Maharashtra State Board 11th Maths Solutions Chapter 8 Continuity Ex 8.1

Question 1.
Examine the continuity of
(i) f(x) = $x^3 + 2x^2 - x - 2$ at x = -2
Solution:
Given, f(x) = $x^3 + 2x^2 - x - 2$
f(x) is a polynomial function and hence it is continuous for all x ∈ R.
∴ f(x) is continuous at x = -2.

(ii) f(x) = sin x, for x ≤ $\frac{\pi}{4}$
= cos x, for x > $\frac{\pi}{4}$, at x = $\frac{\pi}{4}$
Solution:

$$f(x) = \sin x; \quad x \leq \frac{\pi}{4}$$
$$= \cos x; \quad x > \frac{\pi}{4}$$
$$\lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^-} (\sin x)$$
$$= \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}^+} (\cos x)$$
$$= \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$
$$\text{Also, } f\left(\frac{\pi}{4}\right) = \sin \frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}}$$
$$\therefore \lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^+} f(x) = f\left(\frac{\pi}{4}\right)$$
$$\therefore f(x) \text{ is continuous at } x = \frac{\pi}{4}.$$

(iii) f(x) = $\frac{x^2-9}{x-3}$, for x ≠ 3
= 8 for x = 3, at x = 3.
Solution:
f(3) = 8 ....(given)

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3}$$
$$= \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3}$$
$$= \lim_{x \to 3} (x+3)$$
$$\ldots[\because x \to 3, x \neq 3 \therefore x-3 \neq 0]$$
$$= 3 + 3 = 6$$
$$\lim_{x \to 3} f(x) \neq f(3)$$

∴ f(x) is discontinuous at x = 3.

Question 2.
Examine whether the function is continuous at the points indicated against them.
(i) f(x) = $x^3 - 2x + 1$, if x ≤ 2
= 3x – 2, if x > 2, at x = 2.

Solution:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^3 - 2x + 1)$$
$$= (2)^3 - 2(2) + 1$$
$$= 5$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x - 2)$$
$$= 3(2) - 2$$
$$= 4$$
$$\therefore \quad \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$$
$$\therefore \quad f(x) \text{ is discontinuous at } x = 2.$$

(ii) f(x) = $\frac{x^2+18x-19}{x-1}$, for x ≠ 1
= 20, for x = 1, at x = 1.
Solution:

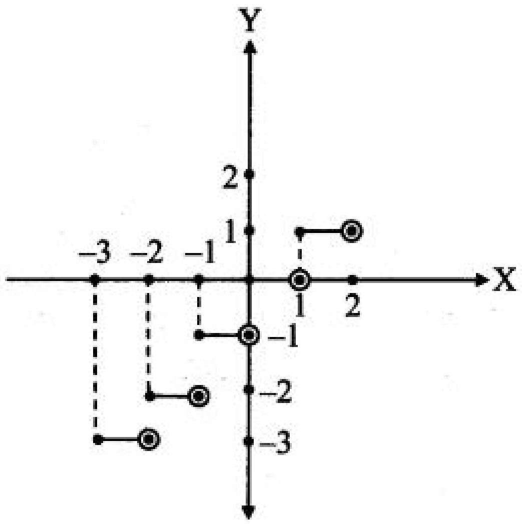
$$f(1) = 20 \qquad \ldots\text{(given)}$$
$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2 + 18x - 19}{x - 1}$$
$$= \lim_{x\to 1} \frac{x^2 + 19x - x - 19}{x - 1}$$
$$= \lim_{x\to 1} \frac{x(x+19) - 1(x+19)}{(x-1)}$$
$$= \lim_{x\to 1} \frac{(x-1)(x+19)}{(x-1)}$$
$$= \lim_{x\to 1} (x + 19) \quad \ldots \begin{bmatrix} \because x \to 1, x \neq 1, \\ \therefore x - 1 \neq 0 \end{bmatrix}$$
$$= 1 + 19 = 20$$
$$\therefore \quad \lim_{x\to 1} f(x) = f(1)$$
$$\therefore \quad f(x) \text{ is continuous at } x = 1.$$

(iii) f(x) = $\frac{x}{\tan 3x} + 2$, for x < 0
= $\frac{7}{3}$, for x ≥ 0, at x = 0.
Solution:

$$\lim_{x\to 0^+} f(x) = \frac{7}{3} \text{ and } f(0) = \frac{7}{3} \qquad \ldots\text{(given)}$$
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left(\frac{x}{\tan 3x} + 2\right)$$
$$= \lim_{x\to 0^-} \frac{x}{\tan 3x} + \lim_{x\to 0^-} 2$$
$$= \lim_{x\to 0^-} \frac{1}{\frac{\tan 3x}{x}} + \lim_{x\to 0^-} 2$$
$$= \lim_{x\to 0^-} \left(\frac{1}{\frac{\tan 3x}{3x} \times 3}\right) + \lim_{x\to 0^-} 2$$
$$= \frac{\lim_{x\to 0^-} 1}{3 \lim_{x\to 0^-} \left(\frac{\tan 3x}{3x}\right)} + \lim_{x\to 0^-} 2$$
$$= \frac{1}{3(1)} + 2 = \frac{7}{3}$$
$$\therefore \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$
$$\therefore \quad f(x) \text{ is continuous at } x = 0.$$

Question 3.
Find all the points of discontinuities of f(x) = [x] on the interval (-3, 2).
Solution:
f(x) = [x], x ∈ (-3, 2)
i.e., f(x) = -3, x ∈ (-3, -2)

= -2, x ∈ [-2, -1)
= -1, x ∈ [- 1, 0)
= 0, x ∈ [0, 1)
= 1, x ∈ [1, 2)



At $x = -2$,

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} \lfloor x \rfloor = -3$$

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \lfloor x \rfloor = -2$$

$$\therefore \quad \lim_{x \to -2^-} f(x) \neq \lim_{x \to -2^+} f(x)$$

∴ **f(x) is discontinuous at $x = -2$.**

Similarly, f(x) is discontinuous at the points x = -1, x = 0, x = 1.
Thus all the integer values of x in the interval (-3, 2),
i.e., the points x = -2, x = -1, x = 0 and x = 1 are the required points of discontinuities.

