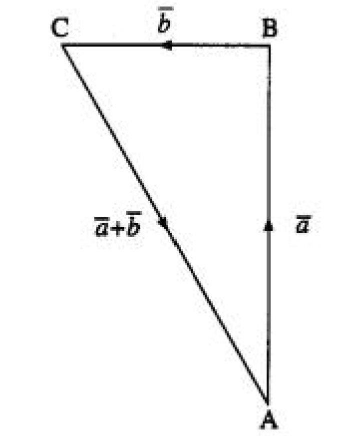
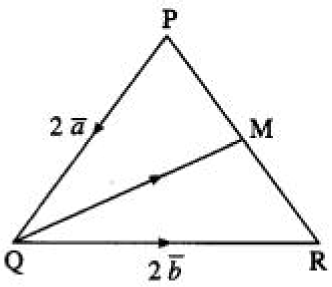
Maharashtra State Board 12th Maths Solutions Chapter 5 Vectors Ex 5.1

Question 1.
The vector $\bar{a}$ is directed due north and $|\bar{a}|$ = 24. The vector $\bar{b}$ is directed due west and $|\bar{b}|$ = 7. Find $|\bar{a}+\bar{b}|$.
Solution:



Let $\overline{AB}$ = $\bar{a}$, $\overline{BC}$ = $\bar{b}$
Then $\overline{AC}$ = $\overline{AB}$ + $\overline{BC}$ = a + b
Given : $|\bar{a}|$ = $|\overline{AB}|$ = l(AB) = 24 and
$|\bar{b}|$ = $|\overline{BC}|$ = l(BC) = 7
∴ ∠ABC = 90°
∴ $[l(AC)]^2 = [l(AB)]^2 + [l(BC)]^2$
$= (24)^2 + (7)^2 = 625$
∴ l(AC) = 25 ∴ $|\overline{AC}|$ = 25
∴ $|\bar{a}+\bar{b}|$ = $|\overline{AC}|$ = 25.

Question 2.
In the triangle PQR, $\overline{PQ}$ = $2\bar{a}$ and $\overline{QR}$ = $2\bar{b}$. The mid-point of PR is M. Find following vectors in terms of $\bar{a}$ and $\bar{b}$.
(i) $\overline{PR}$
Solution:



Given : $\overline{PQ}$ = $2\bar{a}$, $\overline{QR}$ = $2\bar{b}$
(i) $\overline{PR}$ = $\overline{PQ}$ + $\overline{QR}$
= $2\bar{a}$ + $2\bar{a}$.

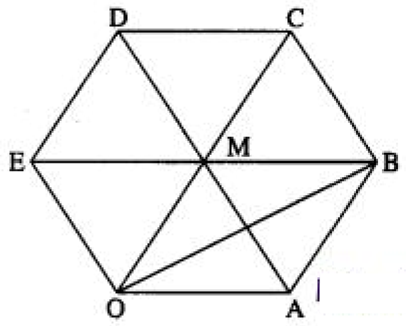
(ii) $\overline{PM}$
Solution:
∵ M is the midpoint of PR
∴ $\overline{PM}$ = $12\overline{PR}$ = $12[2\bar{a} + 2\bar{b}]$
= $\bar{a} + \bar{b}$.

(iii) $\overline{QM}$
Solution:
$\overline{RM}$ = $12(\overline{RP})$ = $-12\overline{PR}$ = $-12(2\bar{a} + 2\bar{b})$
= $-\bar{a} - \bar{b}$
∴ $\overline{QM}$ = $\overline{QR}$ + $\overline{RM}$

$= 2\bar{b} - \bar{a} - \bar{b}$

$= \bar{b} - \bar{a}$.

Question 3.
OABCDE is a regular hexagon. The points A and B have position vectors $\bar{a}$ and $\bar{b}$ respectively, referred to the origin O. Find, in terms of $\bar{a}$ and $\bar{b}$ the position vectors of C, D and E.
Solution:



Given : $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{a}$ Let AD, BE, OC meet at M.
Then M bisects AD, BE, OC.
$\overline{AB} = \overline{AO} + \overline{OB} = -\overline{OA} + \overline{OB} = -\bar{a} + \bar{b} = \bar{b} - \bar{a}$
∵ OABM is a parallelogram

$\therefore\ \overline{OM} = \overline{AB} = \bar{b} - \bar{a}$

$\overline{OC} = 2\overline{OM} = 2(\bar{b} - \bar{a}) = 2\bar{b} - 2\bar{a}$

$\overline{OD} = \overline{OC} + \overline{CD}$

$= \overline{OC} - \overline{DC}$

$= \overline{OC} - \overline{OA}$ ... [∵ OA = DC and OA ∥ DC]

$= 2\bar{b} - 2\bar{a} - \bar{a}$

$= 2\bar{b} - 3\bar{a}$

$\overline{OE} = \overline{OM} + \overline{ME}$

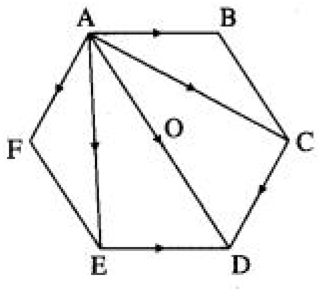
$= (\bar{b} - \bar{a}) - \overline{EM}$

$= \bar{b} - \bar{a} - \bar{a}$ ... [∵ EM = OA and EM ∥ OA]

$= \bar{b} - 2\bar{a}$

Hence, the position vectors of C, D and E are $2\bar{b} - 2\bar{a}$, $2\bar{b} - 3\bar{a}$ and $\bar{b} - 2\bar{a}$ respectively.

Question 4.
If ABCDEF is a regular hexagon, show that $\overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF} = 6\overline{AO}$, where O is the center of the hexagon.
Solution:



ABCDEF is a regular hexagon.
$\therefore \overline{AB} = \overline{ED}$ and $\overline{AF} = \overline{CD}$
∴ by the triangle law of addition of vectors,

$\overline{AC} + \overline{AF} = \overline{AC} + \overline{CD} = \overline{AD}$

$\overline{AE} + \overline{AB} = \overline{AE} + \overline{ED} = \overline{AD}$

$\therefore\ \text{LHS} = \overline{AB} + \overline{AC} + \overline{AD} + \overline{AE} + \overline{AF}$

$= \overline{AD} + (\overline{AC} + \overline{AF}) + (\overline{AE} + \overline{AB})$

$= \overline{AD} + \overline{AD} + \overline{AD}$

$= 3\overline{AD} = 3(2\overline{AO})$ ... [O is midpoint of AD]

$= 6\overline{AO}$.

Question 5.
Check whether the vectors $2\hat{i}+2\hat{j}+3\hat{k}$, + $-3\hat{i}+3\hat{j}+2\hat{k}$, + $3\hat{i}+4\hat{k}$ form a triangle or not.
Solution:
Let, if possible, the three vectors form a triangle ABC
with $\overline{AB}$ = $2\hat{i}+2\hat{j}+3\hat{k}$, $\overline{BC}$ = $3\hat{i}+3\hat{j}+2\hat{k}$, $\overline{AC}$ = $3\hat{i}+4\hat{k}$
Now, $\overline{AB}$ + $\overline{BC}$
= $(2\hat{i}+2\hat{j}+3\hat{k})$ + $(-3\hat{i}+3\hat{j}+2\hat{k})$
= $-\hat{i}+5\hat{j}+5\hat{k} \neq 3\hat{i}+4\hat{k}$ = $\overline{AC}$
Hence, the three vectors do not form a triangle.

Question 6.
In the figure 5.34 express $\bar{c}$ and $\bar{d}$ in terms of $\bar{a}$ and $\bar{b}$. Find a vector in the direction of $\bar{a}$ = $\hat{i}-2\hat{j}$ that has magnitude 7 units.

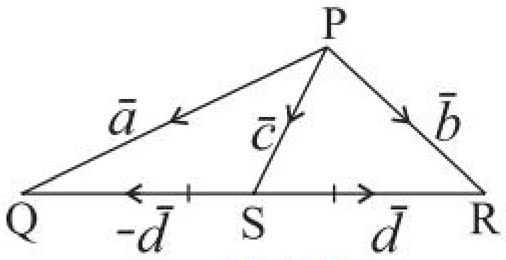


Fig 5.34

Solution:
$\overline{PQ}$ = $\overline{PS}$ + $\overline{SQ}$
∴ $\bar{a} = \bar{c} - \bar{d}$ … (1)
$\overline{PR}$ = $\overline{PS}$ + $\overline{SR}$
∴ $\bar{b} = \bar{c} + \bar{d}$ … (2)
Adding equations (1) and (2), we get
$\bar{a} + \bar{b} = (\bar{c} - \bar{d}) + (\bar{c} + \bar{d}) = 2\bar{c}$

$$\therefore\ \bar{c} = \frac{1}{2}(\bar{a}+\bar{b}) = \frac{1}{2}\bar{a} + \frac{1}{2}\bar{b}$$

Substituting for $\bar{c}$ in (2), we get

$$\bar{d} = \bar{b} - \bar{c} = \bar{b} - \frac{1}{2}(\bar{a}+\bar{b})$$

$$= \frac{2\bar{b} - (\bar{a}+\bar{b})}{2}$$

$$= \frac{1}{2}(\bar{b}-\bar{a}) = \frac{1}{2}\bar{b} - \frac{1}{2}\bar{a}$$

Hence, $\bar{c} = \frac{1}{2}\bar{a} + \frac{1}{2}\bar{b}$ and $\bar{d} = \frac{1}{2}\bar{b} - \frac{1}{2}\bar{a}$.

Question 7.
Find the distance from (4, -2, 6) to each of the following :
(a) The XY-plane
Solution:
Let the point A be (4, -2, 6).
Then,
The distance of A from XY-plane = |z| = 6

(b) The YZ-plane
Solution:
The distance of A from YZ-plane = |x| = 4

(c) The XZ-plane
Solution:
The distance of A from ZX-plane = |y| = 2

(d) The X-axis
Solution:
The distance of A from X-axis
= $\sqrt{y^2+z^2}$ = $\sqrt{(-2)^2+6^2}$ = $\sqrt{40}$ = $2\sqrt{10}$

(e) The Y-axis
Solution:
The distance of A from Y-axis
= $\sqrt{z^2+x^2}$ = $\sqrt{6^2+4^2}$ = $\sqrt{52}$ = $2\sqrt{13}$

(f) The Z-axis
Solution:
The distance of A from Z-axis
= $\sqrt{x^2+y^2}$ = $\sqrt{4^2+(-2)^2}$ = $\sqrt{20}$ = $2\sqrt{5}$

Question 8.
Find the coordinates of the point which is located :
(a) Three units behind the YZ-plane, four units to the right of the XZ-plane and five units above the XY-plane.
Solution:
Let the coordinates of the point be (x, y, z).
Since the point is located 3 units behind the YZ- j plane, 4 units to the right of XZ-plane and 5 units , above the XY-plane,
x = -3, y = 4 and z = 5
Hence, coordinates of the required point are (-3, 4, 5)

(b) In the YZ-plane, one unit to the right of the XZ-plane and six units above the XY-plane.
Solution:
Let the coordinates of the point be (x, y, z).
Since the point is located in the YZ plane, x = 0. Also, the point is one unit to the right of XZ-plane and six units above the XY-plane.
∴ y = 1, z = 6.
Hence, coordinates of the required point are (0, 1, 6).

Question 9.
Find the area of the triangle with vertices (1, 1, 0), (1, 0, 1) and (0, 1, 1).
Solution:
Let A = (1, 1, 0), B = (1, 0, 1), C = (0, 1, 1)

$$l(\mathrm{AB}) = \sqrt{(1-1)^2+(1-0)^2+(0-1)^2} = \sqrt{0+1+1} = \sqrt{2}$$
$$l(\mathrm{BC}) = \sqrt{(1-0)^2+(0-1)^2+(1-1)^2} = \sqrt{1+1+0} = \sqrt{2}$$
$$l(\mathrm{CA}) = \sqrt{(0-1)^2+(1-1)^2+(1-0)^2} = \sqrt{1+0+1} = \sqrt{2}$$

∴ $l(\mathrm{AB}) = l(\mathrm{BC}) = l(\mathrm{CA})$

∴ **the triangle is equilateral**

∴ **its area** $= \dfrac{\sqrt{3}}{4}(\text{side})^2$

$= \dfrac{\sqrt{3}}{4}(\sqrt{2})^2$

$= \dfrac{\sqrt{3}}{2}$ **sq units.**

Question 10.
If $\overline{AB} = 2\hat{i}-4\hat{j}+7\hat{k}$ and initial point A ≡ (1, 5, ,0). Find the terminal point B.
Solution:
Let $\bar{a}$ and $\bar{b}$ be the position vectors of A and B.
Given : A = (1, 5, 0) ∴ $\bar{a} = \hat{i}+5\hat{j}$
Now, $\overline{AB} = 2\hat{i}-4\hat{j}+7\hat{k}$
∴ $\bar{b} - \bar{a} = 2\hat{i}-4\hat{j}+7\hat{k}$
∴ $\bar{b} = (2\hat{i}-4\hat{j}+7\hat{k}) + \bar{a}$
= $(2\hat{i}-4\hat{j}+7\hat{k}) + (\hat{i}+5\hat{j})$

= $3\hat{i}+\hat{j}+7\hat{k}$
Hence, the terminal point B = (3, 1, 7).

Question 11.
Show that the following points are collinear :
(i) A (3, 2, -4), B (9, 8, -10), C (-2, -3, 1).
Solution:
Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the position vectors of the points.
A = (3, 2, -4), B = (9, 8, -10) and C = (-2, -3, 1) respectively.

Then, $\bar{a} = 3\hat{i} + 2\hat{j} - 4\hat{k}$, $\bar{b} = 9\hat{i} + 8\hat{j} - 10\hat{k}$,

$\bar{c} = -2\hat{i} - 3\hat{j} + \hat{k}$

$$\overline{AB} = \bar{b} - \bar{a}$$
$$= (9\hat{i} + 8\hat{j} - 10\hat{k}) - (3\hat{i} + 2\hat{j} - 4\hat{k})$$
$$= 6\hat{i} + 6\hat{j} - 6\hat{k} \quad \ldots (1)$$

and $\overline{BC} = \bar{c} - \bar{b}$

$$= (-2\hat{i} - 3\hat{j} + \hat{k}) - (9\hat{i} + 8\hat{j} - 10\hat{k})$$
$$= -11\hat{i} - 11\hat{j} + 11\hat{k}$$
$$= -11(\hat{i} + \hat{j} - \hat{k}) = -\frac{11}{6}(6\hat{i} + 6\hat{j} - 6\hat{k})$$
$$= -\frac{11}{6}\overline{AB} \quad \ldots [\text{By } (1)]$$

∴ $\overline{BC}$ is a non-zero scalar multiple of $\overline{AB}$
∴ they are parallel to each other.
But they have the point B in common.
∴ $\overline{BC}$ and $\overline{AB}$ are collinear vectors.
Hence, the points A, B and C are collinear.

(ii) P (4, 5, 2), Q (3, 2, 4), R (5, 8, 0).
Solution:
Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the position vectors of the points.
P = (4, 5, 2), Q = (3, 2, 4), R = (5, 8, 0) respectively.

Then $\bar{a} = 4\hat{i} + 5\hat{j} + 2\hat{k}$, $\bar{b} = 3\hat{i} + 2\hat{j} + 4\hat{k}$, $\bar{c} = 5\hat{i} + 8\hat{j} + 0\hat{k}$

$$\overline{AB} = \bar{b} - \bar{a}$$
$$= \left(3\hat{i} + 2\hat{j} + 4\hat{k}\right) - \left(4\hat{i} + 5\hat{j} + 2\hat{k}\right)$$
$$= -\hat{i} - 3\hat{j} + 2\hat{k} \text{ i.e.}$$
$$= -\left(\hat{i} + 3\hat{j} - 2\hat{k}\right) \quad \ldots\ldots(1)$$

and $\overline{BC} = \bar{c} - \bar{b}$

$$= \left(5\hat{i} + 8\hat{j} + 0\hat{k}\right) - \left(3\hat{i} + 2\hat{j} + 4\hat{k}\right)$$
$$= 2\hat{i} + 6\hat{j} - 4\hat{k}$$
$$= 2\left(\hat{i} + 3\hat{j} - 2\hat{k}\right)$$

= $2.\overline{AB}$ ...[By (1)]
∴ $\overline{BC}$ is a non-zero scalar multiple of $\overline{AB}$
∴ they are parallel to each other.
But they have the point B in common.
∴ $\overline{BC}$ and $\overline{AB}$ are collinear vectors.
Hence, the points A, B and C are collinear.

Question 12.
If the vectors $2\hat{i} - q\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear, then find the value of q.
Solution:
The vectors $2\hat{i} - q\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear
∴ the coefficients of $\hat{i}, \hat{j}, \hat{k}$ are proportional

$$\therefore \frac{2}{4} = \frac{-q}{-5} = \frac{3}{6}$$

$$\therefore \frac{q}{5} = \frac{1}{2}$$

$$\therefore q = \frac{5}{2}.$$

Question 13.
Are the four points A(1, -1, 1), B(-1, 1, 1), C(1, 1, 1) and D(2, -3, 4) coplanar? Justify your answer.
Solution:
The position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ of the points A, B, C, D are

$$\bar{a} = \hat{i} - \hat{j} + \hat{k},\ \bar{b} = -\hat{i} + \hat{j} + \hat{k},\ \bar{c} = \hat{i} + \hat{j} + \hat{k},\ \bar{d} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\therefore \overline{AB} = \bar{b} - \bar{a} = (-\hat{i} + \hat{j} + \hat{k}) - (\hat{i} - \hat{j} + \hat{k})$$
$$= -2\hat{i} + 2\hat{j}$$

$$\overline{AC} = \bar{c} - \bar{a} = (\hat{i} + \hat{j} + \hat{k}) - (\hat{i} - \hat{j} + \hat{k}) = 2\hat{j}$$

$$\text{and } \overline{AD} = \bar{d} - \bar{a} = (2\hat{i} - 3\hat{j} + 4\hat{k}) - (\hat{i} - \hat{j} + \hat{k})$$
$$= \hat{i} - 2\hat{j} + 3\hat{k}$$

If A, B, C, D are coplanar, then there exist scalars x, y such that

$$\overline{AB} = x \cdot \overline{AC} + y \cdot \overline{AD}$$

$$\therefore -2\hat{i} + 2\hat{j} = x(2\hat{j}) + y(\hat{i} - 2\hat{j} + 3\hat{k})$$

$$\therefore -2\hat{i} + 2\hat{j} = y\hat{i} + (2x - 2y)\hat{j} + 3y\hat{k}$$

By equality of vectors,
y = -2 ….(1)
2x – 2y = 2 … (2)
3y = 0 … (3)
From (1), y = -2
From (3), y = 0 This is not possible.
Hence, the points A, B, C, D are not coplanar.

Question 14.
Express $-\hat{i} - 3\hat{j} + 4\hat{k}$ as linear combination of the vectors $2\hat{i} + \hat{j} - 4\hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + \hat{j} - 2\hat{k}$.
Solution:

$$\text{Let } \bar{a} = 2\hat{i} + \hat{j} - 4\hat{k},\ \bar{b} = 2\hat{i} - \hat{j} + 3\hat{k},$$
$$\bar{c} = 3\hat{i} + \hat{j} - 2\hat{k} \text{ and } \bar{p} = -\hat{i} - 3\hat{j} + 4\hat{k}.$$

Suppose $\bar{p} = x\bar{a} + y\bar{b} + z\bar{c}$.

$$\text{Then, } -\hat{i} - 3\hat{j} + 4\hat{k} = x(2\hat{i} + \hat{j} - 4\hat{k}) + y(2\hat{i} - \hat{j} + 3\hat{k}) + z(3\hat{i} + \hat{j} - 2\hat{k})$$

$$\therefore -\hat{i} - 3\hat{j} + 4\hat{k} = (2x + 2y + 3z)\hat{i} + (x - y + z)\hat{j} + (-4x + 3y - 2z)\hat{k}$$

By equality of vectors,
2x + 2y + 3 = -1
x – y + z = -3
-4x + 3y – 2z = 4
We have to solve these equations by using Cramer’s Rule

$$D = \begin{vmatrix} 2 & 2 & 3 \\ 1 & -1 & 1 \\ -4 & 3 & -2 \end{vmatrix}$$

= 2(2 – 3) – 2(-2 + 4) + 3(3 – 4)
= -2 – 4 – 3 = -9 ≠ 0

$$D_x = \begin{vmatrix} -1 & 2 & 3 \\ -3 & -1 & 1 \\ 4 & 3 & -2 \end{vmatrix}$$

$$= -1(2-3) - 2(6-4) + 3(-9+4)$$
$$= 1 - 4 - 15 = -18$$

$$D_y = \begin{vmatrix} 2 & -1 & 3 \\ 1 & -3 & 1 \\ -4 & 4 & -2 \end{vmatrix}$$

$$= 2(6-4) + 1(-2+4) + 3(4-12)$$
$$= 4 + 2 - 24 = -18$$

$$D_z = \begin{vmatrix} 2 & 2 & -1 \\ 1 & -1 & -3 \\ -4 & 3 & 4 \end{vmatrix}$$

= 2(-4 + 9) – 2(4 – 12) – 1(3 – 4)
= 10 + 16 + 1 = 27

$$\therefore\ x = \frac{D_x}{D} = \frac{-18}{-9} = 2$$

$$\therefore\ y = \frac{D_y}{D} = \frac{-18}{-9} = 2$$

$$\therefore\ z = \frac{D_z}{D} = \frac{27}{-9} = -3$$

$$\therefore\ \bar{p} = 2\bar{a} + 2\bar{b} - 3\bar{c}.$$

Maharashtra State Board 12th Maths Solutions Chapter 5 Vectors Ex 5.2

Question 1.
Find the position vector of point R which divides the line joining the points P and Q whose position vectors are $2\hat{i}-\hat{j}+3\hat{k}$ and $-5\hat{i}+2\hat{j}-5\hat{k}$ in the ratio 3 : 2
(i) internally
Solution:
It is given that the points P and Q have position vectors $\bar{p}$ = $2\hat{i}-\hat{j}+3\hat{k}$ and $\bar{p}$ = $-5\hat{i}+2\hat{j}-5\hat{k}$ respectively.

(i) If R($\bar{r}$) divides the line segment PQ internally in the ratio 3 : 2, by section formula for internal division,

$$\bar{r} = \frac{3\bar{q}+2\bar{p}}{3+2} = \frac{3(-5\hat{i}+2\hat{j}-5\hat{k})+2(2\hat{i}-\hat{j}+3\hat{k})}{5}$$

$$= \frac{-11\hat{i}+4\hat{j}-9\hat{k}}{5} = \frac{1}{5}(-11\hat{i}+4\hat{j}-9\hat{k})$$

$$\therefore \text{ coordinates of R} = \left(-\frac{11}{5}, \frac{4}{5}, -\frac{9}{5}\right)$$

Hence, the position vector of R is $\frac{1}{5}(-11\hat{i}+4\hat{j}-9\hat{k})$

and the coordinates of R are $\left(-\frac{11}{5}, \frac{4}{5}, -\frac{9}{5}\right)$.

(ii) externally.
Solution:
If R($\bar{r}$) divides the line segment joining P and Q externally in the ratio 3 : 2, by section formula for external division,

$$\bar{r} = \frac{3\bar{q}-2\bar{p}}{3-2} = \frac{3(-5\hat{i}+2\hat{j}-5\hat{k})-2(2\hat{i}-\hat{j}+3\hat{k})}{3-2}$$

$$= -19\hat{i}+8\hat{j}-21\hat{k}$$

∴ coordinates of R = (-19, 8, -21).
Hence, the position vector of R is $-19\hat{i}+8\hat{j}-21\hat{k}$ and coordinates of R are (-19, 8, -21).

Question 2.
Find the position vector of midpoint M joining the points L (7, -6, 12) and N (5, 4, -2).
Solution:
The position vectors $\bar{l}$ and $\bar{n}$ of the points L(7, -6, 12) and N (5, 4, -2) are given by

$$\bar{l} = 7\hat{i}-6\hat{j}+12\hat{k},\ \bar{n} = 5\hat{i}+4\hat{j}-2\hat{k}$$

If M($\bar{m}$) is the midpoint of LN, by midpoint formula,

$$\bar{m} = \frac{\bar{l}+\bar{n}}{2} = \frac{(7\hat{i}-6\hat{j}+12\hat{k})+(5\hat{i}+4\hat{j}-2\hat{k})}{2}$$

$$= \frac{1}{2}(12\hat{i}-2\hat{j}+10\hat{k}) = 6\hat{i}-\hat{j}+5\hat{k}$$

∴ coordinates of M = (6, -1, 5).
Hence, position vector of M is $6\hat{i}-\hat{j}+5\hat{k}$ and the coordinates of M are (6, -1, 5).

