}{3} \quad \ldots \left[\because \cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \tan^{-1}\left[\frac{3-\frac{1}{3}}{1+3\left(\frac{1}{3}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\left(\frac{8}{3}\right)}{1+1}\right]$$

$$= \tan^{-1}\left(\frac{4}{3}\right)$$

$$= \cot^{-1}\left(\frac{3}{4}\right) \quad \ldots \left[\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \text{RHS.}$$

Question 32.
Show that $\tan^{-1}\frac{1}{2} = \frac{1}{3}\tan^{-1}\frac{11}{2}$.
Solution:
We have to show that

$\tan^{-1}\frac{1}{2} = \frac{1}{3}\tan^{-1}\frac{11}{2}$

i.e. to show that $3\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{11}{2}$

$$\text{LHS} = 3\tan^{-1}\frac{1}{2}$$

$$= 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}\left[\frac{2 \times \frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right] + \tan^{-1}\frac{1}{2}$$

$$\ldots \left[\because\ 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left[\frac{1}{(3/4)}\right] + \tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}\left[\frac{\frac{4}{3}+\frac{1}{2}}{1-\frac{4}{3}\times\frac{1}{2}}\right]$$

$$= \tan^{-1}\left(\frac{8+3}{6-4}\right)$$

$$= \tan^{-1}\left(\frac{11}{2}\right) = \text{RHS.}$$

Question 33.
Show that cos-13√2 + 2sin-13√2 = 5π6
Solution:

$$\text{LHS} = \cos^{-1}\frac{\sqrt{3}}{2} + 2\sin^{-1}\frac{\sqrt{3}}{2}$$

$$= \cos^{-1}\left(\cos\frac{\pi}{6}\right) + 2\sin^{-1}\left(\sin\frac{\pi}{3}\right)$$

$$\ldots \left[\because\ \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}\right]$$

$$= \frac{\pi}{6} + 2\left(\frac{\pi}{3}\right)$$

$$\ldots [\because\ \sin^{-1}(\sin x) = x,\ \cos^{-1}(\cos x) = x]$$

$$= \frac{\pi}{6} + \frac{2\pi}{3}$$

$$= \frac{5\pi}{6} = \text{RHS.}$$

Question 34.
Show that 2cot-132 + sec-11312 = π2
Solution:

$$2\cot^{-1}\frac{3}{2} = 2\tan^{-1}\frac{2}{3} \quad \ldots \left[\because \cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right)\right]$$

$$= \tan^{-1}\left[\frac{2\times\frac{2}{3}}{1-\left(\frac{2}{3}\right)^2}\right]$$

$$\ldots \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left[\frac{\left(\frac{4}{3}\right)}{1-\frac{4}{9}}\right]$$

$$= \tan^{-1}\left(\frac{4}{3}\times\frac{9}{5}\right) = \tan^{-1}\frac{12}{5} \quad \ldots (1)$$

Let $\sec^{-1}\frac{13}{12} = \alpha$

Then, $\sec\alpha = \frac{13}{12}$, where $0 < \alpha < \frac{\pi}{2}$

$\therefore \tan\alpha > 0$

Now, $\tan\alpha = \sqrt{\sec^2\alpha - 1}$

$$= \sqrt{\frac{169}{144} - 1} = \sqrt{\frac{25}{144}} = \frac{5}{12}$$

$$\therefore \alpha = \tan^{-1}\frac{5}{12} = \cot^{-1}\frac{12}{5} \ldots \left[\because \tan^{-1}x = \cot^{-1}\left(\frac{1}{x}\right)\right]$$

$$\therefore \sec^{-1}\frac{13}{12} = \cot^{-1}\frac{12}{5} \quad \ldots (2)$$

Now, LHS $= 2\cot^{-1}\frac{3}{2} + \sec^{-1}\frac{13}{12}$

$$= \tan^{-1}\frac{12}{5} + \cot^{-1}\frac{12}{5} \quad \ldots \text{[By (1) and (2)]}$$

$$= \frac{\pi}{2} \quad \ldots \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right]$$

$= \text{RHS}.$

Question 35.
Prove the following :
(i) $\cos^{-1} x = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$, if $x < 0$.
Question is modified
$\cos^{-1} x = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$, if $x > 0$.

Solution:

Let $\cos^{-1}x = \alpha$.

Then, $\cos\alpha = x$, where $0 < \alpha < \pi$

Since, $x > 0$, $0 < \alpha < \frac{\pi}{2}$

$\therefore$ $\sin\alpha > 0$, $\cos\alpha > 0$

$$\text{Now, } \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{\sin^2\alpha}}{\cos\alpha}\right)$$

$$= \tan^{-1}(\tan\alpha)$$

$$= \alpha = \cos^{-1}x$$

Hence, $\cos^{-1}x = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$, if $x > 0$.

(ii) $\cos^{-1} x = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x}$, if $x < 0$.

Solution:

Let $\cos^{-1}x = \alpha$

Then, $\cos\alpha = x$ where $0 < \alpha < \pi$

Since, $x < 0$, $\frac{\pi}{2} < \alpha < \pi$

$$\text{Now, } \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}\right)$$

$$= \tan^{-1}(\tan\alpha) \qquad \dots (1)$$

But $\frac{\pi}{2} < \alpha < \pi$, therefore inverse of tangent does not exist.