Question 4.
Discuss the continuity of the function f(x) = |2x + 3|, at x = −32.
Solution:

$$f(x) = |2x + 3|, \ x = \frac{-3}{2}$$

$$|2x + 3| = 2x + 3 \quad ; \ x \geq \frac{-3}{2}$$

$$= -(2x + 3) \ ; \ x < \frac{-3}{2}$$

$$\lim_{x \to \frac{-3}{2}^-} f(x) = \lim_{x \to \frac{-3}{2}^-} |2x + 3|$$

$$= \lim_{x \to \frac{-3}{2}^-} [-(2x + 3)]$$

$$= -\left[2\left(\frac{-3}{2}\right) + 3\right]$$

$$= 0$$

$$\lim_{x \to \frac{-3}{2}^+} f(x) = \lim_{x \to \frac{-3}{2}^+} |2x + 3|$$

$$= \lim_{x \to \frac{-3}{2}^+} (2x + 3)$$

$$= 2\left(\frac{-3}{2}\right) + 3$$

$$= 0$$

$$f\left(\frac{-3}{2}\right) = \left|2\left(\frac{-3}{2}\right) + 3\right|$$

$$= |0|$$

$$= 0$$

$$\therefore \quad \lim_{x \to \frac{-3}{2}^-} f(x) = \lim_{x \to \frac{-3}{2}^+} f(x) = f\left(\frac{-3}{2}\right)$$

∴ **f(x) is continuous at $x = \frac{-3}{2}$.**

Question 5.
Test the continuity of the following functions at the points or intervals indicated against them.

(i) $f(x)= \dfrac{\sqrt{x-1}-(x-1)^{\frac{1}{3}}}{x-2}$, for $x \neq 2$

$= \dfrac{1}{5}$, for $x = 2$

at $x = 2$

(ii) $f(x)= \dfrac{x^3-8}{\sqrt{x+2}-\sqrt{3x-2}}$ for $x \neq 2$

$= -24$ for $x = 2$ at $x = 2$

(iii) $f(x) = 4x+1$, for $x \leq 8/3$.

$= \dfrac{59-9x}{3}$, for $x > 8/3$, at $x = 8/3$.

(iv) $f(x) = \dfrac{(27-2x)^{\frac{1}{3}}-3}{9-3(243+5x)^{\frac{1}{5}}}$, for $x \neq 0$

$= 2$ for $x = 0$, at $x = 0$

(v) $f(x) = \dfrac{x^2+8x-20}{2x^2-9x+10}$ for $0 < x < 3$; $x \neq 2$

$= 12$, for $x = 2$

$= \dfrac{2-2x-x^2}{x-4}$ for $3 \leq x < 4$

at $x = 2$

Solution:

i. $f(2) = \dfrac{1}{5}$ ...(given)

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{\sqrt{x-1}-(x-1)^{\frac{1}{3}}}{x-2}$$

Put $x - 1 = y$

$\therefore$ $x = 1 + y$

As $x \to 2, y \to 1$

$$\therefore \lim_{x\to 2} f(x) = \lim_{y\to 1} \frac{\sqrt{y}-y^{\frac{1}{3}}}{1+y-2}$$

$$= \lim_{y\to 1} \frac{\left(y^{\frac{1}{2}}-1\right)-\left(y^{\frac{1}{3}}-1\right)}{y-1}$$

$$= \lim_{y\to 1} \left(\frac{y^{\frac{1}{2}}-1}{y-1} - \frac{y^{\frac{1}{3}}-1}{y-1}\right)$$

$$= \lim_{y\to 1} \frac{y^{\frac{1}{2}}-1^{\frac{1}{2}}}{y-1} - \lim_{y\to 1} \frac{y^{\frac{1}{3}}-1^{\frac{1}{3}}}{y-1}$$

$$= \frac{1}{2}(1)^{\frac{-1}{2}} - \frac{1}{3}(1)^{\frac{-2}{3}}$$

$$\ldots\left[\because \lim_{x\to a} \frac{x^n-a^n}{x-a} = n.a^{n-1}\right]$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$\therefore$ $\lim_{x\to 2} f(x) \neq f(2)$

$\therefore$ $f(x)$ is discontinuous at $x = 2$.

ii. $f(2) = -24$ ...(given)

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^3-8}{\sqrt{x+2}-\sqrt{3x-2}}$$

$$= \lim_{x\to 2} \frac{x^3-8}{\sqrt{x+2}-\sqrt{3x-2}} \times \frac{\sqrt{x+2}+\sqrt{3x-2}}{\sqrt{x+2}+\sqrt{3x-2}}$$

$$= \lim_{x\to 2} \frac{(x^3-8)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{(x+2)-(3x-2)}$$

$$= \lim_{x\to 2} \frac{(x^3-2^3)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{-2x+4}$$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{-2(x-2)}$$

$$= \lim_{x\to 2} \frac{(x^2+2x+4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)}{-2}$$

$$\ldots\left[\begin{array}{l}\because x \to 2, x \neq 2 \\ \therefore x-2 \neq 0\end{array}\right]$$

$$= \frac{-1}{2} \lim_{x\to 2} (x^2+2x+4)\left(\sqrt{x+2}+\sqrt{3x-2}\right)$$

$$= \frac{-1}{2} \lim_{x\to 2} (x^2+2x+4) \lim_{x\to 2}\left(\sqrt{x+2}+\sqrt{3x-2}\right)$$

$$= \frac{-1}{2} \times \left[2^2+2(2)+4\right] \times \left(\sqrt{2+2}+\sqrt{3(2)-2}\right)$$

$$= \frac{-1}{2} \times 12 \times (2+2)$$

$$= -24$$

$\therefore$ $\lim_{x\to 2} f(x) = f(2)$

$\therefore$ $f(x)$ is continuous at $x = 2$.