Question 3.
If the points A(3, 0, p), B (-1, q, 3) and C(-3, 3, 0) are collinear, then find
(i) The ratio in which the point C divides the line segment AB.
Solution:
Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be the position vectors of A, B and C respectively.
Then $\bar{a} = 3\hat{i}+0\cdot\hat{j}+p\hat{k}$, $\bar{b} = -\hat{i}+q\hat{j}+3\hat{k}$ and $\bar{c} = -3\hat{i}+3\hat{j}+0\hat{k}$.
As the points A, B, C are collinear, suppose the point C divides line segment AB in the ratio λ : 1.
∴ by the section formula,

$$\bar{c} = \frac{\lambda\cdot\bar{b}+1\cdot\bar{a}}{\lambda+1}$$

$$\therefore\ -3\hat{i}+3\hat{j}+0\cdot\hat{k} = \frac{\lambda(-\hat{i}+q\hat{j}+3\hat{k})+(3\hat{i}+0\cdot\hat{j}+p\hat{k})}{\lambda+1}$$

$$\therefore\ (\lambda+1)(-3\hat{i}+3\hat{j}+0\cdot\hat{k}) = (-\lambda\hat{i}+\lambda q\hat{j}+3\lambda\hat{k})+(3\hat{i}+0\cdot\hat{j}+p\hat{k})$$

$$\therefore\ -3(\lambda+1)\hat{i}+3(\lambda+1)\hat{j}+0\cdot\hat{k} = (-\lambda+3)\hat{i}+\lambda q\hat{j}+(3\lambda+p)\hat{k}$$

By equality of vectors, we have,
-3(λ + 1) = -λ + 3 ... (1)
3(λ + 1) = λ q ... (2)

$0 = 3\lambda + p$ ... (3)
From equation (1), $-3\lambda - 3 = -\lambda + 3$
$\therefore -2\lambda = 6 \therefore \lambda = -3$
$\therefore$ C divides segment AB externally in the ratio 3 : 1.

(ii) The values of p and q.
Solution:
Putting $\lambda = -3$ in equation (2), we get
$3(-3 + 1) = -3q$
$\therefore -6 = -3q \therefore q = 2$
Also, putting $\lambda = -3$ in equation (3), we get
$0 = -9 + p \therefore p = 9$
Hence p = 9 and q = 2.

Question 4.
The position vector of points A and B are $6\bar{a} + 2\bar{b}$ and $\bar{a} - 3\bar{b}$. If the point C divides AB in the ratio 3 : 2 then show that the position vector of C is $3\bar{a} - \bar{b}$.
Solution:
Let $\bar{c}$ be the position vector of C.
Since C divides AB in the ratio 3 : 2,

$$\bar{c} = \frac{3(\bar{a} - 3\bar{b}) + 2(6\bar{a} + 2\bar{b})}{3+2}$$

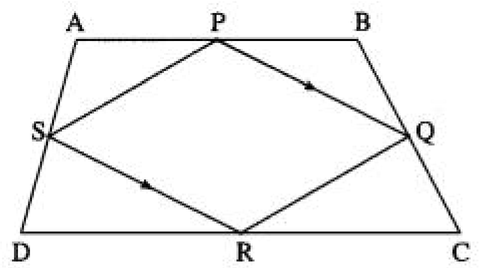
$$= \frac{3\bar{a} - 9\bar{b} + 12\bar{a} + 4\bar{b}}{5}$$

$$= \frac{1}{5}(15\bar{a} - 5\bar{b}) = 3\bar{a} - \bar{b}$$

Hence, the position vector of C is $3\bar{a} - \bar{b}$.

Question 5.
Prove that the line segments joining mid-point of adjacent sides of a quadrilateral form a parallelogram.
Solution:
Let ABCD be a quadrilateral and P, Q, R, S be the midpoints of the sides AB, BC, CD and DA respectively.
Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{p}, \bar{q}, \bar{r}$ and s be the position vectors of the points A, B, C, D, P, Q, R and S respectively.
Since P, Q, R and S are the midpoints of the sides AB, BC, CD and DA respectively,

$$\bar{p} = \frac{\bar{a} + \bar{b}}{2}, \bar{q} = \frac{\bar{b} + \bar{c}}{2}, \bar{r} = \frac{\bar{c} + \bar{d}}{2} \text{ and } \bar{s} = \frac{\bar{d} + \bar{a}}{2}$$



$\therefore \overline{PQ} = \bar{q} - \bar{p}$

$= \left(\frac{\bar{b}+\bar{c}}{2}\right) - \left(\frac{\bar{a}+\bar{b}}{2}\right)$

$= \frac{1}{2}(\bar{b}+\bar{c}-\bar{a}-\bar{b}) = \frac{1}{2}(\bar{c}-\bar{a})$

$\overline{SR} = \bar{r} - \bar{s}$

$= \left(\frac{\bar{c}+\bar{d}}{2}\right) - \left(\frac{\bar{d}+\bar{a}}{2}\right)$

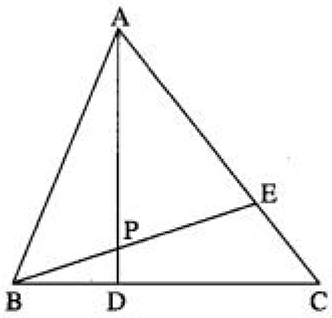
$= \frac{1}{2}(\bar{c}+\bar{d}-\bar{d}-\bar{a}) = \frac{1}{2}(\bar{c}-\bar{a})$

$\therefore \overline{PQ} = \overline{SR} \quad \therefore \overline{PQ} \parallel \overline{SR}$

Similarly, $\overline{QR} \parallel \overline{PS}$

∴ □PQRS is a parallelogram.

Question 6.
D and E divide sides BC and CA of a triangle ABC in the ratio 2 : 3 respectively. Find the position vector of the point of intersection of AD and BE and the ratio in which this point divides AD and BE.
Solution:



Let AD and BE intersect at P.
Let A, B, C, D, E, P have position vectos $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$, $\bar{p}$ respectively.
D and E divide segments BC and CA internally in the ratio 2 : 3.
By the section formula for internal division,

$\bar{d} = \frac{2\bar{c}+3\bar{b}}{2+3}$

$\therefore 5\bar{d} = 2\bar{c}+3\bar{b}$ ... (1)

and $\bar{e} = \frac{2\bar{a}+3\bar{c}}{2+3}$

$\therefore 5\bar{e} = 2\bar{a}+3\bar{c}$ ... (2)

From (1), $5\bar{d}-3\bar{b} = 2\bar{c}$ $\quad \therefore 15\bar{d}-9\bar{b} = 6\bar{c}$

From (2), $5\bar{e}-2\bar{a} = 3\bar{c}$ $\quad \therefore 10\bar{e}-4\bar{a} = 6\bar{c}$

Equating both values of $6\bar{c}$, we get

$15\bar{d}-9\bar{b} = 10\bar{e}-4\bar{a}$

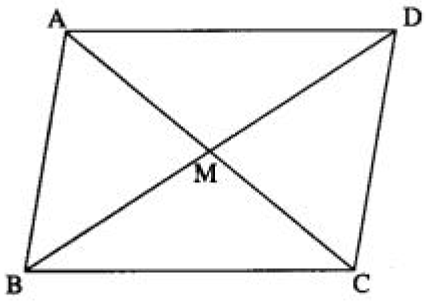
$\therefore 15\bar{d}+4\bar{a} = 10\bar{e}+9\bar{b}$

$\therefore \frac{15\bar{d}+4\bar{a}}{15+4} = \frac{10\bar{e}+9\bar{b}}{10+9}$

LHS is the position vector of the point which divides segment AD internally in the ratio 15 : 4.
RHS is the position vector of the point which divides segment BE internally in the ratio 10 : 9.
But P is the point of intersection of AD and BE.
∴ P divides AD internally in the ratio 15 : 4 and P divides BE internally in the ratio 10 : 9.
Hence, the position vector of the point of interaction of
AD and BE is $\bar{p} = \frac{15\bar{d}+4\bar{a}}{19} = \frac{10\bar{e}+9\bar{b}}{19}$ and it divides AD internally in the ratio 15 : 4 and BE internally in the ratio 10 : 9.

Question 7.
Prove that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.
Solution:

Let $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ be respectively the position vectors of the vertices A, B, C and D of the parallelogram ABCD. Then AB = DC and side AB || side DC.



$\therefore\ \overline{AB} = \overline{DC}$

$\therefore\ \bar{b} - \bar{a} = \bar{c} - \bar{d}$

$\therefore\ \bar{a} + \bar{c} = \bar{b} + \bar{d}$

$\therefore\ \frac{\bar{a} + \bar{c}}{2} = \frac{\bar{b} + \bar{d}}{2}$ ... (1)

The position vectors of the midpoints of the diagonals AC and BD are $(\bar{a} + \bar{c})/2$ and $(\bar{b} + \bar{d})/2$. By (1), they are equal.

∴ the midpoints of the diagonals AC and BD are the same.
This shows that the diagonals AC and BD bisect each other.

(ii) Conversely, suppose that the diagonals AC and BD of □ ABCD bisect each other,
i. e. they have the same midpoint.
∴ the position vectors of these midpoints are equal.

$\therefore \frac{\bar{a}+\bar{c}}{2} = \frac{\bar{b}+\bar{d}}{2}$ $\therefore \bar{a}+\bar{c} = \bar{b}+\bar{d}$

$\therefore \bar{b} - \bar{a} = \bar{c} - \bar{d}$ $\therefore \overline{AB} = \overline{DC}$

$\therefore \overline{AB} \| \overline{DC}$ and $|\overline{AB}| = |\overline{DC}|$

∴ side AB || side DC and AB = DC.
∴ □ ABCD is a parallelogram.

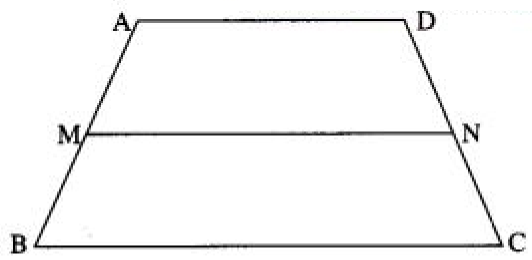
Question 8.
Prove that the median of a trapezium is parallel to the parallel sides of the trapezium and its length is half the sum of parallel sides.
Solution:
Let $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ be respectively the position vectors of the vertices A, B, C and D of the trapezium ABCD, with side AD || side BC.

Then the vectors $\overline{AD}$ and $\overline{BC}$ are parallel.

∴ there exists a scalar k,
such that $\overline{AD} = k \cdot \overline{BC}$

$\therefore \overline{AD} + \overline{BC} = k \cdot \overline{BC} + \overline{BC}$

= (k + 1)BC ...(1)



Let $\bar{m}$ and $\bar{n}$ be the position vectors of the midpoints M and N of the non-parallel sides AB and DC respectively.

Then seg MN is the median of the trapezium.
By the midpoint formula,

$\overline{m} = \frac{\overline{a} + \overline{b}}{2}$ and $\overline{n} = \frac{\overline{d} + \overline{c}}{2}$

$$\therefore \overline{MN} = \overline{n} - \overline{m}$$

$$= \left(\frac{\overline{d} + \overline{c}}{2}\right) - \left(\frac{\overline{a} + \overline{b}}{2}\right)$$

$$= \frac{1}{2}(\overline{d} + \overline{c} - \overline{a} - \overline{b})$$

$$= \frac{1}{2}[(\overline{d} - \overline{a}) + (\overline{c} - \overline{b})]$$

$$= \frac{\overline{AD} + \overline{BC}}{2} \quad \ldots (2)$$

$$= \frac{(k+1)\overline{BC}}{2} \quad \ldots [\text{By } (1)]$$

Thus $\overline{MN}$ is a scalar multiple of $\overline{BC}$

$\therefore$ $\overline{MN}$ and $\overline{BC}$ are parallel vectors

$\therefore$ $\overline{MN} \parallel \overline{BC}$ where $\overline{BC} \parallel \overline{AD}$

$\therefore$ the median MN is parallel to the parallel sides AD and BC of the trapezium.

Now $\overline{AD}$ and $\overline{BC}$ are collinear

$$\therefore |\overline{AD} + \overline{BC}| = |\overline{AD}| + |\overline{BC}| = AD + BC$$

$\therefore$ from (2), we have

$$\therefore \overline{MN} = \frac{\overline{AD} + \overline{BC}}{2}$$

$$\therefore MN = \frac{1}{2}(AD + BC).$$

Question 9.
If two of the vertices of the triangle are A(3, 1, 4) and B(-4, 5, -3) and the centroid of a triangle is G(-1, 2, 1), then find the coordinates of the third vertex C of the triangle.
Solution:
Let $\overline{a}$, $\overline{b}$, $\overline{c}$ and $\overline{g}$ be the position vectors of A, B, C and G respectively.
Then $\overline{a} = 3\hat{i}+\hat{j}+4\hat{k}$, $\overline{b} = -4\hat{i}+5\hat{j}-3\hat{k}$ and $\overline{g} = -\hat{i}+2\hat{j}+\hat{k}$.
Since G is the centroid of the ΔABC, by the centroid formula,

$$\overline{g} = \frac{\overline{a} + \overline{b} + \overline{c}}{3} \quad \therefore 3\overline{g} = \overline{a} + \overline{b} + \overline{c}$$

$$\therefore 3(-\hat{i} + 2\hat{j} + \hat{k}) = (3\hat{i} + \hat{j} + 4\hat{k}) + (-4\hat{i} + 5\hat{j} - 3\hat{k}) + \overline{c}$$

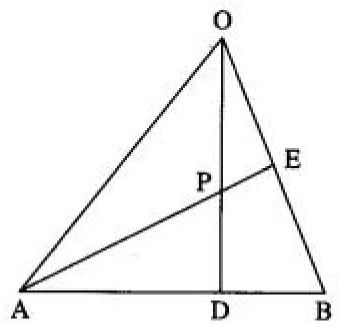
$$\therefore -3\hat{i} + 6\hat{j} + 3\hat{k} = (-\hat{i} + 6\hat{j} + \hat{k}) + \overline{c}$$

$$\therefore \overline{c} = (-3\hat{i} + 6\hat{j} + 3\hat{k}) - (-\hat{i} + 6\hat{j} + \hat{k})$$

$$\therefore \overline{c} = -2\hat{i} + 0\cdot\hat{j} + 2\hat{k}$$

$\therefore$ the coordinates of third vertex C are (-2, 0, 2).

Question 10.
InΔOAB, E is the mid-point of OB and D is the point on AB such that AD : DB = 2 : 1.
If OD and AE intersect at P, then determine the ratio OP : PD using vector methods.
Solution:



Let A, B, D, E, P have position vectors $\bar{a}$, $\bar{b}$, $\bar{d}$, $\bar{e}$, $\bar{p}$ respectively w.r.t. O.
∵ AD : DB = 2 : 1.
∴ D divides AB internally in the ratio 2 : 1.
Using section formula for internal division, we get

$$\bar{d} = \frac{2\bar{b} + \bar{a}}{2+1}$$

$$\therefore\ 3\bar{d} = 2\bar{b} + \bar{a} \qquad \dots (1)$$

Since E is the midpoint of OB, $\bar{e} = \overline{OE} = \frac{1}{2}\overline{OB} = \frac{\bar{b}}{2}$

$$\therefore\ \bar{b} = 2\bar{e} \qquad \dots (2)$$

∴ from (1),

$$3\bar{d} = 2(2\bar{e}) + \bar{a} \qquad \dots [\text{By } (2)]$$

$$= 4\bar{e} + \bar{a}$$

$$\therefore\ \frac{3\bar{d} + 2\cdot\bar{0}}{3+2} = \frac{4\bar{e} + \bar{a}}{4+1}$$

LHS is the position vector of the point which divides OD internally in the ratio 3 : 2.
RHS is the position vector of the point which divides AE internally in the ratio 4 : 1.
But OD and AE intersect at P
∴ P divides OD internally in the ratio 3 : 2.
Hence, OP : PD = 3 : 2.

Question 11.
If the centroid of a tetrahedron OABC is (1, 2, -1) where A = (a, 2, 3), B = (1, b, 2), C = (2, 1, c) respectively, find the distance of P (a, b, c) from the origin.
Solution:
Let G = (1, 2, -1) be the centroid of the tetrahedron OABC.
Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{g}$ be the position vectors of the points A, B, C, G respectively w.r.t. O.

Then $\bar{a} = a\hat{i} + 2\hat{j} + 3\hat{k}$, $\bar{b} = \hat{i} + b\hat{j} + 2\hat{k}$,
$\bar{c} = 2\hat{i} + \hat{j} + c\hat{k}$, $\bar{g} = \hat{i} + 2\hat{j} - \hat{k}$

By formula of centroid of a tetrahedron,

$$\bar{g} = \frac{\bar{0} + \bar{a} + \bar{b} + \bar{c}}{4}$$

$$\therefore\ 4\bar{g} = \bar{a} + \bar{b} + \bar{c}$$

$$\therefore\ 4(\hat{i} + 2\hat{j} - \hat{k}) = (a\hat{i} + 2\hat{j} + 3\hat{k}) + (\hat{i} + b\hat{j} + 2\hat{k}) + (2\hat{i} + \hat{j} + c\hat{k})$$

$$\therefore\ 4\hat{i} + 8\hat{j} - 4\hat{k} = (a+1+2)\hat{i} + (2+b+1)\hat{j} + (3+2+c)\hat{k}$$

$$\therefore\ 4\hat{i} + 8\hat{j} - 4\hat{k} = (a+3)\hat{i} + (b+3)\hat{j} + (c+5)\hat{k}$$

By equality of vectors
a + 3 = 4, b + 3 = 8, c + 5= -4
∴ a = 1, b = 5, c = -9
∴ P = (a, b, c) = (1, 5, -9)
Distance of P from origion = $\sqrt{1^2+5^2+(-9)^2}$
= $\sqrt{1+25+81}$
= $\sqrt{107}$

Question 12.
Find the centroid of tetrahedron with vertices K(5, -7, 0), L(1, 5, 3), M(4, -6, 3), N(6, -4, 2) ?
Solution:
Let $\bar{p}$, $\bar{l}$, $\bar{m}$, $\bar{n}$ be the position vectors of the points K, L, M, N respectively w.r.t. the origin O.

Then $\bar{p} = 5\hat{i} - 7\hat{j} + 0\cdot\hat{k}$, $\bar{l} = \hat{i} + 5\hat{j} + 3\hat{k}$,

$\bar{m} = 4\hat{i} - 6\hat{j} + 3\hat{k}$, $\bar{n} = 6\hat{i} - 4\hat{j} + 2\hat{k}$.

Let G($\bar{g}$) be the centroid of the tetrahedron.

Then by centroid formula

$$\bar{g} = \frac{\bar{p} + \bar{l} + \bar{m} + \bar{n}}{4}$$

$$= \frac{1}{4}[(5\hat{i} - 7\hat{j} + 0\cdot\hat{k}) + (\hat{i} + 5\hat{j} + 3\hat{k}) + (4\hat{i} - 6\hat{j} + 3\hat{k}) + (6\hat{i} - 4\hat{j} + 2\hat{k})]$$

$$= \frac{1}{4}(16\hat{i} - 12\hat{j} + 8\hat{k})$$

$$= 4\hat{i} - 3\hat{j} + 2\hat{k}$$

Hence, the centroid of the tetrahedron is G = (4, -3, 2).

Maharashtra State Board 12th Maths Solutions Chapter 5 Vectors Ex 5.3

Question 1.
Find two unit vectors each of which is perpendicular to both $\bar{u}$ and $\bar{v}$, where $\bar{u} = 2\hat{i}+\hat{j}-2\hat{k}$, $\bar{v} = \hat{i}+2\hat{j}-2\hat{k}$

Solution:
Let $\bar{u} = 2\hat{i}+\hat{j}-2\hat{k}$

$\bar{v} = \hat{i}+2\hat{j}-2\hat{k}$

$$\text{Then } \bar{u} \times \bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= (-2+4)\hat{i} + (-4+2)\hat{j} + (4-1)\hat{k}$$

$$= 2\hat{i} - 2\hat{j} + 3\hat{k}$$

$$\therefore |\bar{u} \times \bar{v}| = \sqrt{(2)^2 + (-2)^2 + (3)^2}$$

$$= \sqrt{4+4+9}$$

$$= \sqrt{17}$$

$$= \pm\frac{\bar{u} \times \bar{v}}{|\bar{u} \times \bar{v}|} = \pm\frac{2\hat{i} - 2\hat{j} + 3\hat{k}}{\sqrt{17}}$$

$$= \pm\left(\frac{2}{\sqrt{17}}\hat{i} - \frac{2}{\sqrt{17}}\hat{j} + \frac{3}{\sqrt{17}}\hat{k}\right)$$

Question 2.
If $\bar{a}$ and $\bar{b}$ are two vectors perpendicular to each other, prove that $(\bar{a} + \bar{b})^2 = (\bar{a} - \bar{b})^2$
Solution:
$\bar{a}$ and $\bar{b}$ are perpendicular to each other.

$$\therefore \ \bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a} = 0 \quad \ldots (1)$$

$$\begin{aligned} \text{LHS} &= (\bar{a}+\bar{b})^2 \\ &= (\bar{a}+\bar{b})\cdot(\bar{a}+\bar{b}) \\ &= \bar{a}\cdot(\bar{a}+\bar{b}) + \bar{b}\cdot(\bar{a}+\bar{b}) \\ &= \bar{a}\cdot\bar{a} + \bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{a} + \bar{b}\cdot\bar{b} \\ &= \bar{a}\cdot\bar{a} + 0 + 0 + \bar{b}\cdot\bar{b} \quad \ldots [\text{By } (1)] \\ &= |\bar{a}|^2 + |\bar{b}|^2 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= (\bar{a}-\bar{b})^2 \\ &= (\bar{a}-\bar{b})\cdot(\bar{a}-\bar{b}) \\ &= \bar{a}\cdot(\bar{a}-\bar{b}) + \bar{b}\cdot(\bar{a}-\bar{b}) \\ &= \bar{a}\cdot\bar{a} - \bar{a}\cdot\bar{b} - \bar{b}\cdot\bar{a} + \bar{b}\cdot\bar{b} \\ &= \bar{a}\cdot\bar{a} + \bar{b}\cdot\bar{b} \quad \ldots [\text{By } (1)] \\ &= |\bar{a}|^2 + |\bar{b}|^2 \end{aligned}$$

∴ LHS = RHS
Hence, $(\bar{a} + \bar{b})^2 = (\bar{a} - \bar{b})^2$.

Question 3.
Find the values of c so that for all real x the vectors $xc\hat{i} - 6\hat{j} + 3\hat{k}$ and $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle.
Solution:
Let $\bar{a} = xc\hat{i} - 6\hat{j} + 3\hat{k}$ and $\bar{b} = x\hat{i} + 2\hat{j} + 2cx\hat{k}$
Consider $\bar{a}\cdot\bar{b} = (xc\hat{i} - 6\hat{j} + 3\hat{k}) \cdot (x\hat{i} + 2\hat{j} + 2cx\hat{k})$
$= (xc)(x) + (-6)(2) + (3)(2cx)$
$= cx^2 - 12 + 6cx$
$= cx^2 + 6cx - 12$
If the angle between $\bar{a}$ and $\bar{b}$ is obtuse, $\bar{a}\cdot\bar{b} < 0$.
$\therefore cx^2 + 6cx - 12 < 0$
$\therefore cx^2 + 6cx < 12$

$$\therefore \ c(x^2+6x) < 12$$

$$\therefore \ c < \frac{12}{x^2+6x}$$

$$\therefore \ c < \frac{12}{(x^2+6x+9)-9} = \frac{12}{(x+3)^2-9}$$

$$\therefore \ c < \min\left\{\frac{12}{(x+3)^2-9}\right\}$$

Now, $\dfrac{12}{(x+3)^2-9}$ is minimum if $(x+3)^2 - 9$ is maximum

i.e. $(x+3)^2 - 9 = \infty - 9 = \infty$

$$\therefore \ c < \min\left\{\frac{12}{\infty}\right\} = 0$$

$\therefore c < 0$.
Hence, the angle between a and b is obtuse if c < 0.

Question 4.
Show that the sum of the length of projections of $p\hat{i} + q\hat{j} + r\hat{k}$ on the coordinate axes, where p = 2, q = 3 and r = 4, is 9.
Solution:
Let $\bar{a} = p\hat{i} + q\hat{j} + r\hat{k}$

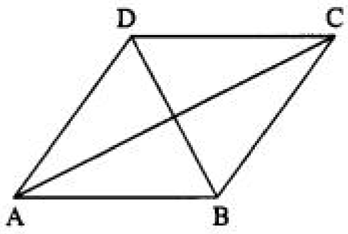
Projection of $\bar{a}$ on X-axis

$$=\frac{\bar{a}\cdot\hat{i}}{|\hat{i}|}=\frac{(p\hat{i}+q\hat{i}+r\hat{k})\cdot\hat{i}}{1}=p=2$$

Similarly, projections of $\bar{a}$ on Y- and Z-axes are 3 and 4 respectively.

∴ sum of these projections = 2 + 3 + 4 = 9.

Question 5.
Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.
Solution:



Let ABCD be a quadrilateral in which

$|\overline{AB}|=|\overline{BC}|=|\overline{CD}|=|\overline{DA}|$ ... (1)

and AB ∥ DC and AD ∥ BC

∴ $\overline{AB}=\overline{DC}$ and $\overline{AD}=\overline{BC}$ ... (2)

Now, $\overline{AC}=\overline{AB}+\overline{BC}$

and $\overline{BD}=\overline{BA}+\overline{AD}=-\overline{AB}+\overline{BC}$ ... [By (2)]

$=\overline{BC}-\overline{AB}$

∴ $\overline{AC}\cdot\overline{BD}=(\overline{AB}+\overline{BC})\cdot(\overline{BC}-\overline{AB})$

$=\overline{AB}\cdot(\overline{BC}-\overline{AB})+\overline{BC}\cdot(\overline{BC}-\overline{AB})$

$=\overline{AB}\cdot\overline{BC}-\overline{AB}\cdot\overline{AB}+\overline{BC}\cdot\overline{BC}-\overline{BC}\cdot\overline{AB}$

$=|\overline{BC}|^2-|\overline{AB}|^2$ ... [∵ $\overline{AB}\cdot\overline{BC}=\overline{BC}\cdot\overline{AB}$]

$=0$ ... [By (1)]

∵ $\overline{AC}$, $\overline{BD}$ are non-zero vectors

∴ $\overline{AC}$ is perpendicular to $\overline{BD}$

Hence, the diagonals are perpendicular.