Consider, $\frac{\pi}{2} - \pi < \alpha - \pi < \pi - \pi$

$\therefore$ $-\frac{\pi}{2} < \alpha - \pi < 0$

and $\tan(\alpha - \pi) = \tan[-(\pi - \alpha)]$

$$= -\tan(\pi - \alpha) \qquad \dots [\because \tan(-\theta) = -\tan\theta]$$

$$= -(-\tan\alpha) = \tan\alpha$$

$\therefore$ from (1), we get

$$\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \tan^{-1}[\tan(\alpha - \pi)]$$

$$= \alpha - \pi \quad \dots [\because \tan^{-1}(\tan x) = x]$$

$$= \cos^{-1}x - \pi$$

$\therefore$ $\cos^{-1}x = \pi + \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$, if $x < 0$.

Question 36.

If $|x| < 1$, then prove that $2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2} = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2}$

Question is modified

If $|x| < 1$, then prove that $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution:

Let $\tan^{-1}x = y$

Then, x = tany

$$\text{Now, } \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan y}{1-\tan^2 y}\right)$$
$$= \tan^{-1}(\tan 2y)$$
$$= 2y = 2\tan^{-1}x \quad \ldots (1)$$

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan y}{1+\tan^2 y}\right) = \sin^{-1}(\sin 2y)$$
$$= 2y = 2\tan^{-1}x \quad \ldots (2)$$

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2 y}{1+\tan^2 y}\right) = \cos^{-1}(\cos 2y)$$
$$= 2y = 2\tan^{-1}x \quad \ldots (3)$$

From (1), (2) and (3), we get

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$
$$= \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

Question 37.
If x, y, z, are positive then prove that $\tan^{-1}\frac{x-y}{1+xy} + \tan^{-1}\frac{y-z}{1+yz} + \tan^{-1}\frac{z-x}{1+zx} = 0$
Solution:

$$\text{LHS} = \tan^{-1}\left(\frac{x-y}{1+xy}\right) + \tan^{-1}\left(\frac{y-z}{1+yz}\right) + \tan^{-1}\left(\frac{z-x}{1+zx}\right)$$
$$= \tan^{-1}x - \tan^{-1}y + \tan^{-1}y - \tan^{-1}z + \tan^{-1}z - \tan^{-1}x$$
$$\ldots [\because x > 0,\ y > 0,\ z > 0]$$
$$= 0 = \text{RHS}.$$

Question 38.
If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$ then, show that xy + yz + zx = 1
Solution:
$\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$

$$\therefore \tan^{-1}\left(\frac{x+y}{1-xy}\right) + \tan^{-1}z = \frac{\pi}{2}$$

$$\therefore \tan^{-1}\left[\frac{\left(\frac{x+y}{1-xy}\right)+z}{1-\left(\frac{x+y}{1-xy}\right)z}\right] = \frac{\pi}{2}$$

$$\therefore \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-xz-yz}\right] = \frac{\pi}{2}$$

$$\therefore \frac{x+y+z-xyz}{1-xy-yz-zx} = \tan\frac{\pi}{2}, \text{ which does not exist}$$

∴ 1 – xy – yz – zx = 0
∴ xy + yz + zx = 1.

Question 39.
If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ then show that $x^2 + y^2 + z^2 + 2xyz = 1$.
Solution:
$0 \le \cos^{-1}x \le \pi$ and
$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$
∴ $\cos^{-1}x = \pi$, $\cos^{-1}y = \pi$ and $\cos^{-1}z = \pi$
∴ x = y = z = cosπ = -1
∴ $x^2 + y^2 + z^2 + 2xyz$
$= (-1)^2 + (-1)^2 + (-1)^2 + 2(-1)(-1)(-1)$
= 1 + 1 + 1 – 2
= 3 – 2 = 1.

Dividing both sides by $\sqrt{(1)^2+(-1)^2}=\sqrt{2}$, we get

$$\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta=-\frac{1}{\sqrt{2}}$$

$$\therefore\ \cos\frac{\pi}{4}\cos\theta-\sin\frac{\pi}{4}\sin\theta=-\cos\frac{\pi}{4}$$

$$\therefore\ \cos\left(\theta-\frac{\pi}{4}\right)=\cos\left(\pi-\frac{\pi}{4}\right)$$

$$\dots\ [\because\ \cos(\pi-\theta)=-\cos\theta]$$

$$\therefore\ \cos\left(\theta-\frac{\pi}{4}\right)=\cos\frac{3\pi}{4} \qquad \dots (1)$$

The general solution of $\cos\theta=\cos\alpha$ is

$\theta=2n\pi\pm\alpha,\ n\in Z$

$\therefore$ the general solution of (1) is given by

$$\theta-\frac{\pi}{4}=2n\pi\pm\frac{3\pi}{4},\ n\in Z$$

Taking positive sign, we get

$$\theta-\frac{\pi}{4}=2n\pi+\frac{3\pi}{4},\ n\in Z.$$

$$\therefore\ \theta=2n\pi+\pi=(2n+1)\pi,\ n\in Z$$

Taking negative sign, we get

$$\theta-\frac{\pi}{4}=2n\pi-\frac{3\pi}{4},\ n\in Z.$$

$$\therefore\ \theta=2n\pi-\frac{\pi}{2},\ n\in Z$$

$\therefore$ the required general solution is

$$\theta=(2n+1)\pi,\ n\in Z \ \text{ or }\ \theta=2n\pi-\frac{\pi}{2},\ n\in Z.$$