iii. $\lim_{x\to \frac{8}{3}^-} f(x) = \lim_{x\to \frac{8}{3}^-} (4x+1)$

$$= 4\left(\frac{8}{3}\right) + 1$$

$$= \frac{35}{3}$$

$\lim_{x\to \frac{8}{3}^+} f(x) = \lim_{x\to \frac{8}{3}^+} \frac{59-9x}{3}$

$$= \frac{59-9\left(\frac{8}{3}\right)}{3}$$

$$= \frac{59-24}{3} = \frac{35}{3}$$

$f(x) = 4x + 1$

$\therefore$ $f\left(\frac{8}{3}\right) = 4\left(\frac{8}{3}\right) + 1 = \frac{35}{3}$

$\therefore$ $\lim_{x\to \frac{8}{3}^-} f(x) = \lim_{x\to \frac{8}{3}^+} f(x) = f\left(\frac{8}{3}\right)$

$\therefore$ $f(x)$ is continuous at $x = \frac{8}{3}$.

iv. $f(0) = 2$ ...(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(27-2x)^{\frac{1}{3}}-3}{9-3(243+5x)^{\frac{1}{5}}}$$

$$= \lim_{x\to 0} \frac{(27-2x)^{\frac{1}{3}}-3}{-3\left[(243+5x)^{\frac{1}{5}}-3\right]}$$

$$= \frac{-1}{3} \lim_{x\to 0} \frac{(27-2x)^{\frac{1}{3}} - (27)^{\frac{1}{3}}}{(243+5x)^{\frac{1}{5}} - (243)^{\frac{1}{5}}}$$

$$= \frac{-1}{3} \lim_{x\to 0} \frac{\frac{(27-2x)^{\frac{1}{3}} - 27^{\frac{1}{3}}}{(27-2x)-27} \times [(27-2x)-27]}{\frac{(243+5x)^{\frac{1}{5}} - (243)^{\frac{1}{5}}}{(243+5x)-243} \times [(243+5x)-243]}$$

$$\ldots \begin{bmatrix} \because x\to 0, -2x\to 0 \text{ and } 5x\to 0 \\ \therefore (27-2x)-27\to 0 \text{ and } (243+5x)-243\to 0 \\ \therefore (27-2x)-27\neq 0 \text{ and } (243+5x)-243\neq 0 \end{bmatrix}$$

$$= \frac{-1}{3} \cdot \frac{\lim\limits_{x\to 0} \frac{(27-2x)^{\frac{1}{3}} - 27^{\frac{1}{3}}}{(27-2x)-27} \times (-2x)}{\lim\limits_{x\to 0} \frac{(243+5x)^{\frac{1}{5}} - (243)^{\frac{1}{5}}}{(243+5x)-243} \times (5x)}$$

$$= \frac{-1}{3} \times \frac{-2}{5} \times \frac{\lim\limits_{x\to 0} \frac{(27-2x)^{\frac{1}{3}} - 27^{\frac{1}{3}}}{(27-2x)-27}}{\lim\limits_{x\to 0} \frac{(243+5x)^{\frac{1}{5}} - (243)^{\frac{1}{5}}}{(243+5x)-243}}$$

$$\ldots[\because x\to 0, x\neq 0]$$

$$= \frac{2}{15} \times \frac{\frac{1}{3}(27)^{\frac{-2}{3}}}{\frac{1}{5}(243)^{\frac{-4}{5}}} \qquad \ldots\left[\because \lim_{x\to a} \frac{x^n - a^n}{x-a} = n.a^{n-1}\right]$$

$$= \frac{2}{15} \times \frac{5}{3} \times \frac{(3^3)^{\frac{-2}{3}}}{(3^5)^{\frac{-4}{5}}}$$

$$= \frac{2}{9} \times \frac{(3)^{-2}}{(3)^{-4}} = \frac{2}{9} \times (3)^2$$

$$= 2$$

$\therefore \quad \lim\limits_{x\to 0} f(x) = f(0)$

$\therefore \quad f(x)$ is continuous at $x = 0$.

v. $\quad f(2) = 12 \qquad \ldots$(given)

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2+8x-20}{2x^2-9x+10}$$

$$= \lim_{x\to 2} \frac{(x+10)(x-2)}{(2x-5)(x-2)}$$

$$= \lim_{x\to 2} \frac{x+10}{2x-5} \qquad \ldots\begin{bmatrix} \because x\to 2, x\neq 2, \\ \therefore x-2\neq 0 \end{bmatrix}$$

$$= \frac{\lim\limits_{x\to 2}(x+10)}{\lim\limits_{x\to 2}(2x-5)} = \frac{2+10}{2(2)-5}$$

$$= \frac{12}{-1} = -12$$

$\therefore \quad \lim\limits_{x\to 2} f(x) \neq f(2)$

$\therefore \quad f(x)$ is discontinuous at $x = 2$.

Question 6.
Identify discontinuities for the following functions as either a jump or a removable discontinuity.
(i) $f(x) = \frac{x^2-10x+21}{x-7}$
Solution:
Given, $f(x) = \frac{x^2-10x+21}{x-7}$
It is a rational function and is discontinuous if
$x - 7 = 0$, i.e., $x = 7$
$\therefore$ f(x) is continuous for all $x \in R$, except at $x = 7$.
$\therefore$ f(7) is not defined.

$$\text{Now, } \lim_{x\to 7} f(x) = \lim_{x\to 7} \frac{x^2-10x+21}{x-7}$$
$$= \lim_{x\to 7} \frac{(x-7)(x-3)}{x-7}$$
$$= \lim_{x\to 7} (x-3) \ldots \begin{bmatrix} \because x\to 7, x\neq 7, \\ \therefore x-7\neq 0 \end{bmatrix}$$
$$= 7-3$$
$$= 4$$

Thus, $\lim_{x\to 7} f(x)$ exist but f(7) is not defined.

∴ f(x) has a removable discontinuity.

(ii) f(x) = x^2 + 3x – 2, for x ≤ 4
= 5x + 3, for x > 4.
Solution:
f(x) = x^2 + 3x – 2, x ≤ 4
= 5x + 3, x > 4
f(x) is a polynomial function for both the intervals.
∴ f(x) is continuous for both the given intervals.
Let us test the continuity at x = 4.