Question 6.
Determine whether $\bar{a}$ and $\bar{b}$ are orthogonal, parallel or neither.

(i) $\bar{a}=-9\hat{i}+6\hat{j}+15\hat{k}$, $\bar{b}=6\hat{i}-4\hat{j}-10\hat{k}$

Solution:

$\bar{a}=-9\hat{i}+6\hat{j}+15\hat{k}=-3(3\hat{i}-2\hat{j}-5\hat{k})$

$=-32(6\hat{i}-4\hat{j}-19\hat{k})$

∴ $\bar{a}=-32\bar{b}$

i.e. $\bar{a}$ is a non-zero scalar multiple of $\bar{b}$

Hence, $\bar{a}$ is parallel to $\bar{b}$.

(ii) $\bar{a}=2\hat{i}+3\hat{j}-\hat{k}$, $\bar{b}=5\hat{i}-2\hat{j}+4\hat{k}$

Solution:

$\bar{a}\cdot\bar{b}=(2\hat{i}+3\hat{j}-\hat{k})\cdot(5\hat{i}-2\hat{j}+4\hat{k})$

= (2)(5) + (3)(-2) + (-1)(4)

= 10 – 6 – 4 = 0

Since, $\bar{a}$, $\bar{b}$ are non-zero vectors and $\bar{a} \cdot \bar{b}$ = 0, $\bar{a}$ is orthogonal to $\bar{b}$.

(iii) $\bar{a}$ = −35i^+12j^+13k^, $\bar{b}$ = 5i^+4j^+3k^.

Solution:

$$\bar{a}.\bar{b} = \left(-\frac{3}{5}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{3}\hat{k}\right).\left(5\hat{i} + 4\hat{j} + 3\hat{k}\right)$$

$$= \left(-\frac{3}{5}\right)(5) + \left(\frac{1}{2}\right)(4) + \left(\frac{1}{3}\right)(3)$$

= -3 + 2 + 1

= 0

Since, $\bar{a}$, $\bar{b}$ are non-zero vectors and $\bar{a} \cdot \bar{b}$ = 0

$\bar{a}$ is orthogonal to $\bar{b}$.

(iv) $\bar{a}$ = 4i^−j^+6k^, $\bar{a}$ = 5i^−2j^+4k^

Solution:

$\bar{a} \cdot \bar{b}$ = (4i^−j^+6k^) · (5i^−2j^+4k^)

= (4)(5) + (-1)(-2) + (6)(4)

= 20 + 2 + 24

= 46 ≠ 0

∴ $\bar{a}$ is not orthogonal to $\bar{b}$.

It is clear that $\bar{a}$ is not a scalar multiple of $\bar{b}$.

∴ $\bar{a}$ is not parallel to $\bar{b}$.

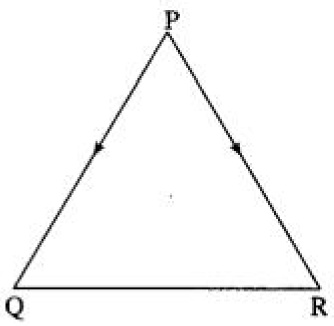
Hence, $\bar{a}$ is neither parallel nor orthogonal to $\bar{b}$.

Question 7.

Find the angle P of the triangle whose vertices are P(0, -1, -2), Q(3, 1, 4) and R(5, 7, 1).

Solution:

The position vectors $\bar{p}$, $\bar{q}$ and $\bar{r}$ of the points P(0, -1, -2), Q(3, 1, 4) and R(5, 7, 1) are



$$\bar{p} = -\hat{j} - 2\hat{k},\ \bar{q} = 3\hat{i} + \hat{j} + 4\hat{k},$$

$$\bar{r} = 5\hat{i} + 7\hat{j} + \hat{k}$$

$$\therefore\ \overline{PQ} = \bar{q} - \bar{p} = (3\hat{i} + \hat{j} + 4\hat{k}) - (-\hat{j} - 2\hat{k}) = 3\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\text{and } \overline{PR} = \bar{r} - \bar{p} = (5\hat{i} + 7\hat{j} + \hat{k}) - (-\hat{j} - 2\hat{k}) = 5\hat{i} + 8\hat{j} + 3\hat{k}$$

$$\therefore\ \overline{PQ} \cdot \overline{PR} = (3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (5\hat{i} + 8\hat{j} + 3\hat{k})$$

$$= (3)(5) + (2)(8) + (6)(3)$$

$$= 15 + 16 + 18 = 49$$

$$|\overline{PQ}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{9 + 4 + 36} = \sqrt{49} = 7$$

$$|\overline{PR}| = \sqrt{5^2 + 8^2 + 3^2} = \sqrt{25 + 64 + 9} = \sqrt{98} = 7\sqrt{2}$$

Using the formula for angle between two vectors,

$$\cos P = \frac{\overline{PQ} \cdot \overline{PR}}{|\overline{PQ}||\overline{PR}|} = \frac{49}{7 \times 7\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos 45^\circ$$

∴ P = 45°

Question 8.
If $\hat{p}$, $\hat{q}$, and $\hat{r}$ are unit vectors, find

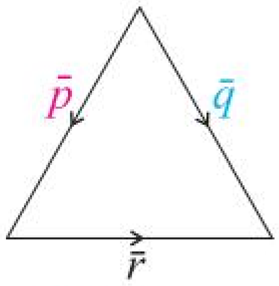
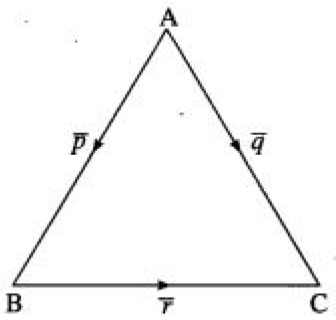


Fig.5.50

(i) $\hat{p} \cdot \hat{q}$
Solution:



Let the triangle be denoted by ABC,
where $\overline{AB} = \bar{p}$, $\overline{AC} = \bar{q}$ and $\overline{BC} = \bar{r}$
∵ $\bar{p}$, $\bar{r}$, $\bar{r}$ are unit vectors.
∴ l(AB) = l(BC) = l(CA) = 1
∴ the triangle is equilateral
∴ ∠A = ∠B = ∠C = 60°
(i) Using the formula for angle between two vectors,

$$\cos A = \frac{\bar{p} \cdot \bar{q}}{|\bar{p}||\bar{q}|}$$

$$\therefore \cos 60° = \frac{\bar{p} \cdot \bar{q}}{1 \times 1}$$

$$\therefore \frac{1}{2} = \bar{p} \cdot \bar{q}$$

$$\therefore \bar{p} \cdot \bar{q} = \frac{1}{2}.$$

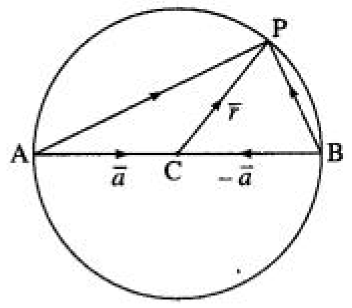
(ii) $\hat{p} \cdot \hat{r}$
Solution:

$$\text{Using } \cos B = \frac{\bar{p} \cdot \bar{r}}{|\bar{p}||\bar{r}|}, \text{ we get } \bar{p} \cdot \bar{r} = \frac{1}{2}$$

$$\cos 60° = \frac{\bar{p} \cdot \bar{r}}{1 \times 1} \qquad \therefore \frac{1}{2} = \bar{p} \cdot \bar{r}$$

$$\therefore \bar{p} \cdot \bar{r} = \frac{1}{2}.$$

Question 9.
Prove by vector method that the angle subtended on semicircle is a right angle.
Solution:
Let seg AB be a diameter of a circle with centre C and P be any point on the circle other than A and B. Then ∠APB is an angle subtended on a semicircle.
Let $\overline{AC} = \overline{CB} = \bar{a}$ and $\overline{CP} = \bar{r}$.
Then $|\bar{a}| = |\bar{r}|$ ...(1)



$\overline{AP} = \overline{AC} + \overline{CP} = \bar{a} + \bar{r} = \bar{r} + \bar{a}$

$\overline{BP} = \overline{BC} + \overline{CP} = -\overline{CB} + \overline{CP} = -\bar{a} + \bar{r}$

$\therefore\ \overline{AP} \cdot \overline{BP} = (\bar{r} + \bar{a}) \cdot (\bar{r} - \bar{a})$

$= \bar{r} \cdot \bar{r} - \bar{r} \cdot \bar{a} + \bar{a} \cdot \bar{r} - \bar{a} \cdot \bar{a}$

$= |\bar{r}|^2 - |\bar{a}|^2 = 0 \qquad \ldots (\because\ \bar{r} \cdot \bar{a} = \bar{a} \cdot \bar{r})$

$\therefore\ \overline{AP} \perp \overline{BP} \quad \therefore\ \angle APB$ is a right angle.

Hence, the angle subtended on a semicircle is the right angle.

Question 10.
If a vector has direction angles 45ºand 60º find the third direction angle.
Solution:
Let α = 45°, β = 60°
We have to find γ.
∴ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
∴ $\cos^2 45° + \cos^2 60° + \cos^2\gamma = 1$

$$\therefore\ \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right) + \cos^2\gamma = 1$$

$$\therefore\ \cos^2\gamma = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\therefore\ \cos\gamma = \pm\frac{1}{2}$$

$$\therefore\ \cos\gamma = \frac{1}{2} \ \text{ or } \ \cos\gamma = -\frac{1}{2}$$

$$\therefore\ \cos\gamma = \cos\frac{\pi}{3} \ \text{ or } \ \cos\gamma = -\cos\frac{\pi}{3}$$

$$= \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\therefore\ \gamma = \frac{\pi}{3} \ \text{ or } \ \gamma = \frac{2\pi}{3}$$

Hence, the third direction angle is $\frac{\pi}{3}$ or $\frac{2\pi}{3}$.

Question 11.
If a line makes angles 90º, 135º, 45º with the X, Y and Z axes respectively, then find its direction cosines.
Solution:
Let l, m, n be the direction cosines of the line.
Then l = cos α, m = cos β, n = cos γ
Here, α = 90°, β = 135° and γ = 45°
∴ l = cos 90° = 0
m = cos 135° = cos (180° – 45°) = -cos 45° = $-\frac{1}{\sqrt{2}}$ and n = cos 45° = $\frac{1}{\sqrt{2}}$
∴ the direction cosines of the line are 0, $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

Question 12.
If a line has the direction ratios, 4, -12, 18 then find its direction cosines.
Solution:
The direction ratios of the line are a = 4, b = -12, c = 18.
Let l, m, n be the direction cosines of the line.

Then $l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{4}{\sqrt{4^2+(-12)^2+(18)^2}}$

$= \frac{4}{\sqrt{16+144+324}} = \frac{4}{22} = \frac{2}{11}$

$m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{-12}{\sqrt{4^2+(-12)^2+(18)^2}}$

$= \frac{-12}{\sqrt{16+144+324}} = \frac{-12}{22} = \frac{-6}{11}$

and $n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{18}{\sqrt{4^2+(-12)^2+(18)^2}}$

$= \frac{18}{\sqrt{16+144+324}} = \frac{18}{22} = \frac{9}{11}$

Hence, the direction cosines of the line are $\frac{2}{11}, \frac{-6}{11}, \frac{9}{11}$.

Question 13.
The direction ratios of $\overline{AB}$ are -2, 2, 1. If A = (4, 1, 5) and l(AB) = 6 units, find B.
Solution:
The direction ratio of $\overline{AB}$ are -2, 2, 1.
∴ the direction cosines of $\overline{AB}$ are

$l = \frac{-2}{\sqrt{(-2)^2+2^2+1^2}} = \frac{-2}{3}$,

$m = \frac{2}{\sqrt{(-2)^2+2^2+1^2}} = \frac{2}{3}$,

$n = \frac{1}{\sqrt{(-2)^2+2^2+1^2}} = \frac{1}{3}$.

i.e. $l = \frac{-2}{3}, m = \frac{2}{3}, n = \frac{1}{3}$

The coordinates of the points which are at a distance of d units from the point (x_1, y_1, z_1) are given by (x_1 ± ld, y_1 ± md, z_1 ± nd)
Here x_1 = 4, y_1 = 1, z_1 = 5, d = 6, l = −23, m = 23, n = 13
∴ the coordinates of the requited points are
(4 ± (−23)6, 1 ± 23(6), 5 ± 13(6))
i.e. (4 – 4, 1 + 4, 5 + 2) and (4 + 4, 1 – 4, 5 – 2)
i.e. (0, 5, 7) and (8, -3, 3).

Question 14.
Find the angle between the lines whose direction cosines l, m, n satisfy the equations 5l + m + 3n = 0 and 5mn – 2nl + 6lm = 0.
Solution:
Given, 5l + m + 3n = 0 …(1)
and 5mn – 2nl + 6lm = 0 …(2)
From (1), m = -(51 + 3n)
Putting the value of m in equation (2), we get,
-5(5l + 3n)n – 2nl – 6l(5l + 3n) = 0
∴ -25ln – 15n^2 – 2nl – 30l^2 – 18ln = 0
∴ – 30l^2 – 45ln – 15n^2 = 0
∴ 2l^2 + 3ln + n^2 = 0
∴ 2l^2 + 2ln + ln + n^2 = 0
∴ 2l(l + n) + n(l + n) = 0
∴ (l + n)(2l + n) = 0
∴ l + n = 0 or 2l + n = 0
l = -n or n = -2l
Now, m = -(5l + 3n), therefore, if l = -n,
m = -(-5n + 3n) = 2n
∴ -l = m2 = n

∴ l−1=m2=n1

∴ the direction ratios of the first line are
$a_1 = -1$, $b_1 = 2$, $c_1 = 1$
If n = -2l, m = -(5l – 6l) – l
∴ l = m = n−2

∴ l1=m1=n−2

∴ the direction ratios of the second line are
$a_2 = -1$, $b_2 = 1$, $c_2 = -2$
Let θ be the angle between the lines.

$$\text{Then } \cos\theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

$$= \left| \frac{(-1)(1) + 2(1) + 1(-2)}{\sqrt{(-1)^2 + 2^2 + 1^2} \cdot \sqrt{1^2 + 1^2 + (-2)^2}} \right|$$

$$= \left| \frac{-1 + 2 - 2}{\sqrt{6} \cdot \sqrt{6}} \right| = \left| \frac{-1}{6} \right| = \frac{1}{6}$$

$$\therefore \ \theta = \cos^{-1}\left(\frac{1}{6}\right).$$

Maharashtra State Board 12th Maths Solutions Chapter 5 Vectors Ex 5.4

Question 1.
If $\bar{a} = 2\hat{i}+3\hat{j}-\hat{k}$, $\bar{b} = \hat{i}-4\hat{j}+2\hat{k}$ find $(\bar{a} + \bar{b}) \times (\bar{a} - \bar{b})$
Solution:

$$\text{Given : } \bar{a} = 2\hat{i} + 3\hat{j} - \hat{k},\ \bar{b} = \hat{i} - 4\hat{j} + 2\hat{k}$$

$$\therefore \ \bar{a} + \bar{b} = (2\hat{i} + 3\hat{j} - \hat{k}) + (\hat{i} - 4\hat{j} + 2\hat{k}) = 3\hat{i} - \hat{j} + \hat{k}$$

$$\text{and } \bar{a} - \bar{b} = (2\hat{i} + 3\hat{j} - \hat{k}) - (\hat{i} - 4\hat{j} + 2\hat{k}) = \hat{i} + 7\hat{j} - 3\hat{k}$$

$$\therefore \ (\bar{a} + \bar{b}) \times (\bar{a} - \bar{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ 1 & 7 & -3 \end{vmatrix}$$

$$= (3 - 7)\hat{i} - (-9 - 1)\hat{j} + (21 + 1)\hat{k}$$

$$= -4\hat{i} + 10\hat{j} + 22\hat{k}.$$

Question 2.
Find a unit vector perpendicular to the vectors $\hat{j}+2\hat{k}$ and $\hat{i}+\hat{j}$.
Solution:

Let $\bar{a} = \hat{j}+2\hat{k}$, $\bar{b} = \hat{i}+\hat{j}$

$$\text{Then } \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= (0-2)\hat{i} - (0-2)\hat{j} + (0-1)\hat{k}$$

$$= -2\hat{i} + 2\hat{j} - \hat{k}$$

$$\therefore |\bar{a} \times \bar{b}| = \sqrt{(-2)^2 + 2^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

Unit vector perpendicular to both $\bar{a}$ and $\bar{b}$

$$= \pm \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|} = \pm \left(\frac{-2\hat{i} + 2\hat{j} - \hat{k}}{3} \right)$$

$$= \pm \left(-\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k} \right)$$

Question 3.
If $\bar{a} \cdot \bar{b} = \sqrt{3}$ and $\bar{a} \times \bar{b} = 2\hat{i}+\hat{j}+2\hat{k}$, find the angle between $\bar{a}$ and $\bar{b}$.
Solution:
Let θ be the angle between $\bar{a}$ and $\bar{b}$

$$\because \bar{a} \times \bar{b} = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\therefore |\bar{a} \times \bar{b}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4+1+4} = 3$$

$$\therefore |\bar{a}||\bar{b}| \sin\theta = 3 \qquad \ldots (1)$$

$$\because \bar{a} \cdot \bar{b} = \sqrt{3}$$

$$\therefore |\bar{a}||\bar{b}| \cos\theta = \sqrt{3} \qquad \ldots (2)$$

Dividing (1) by (2), we get

$$\frac{|\bar{a}||\bar{b}| \sin\theta}{|\bar{a}||\bar{b}| \cos\theta} = \frac{3}{\sqrt{3}} \qquad \therefore \tan\theta = \sqrt{3} = \tan 60^\circ$$

$\therefore$ θ = 60°.

Question 4.
If $\bar{a} = 2\hat{i}+\hat{j}-3\hat{k}$ and $\bar{b} = \hat{i}-2\hat{j}+\hat{k}$, find a vector of magnitude 5 perpendicular to both $\bar{a}$ and $\bar{b}$.
Solution:
Given : $\bar{a} = 2\hat{i}+\hat{j}-3\hat{k}$ and $\bar{b} = \hat{i}-2\hat{j}+\hat{k}$

$$\therefore \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= (1-6)\hat{i} - (2+3)\hat{j} + (-4-1)\hat{k}$$

$$= -5\hat{i} - 5\hat{j} - 5\hat{k}$$

$$\therefore |\bar{a} \times \bar{b}| = \sqrt{(-5)^2 + (-5)^2 + (-5)^2}$$

$$= \sqrt{25+25+25} = \sqrt{75} = 5\sqrt{3}$$

$\therefore$ unit vectors perpendicular to both the vectors $\bar{a}$ and $\bar{b}$.

$$= \frac{\pm(\bar{a} \times \bar{b})}{|\bar{a} \times \bar{b}|} = \frac{\pm(-5\hat{i} - 5\hat{j} - 5\hat{k})}{5\sqrt{3}}$$

$$= \frac{\pm(\hat{i} + \hat{j} + \hat{k})}{\sqrt{3}}$$

$\therefore$ required vectors of magnitude 5 units
$= \pm\frac{5}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$

Question 5.
Find
(i) $\bar{u} \cdot \bar{v}$ if $|\bar{u}| = 2$, $|\vec{v}| = 5$, $|\bar{u} \times \bar{v}| = 8$

Solution:
Let θ be the angle between $\bar{u}$ and $\bar{v}$.
Then $|\bar{u} \times \bar{v}| = 8$ gives
$|\bar{u}||\bar{v}| \sin\theta = 8$
∴ $2 \times 5 \times \sin\theta = 8$

$$\therefore \sin\theta = \frac{4}{5}$$

$$\cos\theta = \pm\sqrt{1-\sin^2\theta} \quad \ldots [\because 0 \leqslant \theta \leqslant \pi]$$

$$= \pm\sqrt{1-\left(\frac{4}{5}\right)^2} = \pm\sqrt{1-\frac{16}{25}}$$

$$= \pm\sqrt{\frac{9}{25}} = \pm\frac{3}{5}$$

Now, $\bar{u}\cdot\bar{v} = |\bar{u}||\bar{v}| \cos\theta$

$$\therefore \bar{u}\cdot\bar{v} = 2 \times 5 \times \left(\pm\frac{3}{5}\right) = \pm 6.$$

(ii) $|\bar{u} \times \bar{v}|$ if $|\bar{u}| = 10$, $|\vec{v}| = 2$, $\bar{u} \cdot \bar{v} = 12$
Solution:
Let θ be the angle between $\bar{u}$ and $\bar{v}$.
Then $\bar{u} \cdot \bar{v} = 12$ gives
$|\bar{u} // \bar{v}| \cos\theta = 12$
∴ $10 \times 2 \times \cos\theta = 12$

$$\therefore \cos\theta = \frac{3}{5}, \text{ where } 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}}$$

$$= \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Now, $|\bar{u} \times \bar{v}| = |\bar{u}||\bar{v}| \sin\theta$

$$\therefore \bar{u} \times \bar{v} = 10 \times 2 \times \frac{4}{5} = 16$$

Question 6.
Prove that $2(\bar{a} - \bar{b}) \times 2(\bar{a} + \bar{b}) = 8(\bar{a} \times \bar{b})$
Solution:

$$\text{LHS} = 2(\bar{a}-\bar{b}) \times 2(\bar{a}+\bar{b})$$

$$= 4[(\bar{a}-\bar{b}) \times (\bar{a}+\bar{b})]$$

$$= 4[\bar{a} \times (\bar{a}+\bar{b}) - \bar{b} \times (\bar{a}+\bar{b})]$$

$$= 4(\bar{a}\times\bar{a} + \bar{a}\times\bar{b} - \bar{b}\times\bar{a} - \bar{b}\times\bar{b})$$

$$= 8(\bar{a}\times\bar{b})$$

$$\ldots [\because \bar{a}\times\bar{a} = \bar{b}\times\bar{b} = \bar{0} \text{ and } -(\bar{b}\times\bar{a}) = \bar{a}\times\bar{b}]$$

$$= \text{RHS}$$

$$\therefore 2(\bar{a}-\bar{b}) \times 2(\bar{a}+\bar{b}) = 8(\bar{a}\times\bar{b}).$$

Question 7.
If $\bar{a} = \hat{i}-2\hat{j}+3\hat{k}$, $\bar{b} = 4\hat{i}-3\hat{j}+\hat{k}$, and $\bar{c} = \hat{i}-\hat{j}+2\hat{k}$, verify that $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$
Solution:
Given : $\bar{a} = \hat{i}-2\hat{j}+3\hat{k}$, $\bar{b} = 4\hat{i}-3\hat{j}+\hat{k}$, $\bar{c} = \hat{i}-\hat{j}+2\hat{k}$

$$\therefore\ \bar{b}+\bar{c} = (4\hat{i}-3\hat{j}+\hat{k})+(\hat{i}-\hat{j}+2\hat{k})$$
$$= 5\hat{i}-4\hat{j}+3\hat{k}$$

and $\bar{a}\times(\bar{b}+\bar{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 4 & -3 & 1 \end{vmatrix}$

$$= (-6+12)\hat{i}-(3-15)\hat{j}+(-4+10)\hat{k}$$
$$= 6\hat{i}+12\hat{j}+6\hat{k} \qquad \ldots (1)$$

Also, $\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 4 & -3 & 1 \end{vmatrix}$

$$= (-2+9)\hat{i}-(1-12)\hat{j}+(-3+8)\hat{k}$$
$$= 7\hat{i}+11\hat{j}+5\hat{k}$$

and $\bar{a}\times\bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 1 & -1 & 2 \end{vmatrix}$

$$= (-4+3)\hat{i}-(2-3)\hat{j}+(-1+2)\hat{k}$$
$$= -\hat{i}+\hat{j}+\hat{k}$$

$$\therefore\ \bar{a}\times\bar{b}+\bar{a}\times\bar{c} = (7\hat{i}+11\hat{j}+5\hat{k})+(-\hat{i}+\hat{j}+\hat{k})$$
$$= 6\hat{i}+12\hat{j}+6\hat{k} \qquad \ldots (2)$$

From (1) and (2), we get

$\bar{a}\times(\bar{b}+\bar{c}) = \bar{a}\times\bar{b}+\bar{a}\times\bar{c}.$

Question 8.

Find the area of the parallelogram whose adjacent sides are the vectors $\bar{a} = 2\hat{i}-2\hat{j}+\hat{k}$ and $\bar{b} = \hat{i}-3\hat{j}-3\hat{k}$.

Solution:

Given : $\bar{a} = 2\hat{i}-2\hat{j}+\hat{k}$, $\bar{b} = \hat{i}-3\hat{j}-3\hat{k}$

$\therefore\ \bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -3 & -3 \end{vmatrix}$

$$= (6+3)\hat{i}-(-6-1)\hat{j}+(-6+2)\hat{k}$$
$$= 9\hat{i}+7\hat{j}-4\hat{k}$$

$$|\bar{a}\times\bar{b}| = \sqrt{9^2+7^2+(-4)^2} = \sqrt{81+49+16} = \sqrt{146}$$

Area of the parallelogram whose adjacent sides are $\bar{a}$ and $\bar{b}$ is $|\bar{a}\times\bar{b}| = \sqrt{146}$ sq units.

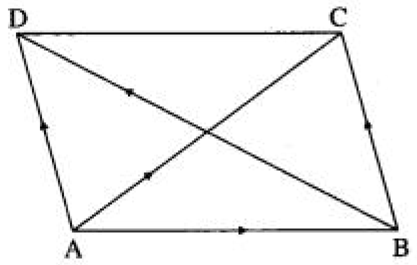
Question 9.