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (x^2+3x-2)$$
$$= (4)^2+3(4)-2$$
$$= 26$$
$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (5x+3)$$
$$= 5(4)+3$$
$$= 23$$
$$\therefore \quad \lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$$
$$\therefore \quad \lim_{x\to 4} f(x) \text{ does not exist.}$$

∴ f(x) is discontinuous at x = 4.
∴ f(x) has a jump discontinuity at x = 4.

(iii) f(x) = x^2 – 3x – 2, for x < -3 = 3 + 8x, for x > -3.
Solution:
f(x) = x^2 – 3x – 2, x < -3 = 3 + 8x, x > -3
f(x) is a polynomial function for both the intervals.
∴ f(x) is continuous for both the given intervals.
Let us test the continuity at x = -3.

$$\lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} (x^2-3x-2)$$
$$= (-3)^2-3(-3)-2$$
$$= 9+9-2$$
$$= 16$$
$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} (3+8x)$$
$$= 3+8(-3)$$
$$= -21$$
$$\therefore \quad \lim_{x\to -3^-} f(x) \neq \lim_{x\to -3^+} f(x)$$
$$\therefore \quad \lim_{x\to -3} f(x) \text{ does not exist.}$$

∴ f(x) is discontinuous at x = -3.
∴ f(x) has a jump discontinuity at x = -3

(iv) f(x) = 4 + sin x, for x < π = 3 – cos x for x > π.
Solution:
f(x) = 4 + sin x, x < π = 3 – cos x, x > π
sin x and cos x are continuous for all x ∈ R.
4 and 3 are constant functions.
∴ 4 + sin x and 3 – cos x are continuous for all x ∈ R.
∴ f(x) is continuous for both the given intervals.
Let us test the continuity at x = π.

$$\lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^-} (4 + \sin x)$$
$$= 4 + \sin \pi$$
$$= 4 + 0$$
$$= 4$$
$$\lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} (3 - \cos x)$$
$$= 3 - \cos \pi$$
$$= 3 - (-1)$$
$$= 4$$
$$\therefore \quad \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x)$$
$$\therefore \quad \lim_{x \to \pi} f(x) = 4$$

But f(π) is not defined.
∴ f(x) has a removable discontinuity at x = π.

Question 7.
Show that the following functions have a continuous extension to the point where f(x) is not defined. Also, find the extension.
(i) f(x) = 1−cos2xsinx, for x ≠ 0.
Solution:
f(x) = 1−cos2xsinx, for x ≠ 0
Here, f(0) is not defined.
Consider,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1 - \cos 2x}{\sin x}$$
$$= \lim_{x \to 0} \frac{2\sin^2 x}{\sin x}$$
$$= 2 \lim_{x \to 0} (\sin x) \quad \ldots \begin{bmatrix} \because x \to 0, x \neq 0 \\ \therefore \sin x \neq 0 \end{bmatrix}$$
$$= 2(\sin 0) = 2 \times 0$$
$$= 0$$
$$\lim_{x \to 0} f(x) \text{ exists.}$$

But f(0) is not defined.
∴ f(x) has a removable discontinuity at x = 0.
∴ The extension of the original function is
f(x) = 1−cos2xsinx for x ≠ 0
= 0 for x = 0
∴ f(x) is continuous at x = 0.

(ii) f(x) = 3sin2x+2cosx(1−cos2x)2(1−cos2x), for x ≠ 0.
Solution:
f(x) = 3sin2x+2cosx(1−cos2x)2(1−cos2x), for x ≠ 0
Here, f(0) is not defined.
Consider,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{3\sin^2 x + 2\cos x\,(1 - \cos 2x)}{2(1 - \cos^2 x)}$$
$$= \lim_{x \to 0} \frac{3\sin^2 x + 2\cos x \cdot (2\sin^2 x)}{2\sin^2 x}$$
$$= \lim_{x \to 0} \frac{\sin^2 x\,(3 + 4\cos x)}{2\sin^2 x}$$
$$= \lim_{x \to 0} \frac{3 + 4\cos x}{2}$$
$$\ldots \begin{bmatrix} \because x \to 0, x \neq 0, \sin x \neq 0 \\ \therefore \sin^2 x \neq 0 \end{bmatrix}$$
$$= \frac{1}{2} \lim_{x \to 0} (3 + 4\cos x) = \frac{1}{3} (3 + 4\cos 0)$$
$$= \frac{1}{2} (3 + 4) = \frac{7}{2}$$
$$\lim_{x \to 0} f(x) \text{ exists.}$$

But f(0) is not defined.
∴ f(x) has a removable discontinuity at x = 0.
∴ The extension of the original function is

f(x) = 3sin2x+2cosx(1−cos2x)2(1−cos2x), x ≠ 0

= 72, x = 0

∴ f(x) is continuous at x = 0.

(iii) f(x) = x2−1x3+1, for x ≠ -1

Solution:

f(x) = x2−1x3+1, for x ≠ -1

Here, f(-1) is not defined.

Consider,

$$\lim_{x\to -1} f(x) = \lim_{x\to -1}\left(\frac{x^2-1}{x^3+1}\right)$$

$$= \lim_{x\to -1}\frac{(x+1)(x-1)}{(x+1)(x^2-x+1)}$$

$$= \lim_{x\to -1}\frac{x-1}{x^2-x+1} \quad \ldots\begin{bmatrix}\because x\to -1, x\neq -1\\ \therefore x+1\neq 0\end{bmatrix}$$

$$= \frac{-1-1}{(-1)^2-(-1)+1} = -\frac{2}{3}$$

But f(-1) is not defined.

∴ f(x) has a removable discontinuity at x = -1.

∴ The extension of the original function is

f(x) = x2−1x3+1, x ≠ -1

= −23, x = -1

∴ f(x) is continuous at x = −23

Question 8.