Show that vector area of a quadrilateral ABCD is $\frac{1}{2}(\overline{AC} \times \overline{BD})$, where AC and BD are its diagonals.

Solution:

Let ABCD be a parallelogram.

Then $\overline{AC} = \overline{AB} + \overline{BC}$



and $\overline{BD} = \overline{BA} + \overline{AD} = -\overline{AB} + \overline{BC}$ ... [$\because \overline{BC} = \overline{AD}$]

$= \overline{BC} - \overline{AB}$

$\therefore \overline{AC} \times \overline{BD} = (\overline{AB} + \overline{BC}) \times (\overline{BC} - \overline{AB})$

$= \overline{AB} \times (\overline{BC} - \overline{AB}) + \overline{BC} \times (\overline{BC} - \overline{AB})$

$= \overline{AB} \times \overline{BC} - \overline{AB} \times \overline{AB} + \overline{BC} \times \overline{BC} - \overline{BC} \times \overline{AB}$

$= \overline{AB} \times \overline{BC} + \overline{AB} \times \overline{BC}$

... [$\overline{AB} \times \overline{AB} = \overline{BC} \times \overline{BC} = \overline{0}$ and $-\overline{BC} \times \overline{AB} = \overline{AB} \times \overline{BC}$]

$\therefore \overline{AC} \times \overline{BD} = 2(\overline{AB} \times \overline{BC})$

= 2 (vector area of parallelogram ABCD)

$\therefore$ vector area of parallelogram ABCD $= \frac{1}{2}(\overline{AC} \times \overline{BD})$.

Question 10.
Find the area of parallelogram whose diagonals are determined by the vectors $\bar{a} = 3\hat{i} - \hat{j} - 2\hat{k}$, and $\bar{b} = -\hat{i} + 3\hat{j} - 3\hat{k}$
Solution:
Given: $\bar{a} = 3\hat{i} - \hat{j} - 2\hat{k}$, $\bar{b} = -\hat{i} + 3\hat{j} - 3\hat{k}$

$$\therefore \bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & -2 \\ -1 & 3 & -3 \end{vmatrix}$$

$= (3 + 6)\hat{i} - (-9 - 2)\hat{j} + (9 - 1)\hat{k}$

$= 9\hat{i} + 11\hat{j} + 8\hat{k}$

and $|\bar{a} \times \bar{b}| = \sqrt{9^2 + 11^2 + 8^2} = \sqrt{81 + 121 + 64} = \sqrt{266}$

Area of the parallelogram having diagonals $\bar{a}$ and $\bar{b}$

$= \frac{1}{2}|\bar{a} \times \bar{b}| = \frac{1}{2}\sqrt{266}$ sq units.

Question 11.
If $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ are four distinct vectors such that $\bar{a} \times \bar{b} = \bar{c} \times \bar{d}$ and $\bar{a} \times \bar{c} = \bar{b} \times \bar{d}$, prove that $\bar{a} - \bar{d}$ is parallel to $\bar{b} - \bar{c}$.
Solution:
$\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ are four distinct vectors

$\therefore \bar{a} \neq \bar{b} \neq \bar{c} \neq \bar{d}$.

$\therefore \bar{a} - \bar{d} \neq \bar{0}$ and $\bar{b} - \bar{c} \neq \bar{0}$ ... (1)

Now, $\bar{a} \times \bar{b} = \bar{c} \times \bar{d}$ ... (2)

and $\bar{a} \times \bar{c} = \bar{b} \times \bar{d}$ ... (3)

Subtracting (3) from (2), we get

$\bar{a} \times \bar{b} - \bar{a} \times \bar{c} = \bar{c} \times \bar{d} - \bar{b} \times \bar{d}$

$\therefore \bar{a} \times (\bar{b} - \bar{c}) = (\bar{c} - \bar{b}) \times \bar{d} = -(\bar{b} - \bar{c}) \times \bar{d}$

$= \bar{d} \times (\bar{b} - \bar{c})$

$\therefore \bar{a} \times (\bar{b} - \bar{c}) - \bar{d} \times (\bar{b} - \bar{c}) = \bar{0}$

$\therefore (\bar{a} - \bar{d}) \times (\bar{b} - \bar{c}) = \bar{0}$

$\therefore \bar{a} - \bar{d}$ and $\bar{b} - \bar{c}$ are parallel to each other.

... [By (1)]

Question 12.
If $\bar{a}$ = $\hat{i}+\hat{j}+\hat{k}$ and, $\bar{c}$ = $\hat{j}-\hat{k}$, find a vector $\bar{b}$ satisfying $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a} \cdot \bar{b}$ = 3
Solution:
Given $\bar{a}$ = $\hat{i}+\hat{j}+\hat{k}$, $\bar{c}$ = $\hat{j}-\hat{k}$

Let $\bar{b} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\bar{a} \cdot \bar{b} = 3$ gives

$(\hat{i}+\hat{j}+\hat{k}) \cdot (x\hat{i}+y\hat{j}+z\hat{k}) = 3$

$\therefore$ $(1)(x)+(1)(y)+(1)(z) = 3$

Also, $x + y + z = 3$ ... (1)

Also, $\bar{c} = \bar{a} \times \bar{b}$

$$\therefore \hat{j}-\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix}$$

$$= (z-y)\hat{i} - (z-x)\hat{j} + (y-x)\hat{k}$$

$$= (z-y)\hat{i} + (x-z)\hat{j} + (y-x)\hat{k}$$

By equality of vectors,
z – y = 0 ....(2)
x – z = 1 ......(3)
y – x = -1 ......(4)
From (2), y = z.
From (3), x = 1 + z
Substituting these values of x and y in (1), we get
1 + z + z + z = 3 ∴ z = $\frac{2}{3}$
∴ y = z = $\frac{2}{3}$
∴ x = 1 + z =1 + $\frac{2}{3}$=$\frac{5}{3}$

$$\therefore \bar{b} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

i.e. $\bar{b} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$.

Question 13.
Find $\bar{a}$, if $\bar{a} \times \hat{i} + 2\bar{a} - 5\hat{j} = \bar{0}$.
Solution:

Let $\bar{a} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\bar{a} \times \hat{i} = (x\hat{i} + y\hat{j} + z\hat{k}) \times \hat{i}$

$= x(\hat{i} \times \hat{i}) + y(\hat{j} \times \hat{i}) + z(\hat{k} \times \hat{i})$

$= z\hat{j} - y\hat{k}$ ... $[\because \hat{i} \times \hat{i} = \bar{0}, \hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{i} = \hat{j}]$

It is given that

$\bar{a} \times \hat{i} + 2\bar{a} - 5\hat{j} = \bar{0}$

$\therefore$ $z\hat{j} - y\hat{k} + 2(x\hat{i} + y\hat{j} + z\hat{k}) - 5\hat{j} = 0$

$\therefore$ $z\hat{j} - y\hat{k} + 2x\hat{i} + 2y\hat{j} + 2z\hat{k} - 5\hat{j} = \bar{0}$

$\therefore$ $2x\hat{i} + (2y + z - 5)\hat{j} + (2z - y)\hat{k} = \bar{0}$

By equality of vectors
2x = 0 i.e. x = 0
2y + z – 5 = 0 … (1)
2z – y = 0 … (2)
From (2), y = 2z
Substituting y = 2z in (1), we get
4z + z = 5 ∴ z = 1
∴ y = 2z = 2(1) = 2
∴ x = 0, y = 2, z = 1
∴ $\bar{a} = 2\hat{j} + \hat{k}$

Question 14.
If $|\bar{a} \cdot \bar{b}| = |\bar{a} \times \bar{b}|$ and $\bar{a} \cdot \bar{b}$ < 0, then find the angle between $\bar{a}$ and $\bar{b}$
Solution:
Let θ be the angle between $\bar{a}$ and $\bar{b}$.

Then $|\bar{a} \cdot \bar{b}| = |\bar{a} \times \bar{b}|$ gives

$|ab\cos\theta| = |ab\sin\theta|$

$\therefore\ -ab\cos\theta = ab\sin\theta \quad \ldots [\because \bar{a}\cdot\bar{b} < 0]$

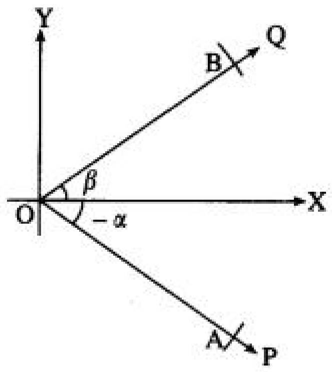
$\therefore\ -1 = \tan\theta$

$\therefore\ \tan\theta = -\tan\frac{\pi}{4} = \tan\left(\pi - \frac{\pi}{4}\right)$

$\therefore\ \tan\theta = \tan\frac{3\pi}{4} \qquad \therefore\ \theta = \frac{3\pi}{4}$

Hence, the angle between $\bar{a}$ and $\bar{b}$ is $\frac{3\pi}{4}$.

Question 15.
Prove by vector method that sin (α + β) = sinα·cosβ+cosα·sinβ.
Solution:



Let ∠XOP and ∠XOQ be in standard position and m∠XOP = -α, m∠XOQ = β.
Take a point A on ray OP and a point B on ray OQ such that
OA = OB = 1.
Since cos (-α) = cos α
and sin (-α) = -sin α,
A is (cos (-α), sin (-α)),
i.e. (cos α, – sin α)
B is (cos β, sin β)

$\therefore\ \overline{OA} = (\cos\alpha)\bar{i} - (\sin\alpha)\cdot\bar{j} + 0\cdot\bar{k}$

$\overline{OB} = (\cos\beta)\cdot\bar{i} + (\sin\beta)\cdot\bar{j} + 0\cdot\bar{k}.$

$$\therefore\ \overline{OA} \times \overline{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\alpha & -\sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix}$$

$$= (\cos\alpha\sin\beta + \sin\alpha\cos\beta)\bar{k} \quad \ldots (1)$$

The angle between $\overline{OA}$ and $\overline{OB}$ is α + β.

Also $\overline{OA}$, $\overline{OB}$ lie in the XY-plane.

∴ the unit vector perpendicular to $\overline{OA}$ and $\overline{OB}$ is $\bar{k}$.

∴ $\overline{OA} \times \overline{OB}$ = [OA·OB sin (α + β)]$\bar{k}$

= sin(α + β)·$\bar{k}$ ...(2)

∴ from (1) and (2),
sin (α + β) = sin α cos β + cos α sin β.

Question 16.
Find the direction ratios of a vector perpendicular to the two lines whose direction ratios are
(i) -2, 1, -1 and -3, -4, 1
Solution:
Let a, b, c be the direction ratios of the vector which is perpendicular to the two lines whose direction ratios are -2, 1, -1 and -3, -4, 1
∴ -2a + b – c = 0 and -3a – 4b + c = 0

$$\therefore \frac{a}{\begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix}} = \frac{b}{\begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix}} = \frac{c}{\begin{vmatrix} -2 & 1 \\ -3 & -4 \end{vmatrix}}$$

$$\therefore \frac{a}{1-4} = \frac{b}{3+2} = \frac{c}{8+3}$$

$$\therefore \frac{a}{-3} = \frac{b}{5} = \frac{c}{11}$$

∴ the required direction ratios are -3, 5, 11
Alternative Method:
Let $\bar{a}$ and $\bar{b}$ be the vectors along the lines whose direction ratios are -2, 1, -1 and -3, -4, 1 respectively.
Then $\bar{a} = -2\hat{i}+\hat{j}-\hat{k}$ and $\bar{b} = -3\hat{i}-4\hat{j}+\hat{k}$
The vector perpendicular to both $\bar{a}$ and $\bar{b}$ is given by

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & -1 \\ -3 & -4 & 1 \end{vmatrix}$$

$$= (1-4)\hat{i} - (-2-3)\hat{j} + (8+3)\hat{k}$$

$$= -3\hat{i} + 5\hat{j} + 11\hat{k}$$

Hence, the required direction ratios are -3, 5, 11.

(ii) 1, 3, 2 and -1, 1, 2
Solution:

Question 17.
Prove that two vectors whose direction cosines are given by relations al + bm + cn = 0 and fmn + gnl + hlm = 0 are perpendicular if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c} = 0$
Solution:
Given, al + bm + cn = 0 ...(1)
and fmn + gnl + hlm = 0 .....(2)
From (1), n = $-\left(\frac{al+bm}{c}\right)$ .....(3)
Substituting this value of n in equation (2), we get
(fm + gl)·[latex]-\left(\frac{a l+b m}{c}\right)[/latex] + hlm = 0
∴ -(aflm + bfm^2 + agl^2 + bglm) + chlm = 0
∴ agl^2 + (af + bg – ch)lm + bfm^2 = 0 ... (4)
Note that both l and m cannot be zero, because if l = m = 0, then from (3), we get
n = 0, which is not possible as $l^2 + m^2 + n^2 = 1$.
Let us take m # 0.
Dividing equation (4) by m^2, we get
ag$\left(\frac{l}{m}\right)^2$ + (af + bg – ch)$\left(\frac{l}{m}\right)$ + bf = 0 ... (5)
This is quadratic equation in $\left(\frac{l}{m}\right)$.
If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of the two lines given by the equation (1) and (2), then $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ are the roots of the equation (5).
From the quadratic equation (5), we get

product of roots $= \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{bf}{ag}$

$\therefore \ \frac{l_1 l_2}{m_1 m_2} = \frac{(f/a)}{(g/b)}$

$\therefore \ \frac{l_1 l_2}{(f/a)} = \frac{m_1 m_2}{(g/b)}$

Similarly, we can show that,

$\frac{l_1 l_2}{(f/a)} = \frac{n_1 n_2}{(h/c)}$

$\therefore \ \frac{l_1 l_2}{(f/a)} = \frac{m_1 m_2}{(g/b)} = \frac{n_1 n_2}{(h/c)} = \lambda$ ... (Say)

$\therefore \ l_1 l_2 = \lambda\left(\frac{f}{a}\right),\ m_1 m_2 = \lambda\left(\frac{g}{b}\right),\ n_1 n_2 = \lambda\left(\frac{h}{c}\right)$

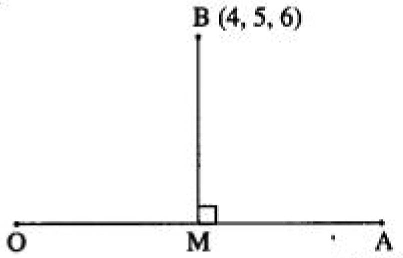
Now, the lines are perpendicular if

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

i.e. if $\lambda\left(\frac{f}{a}\right) + \lambda\left(\frac{g}{b}\right) + \lambda\left(\frac{h}{c}\right) = 0$

i.e. if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c} = 0$.

Question 18.
If A(1, 2, 3) and B(4, 5, 6) are two points, then find the foot of the perpendicular from the point B to the line joining the origin and point A.
Solution:



Let M be the foot of the perpendicular drawn from B to the line joining O and A.
Let M = (x, y, z)
OM has direction ratios x – 0, y – 0, z – 0 = x, y, z
OA has direction ratios 1 – 0, 2 – 0, 3 – 0 = 1, 2, 3
But O, M, A are collinear.
∴ $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ = k ...(Let)
∴ x = k, y = 2k, z = 3k
∴ M = (k, 2k, 3k)
∵ BM has direction ratios
k – 4, 2k – 5, 3k – 6
BM is perpendicular to OA
∴ (l)(k – 4) + 2(2k – 5) + 3(3k – 6)
∴ = k – 4 + 4k – 10 + 9k – 18 = 0
∴ 14k = 32
∴ k = $\frac{16}{7}$
∴ M = (k, 2k, 3k) = $\left(\frac{16}{7}, \frac{32}{7}, \frac{48}{7}\right)$

Maharashtra State Board 12th Maths Solutions Chapter 5 Vectors Ex 5.5

Question 1.
Find $\bar{a} \cdot (\bar{b} \times \bar{c})$, if $\bar{a} = 3\hat{i} - \hat{j} + 4\hat{k}$, $\bar{b} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\bar{c} = -5\hat{i} + 2\hat{j} + 3\hat{k}$
Solution:
$\bar{a} \cdot (\bar{b} \times \bar{c}) = \begin{vmatrix} 3 & -1 & 4 \\ 2 & 3 & -1 \\ -5 & 2 & 3 \end{vmatrix}$
= 3(9 + 2) + 1 (6 – 5) + 4(4 + 15)
= 33 + 1 + 76
= 110.

Question 2.
If the vectors $3\hat{i} + 5\hat{k}, 4\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + \hat{j} + 4\hat{k}$ are to co-terminus edges of the parallelo piped, then find the volume of the parallelopiped.
Solution:
Let $\bar{a} = 3\hat{i} + 5\hat{k}$, $\bar{b} = 4\hat{i} + 2\hat{j} - 3\hat{k}$, $\bar{c} = 3\hat{i} + \hat{j} + 4\hat{k}$

$$\therefore\ [\bar{a}\ \bar{b}\ \bar{c}] = = \begin{vmatrix} 3 & 0 & 5 \\ 4 & 2 & -3 \\ 3 & 1 & 4 \end{vmatrix}$$

= 3(8 + 3) – 0(16 + 9) + 5(4 – 6)
= 33 – 0 – 10 = 23
∴ volume of the parallelopiped = $[\bar{a}\ \bar{b}\ \bar{c}]$
= 23 cubic units.

Question 3.
If the vectors $-3\hat{i} + 4\hat{j} - 2\hat{k}, \hat{i} + 2\hat{k}$ and $\hat{i} - p\hat{j}$ are coplanar, then find the value of p.
Solution:
Let $\bar{a} = -3\hat{i} + 4\hat{j} - 2\hat{k}$, $\bar{b} = \hat{i} + 2\hat{k}$, $\bar{c} = \hat{i} - p\hat{j}$

Then, $\bar{a}, \bar{b}, \bar{c}$ are coplanar

$$\therefore\ [\bar{a}\ \bar{b}\ \bar{c}] = 0$$

$$\therefore\ \begin{vmatrix} -3 & 4 & -2 \\ 1 & 0 & 2 \\ 1 & -p & 0 \end{vmatrix} = 0$$

∴ -3(0 + 2p) – 4(0 – 2) – 2(-p – 0) = 0
∴ -6p + 8 + 2p = 0
∴ -4p = -8
P = 2.

Question 4.
Prove that :
(i) [latex]\bar{a} \bar{b}+\bar{c} \bar{a}+\bar{b}+\bar{c}[/latex] = 0
Solution:

$$[\bar{a}\quad \bar{b}+\bar{c}\quad \bar{a}+\bar{b}+\bar{c}]$$
$$= \bar{a} \cdot [(\bar{b}+\bar{c}) \times (\bar{a}+\bar{b}+\bar{c})]$$
$$= \bar{a} \cdot (\bar{b} \times \bar{a} + \bar{b} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{c} \times \bar{b} + \bar{c} \times \bar{c})$$
$$= \bar{a} \cdot (\bar{b} \times \bar{a}) + \bar{a} \cdot (\bar{b} \times \bar{b}) + \bar{a} \cdot (\bar{b} \times \bar{c}) + \bar{a} \cdot (\bar{c} \times \bar{a}) + \bar{a} \cdot (\bar{c} \times \bar{b}) + \bar{a} \cdot (\bar{c} \times \bar{c})$$
$$= 0 + 0 + \bar{a} \cdot (\bar{b} \times \bar{c}) + 0 - \bar{a} \cdot (\bar{b} \times \bar{c}) + 0$$
$$= 0.$$

(ii) $(\bar{a} - 2\bar{b} - \bar{c}) \cdot [(\bar{a} - \bar{b}) \times \bar{a} - \bar{b} - \bar{c}] = 3[\bar{a} - \bar{b} - \bar{c}]$
Question is modified.
$(\bar{a} - 2\bar{b} - \bar{c})\ [(\bar{a} - \bar{b}) \times \bar{a} - \bar{b} - \bar{c}] = 3[\bar{a}\ \bar{b}\ \bar{c}]$

Solution:

$$(\bar{a}+2\bar{b}-\bar{c})\cdot[(\bar{a}-\bar{b})\times(\bar{a}-\bar{b}-\bar{c})]$$
$$=(\bar{a}+2\bar{b}-\bar{c})\cdot(\bar{a}\times\bar{a}-\bar{a}\times\bar{b}-\bar{a}\times\bar{c}-\bar{b}\times\bar{a}+\bar{b}\times\bar{b}+\bar{b}\times\bar{c})$$
$$=(\bar{a}+2\bar{b}-\bar{c})\cdot(\bar{0}-\bar{a}\times\bar{b}+\bar{c}\times\bar{a}+\bar{a}\times\bar{b}+\bar{0}+\bar{b}\times\bar{c})$$
$$=(\bar{a}+2\bar{b}-\bar{c})\cdot(\bar{c}\times\bar{a}+\bar{b}\times\bar{c})$$
$$=\bar{a}\cdot(\bar{c}\times\bar{a})+\bar{a}\cdot(\bar{b}\times\bar{c})+2\bar{b}\cdot(\bar{c}\times\bar{a})+2\bar{b}\cdot(\bar{b}\times\bar{c})-\bar{c}\cdot(\bar{c}\times\bar{a})-\bar{c}\cdot(\bar{b}\times\bar{c})$$
$$=0+\bar{a}\cdot(\bar{b}\times\bar{c})+2\bar{b}\cdot(\bar{c}\times\bar{a})+2\times0-0-0$$
$$=[\bar{a}\ \bar{b}\ \bar{c}]+2[\bar{b}\ \bar{c}\ \bar{a}]$$
$$=[\bar{a}\ \bar{b}\ \bar{c}]+2[\bar{a}\ \bar{b}\ \bar{c}]=3[\bar{a}\ \bar{b}\ \bar{c}].$$

Question 5.
If $\bar{c}=3\bar{a}-2\bar{b}$ prove that $[\bar{a}\ \bar{b}\ \bar{c}]=0$
Solution:
We use the results : $\bar{b}\times\bar{b}=0$ and if in a scalar triple product, two vectors are equal, then the scalar triple product is zero.

$$[\bar{a}\ \ \bar{b}\ \ \bar{c}]=\bar{a}\cdot(\bar{b}\times\bar{c})$$
$$=\bar{a}\cdot[\bar{b}\times(3\bar{a}-2\bar{b})]$$
$$=\bar{a}\cdot(3\bar{b}\times\bar{a}-2\bar{b}\times\bar{b})$$
$$=\bar{a}\cdot(3\bar{b}\times\bar{a}-\bar{0})$$
$$=3\bar{a}\cdot(\bar{b}\times\bar{a})=3\times0=0.$$

Alternative Method :

$\bar{c}=3\bar{a}-2\bar{b}$

$\therefore$ $\bar{c}$ is a linear combination of $\bar{a}$ and $\bar{b}$

$\therefore$ $\bar{a}, \bar{b}, \bar{c}$ are coplanar

$\therefore$ $[\bar{a}\ \bar{b}\ \bar{c}]=0.$

Question 6.
If u = $\hat{i}-2\hat{j}+\hat{k}$, $\bar{v}=3\hat{i}+\hat{k}$ and $\bar{w}=\hat{j}-\hat{k}$ are given vectors, then find
(i) $[\bar{u}+\bar{w}]\cdot[(\bar{w}\times\bar{r})\times(\bar{r}\times\bar{w})]$
Question is modified.
If $\bar{u}=\hat{i}-2\hat{j}+\hat{k}$, $\bar{r}=3\hat{i}+\hat{k}$ and $\bar{w}=\hat{j}-\hat{k}$ are given vectors, then find $[\bar{u}+\bar{w}]\cdot[(\bar{u}\times\bar{r})\times(\bar{r}\times\bar{w})]$
Solution:

$$\bar{u}+\bar{w}=(\hat{i}-2\hat{j}+\hat{k})+(\hat{j}-\hat{k})$$
$$=\hat{i}-\hat{j}$$
$$\bar{u}\times\bar{r}==\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 1\\ 3 & 0 & 1\end{vmatrix}$$
$$=(-2-0)\hat{i}-(1-3)\hat{j}+(0+6)\hat{k}$$
$$=-2\hat{i}+2\hat{j}+6\hat{k}$$
$$\bar{r}\times\bar{w}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 0 & 1\\ 0 & 1 & -1\end{vmatrix}$$
$$=(0-1)\hat{i}-(-3-0)\hat{j}+(3-0)\hat{k}$$
$$=-\hat{i}+3\hat{j}+3\hat{k}$$
$$\text{Now, }(\bar{u}+\bar{w})\cdot[(\bar{u}\times\bar{r})\times(\bar{r}\times\bar{w})]=\begin{vmatrix}1 & -1 & 0\\ -2 & 2 & 6\\ -1 & 3 & 3\end{vmatrix}$$

= 1(6 – 18) + 1 (-6 + 6) + 0
= -12 + 0 + 0 = -12.