Discuss the continuity of the following functions at the points indicated against them.

$$\text{(i) } f(x) = \frac{\sqrt{3}-\tan x}{\pi-3x}, x\neq\frac{\pi}{3}$$
$$= \frac{3}{4}, \quad \text{for } x=\frac{\pi}{3}, \text{ at } x=\frac{\pi}{3}.$$

$$\text{(ii) } f(x) = \frac{e^{1/x}-1}{e^{1/x}+1}, \quad \text{for } x\neq 0$$
$$= 1, \quad \text{for } x=0, \text{ at } x=0.$$

$$\text{(iii) } f(x) = \frac{4^x-2^{x+1}+1}{1-\cos 2x}, \quad \text{for } x\neq 0$$
$$= \frac{(\log 2)^2}{2}, \quad \text{for } x=0, \text{ at } x=0.$$

Solution:

i. $f\left(\frac{\pi}{3}\right) = \frac{3}{4}$ …(given)

$$\lim_{x\to\frac{\pi}{3}} f(x) = \lim_{x\to\frac{\pi}{3}}\frac{\sqrt{3}-\tan x}{\pi-3x}$$

Put $\frac{\pi}{3}-x = h$,

$\therefore \quad x = \frac{\pi}{3}-h$

As $x \to \frac{\pi}{3}$, $h \to 0$

$$\therefore \quad \lim_{x \to \frac{\pi}{3}} f(x) = \lim_{h \to 0} \frac{\sqrt{3} - \tan\left(\frac{\pi}{3} - h\right)}{\pi - 3\left(\frac{\pi}{3} - h\right)}$$

$$= \lim_{h \to 0} \frac{\sqrt{3} - \frac{\tan\frac{\pi}{3} - \tan h}{1 + \tan\frac{\pi}{3}\tan h}}{3h}$$

$$= \lim_{h \to 0} \frac{\sqrt{3} - \frac{\sqrt{3} - \tan h}{1 + \sqrt{3}\tan h}}{3h}$$

$$= \lim_{h \to 0} \frac{\sqrt{3}\left(1 + \sqrt{3}\tan h\right) - \left(\sqrt{3} - \tan h\right)}{3h\left(1 + \sqrt{3}\tan h\right)}$$

$$= \lim_{h \to 0} \frac{\sqrt{3} + 3\tan h - \sqrt{3} + \tan h}{3h\left(1 + \sqrt{3}\tan h\right)}$$

$$= \lim_{h \to 0} \frac{4\tan h}{3h\left(1 + \sqrt{3}\tan h\right)}$$

$$= \lim_{h \to 0} \frac{4}{3\left(1 + \sqrt{3}\tan h\right)} \times \frac{\tan h}{h}$$

$$= \frac{4}{3}\left(\lim_{h \to 0} \frac{1}{1 + \sqrt{3}\tan h}\right)\left(\lim_{h \to 0} \frac{\tan h}{h}\right)$$

$$= \frac{4}{3}\left[\frac{1}{1 + \sqrt{3}(0)}\right](1)$$

$$= \frac{4}{3}$$

$$\therefore \quad \lim_{x \to \frac{\pi}{3}} f(x) \neq f\left(\frac{\pi}{3}\right)$$

$\therefore$ f(x) is discontinuous at $x = \frac{\pi}{3}$.

ii. f(0) = 1 …(given)

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} \frac{e^{\frac{1}{(0-h)}} - 1}{e^{\frac{1}{(0-h)}} + 1}$$

$$= \lim_{h \to 0} \frac{e^{\frac{-1}{h}} - 1}{e^{\frac{-1}{h}} + 1}$$

$$= \lim_{h \to 0} \frac{\frac{1}{e^{\frac{1}{h}}} - 1}{\frac{1}{e^{\frac{1}{h}}} + 1}$$

$$= \frac{0 - 1}{0 + 1}$$

$$= -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} \frac{e^{\frac{1}{(0+h)}} - 1}{e^{\frac{1}{0+h}} + 1}$$

$$= \lim_{h \to 0} \frac{e^{\frac{1}{h}} - 1}{e^{\frac{1}{h}} + 1}$$

$$= \lim_{h \to 0} \frac{e^{\frac{1}{h}}\left(1 - \frac{1}{e^{\frac{1}{h}}}\right)}{e^{\frac{1}{h}}\left(1 + \frac{1}{e^{\frac{1}{h}}}\right)}$$

$$= \frac{1-0}{1+0} = 1$$

$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\therefore \quad$ **f(x) is discontinuous at $x = 0$.**

iii. $f(0) = \frac{[illegible]}{2}$ $\quad$ …(given)

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{4^x - 2^{x+1} + 1}{1 - \cos 2x}$$

$$= \lim_{x \to 0} \frac{\left(2^2\right)^x - 2^x \cdot 2^1 + 1}{2\sin^2 x}$$

$$= \lim_{x \to 0} \frac{\left(2^x\right)^2 - 2\left(2^x\right) + 1}{2\sin^2 x}$$

$$= \lim_{x \to 0} \frac{\left(2^x - 1\right)^2}{2\sin^2 x}$$

$\quad$ …[∵ $a^2 - 2ab + b^2 = (a - b)^2$]

$$= \lim_{x \to 0} \frac{\frac{\left(2^x - 1\right)^2}{x^2}}{\frac{2\sin^2 x}{x^2}}$$

$\quad$ …$\left[\begin{array}{l} \because x \to 0, x \neq 0, \\ \therefore x^2 \neq 0 \end{array}\right]$

$$= \frac{\lim_{x \to 0}\left(\frac{2^x - 1}{x}\right)^2}{2\lim_{x \to 0}\left(\frac{\sin x}{x}\right)^2}$$

$$= \frac{(\log 2)^2}{2(1)^2} = \frac{(\log 2)^2}{2}$$

$\therefore \quad \lim_{x \to 0} f(x) = f(0)$

$\therefore \quad$ **f(x) is continuous at $x = 0$.**

Question 9.
Which of the following functions has a removable discontinuity? If it has a removable discontinuity, redefine the function so that it becomes continuous.

(i) $f(x) = \frac{e^{5\sin x} - e^{2x}}{5\tan x - 3x}$, for $x \neq 0$

$\quad = 3/4$, $\quad$ for $x = 0$, at $x = 0$.

(ii) $f(x) = \log_{(1+3x)}(1+5x)$ for $x > 0$

$= \dfrac{32^x - 1}{8^x - 1}$, for $x < 0$, at $x = 0$.