Question 7.
Find the volume of a tetrahedron whose vertices are A(-1, 2, 3) B(3, -2, 1), C (2, 1, 3) and D(-1, -2, 4).
Solution:
The position vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ of the points A, B, C and D w.r.t. the origin are $\bar{a} = -\hat{i}+2\hat{j}+3\hat{k}$, $\bar{b} = 3\hat{i}-2\hat{j}+\hat{k}$, $\bar{c} = 2\hat{i}+\hat{j}+3\hat{k}$ and
$\bar{d} = -\hat{i}-2\hat{j}+4\hat{k}$.

$$\therefore \overline{AB} = \bar{b}-\bar{a} = (3\hat{i}-2\hat{j}+\hat{k})-(-\hat{i}+2\hat{j}+3\hat{k})$$
$$= 4\hat{i}-4\hat{j}-2\hat{k}$$
$$\overline{AC} = \bar{c}-\bar{a} = (2\hat{i}+\hat{j}+3\hat{k})-(-\hat{i}+2\hat{j}+3\hat{k})$$
$$= 3\hat{i}-\hat{j}$$
and $\overline{AD} = \bar{d}-\bar{a} = (-\hat{i}-2\hat{j}+4\hat{k})-(-\hat{i}+2\hat{j}+3\hat{k})$
$$= -4\hat{j}+\hat{k}$$
$$\therefore [\overline{AB}\ \overline{AC}\ \overline{AD}] = \begin{vmatrix} 4 & -4 & -2 \\ 3 & -1 & 0 \\ 0 & -4 & 1 \end{vmatrix}$$
$$= 4(-1+0)+4(3-0)-2(-12+0)$$
$$= -4+12+24 = 32$$
$$\therefore \text{ volume of the tetrahedron} = \frac{1}{6}\left|[\overline{AB}\ \overline{AC}\ \overline{AD}]\right|$$
$$= \frac{1}{6}(32) = \frac{16}{3} \text{ cu units.}$$

Question 8.
If $\bar{a} = \hat{i}+2\hat{j}+3$, $\bar{b} = 3\hat{i}+2\hat{j}$ and $\bar{c} = ,2\hat{i}+\hat{j}+3$ then verify that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$
Solution:
$$\bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 2 & 1 & 3 \end{vmatrix}$$
$$= (6-0)\hat{i}-(9-0)\hat{j}+(3-4)\hat{k}$$
$$= 6\hat{i}-9\hat{j}-\hat{k}$$
$$\therefore \bar{a} \times (\bar{b} \times \bar{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 6 & -9 & -1 \end{vmatrix}$$
$$= (-2+27)\hat{i}-(-1-18)\hat{j}+(-9-12)\hat{k}$$
$$= 25\hat{i}+19\hat{j}-21\hat{k} \quad \dots (1)$$
$$\bar{a} \cdot \bar{c} = (\hat{i}+2\hat{j}+3\hat{k}) \cdot (2\hat{i}+\hat{j}+3\hat{k})$$
$$= (1)(2)+(2)(1)+(3)(3)$$
$$= 2+2+9 = 13$$
$$\therefore (\bar{a} \cdot \bar{c})\bar{b} = 13(3\hat{i}+2\hat{j}) = 39\hat{i}+26\hat{j}$$
Also, $\bar{a} \cdot \bar{b} = (\hat{i}+2\hat{j}+3\hat{k}) \cdot (3\hat{i}+2\hat{j})$
$$= (1)(3)+(2)(2)+(3)(0)$$
$$= 3+4+0 = 7$$
$$\therefore (\bar{a} \cdot \bar{b})\bar{c} = 7(2\hat{i}+\hat{j}+3\hat{k}) = 14\hat{i}+7\hat{j}+21\hat{k}$$
$$\therefore (\bar{a} \cdot \bar{c})\bar{b}-(\bar{a} \cdot \bar{b})\bar{c}$$
$$= (39\hat{i}+26\hat{j})-(14\hat{i}+7\hat{j}+21\hat{k})$$
$$= 25\hat{i}+19\hat{j}-21\hat{k} \quad \dots (2)$$
From (1) and (2), we get
$\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

Question 9.

If, $\bar{a} = \hat{i} - 2\hat{j}$, $\bar{b} = \hat{i} + 2\hat{j}$ and $\bar{c} = 2\hat{i} + \hat{j} - 2$ then find

(i) $\bar{a} \times (\bar{b} \times \bar{c})$

Solution:

$$\bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= (-4 - 0)\hat{i} - (-2 - 0)\hat{j} + (1 - 4)\hat{k}$$

$$= -4\hat{i} + 2\hat{j} - 3\hat{k}$$

$$\therefore \bar{a} \times \left(\bar{b} \times \bar{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ -4 & 2 & -3 \end{vmatrix}$$

$$= (6 - 0)\hat{i} - (-3 - 0)\hat{j} + (2 - 8)\hat{k}$$

$$= 6\hat{i} + 3\hat{j} - 6\hat{k}$$

(ii) $(\bar{a} \times \bar{b}) \times \bar{c}$ Are the results same? Justify.

Solution:

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= (0 - 0)\hat{i} - (0 - 0)\hat{j} + (2 - (-2))\hat{k}$$

$$= 4\hat{k}$$

$$\therefore \left(\bar{a} \times \bar{b}\right) \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 4 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= (0 - 4)\hat{i} - (0 - 8)\hat{j} + (0 - 0)\hat{k}$$

$$= -4\hat{i} + 8\hat{j}$$

$\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$

Question 10.

Show that $\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) = 0$

Solution:

$$\text{LHS} = \bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b})$$

$$= (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c} + (\bar{b} \cdot \bar{a})\bar{c} - (\bar{b} \cdot \bar{c})\bar{a} + (\bar{c} \cdot \bar{b})\bar{a} - (\bar{c} \cdot \bar{a})\bar{b}$$

$$= (\bar{c} \cdot \bar{a})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c} + (\bar{a} \cdot \bar{b})\bar{c} - (\bar{b} \cdot \bar{c})\bar{a} + (\bar{b} \cdot \bar{c})\bar{a} - (\bar{c} \cdot \bar{a})\bar{b}$$

$$\ldots \ [\because \bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}]$$

$$= \bar{0} = \text{RHS}.$$

Maharashtra State Board 12th Maths Solutions Chapter 5 Vectors Miscellaneous Exercise 5

I) Select the correct option from the given alternatives :
Question 1.
If $|\bar{a}|$ = 2, $|\bar{b}|$ = 3 $|\bar{c}|$ = 4 then [$\bar{a}$ + $\bar{b}$ $\bar{b}$ + $\bar{c}$ $\bar{c}$ – $\bar{a}$] is equal to
(A) 24
(B) -24
(C) 0
(D) 48
Solution:
(C) 0

Question 2.
If $|\bar{a}|$ = 3, $|\bar{b}|$ = 4, then the value of λ for which $\bar{a}$ + λ$\bar{b}$ is perpendicular to $\bar{a}$ – λ$\bar{b}$, is

A) $\frac{9}{16}$ B) $\frac{3}{4}$ C) $\frac{3}{2}$ D) $\frac{4}{3}$

Solution:
(b) 34

Question 3.
If sum of two unit vectors is itself a unit vector, then the magnitude of their difference is
(A) 2–√
(B) 3–√
(C) 1
(D) 2
Solution:
(B) 3–√

Question 4.
If $|\bar{a}|$ = 3, $|\bar{b}|$ = 5, $|\bar{c}|$ = 7 and $\bar{a}$ + $\bar{b}$ + $\bar{c}$ = 0, then the angle between $\bar{a}$ and $\bar{b}$ is

A) $\frac{\pi}{2}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{4}$ D) $\frac{\pi}{6}$

Solution:
(b) π3

Question 5.
The volume of tetrahedron whose vertices are (1, -6, 10), (-1, -3, 7), (5, -1, λ) and (7, -4, 7) is 11 cu. units then the value of λ is
(A) 7
(B) π3
(C) 1
(D) 5
Solution:
(A) 7

Question 6.
If α, β, γ are direction angles of a line and α = 60°, β = 45°, the γ =
(A) 30° or 90°
(B) 45° or 60°
(C) 90° or 30°
(D) 60° or 120°
Solution:
(D) 60° or 120°

Question 7.
The distance of the point (3, 4, 5) from Y- axis is
(A) 3
(B) 5
(C) 34−−√
(D) 41−−√

Solution:
(C) 34−−√

Question 8.
The line joining the points (-2, 1, -8) and (a, b, c) is parallel to the line whose direction ratios are 6, 2, 3. The value of a, b, c are
(A) 4, 3, -5
(B) 1, 2, -132
(C) 10, 5, -2
(D) 3, 5, 11
Solution:
(A) 4, 3, -5

Question 9.
If cos α, cos β, cos γ are the direction cosines of a line then the value of $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma$ is
(A) 1
(B) 2
(C) 3
(D) 4
Solution:
(B) 2

Question 10.
If l, m, n are direction cosines of a line then $l\hat{i}+m\hat{j}+n\hat{k}$ is
(A) null vector
(B) the unit vector along the line
(C) any vector along the line
(D) a vector perpendicular to the line
Solution:
(B) the unit vector along the line

Question 11.
If $|\bar{a}| = 3$ and $-1 \le k \le 2$, then $|k\bar{a}|$ lies in the interval
(A) [0, 6]
(B) [-3, 6]
(C) [3, 6]
(D) [1, 2]
Solution:
(A) [0, 6]

Question 12.
Let α, β, γ be distinct real numbers. The points with position vectors $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$, $\beta\hat{i}+\gamma\hat{j}+\alpha\hat{k}$, $\gamma\hat{i}+\alpha\hat{j}+\beta\hat{k}$
(A) are collinear
(B) form an equilateral triangle
(C) form a scalene triangle
(D) form a right angled triangle
Solution:
(B) form an equilateral triangle

Question 13.
Let $\bar{p}$ and $\bar{q}$ be the position vectors of P and Q respectively, with respect to O and $|\bar{p}| = p$, $|\bar{q}| = q$. The points R and S divide PQ internally and externally in the ratio 2 : 3 respectively. If OR and OS are perpendicular then.
(A) $9p^2 = 4q^2$
(B) $4p^2 = 9q^2$
(C) $9p = 4q$
(D) $4p = 9q$
Solution:
(A) $9p^2 = 4q^2$

Question 14.
The 2 vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represents the two sides AB and AC, respectively of a ΔABC. The length of the median through A is
(A) 34√2
(B) 48√2

(C) $18--\sqrt{}$
(D) None of these
Solution:
(A) 34√2

Question 15.
If $\bar{a}$ and $\bar{b}$ are unit vectors, then what is the angle between $\bar{a}$ and $\bar{b}$ for $3-\sqrt{}\bar{a}-\bar{b}$ to be a unit vector ?
(A) 30°
(B) 45°
(C) 60°
(D) 90°
Solution:
(A) 30°

Question 16.
If θ be the angle between any two vectors $\bar{a}$ and $\bar{b}$, then $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$, when θ is equal to
(A) 0
(B) π4
(C) π2
(D) π
Solution:
(B) π4

Question 17.
The value of $\hat{i} \cdot (\hat{j}\times\hat{k})+\hat{j} \cdot (\hat{i}\times\hat{k})+\hat{k} \cdot (\hat{i}\times\hat{j})$
(A) 0
(B) -1
(C) 1
(D) 3
Solution:
(C) 1

Question 18.
Let a, b, c be distinct non-negative numbers. If the vectors $a\hat{i}+a\hat{j}+c\hat{k}$, $\hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ lie in a plane, then c is
(A) The arithmetic mean of a and b
(B) The geometric mean of a and b
(C) The harmonic man of a and b
(D) 0
Solution:
(B) The geometric mean of a and b

Question 19.
Let $\bar{a} = \hat{i}\hat{j}$, $\bar{b} = \hat{j}\hat{k}$, $\bar{c} = \hat{k}\hat{i}$. If $\bar{d}$ is a unit vector such that $\bar{a}.\bar{d}=0=[\bar{b}\,\bar{c}\,\bar{d}]$, then $\bar{d}$ equals.

A) $\pm\frac{\hat{i}+\hat{j}-2\hat{k}}{\sqrt{6}}$ B) $\pm\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ C) $\pm\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ D) $\pm\hat{k}$

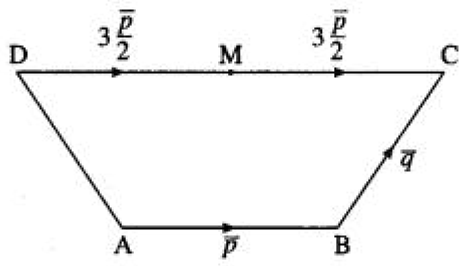
Solution:
(a) $\pm\hat{i}+\hat{j}-2\hat{k}6\sqrt{}$

Question 20.
If $\bar{a}\ \bar{b}\ \bar{c}$ are non coplanar unit vectors such that $\bar{a}\times(\bar{b}\times\bar{c}) = (\bar{b}+\bar{c})2\sqrt{}$ then the angle between $\bar{a}$ and $\bar{b}$ is
(A) 3π4
(B) π4
(C) π2
(D) π
Solution:
(A) 3π4

II Answer the following :
1) ABCD is a trapezium with AB parallel to DC and DC = 3AB. M is the mid-point of DC, $\overline{AB} = \bar{p}$ and $\overline{BC} = \bar{q}$. Find in terms of $\bar{p}$ and $\bar{q}$.

(i) $\overline{AM}$

Solution:



DC is parallel to AB and DC = 3AB.

$\because\ \overline{AB} = \bar{p} \qquad \therefore\ \overline{DC} = 3\bar{p}$

M is the midpoint of DC

$\therefore\ \overline{DM} = \overline{MC} = \frac{1}{2}\overline{DC} = \frac{3}{2}\bar{v}$

(i) $\overline{AM} = \overline{AB} + \overline{BC} + \overline{CM}$

$= \overline{AB} + \overline{BC} - \overline{MC}$

$= \bar{p} + \bar{q} - \frac{3p}{2}$

$= \bar{q} - \frac{1}{2}\bar{p}$

(ii) $\overline{BD}$

Solution:

$\overline{BD} = \overline{BC} + \overline{CD} = \overline{BC} - \overline{DC} = \bar{q} - 3\bar{p}$

(iii) $\overline{MB}$

Solution:

$\overline{MB} = \overline{MC} + \overline{CB} = \overline{MC} - \overline{BC} = \frac{3}{2}\bar{p} - \bar{q}$

(iv) $\overline{DA}$

Solution:

$\overline{DA} = \overline{DC} + \overline{CB} + \overline{BA} = \overline{DC} - \overline{BC} - \overline{AB}$

$= 3\bar{p} - \bar{q} - \bar{p} = 2\bar{p} - \bar{q}$

Question 2.

The points A, B and C have position vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively. The point P is midpoint of AB. Find in terms of $\bar{a}$, $\bar{b}$ and $\bar{c}$ the vector $\overline{PC}$

Solution:

P is the mid-point of AB.

$\therefore \bar{p} = \frac{\bar{a}+\bar{b}}{2}$, where $\bar{p}$ is the position vector of P.

Now, $\overline{PC} = \bar{c} - \bar{p} = \bar{c} - \frac{1}{2}(\bar{a} + \bar{b})$

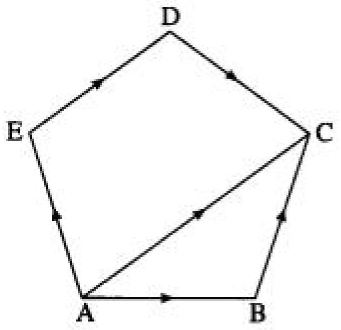
$= -\frac{1}{2}(\bar{a} + \bar{b}) + \bar{c} = -\frac{1}{2}\bar{a} - \frac{1}{2}\bar{b} + \bar{c}.$

Question 3.

In a pentagon ABCDE

Show that $\overline{AB} + \overline{AE} + \overline{BC} + \overline{DC} + \overline{ED} = 2\overline{AC}$

Solution:



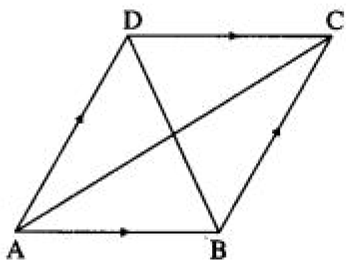
$$\begin{aligned}\text{LHS} &= \overline{AB} + \overline{AE} + \overline{BC} + \overline{DC} + \overline{ED} \\ &= (\overline{AB} + \overline{BC}) + (\overline{AE} + \overline{ED} + \overline{DC}) \\ &= \overline{AC} + \overline{AC} \\ &= 2\overline{AC} = \text{RHS}\end{aligned}$$

Question 4.

If in parallelogram ABCD, diagonal vectors are $\overline{AC} = 2\hat{i}+3\hat{j}+4\hat{k}$ and $\overline{BD} = -6\hat{i}+7\hat{j}-2\hat{k}$, then find the adjacent side vectors $\overline{AB}$ and $\overline{AD}$

Solution:

ABCD is a parallelogram



$\therefore\ \overline{AB} = \overline{DC},\ \overline{AD} = \overline{BC}$

$$\begin{aligned}\overline{AC} &= \overline{AB} + \overline{BC} \\ &= \overline{AB} + \overline{AD} \qquad \ldots (1)\end{aligned}$$

$$\overrightarrow{BD} = \overline{BA} + \overline{AD} = -\overline{AB} + \overline{AD} \qquad \ldots (2)$$

Adding (1) and (2), we get

$$\begin{aligned}2\overline{AD} &= \overline{AC} + \overline{BD} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + (-6\hat{i} + 7\hat{j} - 2\hat{k}) \\ &= -4\hat{i} + 10\hat{j} + 2\hat{k}\end{aligned}$$

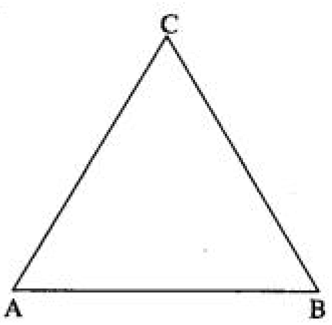
$$\begin{aligned}\therefore\ \overline{AD} &= \frac{1}{2}(-4\hat{i} + 10\hat{j} + 2\hat{k}) \\ &= -2\hat{i} + 5\hat{j} + \hat{k}\end{aligned}$$

$$\begin{aligned}\text{From (1), } \overline{AB} &= \overline{AC} - \overline{AD} \\ &= (2\hat{i} + 3\hat{j} + 4\hat{k}) - (-2\hat{i} + 5\hat{j} + \hat{k}) \\ &= 4\hat{i} - 2\hat{j} + 3\hat{k}.\end{aligned}$$

Question 5.

If two sides of a triangle are $\hat{i}+2\hat{j}$ and $\hat{i}+\hat{k}$, then find the length of the third side.

Solution:



Let ABC be a triangle with $\overline{AB} = \hat{i}+2\hat{j}$, $\overline{BC} = \hat{i}+\hat{k}$.

By triangle law of vectors

$$\overline{AC} = \overline{AB} + \overline{BC}$$
$$= (\hat{i} + 2\hat{j}) + (\hat{i} + \hat{k}) = 2\hat{i} + 2\hat{j} + \hat{k}$$
$$\therefore\ l(AC) = |\overline{AC}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3 \text{ units}$$

Hence, the length of third side is 3 units.

Question 6.
If $|\bar{a}| = |\bar{b}| = 1$ $\bar{a}.\bar{b} = 0$ and $\bar{a} + \bar{b} + \bar{c} = 0$ then find $|\bar{c}|$
Solution:
$\bar{a} + \bar{b} + \bar{c} = 0$
$\therefore -\bar{c} = \bar{a} + \bar{b}$
Taking dot product of both sides with itself, we get

$$(-\bar{c})\cdot(-\bar{c}) = (\bar{a} + \bar{b})\cdot(\bar{a} + \bar{b})$$
$$\therefore\ |\bar{c}|^2 = \bar{a}\cdot(\bar{a} + \bar{b}) + \bar{b}\cdot(\bar{a} + \bar{b})$$
$$= \bar{a}\cdot\bar{a} + \bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{a} + \bar{b}\cdot\bar{b}$$
$$= |\bar{a}|^2 + 0 + 0 + |\bar{b}|^2 \qquad \ldots\ [\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a} = 0]$$
$$= 1 + 1 = 2 \qquad \ldots\ [|\bar{a}| = |\bar{b}| = 1]$$
$$\therefore\ |\bar{c}| = \sqrt{2}.$$

Question 7.
Find the lengths of the sides of the triangle and also determine the type of a triangle.
(i) A(2, -1, 0), B(4, 1, 1,), C(4, -5, 4)
Solution:
The position vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ of the points A, B, C are

$$\bar{a} = 2\hat{i} - \hat{j},\ \bar{b} = 4\hat{i} + \hat{j} + \hat{k},\ \bar{c} = 4\hat{i} - 5\hat{j} + 4\hat{k}$$
$$\overline{AB} = \bar{b} - \bar{a} = (4\hat{i} + \hat{j} + \hat{k}) - (2\hat{i} - \hat{j}) = 2\hat{i} + 2\hat{j} + \hat{k}$$
$$\overline{BC} = \bar{c} - \bar{b} = (4\hat{i} - 5\hat{j} + 4\hat{k}) - (4\hat{i} + \hat{j} + \hat{k}) = -6\hat{j} + 3\hat{k}$$
$$\overline{CA} = \bar{a} - \bar{c} = (2\hat{i} - \hat{j}) - (4\hat{i} - 5\hat{j} + 4\hat{k}) = -2\hat{i} + 4\hat{j} - 4\hat{k}$$
$$\therefore\ l(AB) = |\overline{AB}| = \sqrt{2^2 + 2^2 + 1^2}$$
$$= \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$
$$l(BC) = |\overline{BC}| = \sqrt{(-6)^2 + 3^2}$$
$$= \sqrt{36 + 9} = \sqrt{45} = 3\sqrt{5}$$
$$l(CA) = |\overline{CA}| = \sqrt{(-2)^2 + 4^2 + (-4)^2}$$
$$= \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$
$$\therefore\ [l(AB)]^2 + [l(CA)]^2 = 3^2 + 6^2$$
$$= 9 + 36 = 45 = (3\sqrt{5})^2$$
$$= [l(BC)]^2$$

$\therefore$ Δ ABC is right angled at A.

(ii) L(3, -2, -3), M(7, 0, 1), N (1, 2, 1)
Solution:
The position vectors bar $\bar{a}$, $\bar{b}$, $\bar{c}$ of the points L M, N are

$\bar{a} = 3\hat{i} - 2\hat{j} - 3\hat{k},\ \bar{b} = 7\hat{i} + \hat{k},\ \bar{c} = \hat{i} + 2\hat{j} + \hat{k}$

$\overline{LM} = \bar{b} - \bar{a} = \left(7\hat{i} + \hat{k}\right) - \left(3\hat{i} - 2\hat{j} - 3\hat{k}\right) = 4\hat{i} + 2\hat{j} + 4\hat{k}$

$\overline{MN} = \bar{c} - \bar{b} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) - \left(7\hat{i} + \hat{k}\right) = -6\hat{i} + 2\hat{j}$

$\overline{NL} = \bar{a} - \bar{c} = \left(3\hat{i} - 2\hat{j} - 3\hat{k}\right) - \left(\hat{i} + 2\hat{j} + \hat{k}\right) = 2\hat{i} - 4\hat{j} - 4\hat{k}$

$\therefore$ l(LM) = $\left|\overline{AB}\right| = \sqrt{4^2 + 2^2 + 4^2} = \sqrt{16 + 4 + 16} = \sqrt{36} = 6$

l(MN) = $\left|\overline{MN}\right| = \sqrt{(-6)^2 + 2^2} = \sqrt{36 + 4} = \sqrt{40} = \sqrt{10 \times 4} = 2\sqrt{10}$

l(NL) = $\left|\overline{NL}\right| = \sqrt{(2)^2 + (-4)^2 + (-4)^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$

l(LM) = 6, l(MN) = $2\sqrt{10}$, l(NL) = 6
ΔLMN is sosceles

Question 8.
Find the component form of if a if
(i) It lies in YZ plane and makes 60° with positive Y-axis and $|\bar{a}| = 4$
Solution:
Let α, β, γ be the direction angles of $\bar{a}$
Since $\bar{a}$ lies in YZ-plane, it is perpendicular to X-axis
∴ α = 90°
It is given that β= 60°
∵ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
∴ $\cos^2 90° + \cos^2 60° + \cos^2\gamma = 1$
∴ $0 + \left(\frac{1}{2}\right)^2 + \cos^2\gamma = 1$
∴ $\cos^2\gamma = 1 - \frac{1}{4} = \frac{3}{4}$
∴ $\cos\gamma = \pm\frac{\sqrt{3}}{2}$
Unit vector along a is given by

$$\hat{a} = (\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k} = 0\cdot\hat{i} + \frac{1}{2}\hat{j} + \frac{\sqrt{3}}{2}\hat{k}$$

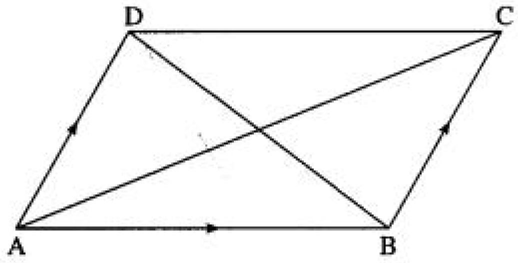
$$= \frac{1}{2}\hat{j} \pm \frac{\sqrt{3}}{2}\hat{k}$$

$$\therefore\ \bar{a} = |\bar{a}|\,\hat{a} = 4\left(\frac{1}{2}\hat{j} \pm \frac{\sqrt{3}}{2}\hat{k}\right) \qquad \ldots [\because |\bar{a}| = 4]$$

$$\therefore\ \bar{a} = 2\hat{j} \pm 2\sqrt{3}\,\hat{k}$$

(ii) It lies in XZ plane and makes 45° with positive Z-axis and $|\bar{a}| = 10$
Solution:

Question 9.
Two sides of a parallelogram are $3\hat{i}+4\hat{j}-5\hat{k}$ and $-2\hat{j}+7\hat{k}$. Find the unit vectors parallel to the diagonals.
Solution:



Let ABCD be a parallelogram with

$\overline{AB} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\overline{BC} = -2\hat{j} + 7\hat{k}$

Then $\overline{AC} = \overline{AB} + \overline{BC} = (3\hat{i} + 4\hat{j} - 5\hat{k}) + (-2\hat{j} + 7\hat{k})$

$$= 3\hat{i} + 2\hat{j} + 2\hat{k}$$

$\therefore\ |\overline{AC}| = \sqrt{3^2 + 2^2 + 2^2} = \sqrt{9+4+4} = \sqrt{17}$

$\therefore$ unit vector along $\overline{AC} = \dfrac{\overline{AC}}{|\overline{AC}|}$

$$= \frac{1}{\sqrt{17}}(3\hat{i} + 2\hat{j} + 2\hat{k})$$

Also, $\overline{BD} = \overline{BA} + \overline{AD} = -\overline{AB} + \overline{BC} = \overline{BC} - \overline{AB}$

$$= (-2\hat{j} + 7\hat{k}) - (3\hat{i} + 4\hat{j} - 5\hat{k})$$

$$= -3\hat{i} - 6\hat{j} + 12\hat{k}$$

$$= 3(-\hat{i} - 2\hat{j} + 4\hat{k})$$

$\therefore\ |\overline{BD}| = 3\sqrt{(-1)^2 + (-2)^2 + 4^2} = 3\sqrt{1+4+16} = 3\sqrt{21}$

$\therefore$ unit vector along $\overline{BD} = \dfrac{\overline{BD}}{|\overline{BD}|}$

$$= \frac{3(-\hat{i} - 2\hat{j} + 4\hat{k})}{3\sqrt{21}}$$

$$= \frac{1}{\sqrt{21}}(-\hat{i} - 2\hat{j} + 4\hat{k})$$

Hence, the unit vectors parallel to the diagonals are

$\dfrac{1}{\sqrt{17}}(3\hat{i} + 2\hat{j} + 2\hat{k})$ and $\dfrac{1}{\sqrt{21}}(-\hat{i} - 2\hat{j} + 4\hat{k})$ respectively.

Question 10.
If D, E, F are the mid-points of the sides BC, CA, AB of a triangle ABC , prove that $\overline{AD} + \overline{BE} + \overline{CF} = 0$

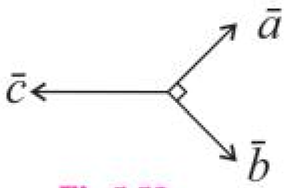


Fig.5.59

Solution:
Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ be the position vectors of the points A, B, C, D, E, F respectively.
Since D, E, F are the midpoints of BC, CA, AB respec-tively, by the midpoint formula

$\bar{d} = \dfrac{\bar{b}+\bar{c}}{2}, \bar{e} = \dfrac{\bar{c}+\bar{a}}{2}, \bar{f} = \dfrac{\bar{a}+\bar{b}}{2}$

$\therefore\ \overline{AD} + \overline{BE} + \overline{CF} = (\bar{d} - \bar{a}) + (\bar{e} - \bar{b}) + (\bar{f} - \bar{c})$

$$= \left(\frac{\bar{b}+\bar{c}}{2} - \bar{a}\right) + \left(\frac{\bar{c}+\bar{a}}{2} - \bar{b}\right) + \left(\frac{\bar{a}+\bar{b}}{2} - \bar{c}\right)$$

$$= \frac{1}{2}\bar{b} + \frac{1}{2}\bar{c} - \bar{a} + \frac{1}{2}\bar{c} + \frac{1}{2}\bar{a} - \bar{b} + \frac{1}{2}\bar{a} + \frac{1}{2}\bar{b} - \bar{c}$$

$$= (\bar{a} + \bar{b} + \bar{c}) - (\bar{a} + \bar{b} + \bar{c}) = \bar{0}.$$

Question 11.
Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point (2, 4)
Solution:
Differentiating $y = x^2$ w.r.t. x, we get = 2x
Slope of tangent at P(2, 4) = $\left(\frac{dy}{dx}\right)_{at\ P(2,4)}$ = 2 × 2 = 4
∴ the equation of tangent at P is
y – 4 = 4(-2)
∴ y = 4x – 4
∴ y = 4x is equation of line parallel to the tangent at P and passing through the origin O.
4x = y, z = 0 ∴ $\frac{x}{1}=\frac{y}{4}$, z = 0
∴ the direction ratios of this line are 1, 4, 0
∴ its direction cosines are

$$\pm\frac{1}{\sqrt{1^2+4^2+0^2}},\ \pm\frac{4}{\sqrt{1^2+4^2+0^2}},\ 0$$

i.e. $\pm\frac{1}{\sqrt{17}},\ \pm\frac{4}{\sqrt{17}},\ 0$

∴ unit vectors parallel to tangent line at P(2, 4) is

$$\pm\frac{1}{\sqrt{17}}(\hat{i}+4\hat{j}).$$

Question 12.
Express the vector $\hat{i}+4\hat{j}-4\hat{k}$ as a linear combination of the vectors $2\hat{i}-\hat{j}+3\hat{k}$, $\hat{i}-2\hat{j}+4\hat{k}$ and $-\hat{i}+3\hat{j}-5\hat{k}$
Solution:

Let $\bar{a} = 2\hat{i} - \hat{j} + 3\hat{k}$,
$\bar{b} = \hat{i} - 2\hat{j} + 4\hat{k}$,
$\bar{c} = -\hat{i} + 3\hat{j} - 5\hat{k}$
$\bar{p} = \hat{i} + 4\hat{j} - 4\hat{k}$

Suppose $\bar{p} = x\bar{a} + y\bar{b} + z\bar{c}$.

Then, $\hat{i} + 4\hat{j} - 4\hat{k} = x\left(2\hat{i} - \hat{j} + 3\hat{k}\right) + y\left(\hat{i} - 2\hat{j} + 4\hat{k}\right) + z\left(-\hat{i} + 3\hat{j} - 5\hat{k}\right)$

$\therefore \hat{i} + 4\hat{j} - 4\hat{k} = (2x + 2y - z)\hat{i} + (-x - 2y + 3z)\hat{j} + (3x + 4y - 5z)\hat{k}$

By equality of vectors,
2x + 2y – z = 1
-x – 2y + 3z = 4
3x + 4y – 5z = -4
We have to solve these equations by using Cramer's Rule.
D = $\begin{vmatrix} 2 & 2 & -1 \\ -1 & -2 & 3 \\ 3 & 4 & -5 \end{vmatrix}$
= 2(10 – 12) – 2(5 – 9) – 1(-4 + 6)
= -4 + 8 – 2
= 2 ≠ 0
D_x = $\begin{vmatrix} 1 & 2 & -1 \\ 4 & -2 & 3 \\ -4 & 4 & -5 \end{vmatrix}$
= 1(10 – 12) – 2(-20 + 12) – 1 (16 – 8)
= -2 + 16 – 8
= 6
D_y = $\begin{vmatrix} 2 & 1 & -1 \\ -1 & 4 & 3 \\ 3 & -4 & -5 \end{vmatrix}$
= 2(-20 + 12) – 1(5 – 9) – 1(4 – 12)
= -16 – 4 – 8
= -28
D_z = $\begin{vmatrix} 2 & 2 & 1 \\ -1 & -2 & 4 \\ 3 & 4 & -4 \end{vmatrix}$
= 2(8 – 16) – 2(4 – 12) + 1(-4 + 6)
= -16 – 16 + 2

= -30

$\therefore x = \frac{D_x}{D} = \frac{6}{2} = 3$

$\therefore y = \frac{D_y}{D} = \frac{-28}{2} = -14$

$\therefore z = \frac{D_z}{D} = \frac{-30}{2} = -15$

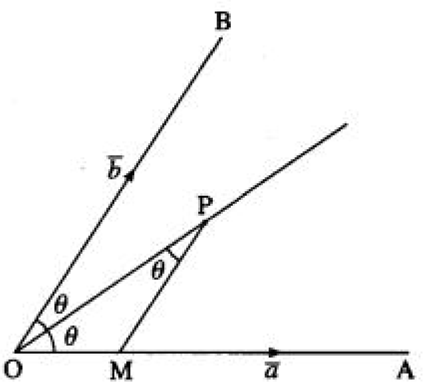
$\therefore \bar{p} = 3\bar{a} - 14\bar{b} - 3\bar{c}$

Question 13.
If $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$ then show that the vector along the angle bisector of angle AOB is given by $\bar{d} = \lambda\left(\frac{\bar{a}}{|\bar{b}|}+\frac{\bar{b}}{|\bar{b}|}\right)$

Question is modified

If $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$ then show that the vector along the angle bisector of ∠AOB is given by $\bar{d} = \lambda\left(\frac{\bar{a}}{|\bar{a}|}+\frac{\bar{b}}{|\bar{b}|}\right)$

Solution:



Choose any point P on the angle bisector of ∠AOB. Draw PM parallel to OB.

∴ ∠OPM = ∠POM

= ∠POB

Hence, OM = MP

∴ OM and MP is the same scalar multiple of unit vectors $\hat{a}$ and $\hat{b}$ along these directions,

where $\hat{a} = \frac{\bar{a}}{|\bar{a}|}$ and $\hat{b} = \frac{\bar{b}}{|\bar{b}|}$

$\therefore \overline{OM} = \lambda\hat{a}$ and $\overline{MP} = \lambda\hat{b}$

$\therefore \overline{OP} = \overline{OM} + \overline{MP} = \lambda\hat{a} + \lambda\hat{b}$

$= \lambda(\hat{a} + \hat{b})$

Hence, the vector along angle bisector of ∠AOB is given by

$\bar{d} = \overline{OP} = \lambda\left(\frac{\bar{a}}{|\bar{a}|} + \frac{\bar{b}}{|\bar{b}|}\right).$

Question 14.
The position vectors f three consecutive vertices of a parallelogram are $\hat{i}+\hat{j}+\hat{k}$, $\hat{i}+3\hat{j}+5\hat{k}$ and $7\hat{i}+9\hat{j}+11\hat{k}$ Find the position vector of the fourth vertex.

Solution:

Let ABCD be a parallelogram.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be the position vectors of the vertices A, B, C, D of the parallelogram, where $\bar{a} = \hat{i} + \hat{j} + \hat{k}$, $\bar{b} = \hat{i} + 3\hat{j} + 5\hat{k}$, $\bar{c} = 7\hat{i} + 9\hat{j} + 11\hat{k}$.

Since ABCD is a parallelogram, $\overline{AB} = \overline{DC}$

$\therefore\ \bar{b} - \bar{a} = \bar{c} - \bar{d}$

$$\therefore\ \bar{d} = \bar{a} + \bar{c} - \bar{b}$$
$$= (\hat{i} + \hat{j} + \hat{k}) + (7\hat{i} + 9\hat{j} + 11\hat{k}) - (\hat{i} + 3\hat{j} + 5\hat{k})$$
$$= 7\hat{i} + 7\hat{j} + 7\hat{k} = 7(\hat{i} + \hat{j} + \hat{k})$$

Hence, the position vector of the fourth vertex is 7(i^ + j^ + k^).

Question 15.
A point P with position vector $\frac{-14i\hat{}+39j\hat{}+28k\hat{}}{5}$ divides the line joining A(-1, 6, 5) and B in the ratio 3 : 2 then find the point B.
Solution:
Let A, B and P have position vectors $\bar{a}$, $\bar{b}$ and $\bar{p}$ respectively.

Then $\bar{a} = -\hat{i} + 6\hat{j} + 5\hat{k}$, $\bar{p} = \dfrac{-14\hat{i} + 39\hat{j} + 28\hat{k}}{5}$

Now, P divides AB internally in the ratio 3 : 2

$$\therefore\ \bar{p} = \frac{3\bar{b} + 2\bar{a}}{5}$$

$$\therefore\ 5\bar{p} = 3\bar{b} + 2\bar{a} \qquad \therefore\ 3\bar{b} = 5\bar{p} - 2\bar{a}$$

$$\therefore\ 3\bar{b} = 5\left(\frac{-14\hat{i} + 39\hat{j} + 28\hat{k}}{5}\right) - 2(-\hat{i} + 6\hat{j} + 5\hat{k})$$
$$= -14\hat{i} + 39\hat{j} + 28\hat{k} + 2\hat{i} - 12\hat{j} - 10\hat{k}$$
$$= -12\hat{i} + 27\hat{j} + 18\hat{k}$$
$$\therefore\ \bar{b} = -4\hat{i} + 9\hat{j} + 6\hat{k}$$

∴ coordinates of B are (-4, 9, 6).

Question 16.
Prove that the sum of the three vectors determined by the medians of a triangle directed from the vertices is zero.
Solution:
Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of the vertices A, B and C respectively.
Then we know that the position vector of the centroid O of the triangle is $\frac{\vec{a}+\vec{b}+\vec{c}}{3}$
Therefore sum of the three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, is

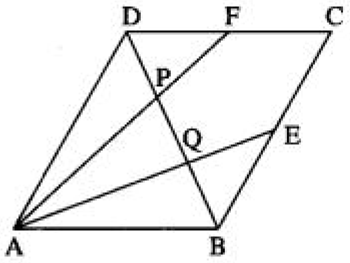
$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \vec{a} - \left(\frac{\vec{a} + \vec{b} + \vec{c}}{3}\right) + \vec{b} - \left(\frac{\vec{a} + \vec{b} + \vec{c}}{3}\right) + \vec{c} - \left(\frac{\vec{a} + \vec{b} + \vec{c}}{3}\right)$$

$$= \left(\vec{a} + \vec{b} + \vec{c}\right) - 3\left(\frac{\vec{a} + \vec{b} + \vec{c}}{3}\right)$$

$$= \vec{0}$$

Hence, Sum os the three vectors determined by the medians of a triangle directed from the vertices is zero.

Question 17.
ABCD is a parallelogram E, F are the mid points of BC and CD respectively. AE, AF meet the diagonal BD at Q and P respectively. Show that P and Q trisect DB.
Solution:



Let A, B, C, D, E, F, P, Q have position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{p}, \bar{q}$ respectively.

$\because$ ABCD is a parallelogram

$\therefore \overline{AB} = \overline{DC}$

$\therefore \bar{b} - \bar{a} = \bar{c} - \bar{d}$

$\therefore \bar{c} = \bar{b} + \bar{d} - \bar{a}$ ... (1)

E is the midpoint of BC

$\therefore \bar{e} = \frac{\bar{b}+\bar{c}}{2} \quad \therefore 2\bar{e} = \bar{b} + \bar{c}$ ... (2)

F is the mid-point of CD

$\therefore \bar{f} = \frac{\bar{c}+\bar{d}}{2} \quad \therefore 2\bar{f} = \bar{c} + \bar{d}$ ... (3)

$2\bar{e} = \bar{b} + \bar{c}$ ... [By (2)]

$= \bar{b} + (\bar{b} + \bar{d} - \bar{a})$ ... [By (1)]

$\therefore 2\bar{e} + \bar{a} = 2\bar{b} + \bar{d}$

$\therefore \frac{2\bar{e}+\bar{a}}{2+1} = \frac{2\bar{b}+\bar{d}}{2+1}$

LHS is the position vector of the point on AE and RHS is the position vector of the point on DB. But AE and DB meet at Q.

$\therefore \bar{q} = \frac{2\bar{b}+\bar{d}}{2+1}$

$\therefore$ Q divides DB in the ratio 2 : 1 ... (4)

$2\bar{f} = \bar{c} + \bar{d}$ ... [By (3)]

$= (\bar{b} + \bar{d} - \bar{a}) + \bar{d}$ ... [By (1)]

$\therefore 2\bar{f} + \bar{a} = 2\bar{d} + \bar{b}$

$\therefore \frac{\bar{a}+2\bar{f}}{1+2} = \frac{\bar{b}+2\bar{d}}{1+2}$

LHS is the position vector of the point on AF and RHS is the position vector of the point on DB.
But AF and DB meet at P.

$\therefore \bar{p} = \frac{\bar{b}+2\bar{d}}{1+2}$

$\therefore$ P divides DB in the ratio 1 : 2 ... (5)

From (4) and (5), if follows that P and Q trisect DB.

Question 18.

If aBC is a triangle whose orthocenter is P and the circumcenter is Q, then prove that $\overline{PA} + \overline{PC} + \overline{PB} = 2\,\overline{PQ}$

Solution:

Let G be the centroid of the Δ ABC.

Let A, B, C, G, Q have position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{g}, \bar{q}$ w.r.t. P. We know that Q, G, P are collinear and G divides segment QP internally in the ratio 1 : 2.

$\therefore \bar{g} = \frac{1 \cdot \bar{p} + 2\bar{q}}{1+2} = \frac{2\bar{q}}{3}$ $[\because \bar{p} = \bar{0}]$

$\therefore 3\bar{g} = 2\bar{q}$

$\therefore \frac{3(\bar{a}+\bar{b}+\bar{c})}{3} = 2\bar{q}$

$\therefore \bar{a} + \bar{b} + \bar{c} = 2\bar{q}$

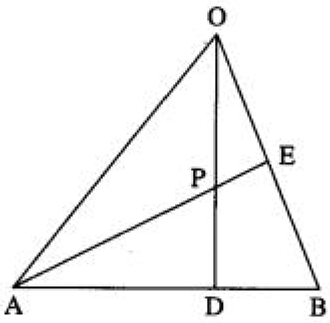
$\therefore \overline{PA} + \overline{PB} + \overline{PC} = 2\overline{PQ}$.

Question 19.
If P is orthocenter, Q is circumcenter and G is centroid of a triangle ABC, then prove that $\overline{QP} = 3\overline{QG}$
Solution:
Let $\bar{p}$ and $\bar{g}$ be the position vectors of P and G w.r.t. the circumcentre Q.
i.e. $\overline{QP}$ = p and $\overline{QG}$ = g.
We know that Q, G, P are collinear and G divides segment QP internally in the ratio 1 : 2
∴ by section formula for internal division,

$$\bar{g} = \frac{1 \cdot \bar{p} + 2\bar{q}}{1+2} = \frac{\bar{p}}{3} \quad \ldots [\because \bar{q} = 0]$$

$$\therefore \bar{p} = 3\bar{g}$$

$$\therefore \overline{QP} = 3\overline{QG}.$$

Question 20.
In a triangle OAB, E is the midpoint of BO and D is a point on AB such that AD: DB = 2:1. If OD and AE intersect at P, determine the ratio OP:PD using vector methods.
Solution:


Let A, B, D, E, P have position vectors $\bar{a}, \bar{b}, \bar{d}, \bar{e}, \bar{p}$ respectively w.r.t. O.
∵ AD : DB = 2 : 1.
∴ D divides AB internally in the ratio 2 : 1.
Using section formula for internal division, we get

$$\bar{d} = \frac{2\bar{b} + \bar{a}}{2+1}$$

$$\therefore 3\bar{d} = 2\bar{b} + \bar{a} \quad \ldots (1)$$

Since E is the midpoint of OB, $\bar{e} = \overline{OE} = \frac{1}{2}\overline{OB} = \frac{\bar{b}}{2}$

$$\therefore \bar{b} = 2\bar{e} \quad \ldots (2)$$

∴ from (1),

$$3\bar{d} = 2(2\bar{e}) + \bar{a} \quad \ldots [\text{By } (2)]$$

$$= 4\bar{e} + \bar{a}$$

$$\therefore \frac{3\bar{d} + 2 \cdot \bar{0}}{3+2} = \frac{4\bar{e} + \bar{a}}{4+1}$$

LHS is the position vector of the point which divides OD internally in the ratio 3 : 2.
RHS is the position vector of the point which divides AE internally in the ratio 4 : 1.
But OD and AE intersect at P
∴ P divides OD internally in the ratio 3 : 2.
Hence, OP : PD = 3 : 2.

Question 21.
Dot-product of a vector with vectors $3\hat{i} - 5\hat{k}, 2\hat{i} + 7\hat{j}$ and $\hat{i} + \hat{j} + \hat{k}$ are respectively -1, 6 and 5. Find the vector.
Solution:
Let $\bar{a} = 3\hat{i} - 5\hat{k}$, $\bar{b} = 2\hat{i} + 7\hat{j}$, $\bar{c} = \hat{i} + \hat{j} + \hat{k}$
Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the required vector.
Then $\bar{r} \cdot \bar{a} = -1$, $\bar{r} \cdot \bar{b} = 6$, $\bar{r} \cdot \bar{c} = 5$

$$\therefore (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (3\hat{i} - 5\hat{k}) = -1,$$
$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 7\hat{j}) = 6 \text{ and}$$
$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 5$$

∴ 3x – 5z= -1 ... (1)
∴ 2x + 7y = 6 ... (2)
∴ x + y + z = 5 ... (3)

From (3), z = 5 – x – y
Substituting this value of z in (1), we get
∴ 3x – 5(5 – x – y)= -1
∴ 8x + 5y = 24 … (4)
Multiplying (2) by 4 and subtracting from (4), we get
8x + 5y – 4(2x + 7y) = 24 – 6 × 4
∴ -23y = 0 ∴ y = 0
Substituting y = 0 in (2), we get
∴ 2x = 6 ∴ x = 3
Substituting x = 3 in (1), we get
∴ 3(3) – 5z = -1
∴ 5z = -10 ∴ z = 2
∴ $\bar{r} = 3\hat{i} + 0 \cdot \hat{j} + 2\hat{k} = 3\hat{i} + 2\hat{k}$
Hence, the required vector is $3\hat{i} + 2\hat{k}$

Question 22.
If $\bar{a}$, $\bar{b}$, $\bar{c}$ are unit vectors such that $\bar{a} + \bar{b} + \bar{c} = 0$, then find the value of $\bar{a}.\bar{b} + \bar{b}.\bar{c} + \bar{c}.\bar{a}$
Solution:
$\bar{a}$, $\bar{b}$, $\bar{c}$ are unit vectors

$\therefore\ |\bar{a}| = |\bar{b}| = |\bar{c}| = 1.$

Also, $\bar{a}\cdot\bar{a} = \bar{b}\cdot\bar{b} = \bar{c}\cdot\bar{c} = 1$

Now, $\bar{a} + \bar{b} + \bar{c} = \bar{0}$ … (1)

Taking scalar product of both sides with $\bar{a}$, we get

$\bar{a}\cdot(\bar{a} + \bar{b} + \bar{c}) = \bar{a}\cdot\bar{0}$

$\therefore\ \bar{a}\cdot\bar{a} + \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c} = 0$

$\therefore\ \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c} = -\bar{a}\cdot\bar{a} = -1$ … (2)

Similarly taking scalar product of both sides of (1) with $\bar{b}$ and $\bar{c}$, we get

$\bar{b}\cdot\bar{a} + \bar{b}\cdot\bar{c} = -1$ … (3)

$\bar{c}\cdot\bar{a} + \bar{c}\cdot\bar{b} = -1$ … (4)

Adding (2), (3), (4) and using the fact that scalar product commutative, we get

$2(\bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{c} + \bar{c}\cdot\bar{a}) = -3$

$\therefore\ \bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{c} + \bar{c}\cdot\bar{a} = -\frac{3}{2}$

Question 23.
If a parallelogram is constructed on the vectors $\bar{a} = 3\bar{p} - \bar{q}$, $\bar{b} = \bar{p} + 3\bar{q}$ and $|\bar{p}| = |\bar{q}| = 2$ and angle between $\bar{p}$ and $\bar{q}$ is $\frac{\pi}{3}$ show that the ratio of the lengths of the sides is $\sqrt{7} : \sqrt{13}$
Solution:

$|\bar{p}| = |\bar{q}| = 2$ and angle between $\bar{p}$ and $\bar{q}$ *is* $\frac{\pi}{3}$.

$\therefore\ \bar{p}\cdot\bar{q} = |\bar{p}||\bar{q}| \cos\frac{\pi}{3} = 2 \times 2 \times \frac{1}{2} = 2$

Now, $\bar{a} = 3\bar{p} - \bar{q}$

$$\begin{aligned}\therefore\ |\bar{a}|^2 &= |3\bar{p} - \bar{q}|^2 \\ &= (3\bar{p} - \bar{q})\cdot(3\bar{p} - \bar{q}) \\ &= 3\bar{p}\cdot(3\bar{p} - \bar{q}) - \bar{q}\cdot(3\bar{p} - \bar{q}) \\ &= 9\bar{p}\cdot\bar{p} - 3\bar{p}\cdot\bar{q} - 3\bar{q}\cdot\bar{p} + \bar{q}\cdot\bar{q}\end{aligned}$$

$= 9|\bar{p}|^2 - 6\bar{p}\cdot\bar{q} + |\bar{q}|^2$ ... $[\because \bar{q}\cdot\bar{p} = \bar{p}\cdot\bar{q}]$

$= 9 \times 4 - 6 \times 2 + 4$ ... $[\because \bar{p}\cdot\bar{q} = 2]$

$= 28$

$\therefore |\bar{a}| = \sqrt{28}$

Also $\bar{b} = \bar{p} + 3\bar{q}$

$\therefore |\bar{b}|^2 = |\bar{p} + 3\bar{q}|^2$

$= (\bar{p} + 3\bar{q})\cdot(\bar{p} + 3\bar{q})$

$= \bar{p}\cdot(\bar{p} + 3\bar{q}) + 3\bar{q}\cdot(\bar{p} + 3\bar{q})$

$= \bar{p}\cdot\bar{p} + 3\bar{p}\cdot\bar{q} + 3\bar{q}\cdot\bar{p} + 9\bar{q}\cdot\bar{q}$ ... $[\because \bar{p}\cdot\bar{q} = \bar{q}\cdot\bar{p}]$

$= |\bar{p}|^2 + 3\bar{p}\cdot\bar{q} + 3\bar{p}\cdot\bar{q} + 9|\bar{q}|^2$

$= 4 + 12 + 36$ ... $[\because \bar{p}\cdot\bar{q} = 2]$

$= 52$

$\therefore |\bar{b}| = \sqrt{52}$

Ratio of lengths of the sides

$$= \frac{|\bar{a}|}{|\bar{b}|} = \frac{\sqrt{28}}{\sqrt{52}} = \frac{2\sqrt{7}}{2\sqrt{13}} = \frac{\sqrt{7}}{\sqrt{13}}.$$

Hence, the ratio of the lengths of the sides is $\sqrt{7} : \sqrt{13}$.