(iii) $f(x) = \left(\dfrac{3-8x}{3-2x}\right)^{\frac{1}{x}}$, for $x \neq 0$.

(iv) $f(x) = 3x + 2$, for $-4 \le x \le -2$

$= 2x - 3$, for $-2 < x \le 6$.

(v) $f(x) = \dfrac{x^3 - 8}{x^2 - 4}$, for x > 2

$= 3$, for $x = 2$

$= \dfrac{e^{3(x-2)^2} - 1}{2(x-2)^2}$, for $x < 2$

Solution:

i. $f(0) = \dfrac{3}{4}$ ...(given)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{5\sin x} - e^{2x}}{5\tan x - 3x}$$

$$= \lim_{x\to 0} \frac{\left(e^{5\sin x} - 1\right) - \left(e^{2x} - 1\right)}{5\tan x - 3x}$$

$$= \lim_{x\to 0} \left[\frac{\frac{\left(e^{5\sin x} - 1\right) - \left(e^{2x} - 1\right)}{x}}{\frac{5\tan x - 3x}{x}}\right] \quad \ldots[\because x \to 0, x \neq 0]$$

$$= \frac{\lim_{x\to 0}\left(\frac{e^{5\sin x} - 1}{x} - \frac{e^{2x} - 1}{x}\right)}{\lim_{x\to 0}\left(\frac{5\tan x}{x} - 3\right)}$$

$$= \frac{\lim_{x\to 0}\left(\frac{e^{5\sin x} - 1}{5\sin x} \times \frac{5\sin x}{x}\right) - \lim_{x\to 0}\left(\frac{e^{2x} - 1}{2x} \times 2\right)}{\lim_{x\to 0}\frac{5\tan x}{x} - \lim_{x\to 0} 3}$$

$$= \frac{5\lim_{x\to 0}\left(\frac{e^{5\sin x} - 1}{5\sin x}\right) \cdot \lim_{x\to 0}\left(\frac{\sin x}{x}\right) - 2\lim_{x\to 0}\frac{e^{2x} - 1}{2x}}{5\lim_{x\to 0}\frac{\tan x}{x} - \lim_{x\to 0}(3)}$$

$$= \frac{5(1)(1) - 2(1)}{5(1) - 3}$$

$$\ldots\left[\begin{array}{l}\because x\to 0,\ 2x\to 0,\ \sin x\to 0,\ 5\sin x\to 0 \\ \text{and } \lim_{x\to 0}\left(\frac{e^x - 1}{x}\right) = 1\end{array}\right]$$

$$= \frac{3}{2}$$

$\therefore$ $\lim_{x\to 0} f(x) \neq f(0)$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

$\therefore$ $f(x)$ has a removable discontinuity at $x = 0$.
This discontinuity can be removed by redefining $f(0) = \dfrac{3}{2}$.

$\therefore$ $f(x)$ can be redefined as

$$f(x) = \frac{e^{5\sin x} - e^{2x}}{5\tan x - 3x}, \quad x \neq 0$$

$$= \frac{3}{2}, \quad x = 0$$

ii. $f(x) = \log_{(1+3x)}(1+5x)$, $x > 0$

$= \dfrac{32^x - 1}{8^x - 1}$, $x < 0$

Here, f(0) is not defined.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \log_{(1+3x)}(1+5x)$$

$$= \lim_{x\to 0^+} \frac{\log(1+5x)}{\log(1+3x)}$$

$$= \lim_{x\to 0^+} \left[\frac{\frac{\log(1+5x)}{x}}{\frac{\log(1+3x)}{x}}\right] \quad \ldots[\because x \to 0, x \neq 0]$$

$$= \frac{\lim\limits_{x\to 0^+} \frac{\log(1+5x)}{5x} \times 5}{\lim\limits_{x\to 0^+} \frac{\log(1+3x)}{3x} \times 3}$$

$$= \frac{1\times 5}{1\times 3} \quad \ldots\left[\begin{array}{l}\because x\to 0, 3x\to 0, 5x\to 0 \\ \text{and } \lim\limits_{x\to 0} \frac{\log(1+x)}{x} = 1\end{array}\right]$$

$$= \frac{5}{3}$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{32^x - 1}{8^x - 1}$$

$$= \lim_{x\to 0^-} \left[\frac{\frac{32^x - 1}{x}}{\frac{8^x - 1}{x}}\right] \quad \ldots[\because x \to 0, x \neq 0]$$

$$= \frac{\lim\limits_{x\to 0^-} \frac{32^x - 1}{x}}{\lim\limits_{x\to 0^-} \frac{8^x - 1}{x}}$$

$$= \frac{\log 32}{\log 8} \quad \ldots\left[\because \lim_{x\to 0} \frac{a^x - 1}{x} = \log a\right]$$

$$= \frac{\log(2)^5}{\log(2)^3}$$

$$= \frac{5\log 2}{3\log 2}$$

$$= \frac{5}{3}$$

$\therefore$ $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x)$

$\therefore$ $\lim\limits_{x\to 0} f(x)$ exists.

But f(0) is not defined.

$\therefore$ $f(x)$ has a removable discontinuity at $x = 0$.

This discontinuity can be removed by defining $f(0) = \dfrac{5}{3}$.

$\therefore$ $f(x)$ can be redefined as

$f(x) = \log_{(1+3x)}(1+5x)$; $x > 0$

$= \dfrac{5}{3}$; $x = 0$

$= \dfrac{32^x - 1}{8^x - 1}$; $x < 0$

iii. $f(x) = \left(\dfrac{3-8x}{3-2x}\right)^{\frac{1}{x}}$; $x \neq 0$

Here, f(0) is not defined.