Question 24.
Express the vector $\bar{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as a sum of two vectors such that one is parallel to the vector $\bar{b} = 3\hat{i} + \hat{k}$ and other is perpendicular to $\bar{b}$.
Solution:

Let $\bar{a} = \bar{c} + \bar{d}$, where $\bar{c}$ is parallel to $\bar{b}$ and $\bar{d}$ is perpendicular to $\bar{b}$.

Since, $\bar{c}$ is parallel to $\bar{b}$, $\bar{c} = m\bar{b}$, where m is a scalar.

$\therefore \bar{c} = m(3\hat{i} + \hat{k})$

i.e. $\bar{c} = 3m\hat{i} + m\hat{k}$

Let $\bar{d} = x\hat{i} + y\hat{j} + z\hat{k}$

Since, $\bar{d}$ is perpendicular to $\bar{b} = 3\hat{i} + \hat{k}$, $\bar{d}\cdot\bar{b} = 0$

$\therefore (x\hat{i} + y\hat{j} + z\hat{k})\cdot(3\hat{i} + \hat{k}) = 0$

$\therefore 3x + z = 0 \qquad \therefore z = -3x$

$\therefore \bar{d} = x\hat{i} + y\hat{k} - 3x\hat{k}$

Now, $\bar{a} = \bar{c} + \bar{d}$ gives

$\therefore 5\hat{i} - 2\hat{j} + 5\hat{k} = (3m\hat{i} + m\hat{k}) + (x\hat{i} + y\hat{j} - 3x\hat{k})$

$= (3m + x)\hat{i} + y\hat{j} + (m - 3x)\hat{k}$

By equality of vectors
3m + x = 5 ... (1)
y = -2
and m – 3x = 5
From (1) and (2)
3m + x = m – 3x
∴ 2m = -4x m ∴ m = -2x
Substituting m = -2x in (1), we get
∴ -6x + x = 5 ∴ -5x = 5 ∴ x = -1
∴ m = -2x = 2

$\therefore \bar{c} = 6\hat{i} + 2\hat{k}$ is parallel to $\bar{b}$ and

$\bar{d} = -\hat{i} - 2\hat{j} + 3\hat{k}$ is perpendicular to $\bar{b}$

Hence, $\bar{a} = \bar{c} + \bar{d}$, where $\bar{c} = 6\hat{i} + 2\hat{k}$ and $\bar{d} = -\hat{i} - 2\hat{j} + 3\hat{k}$.

Question 25.
Find two unit vectors each of which makes equal angles with $\bar{u}$, $\bar{v}$ and $\bar{w}$. $\bar{u}=2\hat{i}+\hat{j}-2\hat{k}$, $\bar{v}=\hat{i}+2\hat{j}-2\hat{k}$ and $\bar{w}=2\hat{i}-2\hat{j}+\hat{k}$
Solution:
Let $\bar{r}=x\hat{i}+y\hat{j}+z\hat{k}$ be the unit vector which makes angle θ with each of the vectors

Then $|\bar{r}|=1$

Also, $\bar{u}=2\hat{i}+\hat{j}-2\hat{k}$, $\bar{v}=\hat{i}+2\hat{j}-2\hat{k}$, $\bar{w}=2\hat{i}-2\hat{j}+\hat{k}$

$|\bar{u}|=\sqrt{2^2+1^2+(-2)^2}=\sqrt{4+1+4}=\sqrt{9}=3$

$|\bar{v}|=\sqrt{1^2+2^2+(-2)^2}=\sqrt{1+4+4}=\sqrt{9}=3$

$|\bar{w}|=\sqrt{2^2+(-2)^2+1^2}=\sqrt{4+4+1}=\sqrt{9}=3$

Angle between $\bar{r}$ and $\bar{u}$ is θ.

$$\therefore \cos\theta=\frac{\bar{r}\cdot\bar{u}}{|\bar{r}||\bar{u}|}$$

$$=\frac{(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}+\hat{j}-2\hat{k})}{1\times 3}$$

$$=\frac{2x+y-2z}{3} \quad \ldots (1)$$

Also, the angle between $\bar{r}$ and $\bar{v}$ and between $\bar{r}$ and $\bar{w}$ is θ.

$$\therefore \cos\theta=\frac{\bar{r}\cdot\bar{v}}{|\bar{r}||\bar{v}|}$$

$$=\frac{(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+2\hat{j}-2\hat{k})}{1\times 3}$$

$$=\frac{x+y-2z}{3} \quad \ldots (2)$$

and $\cos\theta=\dfrac{\bar{r}\cdot\bar{w}}{|\bar{r}||\bar{w}|}$

$$=\frac{(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}-2\hat{j}+\hat{k})}{1\times 3}$$

$$= \frac{2x - 2y + z}{3} \quad \dots (3)$$

From (1) and (2), we get

$$\frac{2x + y - 2z}{3} = \frac{x + 2y - 2z}{3}$$

$\therefore\ 2x + y - 2z = x + 2y - 2z$

$\therefore\ x = y$

From (2) and (3), we get

$$\frac{x + 2y - 2z}{3} = \frac{2x - 2y + z}{3}$$

$\therefore\ x + 2y - 2z = 2x - 2y + z.$

$\therefore\ 3y = 3z \quad \dots [\because x = y]$

$\therefore\ y = z$

$\therefore\ x = y = z$

$\therefore\ \bar{r} = x\hat{i} + y\hat{j} + z\hat{k} = x\hat{i} + x\hat{j} + x\hat{k}$

$\therefore\ |\bar{r}| = \sqrt{x^2 + x^2 + x^2} = 1$

$\therefore\ x^2 + x^2 + x^2 = 1 \qquad \therefore\ 3x^2 = 1$

$\therefore\ x^2 = \frac{1}{3} \qquad \therefore\ x = \pm \frac{1}{\sqrt{3}}$

$\therefore\ \bar{r} = \pm \frac{1}{\sqrt{3}}\hat{i} \pm \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{3}}\hat{k}$

$= \pm \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

Hence, the required unit vectors are $\pm \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$.

Question 26.
Find the acute angles between the curves at their points of intersection. $y = x^2$, $y = x^3$
Solution:
The angle between the curves is same as the angle between their tangents at the points of intersection. We find the points of intersection of $y = x^2$ ... (1)
and $y = x^3$ ... (2)
From (1) and (2)
$x^3 = x^2$
$\therefore x^3 - x^2 = 0$
$\therefore x^2(x - 1) = 0$
$\therefore x = 0$ or $x = 1$
When x = 0, y = 0.
When x = 1, y = 1.

For $y = x^2$, $\frac{dy}{dx} = 2x$

For $y = x^3$, $\frac{dy}{dx} = 3x^2$

Angle at O = (0, 0)

Slope of tangent to $y = x^2$ at O

$= \left(\frac{dy}{dx}\right)_{\text{at } O(0,\,0)} = 2 \times 0 = 0$

$\therefore$ equation of tangent to $y = x^2$ at O is $y = 0$.

Slope of tangent to $y = x^3$ at O $= \left(\frac{dy}{dx}\right)_{\text{at } O(0,\,0)} = 3 \times 0 = 0$

$\therefore$ equation of tangent to $y = x^3$ at P is $y = 0$.
$\therefore$ the tangents to both curves at (0, 0) are $y = 0$
$\therefore$ angle between them is 0.
Angle at P = (1, 1)

Slope of tangent to y = x2 at P

$$= \left(\frac{dy}{dx}\right)_{\text{at } P(1,\,1)} = 2 \times 1 = 2$$

$\therefore$ equation of tangent to $y = x^2$ at P is $y - 1 = 2(x - 1)$

$\therefore\ y = 2x - 1$

Slope of tangent to $y = x^3$ at $P = \left(\frac{dy}{dx}\right)_{\text{at } P(1,\,1)} = 3 \times 1^2 = 3$

∴ equation of tangent to y = x3 at P is y – 1 = 3(x – 1) y = 3x – 2
We have to find angle between y = 2x – 1 and y = 3x – 2
Lines through origin parallel to these tagents are y = 2x and y = 3x
∴ x1=y2 and x1=y3
These lines lie in XY-plane.
∴ the direction ratios of these lines are 1, 2, 0 and 1, 3, 0.
The angle θ between them is given by

$$\cos\theta = \frac{(1)(1) + (2)(3) + (0)(0)}{\sqrt{1^2 + 2^2 + 0^2}\sqrt{1^2 + 3^2 + 0^2}}$$

$$= \frac{1 + 6 + 0}{\sqrt{5}\sqrt{10}} = \frac{7}{\sqrt{50}} = \frac{7}{5\sqrt{2}}$$

$$\therefore\ \theta = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right)$$

Hence, the required angles are 0 and $\cos^{-1}\left(\frac{7}{5\sqrt{2}}\right)$.

Question 27.
Find the direction cosines and direction angles of the vector.
(i) 2i^+j^+2k^
Solution:
Let a¯ = 2i^+j^+2k^

$$|\bar{a}| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4 + 1 + 4} = \sqrt{9} = 3$$

$\therefore$ unit vector along $\bar{a}$

$$= \hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

$\therefore$ its direction cosines are $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$.

If α, β, γ are the direction angles, then $\cos\alpha = \frac{2}{3}$, $\cos\beta = \frac{1}{3}$,

$\cos\gamma = \frac{2}{3}$

$$\therefore\ \alpha = \cos^{-1}\left(\frac{2}{3}\right),\ \beta = \cos^{-1}\left(\frac{1}{3}\right),\ \gamma = \cos^{-1}\left(\frac{2}{3}\right)$$

Hence, direction cosines are $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$ and direction angles are $\cos^{-1}\left(\frac{2}{3}\right), \cos^{-1}\left(\frac{1}{3}\right), \cos^{-1}\left(\frac{2}{3}\right)$.

(ii) (1/2)i^+j^+k^
Solution:

Question 28.
Let b¯ = 4i^+3j^ and c¯ be two vectors perpendicular to each other in the XY-plane. Find vectors in the same plane having projection 1 and 2 along b¯ and c¯, respectively, are given y.
Solution:

$\bar{b} = 4\hat{i}+3\hat{j}$

$\therefore\ |\bar{b}| = \sqrt{4^2+3^2} = \sqrt{16+9} = 5$

Let $\bar{c} = m\hat{i} + n\hat{j}$ be perpendicular to $\bar{b}$

Then $\bar{b}\cdot\bar{c} = 0 \qquad \therefore\ (4\hat{i}+3\hat{j})\cdot(m\hat{i}+n\hat{j}) = 0$

$\therefore\ 4m+3n = 0 \qquad \therefore\ n = -\dfrac{4m}{3}$

$\therefore\ \bar{c} = m\hat{i} - \dfrac{4m}{3}\hat{j} = \dfrac{m}{3}(3\hat{i}-4\hat{j})$

$\therefore\ \bar{c} = p(3\hat{i}-4\hat{j}) \qquad \ldots \left[p = \dfrac{m}{3}\right]$

$\therefore\ |\bar{c}| = p\sqrt{3^2+(-4)^2} = p\sqrt{9+16} = 5p$

Let $\bar{d} = x\hat{i} + y\hat{j}$ be the vector having projections 1 and 2 along $\bar{b}$ and $\bar{c}$.

$\therefore\ \dfrac{\bar{b}\cdot\bar{d}}{|\bar{b}|} = 1$

$\therefore\ \dfrac{(4\hat{i}+3\hat{j})\cdot(x\hat{i}+y\hat{j})}{5} = 1$

$\therefore\ 4x+3y = 5 \qquad \ldots (1)$

Also, $\dfrac{\bar{c}\cdot\bar{d}}{|\bar{c}|} = 2$

$\therefore\ \dfrac{(3p\hat{i}-4p\hat{j})\cdot(x\hat{i}+y\hat{j})}{5p} = 2$

$\therefore\ 3px-4py = 10p$

$\therefore\ 3x-4y = 10 \qquad \ldots (2)$

From (1), $3y = 5-4x \qquad \therefore\ y = \dfrac{5-4x}{3}$

Substituting for y in (2), we get

$3x - 4\left(\dfrac{5-4x}{3}\right) = 10$

$\therefore\ 9x-20+16x = 30$

$\therefore\ 25x = 50 \qquad \therefore\ x = 2$

$y = \dfrac{5-4x}{3} = \dfrac{5-4(2)}{3} = -1$

$\therefore\ \bar{d} = 2\hat{i}-\hat{j}$

Hence, the required vector is $2\hat{i}-\hat{j}$.

Question 29.
Show that no line in space can make angle $\pi 6$ and $\pi 4$ with X- axis and Y-axis.
Solution:
Let, if possible, a line in space make angles $\pi 6$ and $\pi 4$ with X-axis and Y-axis.

$\therefore\ \alpha = \dfrac{\pi}{6},\ \beta = \dfrac{\pi}{4}$

Let the line make angle γ with Z-axis.

$\because\ \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\therefore\ \cos^2\left(\dfrac{\pi}{6}\right) + \cos^2\left(\dfrac{\pi}{4}\right) + \cos^2\gamma = 1$

$\therefore\ \left(\dfrac{\sqrt{3}}{2}\right)^2 + \left(\dfrac{1}{\sqrt{2}}\right)^2 + \cos^2\gamma = 1$

∴ $\cos^2\gamma = 1 - \frac{3}{4} - \frac{1}{2} = -\frac{1}{4}$

This is not possible, because cos γ is real

∴ $\cos^2\gamma$ cannot be negative.

Hence, there is no line in space which makes angles $\frac{\pi}{6}$ and $\frac{\pi}{4}$ with X-axis and Y-axis.

Question 30.

Find the angle between the lines whose direction cosines are given by the equation 6mn – 2nl + 5lm = 0, 3l + m + 5n = 0

Solution:

Given 6mn – 2nl + 5lm = o

3l + m +5n = 0.

From (2), m = 3l – 5n

Putting the value of m in equation (1), we get,

⇒ 6n(-3l – 5n) – 2nl + 5l(-3l – 5n) = 0

⇒ -18nl- 30n – 2nl- 15l2 – 25nl = 0

⇒ – $30n^2$ – 45nl – $15l^2$ = 0

⇒ $2n^2$ + 3nl + l^2 = 0

⇒ $2n^2$ + 2nl + nl + l^2 = 0

⇒ (2n + l) (n + l) = 0

∴ 2n + l = 0 OR n + l = 0

∴ l = -2n OR l = -n

∴ l = -2n

From (2), 3l + m + 5n = 0

∴ -6n + m + 5n = 0

∴ m = n

i.e. (-2n, n, n) = (-2, 1, 1)

∴ l = -n

∴ -3n + m + 5n = 0

∴ m = -2n

i.e. (-n, -2n, n) = (1, 2, -1)

(a_1, b_1, c_1) = (-2, 1, 1) and (a_2, b_3, c_3) = (1, 2, -1)

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}.\sqrt{a_2^2 + b^2{}_{-}2 + c_2^2}}\right|$$

$$= \left|\frac{(2)(1) + (-1)(2) + (-1)(-1)}{\sqrt{(2)^2 + 1^2 + 1^2}.\sqrt{1^2 + 2^2 + (1)^2}}\right|$$

$$= \left|\frac{2 - 2 + 1}{\sqrt{6}.\sqrt{6}}\right|$$

$$= \left|-\frac{1}{6}\right| = \frac{1}{6}$$

$$\theta = \cos^{-1}\left(\frac{1}{6}\right)$$

Question 31.

If Q is the foot of the perpendicular from P(2, 4, 3) on the line joining the points A(1, 2, 4) and B(3, 4, 5), find coordinates of Q.

Solution:

Let PQ be the perpendicular drawn from point P(2, 4, 3) to the line joining the points A(1, 2, 4) and B (3, 4, 5).

Let Q divides AB internally in the ratio λ : 1

$$\therefore Q \equiv \left(\frac{3\lambda + 1}{\lambda + 1}, \frac{44\lambda + 2}{\lambda + 1}, \frac{5\lambda + 4}{\lambda + 1}\right) \quad ....(i)$$

Direction ratios of PQ are

$$\frac{3\lambda + 1}{\lambda + 1} - 2, \frac{4\lambda + 2}{\lambda + 1} - 4, \frac{5\lambda + 4}{\lambda + 1} - 3$$

i.e., $\frac{\lambda - 1}{\lambda + 1}, \frac{-2}{\lambda + 1}, \frac{2\lambda + 1}{\lambda + 1}$

Now, direction ratios of AB are, 3 – 1, 4 – 2, 5 – 4 i.e., 2, 2, 1.

Since PQ is perpendicular to AB,

$$2\left(\frac{\lambda-1}{\lambda+1}\right)+\frac{2(-2)}{\lambda+1}+1\left(\frac{2\lambda+1}{\lambda+1}\right)=0$$

$$\therefore \frac{2\lambda-2-4+2\lambda+1}{\lambda+1}=0$$

$\therefore 4\lambda - 5 = 0$

$\therefore 4\lambda = 5$

$\therefore \lambda = \frac{5}{4}$

Putting $\lambda = \frac{5}{4}$ in (i),

Coordinates of Q are,

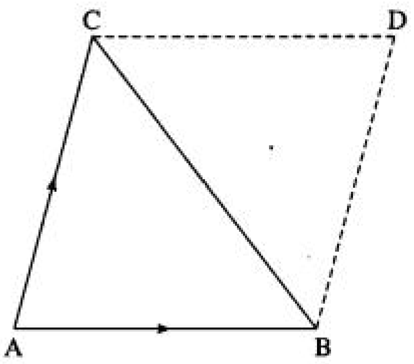
$$\frac{3\left(\frac{5}{4}\right)+1}{\left(\frac{5}{4}\right)+1}=\frac{19}{19}$$

$$\frac{\left(\frac{5}{4}\right)+2}{\left(\frac{5}{4}\right)+1}=\frac{28}{9}$$

$$\frac{5\left(\frac{5}{4}\right)+4}{\left(\frac{5}{4}\right)+1}=\frac{41}{9}$$

$$\therefore Q \equiv \left(\frac{19}{9}, \frac{28}{9}, \frac{41}{9}\right)$$

Question 32.
Show that the area of a triangle ABC, the position vectors of whose vertices are a, b and c is $12[\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}]$
Question is modified.
Show that the area of a triangle ABC, the position vectors of whose vertices are $\bar{a}$, $\bar{b}$ and $\bar{c}$ is $12[\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a}]$.
Solution:



Consider the triangle ABC.
Complete the parallelogram ABDC.
Vector area of ΔABC

$=\frac{1}{2}$(vector area of parallelogram ABDC)

$=\frac{1}{2}(\overline{AB} \times \overline{AC})$

$=\frac{1}{2}[(\bar{b}-\bar{a}) \times (\bar{c}-\bar{a})]$ ... $[\because \overline{AB}=\bar{b}-\bar{a}$ and $\overline{AC}=\bar{c}-\bar{a}]$

$=\frac{1}{2}[\bar{b} \times \bar{c} - \bar{b} \times \bar{a} - \bar{a} \times \bar{c} + \bar{a} \times \bar{a}]$

$=\frac{1}{2}[\bar{b} \times \bar{c} + \bar{a} \times \bar{b} + \bar{c} \times \bar{a} + \bar{0}]$

$=\frac{1}{2}[\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a}]$

Question 33.
Find a unit vector perpendicular to the plane containing the point (a, 0, 0), (0, b, 0), and (0, 0, c). What is the area of the triangle with

these vertices?
Solution:

The position vectors $\bar{p}, \bar{q}, \bar{r}$ of the points A$(a, 0, 0)$, B$(0, b, 0)$, C$(0, 0, c)$ are

$\bar{p} = a\hat{i}, \bar{q} = b\hat{j}, \bar{r} = c\hat{k}$

$\overline{AB} = \bar{q} - \bar{p} = b\hat{j} - a\hat{i} = -a\hat{i} + b\hat{j}$

$\overline{BC} = \bar{r} - \bar{q} = c\hat{k} - b\hat{j} = -b\hat{j} + c\hat{k}$

$$\overline{AB} \times \overline{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ 0 & -b & c \end{vmatrix}$$

$$= (bc - 0)\hat{i} - (-ac - 0)\hat{j} + (ab - 0)\hat{k}$$

$$= bc\hat{i} + ac\hat{j} + ab\hat{k}$$

$$|\overline{AB} \times \overline{BC}| = \sqrt{(bc)^2 + (ac)^2 + (ab)^2}$$

$$= \sqrt{b^2c^2 + a^2c^2 + a^2b^2}$$

$\overline{AB} \times \overline{BC}$ is perpendicular to the plane containing A, B, C.

$\therefore$ the required unit vector

$$= \frac{\overline{AB} \times \overline{BC}}{|\overline{AB} \times \overline{BC}|} = \frac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}$$

Area of $\triangle ABC = \frac{1}{2}|\overline{AB} \times \overline{BC}|$

$$= \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2} \text{ sq units.}$$

Question 34.
State whether each expression is meaningful. If not, explain why ? If so, state whether it is a vector or a scalar.
(a) $\bar{a} \cdot (\bar{b} \times \bar{c})$
Solution:
This is the scalar product of two vectors. Therefore, this expression is meaningful and it is a scalar.

(b) $\bar{a} \times (\bar{b} \cdot \bar{c})$
Solution:
This expression is meaningless because $\bar{a}$ is a vector, $\bar{b} \cdot \bar{c}$ is a scalar and vector product of vector and scalar is not defined.

(c) $\bar{a} \times (\bar{b} \times \bar{c})$
Solution:
This is vector product of two vectors. Therefore, this expression is meaningful and it is a vector.

(d) $\bar{a} \cdot (\bar{b} \cdot \bar{c})$
Solution:
This is meaningless because $\bar{a}$ is a vector, $\bar{b} \cdot \bar{c}$ is a scalar and scalar product of vector and scalar is not defined.

(e) $(\bar{a} \cdot \bar{b}) \times (\bar{c} \cdot \bar{d})$
Solution:
This is meaningless because $\bar{a} \cdot \bar{b}, \bar{c} \cdot \bar{d}$ are scalars and cross product of two scalars is not defined.

(f) $(\bar{a} \times \bar{b}) \cdot (\bar{c} \times \bar{d})$
Solution:
This is scalar product of two vectors. Therefore, this expression is meaningful and it is a scalar.

(g) $(\bar{a} \cdot \bar{b}) \cdot \bar{c}$
Solution:
This is meaningless because $\bar{c}$ is a vector, $\bar{a} \cdot \bar{b}$ scalar and scalar product of vector and scalar is not defined.

(h) $(\bar{a} \cdot \bar{b})\bar{c}$
Solution:
This is a scalar multiplication of a vector. Therefore, this expression is meaningful and it is a vector.

(i) $(|\bar{a}|)(\bar{b} \cdot \bar{c})$
Solution:
This is the product of two scalars. Therefore, this expression is meaningful and it is a scalar.

(j) $\bar{a} \cdot (\bar{b}+\bar{c})$
Solution:
This is the scalar product of two vectors. Therefore, this expression is meaningful and it is a scalar.

(k) $\bar{a} \cdot \bar{b}+\bar{c}$
Solution:
This is the sum of scalar and vector which is not defined. Therefore, this expression is meaningless.

(l) $|\bar{a}| \cdot (\bar{b}+\bar{c})$
Solution:
This is meaningless because $\bar{a}$ is a vector, $\bar{b}+\bar{c}$ is a scalar and the scalar product of vector and scalar is not defined.

Question 35.
Show that, for any vectors $\bar{a}, \bar{b}, \bar{c}$
$(\bar{a}+\bar{b}+\bar{c})\times\bar{c}+(\bar{a}+\bar{b}+\bar{c})\times\bar{b}+(\bar{b}+\bar{c})\times\bar{a}=2\bar{a}\times\bar{c}$
Question is modified.
For any vectors $\bar{a}, \bar{b}, \bar{c}$ show that
$(\bar{a}+\bar{b}+\bar{c})\times\bar{c}+(\bar{a}+\bar{b}+\bar{c})\times\bar{b}+(\bar{b}-\bar{c})\times\bar{a}=2\bar{a}\times\bar{c}$.
Solution:

$$\text{LHS} = (\bar{a}+\bar{b}+\bar{c}) \times \bar{c} + (\bar{a}+\bar{b}+\bar{c}) \times \bar{b} + (\bar{b}-\bar{c}) \times \bar{a}$$
$$= \bar{a}\times\bar{c}+\bar{b}\times\bar{c}+\bar{c}\times\bar{c}+\bar{a}\times\bar{b}+\bar{b}\times\bar{b}+\bar{c}\times\bar{b}+\bar{b}\times\bar{a}-\bar{c}\times\bar{a}$$
$$= \bar{a}\times\bar{c}+\bar{b}\times\bar{c}+\bar{0}+\bar{a}\times\bar{b}+\bar{0}-\bar{b}\times\bar{c}-\bar{a}\times\bar{b}+\bar{a}\times\bar{c}$$
$$\ldots [\because \bar{a}\times\bar{b} = -\bar{b}\times\bar{a}]$$
$$= 2\bar{a}\times\bar{c} = \text{RHS}.$$

Question 36.
Suppose that $\bar{a}$ = 0.
(a) If $\bar{a} \cdot \bar{b}=\bar{a} \cdot \bar{c}$ then is $\bar{b}=\bar{c}$?
Solution:
No. Take $\bar{b}=\hat{i}, \bar{c}=\hat{j}$.
Then $\bar{a}\cdot\bar{b}=\bar{a}\cdot\bar{c}=0$, but $\bar{b}\neq\bar{c}$

(b) If $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}$ then is $\bar{b}=\bar{c}$?
Solution:
No. Take $\bar{b}=\hat{i}, \bar{c}=\hat{j}$
Then $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}=\bar{0}$, but $\bar{b}\neq\bar{c}$.