Consider, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \left(\dfrac{3-8x}{3-2x}\right)^{\frac{1}{x}}$

$$= \lim_{x\to 0} \left[\frac{3\left(1-\frac{8x}{3}\right)}{3\left(1-\frac{2x}{3}\right)}\right]^{\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{\left(1-\frac{8x}{3}\right)^{\frac{1}{x}}}{\left(1-\frac{2x}{3}\right)^{\frac{1}{x}}}$$

$$= \frac{\lim\limits_{x\to 0}\left[\left(1-\frac{8x}{3}\right)^{\frac{-3}{8x}}\right]^{\frac{-8}{3}}}{\lim\limits_{x\to 0}\left[\left(1-\frac{2x}{3}\right)^{\frac{-3}{2x}}\right]^{\frac{-2}{3}}}$$

$$= \frac{e^{\frac{-8}{3}}}{e^{\frac{-2}{3}}} \qquad \ldots \left[\begin{array}{l}\because x\to 0, \dfrac{-8x}{3}\to 0, \dfrac{-2x}{3}\to 0 \\ \text{and } \lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e\end{array}\right]$$

$$= e^{\frac{-6}{3}} = e^{-2}$$

$\therefore$ $\lim\limits_{x\to 0} f(x)$ exists.

But f(0) is not defined.

$\therefore$ $f(x)$ has a removable discontinuity at $x = 0$.

This discontinuity can be removed by defining

$\therefore$ $f(x)$ can be redefined as

$$f(x) = \left(\frac{3-8x}{3-2x}\right)^{\frac{1}{x}} \quad ; x \neq 0$$
$$= e^{-2} \quad ; x = 0$$

iv. $f(x) = 3x + 2$, for $-4 \le x \le -2$

$= 2x - 3$, for $-2 < x \le 6$.

$\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^-} (3x+2)$

$= 3(-2) + 2 = -4$

$\lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2^+} (2x-3)$

$= 2(-2) - 3$

$= -7$

$\therefore$ $\lim\limits_{x\to -2^-} f(x) \neq \lim\limits_{x\to -2^+} f(x)$

$\therefore$ $\lim\limits_{x\to -2} f(x)$ does not exist.

$\therefore$ $f(x)$ is discontinuous at $x = -2$.

This discontinuity is irremovable.

v. $f(2) = 3$ ...(given)

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{x^3-8}{x^2-4} = \lim_{x\to 2^+} \frac{x^3-2^3}{x^2-2^2}$$

$$= \lim_{x\to 2^+} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$$

$$= \lim_{x\to 2^+} \frac{x^2+2x+4}{x+2} \quad \ldots\begin{bmatrix}\because x\to 2, x\neq 2\\ \therefore x-2\neq 0\end{bmatrix}$$

$$= \frac{\lim_{x\to 2^+}(x^2+2x+4)}{\lim_{x\to 2^+}(x+2)}$$

$$= \frac{(2)^2+2(2)+4}{2+2}$$

$$= \frac{12}{4}$$

$$= 3$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{e^{3(x-2)^2}-1}{2(x-2)^2}$$

Put $x - 2 = h$

$\therefore$ $x = 2 + h$

As $x \to 2$, $h \to 0$

$$\therefore \lim_{x\to 2^-} f(x) = \lim_{h\to 0} \frac{e^{3h^2}-1}{2h^2}$$

$$= \frac{1}{2}\lim_{h\to 0} \frac{e^{3h^2}-1}{3h^2} \times 3$$

$$= \frac{1}{2} \times 1 \times 3 \quad \ldots\begin{bmatrix}\because h\to 0, 3h^2\to 0\\ \text{and } \lim_{x\to 0}\frac{e^x-1}{x}=1\end{bmatrix}$$

$$= \frac{3}{2}$$

$\therefore$ $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

$\therefore$ $\lim_{x\to 2} f(x)$ does not exist.

$\therefore$ $f(x)$ is discontinuous at $x = 2$.

This discontinuity is irremovable.

Question 10.

(i) If f(x) = 2+sinx√−3√cos2x, for x ≠ π2, is continuous at x = π2 then find f(π2).

Solution:

f(x) is continuous at x = π2, ...(given)

$$f\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}} f(x)$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\sqrt{2+\sin x}-\sqrt{3}}{\cos^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}} \left[\frac{\sqrt{2+\sin x}-\sqrt{3}}{1-\sin^2 x} \times \frac{\sqrt{2+\sin x}+\sqrt{3}}{\sqrt{2+\sin x}+\sqrt{3}}\right]$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{2+\sin x-3}{(1-\sin x)(1+\sin x)\left(\sqrt{2+\sin x}+\sqrt{3}\right)}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\sin x-1}{-(\sin x-1)(1+\sin x)\left(\sqrt{2+\sin x}+\sqrt{3}\right)}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{1}{-(1+\sin x)\left(\sqrt{2+\sin x}+\sqrt{3}\right)}$$

$$\ldots\left[\begin{array}{l}\because x\to\frac{\pi}{2}, \sin x\to 1,\\ \sin x\neq 1, \therefore \sin x-1\neq 0\end{array}\right]$$

$$= \frac{-1}{\lim_{x\to\frac{\pi}{2}}(1+\sin x)\left(\sqrt{2+\sin x}+\sqrt{3}\right)}$$

$$= \frac{-1}{\lim_{x\to\frac{\pi}{2}}(1+\sin x)\cdot\lim_{x\to\frac{\pi}{2}}\left(\sqrt{2+\sin x}+\sqrt{3}\right)}$$

$$= \frac{-1}{(1+1)\left(\sqrt{2+1}+\sqrt{3}\right)} = \frac{-1}{2\times 2\sqrt{3}}$$

$$f\left(\frac{\pi}{2}\right) = \frac{-1}{4\sqrt{3}}$$

(ii) If f(x) = $\frac{\cos^2x-\sin^2x-1}{\sqrt{3x^2+1}-1}$ for x ≠ 0, is continuous at x = 0 then find f(0).
Solution:
f(x) is continuous at x = 0, …..(given)

$$f(0) = \lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0} \frac{\cos^2 x-\sin^2 x-1}{\sqrt{3x^2+1}-1}$$

$$= \lim_{x\to 0} \frac{\cos 2x-1}{\sqrt{3x^2+1}-1} \times \frac{\sqrt{3x^2+1}+1}{\sqrt{3x^2+1}+1}$$

$$= \lim_{x\to 0} \frac{-(1-\cos 2x)\left(\sqrt{3x^2+1}+1\right)}{\left(3x^2+1\right)-1}$$