(c) If $\bar{a} \cdot \bar{b}=\bar{a} \cdot \bar{c}$ and $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}$ then is $\bar{b}=\bar{c}$?
Solution:
No. Take $\bar{a}=\bar{0}, \bar{b}=\hat{i}, \bar{c}=\hat{j}$
Then $\bar{a}\cdot\bar{b}=\bar{a}\cdot\bar{c}=0$ and $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}=\bar{0}$
but $\bar{b}\neq\bar{c}$.

Question 37.
If A(3, 2, -1), B(-2, 2, -3), C(3, 5, -2), D(-2, 5, -4) then
(i) verify that the points are the vertices of a parallelogram and
Solution:

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ be the position vectors of A, B, C, D respectively w.r.t. the origin O.

Then $\bar{a} = 3\hat{i} + 2\hat{j} - \hat{k}$, $\bar{b} = -2\hat{i} + 2\hat{j} - 3\hat{k}$,
$\bar{c} = 3\hat{i} + 5\hat{j} - 2\hat{k}$, $\bar{d} = -2\hat{i} + 5\hat{j} - 4\hat{k}$.

(i) $\therefore\ \overline{AB} = \bar{b} - \bar{a} = (-2\hat{i} + 2\hat{j} - 3\hat{k}) - (3\hat{i} + 2\hat{j} - \hat{k})$
$= -5\hat{i} - 2\hat{k}$

$\overline{DC} = \bar{c} - \bar{d} = (3\hat{i} + 5\hat{j} - 2\hat{k}) - (-2\hat{i} + 5\hat{j} - 4\hat{k})$
$= 5\hat{i} + 2\hat{k} = -(-5\hat{i} - 2\hat{k})$

$\therefore\ \overline{DC} = -\overline{AB}$

$\therefore\ \overline{DC}$ is scalar multiple of $\overline{AB}$

$\therefore\ \overline{DC}$ is parallel to $\overline{AB}$

Also, $|\overline{DC}| = \sqrt{5^2 + 2^2} = \sqrt{25 + 4} = \sqrt{29}$

and $|\overline{AB}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{25 + 4} = \sqrt{29}$

$\therefore\ |\overline{DC}| = |\overline{AB}|$

$\therefore\ l(AB) = l(DC)$

∴ opposite sides AB and DC of ABCD are parallel and equal.
∴ ABCD is a parallelogram.

(ii) find its area.
Solution:

$\overline{AB} = \bar{b} - \bar{a} = (-2\hat{i} + 2\hat{j} - 3\hat{k}) - (3\hat{i} + 2\hat{j} - \hat{k})$
$= -5\hat{i} - 2\hat{k}$

$\overline{AD} = \bar{d} - \bar{a} = (-2\hat{i} + 5\hat{j} - 4\hat{k}) - (3\hat{i} + 2\hat{j} - \hat{k})$
$= -5\hat{i} + 3\hat{j} - 3\hat{k}$

$$\therefore\ \overline{AB} \times \overline{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 0 & -2 \\ -5 & 3 & -3 \end{vmatrix}$$
$$= (0 + 6)\hat{i} - (15 - 10)\hat{j} + (-15 + 0)\hat{k}$$
$$= 6\hat{i} - 5\hat{j} - 15\hat{k}$$

$\therefore$ area of parallelogram $= |\overline{AB} \times \overline{AD}|$
$= \sqrt{6^2 + (-5)^2 + (-15)^2} = \sqrt{36 + 25 + 225}$
$= \sqrt{286}$ sq units.

Question 38.
Let A, B, C, D be any four points in space. Prove that $\left|\overline{AB} \times \overline{CD} + \overline{BC} \times \overline{AD} + \overline{CA} \times \overline{BD}\right|$ = 4 (area of ΔABC)
Solution:
Let A, B, C, D have position vectors $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ respectively.

Consider $\left|\overline{AB} \times \overline{CD} + \overline{BC} \times \overline{AD} + \overline{CA} \times \overline{BD}\right|$

$$= (\bar{b} - \bar{a}) \times (\bar{d} - \bar{c}) + (\bar{c} - \bar{b}) \times (\bar{d} - \bar{a}) + (\bar{a} - \bar{c}) \times (\bar{d} - \bar{b})$$

$$= \bar{b} \times (\bar{d} - \bar{c}) - \bar{a} \times (\bar{d} - \bar{c}) + \bar{c} \times (\bar{d} - \bar{a}) - \bar{b} \times (\bar{d} - \bar{a}) + \bar{a} \times (\bar{d} - \bar{b}) - \bar{c} \times (\bar{d} - \bar{b})$$

$$= \bar{b} \times \bar{d} - \bar{b} \times \bar{c} - \bar{a} \times \bar{d} + \bar{a} \times \bar{c} + \bar{c} \times \bar{d} - \bar{c} \times \bar{a} - \bar{b} \times \bar{d} + \bar{b} \times \bar{a} + \bar{a} \times \bar{d} - \bar{a} \times \bar{b} - \bar{c} \times \bar{d} + \bar{c} \times \bar{b}$$

$$= \bar{b} \times \bar{d} - \bar{b} \times \bar{c} - \bar{a} \times \bar{d} - \bar{c} \times \bar{a} + \bar{c} \times \bar{d} - \bar{c} \times \bar{a} - \bar{b} \times \bar{d} - \bar{a} \times \bar{b} + \bar{a} \times \bar{d} - \bar{a} \times \bar{b} - \bar{c} \times \bar{d} - \bar{b} \times \bar{c}$$

$$\ldots [\because \bar{p} \times \bar{q} = -\bar{q} \times \bar{p}]$$

$$= -2(\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a})$$

$$\therefore |\overline{AB} \times \overline{CD} + \overline{BC} \times \overline{AD} + \overline{CA} \times \overline{BD}|$$

$$= |-2(\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a})|$$

$$= 4\left[\frac{1}{2}|\bar{a} \times \bar{b} + \bar{b} \times \bar{c} + \bar{c} \times \bar{a}|\right]$$

$$= 4\,(\text{area of } \triangle ABC).$$

Question 39.
Let $\hat{a}, \hat{b}, \hat{c}$ be unit vectors such that $\hat{a} \cdot \hat{b} = \hat{a} \cdot \hat{c} = 0$ and the angle between $\hat{b}$ and $\hat{c}$ be $\frac{\pi}{6}$.
Prove that $\hat{a} = \pm 2(\hat{b} \times \hat{c})$
Solution:
$\hat{a} \cdot \hat{b} = \hat{a} \cdot \hat{c} = 0$
$\therefore$ $\hat{a}$ is perpendicular to $\hat{b}$ and $\hat{c}$ both
$\therefore$ $\hat{a}$ is parallel to $\hat{b} \times \hat{c}$
$\therefore$ $\hat{a}$ = m($\hat{b} \times \hat{c}$), m is a scalar.

$$\therefore |\hat{a}| = |m|\,|\hat{b} \times \hat{c}|$$

$$\therefore |\hat{a}| = |m|\,|\hat{b}||\hat{c}| \sin\frac{\pi}{6}$$

$$\therefore 1 = |m| \times 1 \times 1 \times \frac{1}{2} = \frac{|m|}{2} \quad \ldots [\because |\hat{a}| = |\hat{b}| = |\hat{c}| = 1]$$

$$\therefore 2 = |m|$$

$$\therefore m = \pm 2$$

$$\therefore \hat{a} = \pm 2(\hat{b} \times \hat{c}).$$

Question 40.
Find the value of 'a' so that the volume of parallelopiped a formed by $\hat{i} + \hat{j} + \hat{k} + a\hat{k}$ aand $a\hat{j} + \hat{k}$ becomes minimum.
Question is modified.
Find the value of 'a' so that the volume of parallelopiped formed by $\hat{i} + a\hat{j} + \hat{k}, \hat{j} + a\hat{k}$ and $a\hat{i} + \hat{k}$ becomes minimum.
Solution:
Let $\bar{p} = \hat{i} + a\hat{j} + \hat{k}$, $\bar{q} = \hat{j} + a\hat{k}$, $\bar{r} = a\hat{i} + \hat{k}$
Let V be the volume of the parallelopiped formed by $\bar{p}, \bar{q}, \bar{r}$.

Then $V = [\bar{p}\ \bar{q}\ \bar{r}]$

$$= \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix}$$

$$= 1(1-0) - a(0-a^2) + 1(0-a)$$

$$= 1 + a^3 - a$$

$$\therefore\ \frac{dV}{da} = \frac{d}{da}(1 + a^3 - a)$$

$$= 0 + 3a^2 - 1 = 3a^2 - 1$$

and $\dfrac{d^2V}{da^2} = \dfrac{d}{da}(3a^2 - 1)$

$$= 3 \times 2a - 0 = 6a$$

For maximum and minimum V, $\dfrac{dV}{da} = 0$

$\therefore\ 3a^2 - 1 = 0$

$\therefore\ a^2 = \dfrac{1}{3} \qquad \therefore\ a = \pm\dfrac{1}{\sqrt{3}}$

Now, $\left(\dfrac{d^2V}{da^2}\right)_{\text{at } a = \frac{1}{\sqrt{3}}} = 6\left(\dfrac{1}{\sqrt{3}}\right) = 2\sqrt{3} > 0$

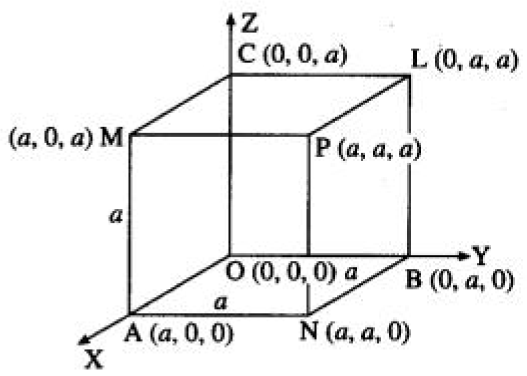
$\therefore\ V$ is minimum when $a = \dfrac{1}{\sqrt{3}}$

Also, $\left(\dfrac{d^2V}{da^2}\right)_{\text{at } a = -\frac{1}{\sqrt{3}}} = 6\left(-\dfrac{1}{\sqrt{3}}\right) = -2\sqrt{3} < 0$

$\therefore\ V$ is maximum when $a = -\dfrac{1}{\sqrt{3}}$

Hence, $a = \dfrac{1}{\sqrt{3}}$

Question 41.
Find the volume of the parallelepiped spanned by the diagonals of the three faces of a cube of side a that meet at one vertex of the cube.
Solution:



Take origin O as one vertex of the cube and OA, OB and OC as the positive directions of the X-axis, the Y-axis and the Z-axis respectively.
Here, the sides of the cube are
OA = OB = OC = a
∴ the coordinates of all the vertices of the cube will be
O = (0, 0, 0) A = (a, 0, 0)
B = (0, a, 0) C = (0, 0, a)
N = (a, a, 0) L = (0, a, a)
M = (a, 0, a) P = (a, a, a)

ON, OL, OM are the three diagonals which meet at the vertex O

$$\overline{ON}=\bar{a}\hat{i}+\bar{a}\hat{j},\quad \overline{OL}=\bar{a}\hat{j}+a\hat{k}$$

$$\overline{OM}=\bar{a}\hat{i}+a\hat{k}.$$

$$[\overline{ON}\ \overline{OL}\ \overline{OM}]=\begin{vmatrix} a & a & 0\\ 0 & a & a\\ a & 0 & a\end{vmatrix}$$

$$=a(a^2-0)-a(0-a^2)+0$$

$$=a^3+a^3=2a^3$$

$$\therefore\ \text{required volume}=[\overline{ON}\ \overline{OL}\ \overline{OM}]$$

$$=2a^3\ \text{cubic units.}$$

Question 42.

If $\bar{a},\bar{b},\bar{c}$ are three non-coplanar vectors, then show that $\frac{\bar{a}\cdot(\bar{b}\times\bar{c})}{(\bar{c}\times\bar{a})\cdot\bar{b}}+\frac{\bar{b}\cdot(\bar{a}\times\bar{c})}{(\bar{c}\times\bar{a})\cdot\bar{b}}=0$

Solution:

$$\text{LHS}=\frac{\bar{a}\cdot(\bar{b}\times\bar{c})}{(\bar{c}\times\bar{a})\cdot\bar{b}}+\frac{\bar{b}\cdot(\bar{a}\times\bar{c})}{(\bar{c}\times\bar{a})\cdot\bar{b}}$$

$$=\frac{[\bar{a}\ \bar{b}\ \bar{c}]}{[\bar{c}\ \bar{a}\ \bar{b}]}+\frac{[\bar{b}\ \bar{a}\ \bar{c}]}{[\bar{c}\ \bar{a}\ \bar{b}]}$$

$$=\frac{[\bar{a}\ \bar{b}\ \bar{c}]}{[\bar{a}\ \bar{b}\ \bar{c}]}-\frac{[\bar{a}\ \bar{b}\ \bar{c}]}{[\bar{a}\ \bar{b}\ \bar{c}]}$$

$$=0=\text{RHS.}$$

Question 43.

Prove that $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})=\begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}$

Solution:

$$\text{Let } \bar{a}\times\bar{b}=\bar{m}$$

$$\text{LHS}=(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})$$

$$=\bar{m}\cdot(\bar{c}\times\bar{d})$$

$$=(\bar{m}\times\bar{c})\cdot\bar{d}$$

$$=[(\bar{a}\times\bar{b})\times\bar{c}]\cdot\bar{d}$$

$$=[(\bar{c}\cdot\bar{a})\bar{b}-(\bar{c}\cdot\bar{b})\bar{a}]\cdot\bar{d}$$

$$=(\bar{c}\cdot\bar{a})(\bar{b}\cdot\bar{d})-(\bar{c}\cdot\bar{b})(\bar{a}\cdot\bar{d})$$

$$=\begin{vmatrix}\bar{c}\cdot\bar{a} & \bar{c}\cdot\bar{b}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}$$

$$=\begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}\qquad \ldots\ [\text{Dot product is commutative}]$$

$$=\text{RHS.}$$

Question 44.

Find the volume of a parallelopiped whose coterminus edges are represented by the vector $\hat{j}+\hat{k}$, $\hat{i}+\hat{k}$ and $\hat{i}+\hat{j}$. Also find volume of tetrahedron having these coterminous edges.

Solution:

Let $\bar{a}=\hat{j}+\hat{k}$, $\bar{b}=\hat{i}+\hat{k}$ and $\bar{c}=\hat{i}+\hat{j}$ be the co-terminus edges of a parallelopiped.

Then volume of the parallelopiped = $[\bar{a}\ \bar{b}\ \bar{c}]$

$=\begin{vmatrix} 0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{vmatrix}$

= 0(0 – 1) – 1(0 – 1) + 1(1 – 0)

= 0 + 1 + 1 = 2cu units.

Also, volume of tetrahedron = $\frac{1}{6}[\bar{a}\ \bar{b}\ \bar{c}]$

= $\frac{1}{6}(2)=\frac{1}{3}$ cubic units.

Question 45.
Using properties of scalar triple product, prove that $[\bar{a}+\bar{b}\ \bar{b}+\bar{c}\ \bar{c}+\bar{a}]=2[\bar{a}\ \bar{b}\ \bar{c}]$.
Solution:

$$\begin{aligned}
\text{LHS} &= [\bar{a}+\bar{b}\ \ \bar{b}+\bar{c}\ \ \bar{c}+\bar{a}] \\
&= (\bar{a}+\bar{b})\cdot\{(\bar{b}+\bar{c})\times(\bar{c}+\bar{a})\} \\
&= (\bar{a}+\bar{b})\cdot\{\bar{b}\times\bar{c}+\bar{b}\times\bar{a}+\bar{c}\times\bar{c}+\bar{c}\times\bar{a}\} \\
&= (\bar{a}+\bar{b})\cdot(\bar{b}\times\bar{c}+\bar{b}\times\bar{a}+\bar{c}\times\bar{a}) \ \ \dots\ [\because\ \bar{c}\times\bar{c}=\bar{0}] \\
&= \bar{a}\cdot\{(\bar{b}\times\bar{c})+(\bar{b}\times\bar{a})+(\bar{c}\times\bar{a})\}+ \\
&\qquad \bar{b}\cdot\{(\bar{b}\times\bar{c})+(\bar{b}\times\bar{a})+(\bar{c}\times\bar{a})\} \\
&= \bar{a}\cdot(\bar{b}\times\bar{c})+\bar{a}\cdot(\bar{b}\times\bar{a})+\bar{a}\cdot(\bar{c}\times\bar{a})+ \\
&\qquad \bar{b}\cdot(\bar{b}\times\bar{c})+\bar{b}\cdot(\bar{b}\times\bar{a})+\bar{b}\cdot(\bar{c}\times\bar{a}) \\
&= [\bar{a}\ \bar{b}\ \bar{c}]+[\bar{a}\ \bar{b}\ \bar{a}]+[\bar{a}\ \bar{c}\ \bar{a}]+ \\
&\qquad [\bar{b}\ \bar{b}\ \bar{c}]+[\bar{b}\ \bar{b}\ \bar{a}]+[\bar{b}\ \bar{c}\ \bar{a}] \\
&= [\bar{a}\ \bar{b}\ \bar{c}]+0+0+0+0+[\bar{a}\ \bar{b}\ \bar{c}] \\
&= 2[\bar{a}\ \bar{b}\ \bar{c}] \\
&= \text{RHS.}
\end{aligned}$$

Question 46.
If four points A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) and D($\bar{d}$) are coplanar then show that $[\bar{a}\ \bar{b}\ \bar{d}]+[\bar{b}\ \bar{c}\ \bar{d}]+[\bar{c}\ \bar{a}\ \bar{d}]=[\bar{a}\ \bar{b}\ \bar{c}]$
Solution:

$\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ are the position vectors of the points A, B, C and D respectively.

$\therefore\ \overline{AB}=\bar{b}-\bar{a},\ \overline{AC}=\bar{c}-\bar{a},\ \overline{AD}=\bar{d}-\bar{a}$

The points A, B, C, D are coplanar.

$\therefore$ the vectors $\overline{AB}, \overline{AC}, \overline{AD}$ are coplanar.

$\therefore\ [\overline{AB}\ \ \overline{AC}\ \ \overline{AD}]=0$

$\therefore\ [\bar{b}-\bar{a}\ \ \bar{c}-\bar{a}\ \ \bar{d}-\bar{a}]=0$

$\therefore\ (\bar{b}-\bar{a})\cdot[(\bar{c}-\bar{a})\times(\bar{d}-\bar{a})]=0$

$\therefore\ (\bar{b}-\bar{a})\cdot(\bar{c}\times\bar{d}-\bar{c}\times\bar{a}-\bar{a}\times\bar{d}+\bar{a}\times\bar{a})=0,$ where $\bar{a}\times\bar{a}=\bar{0}$

$\therefore\ (\bar{b}-\bar{a})\cdot(\bar{c}\times\bar{d}-\bar{c}\times\bar{a}-\bar{a}\times\bar{d})=0$

$\therefore\ \bar{b}\cdot(\bar{c}\times\bar{d})-\bar{b}\cdot(\bar{c}\times\bar{a})-\bar{b}\cdot(\bar{a}\times\bar{d})-\bar{a}\cdot(\bar{c}\times\bar{d})+\bar{a}\cdot(\bar{c}\times\bar{a})+\bar{a}\cdot(\bar{a}\times\bar{d})=0 \quad \dots (1)$

Now, $\bar{a}\cdot(\bar{c}\times\bar{a})=0,\ \bar{a}\cdot(\bar{a}\times\bar{d})=0,$

$-\bar{b}\cdot(\bar{a}\times\bar{d})=\bar{b}\cdot(\bar{d}\times\bar{a})=[\bar{b}\ \bar{d}\ \bar{a}]=[\bar{a}\ \bar{b}\ \bar{d}],$

$-\bar{a}\cdot(\bar{c}\times\bar{d})=\bar{a}\cdot(\bar{d}\times\bar{c})=[\bar{a}\ \bar{d}\ \bar{c}]=[\bar{c}\ \bar{a}\ \bar{d}]$

Also, $\bar{b}\cdot(\bar{c}\times\bar{d})=[\bar{b}\ \bar{c}\ \bar{d}],$

$-\bar{b}\cdot(\bar{c}\times\bar{a})=-\bar{a}\cdot(\bar{b}\times\bar{c})=-[\bar{a}\ \bar{b}\ \bar{c}]$

$\therefore$ from (1)

$[\bar{b}\ \bar{c}\ \bar{d}]-[\bar{a}\ \bar{b}\ \bar{c}]+[\bar{a}\ \bar{b}\ \bar{d}]+[\bar{c}\ \bar{a}\ \bar{d}]+0+0=0$

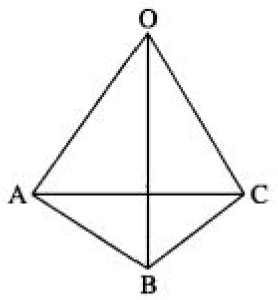
$\therefore\ [\bar{a}\ \bar{b}\ \bar{d}]+[\bar{b}\ \bar{c}\ \bar{d}]+[\bar{c}\ \bar{a}\ \bar{d}]=[\bar{a}\ \bar{b}\ \bar{c}].$

Question 47.
If $\bar{a}$ $\bar{b}$ and $\bar{c}$ are three non coplanar vectors, then $(\bar{a}+\bar{b}+\bar{c})\cdot[(\bar{a}+\bar{b})\times(\bar{a}+\bar{c})]=-[\bar{a}\ \bar{b}\ \bar{c}]$.

Solution:

$$\text{LHS} = (\bar{a}+\bar{b}+\bar{c})\cdot[(\bar{a}+\bar{b})\times(\bar{a}+\bar{c})]$$
$$= (\bar{a}+\bar{b}+\bar{c})\cdot[\bar{a}\times(\bar{a}+\bar{c})+\bar{b}\times(\bar{a}+\bar{c})]$$
$$= (\bar{a}+\bar{b}+\bar{c})[\bar{a}\times\bar{a}+\bar{a}\times\bar{c}+\bar{b}\times\bar{a}+\bar{b}\times\bar{c}]$$
$$= \bar{a}\cdot(\bar{a}\times\bar{c})+\bar{a}\cdot(\bar{b}\times\bar{a})\times\bar{a}\cdot(\bar{b}\times\bar{c})+$$
$$\bar{b}\cdot(\bar{a}\times\bar{c})+\bar{b}\cdot(\bar{b}\times\bar{a})+\bar{b}\cdot(\bar{b}\times\bar{c})+$$
$$\bar{c}\cdot(\bar{a}\times\bar{c})+\bar{c}\cdot(\bar{b}\times\bar{a})\times\bar{c}\cdot(\bar{b}\times\bar{c}) \quad \ldots [\because \bar{a}\times\bar{a}=\bar{0}]$$
$$= [\bar{a}\ \bar{a}\ \bar{c}]+[\bar{a}\ \bar{b}\ \bar{a}]+[\bar{a}\ \bar{b}\ \bar{c}]+[\bar{b}\ \bar{a}\ \bar{c}]+[\bar{b}\ \bar{b}\ \bar{a}]+$$
$$[\bar{b}\ \bar{b}\ \bar{c}]+[\bar{c}\ \bar{a}\ \bar{c}]+[\bar{c}\ \bar{b}\ \bar{a}]+[\bar{c}\ \bar{b}\ \bar{c}]$$
$$= 0+0+[\bar{a}\ \bar{b}\ \bar{c}]-[\bar{a}\ \bar{b}\ \bar{c}]+0+0+0-[\bar{a}\ \bar{b}\ \bar{c}]+0$$
$$= -[\bar{a}\ \bar{b}\ \bar{c}] = \text{RHS}.$$

Question 48.
If in a tetrahedron, edges in each of the two pairs of opposite edges are perpendicular, then show that the edges in the third pair are also perpendicular.
Solution:



Let O-ABC be a tetrahedron. Then o
(OA, BC), (OB, CA) and (OC, AB) are the pair of opposite edges.

$\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$, $\overline{OC} = \bar{c}$,

$\overline{AB} = \bar{b} - \bar{a}$, $\overline{BC} = \bar{c} - \bar{b}$ and $\overline{CA} = \bar{a} - \bar{c}$.

Now, suppose the pairs (OA, BC) and (OB, CA) are perpendicular to each other.

Then $\overline{OA}\cdot\overline{BC} = 0$, i.e. $\bar{a}\cdot(\bar{c}-\bar{b}) = 0$

$\therefore\ \bar{a}\cdot\bar{c} - \bar{a}\cdot\bar{b} = 0 \quad \ldots (1)$

and $\overline{OB}\cdot\overline{CA} = 0$, i.e. $\bar{b}\cdot(\bar{a}-\bar{c}) = 0$

$\therefore\ \bar{b}\cdot\bar{a} - \bar{b}\cdot\bar{c} = 0$

$\therefore\ \bar{a}\cdot\bar{b} - \bar{b}\cdot\bar{c} = 0 \quad \ldots (2)$

Adding (1) and (2), we get

$\bar{a}\cdot\bar{c} - \bar{b}\cdot\bar{c} = 0$

$\therefore\ \bar{c}\cdot\bar{b} - \bar{c}\cdot\bar{a} = 0$

i.e. $\bar{c}\cdot(\bar{b}-\bar{a}) = 0$

$\therefore\ \overline{OC}\cdot\overline{AB} = 0$

Take O as the origin of reference and let $\bar{a}$ $\bar{b}$ and
∴ the third pair (OC, AB) is perpendicular.