$$= \lim_{x\to 0} \frac{-2\sin^2 x\cdot\left(\sqrt{3x^2+1}+1\right)}{3x^2}$$

$$= \frac{-2}{3}\lim_{x\to 0} \frac{\sin^2 x}{x^2}\left(\sqrt{3x^2+1}+1\right)$$

$$= \frac{-2}{3}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \times \lim_{x\to 0}\left(\sqrt{3x^2+1}+1\right)$$

$$= \frac{-2}{3}(1)^2 \times \left(\sqrt{3(0)+1}+1\right)$$

$$= \frac{-2}{3} \times (1+1)$$

$$f(0) = \frac{-4}{3}$$

(iii) If f(x) = $\frac{4^{x-\pi}+4^{\pi-x}-2}{(x-\pi)^2}$ for x ≠ π, is continuous at x = π, then find f(π).
Solution:

f(x) is continuous at x = π, ...(given)

$$\therefore \quad f(\pi) = \lim_{x \to \pi} f(x) = \lim_{x \to \pi} \frac{4^{x-\pi} + 4^{\pi-x} - 2}{(x-\pi)^2}$$

Put $x - \pi = h$

As $x \to \pi$, $h \to 0$

$$\therefore \quad f(\pi) = \lim_{h \to 0} \frac{4^h + 4^{-h} - 2}{h^2}$$

$$= \lim_{h \to 0} \frac{4^h + \frac{1}{4^h} - 2}{h^2}$$

$$= \lim_{h \to 0} \frac{(4^h)^2 + 1 - 2(4^h)}{4^h \cdot (h^2)}$$

$$= \lim_{h \to 0} \frac{(4^h - 1)^2}{4^h \cdot h^2} \quad \ldots[\because a^2 - 2ab + b^2 = (a-b)^2]$$

$$= \lim_{h \to 0} \left(\frac{4^h - 1}{h}\right)^2 \times \frac{1}{4^h}$$

$$= \lim_{h \to 0} \left(\frac{4^h - 1}{h}\right)^2 \times \lim_{h \to 0} \frac{1}{4^h}$$

$$= (\log 4)^2 \times \frac{1}{4^0} \quad \ldots\left[\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log a\right]$$

$$= (\log 2^2)^2 \times \frac{1}{1} = (2 \log 2)^2$$

$$\therefore \quad f(\pi) = 4(\log 2)^2$$

Question 11.

(i) If f(x) = 24x−8x−3x+112x−4x−3x+1, for x ≠ 0

= k, for x = 0

is continuous at x = 0, then find k.

Solution:

f(x) is continuous at x = 0 .....(given)

$$\therefore \quad f(0)=\lim_{x\to 0} f(x)$$

$$\therefore \quad k=\lim_{x\to 0} \frac{24^x-8^x-3^x+1}{12^x-4^x-3^x+1}$$

$$=\lim_{x\to 0} \frac{8^x\cdot 3^x-8^x-3^x+1}{4^x\cdot 3^x-4^x-3^x+1}$$

$$=\lim_{x\to 0} \frac{8^x\left(3^x-1\right)-1\left(3^x-1\right)}{4^x\left(3^x-1\right)-1\left(3^x-1\right)}$$

$$=\lim_{x\to 0} \frac{\left(3^x-1\right)\left(8^x-1\right)}{\left(3^x-1\right)\left(4^x-1\right)}$$

$$=\lim_{x\to 0} \frac{8^x-1}{4^x-1} \qquad \ldots\left[\begin{array}{l}\text{As } x\to 0,\ 3^x\to 3^0,\\ 3^x\to 1, 3^x\neq 1\\ \therefore 3^x-1\neq 0\end{array}\right]$$

$$=\lim_{x\to 0}\left(\frac{\frac{8^x-1}{x}}{\frac{4^x-1}{x}}\right) \qquad \ldots[\because x\to 0, x\neq 0]$$

$$=\frac{\lim_{x\to 0}\frac{8^x-1}{x}}{\lim_{x\to 0}\frac{4^x-1}{x}}$$

$$=\frac{\log 8}{\log 4} \qquad \ldots\left[\because \lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log a\right]$$

$$=\frac{\log (2)^3}{\log (2)^2}$$

$$=\frac{3\log 2}{2\log 2}$$

$$k=\frac{3}{2}$$

(ii) If f(x) = $\frac{5^x+5^{-x}-2}{x^2}$, for x ≠ 0
= k, for x = 0
is continuous at x = 0, then find k.
Solution:
f(x) is continuous at x = 0 …..(given)

$$\therefore \quad f(0)=\lim_{x\to 0} f(x)$$

$$=\lim_{x\to 0} \frac{5^x+5^{-x}-2}{x^2}$$

$$=\lim_{x\to 0} \frac{5^x+\frac{1}{5^x}-2}{x^2}$$

$$=\lim_{x\to 0} \frac{\left(5^x\right)^2+1-2\left(5^x\right)}{5^x\cdot x^2}$$

$$=\lim_{x\to 0} \frac{\left(5^x-1\right)^2}{5^x\cdot x^2}$$

$$\ldots[\because a^2-2ab+b^2=(a-b)^2]$$

$$=\lim_{x\to 0}\left(\frac{5^x-1}{x}\right)^2\cdot\frac{1}{5^x}$$

$$=\lim_{x\to 0}\left(\frac{5^x-1}{x}\right)^2\times\lim_{x\to 0}\frac{1}{5^x}$$

$$=(\log 5)^2\times\frac{1}{5^0} \qquad \ldots\left[\because \lim_{x\to 0}\frac{a^x-1}{x}=\log a\right]$$

$$\therefore \quad k=(\log 5)^2$$

(iii) If f(x) = $\frac{\sin 2x}{5x}$ – a, for x > 0
= 4 for x = 0
= x^2 + b – 3, for x < 0
is continuous at x = 0, find a and b.

Solution:
f(x) is continuous at x = 0 ……(given)

$\therefore \quad \lim_{x\to 0^+} f(x) = f(0)$

$\therefore \quad \lim_{x\to 0^+}\left(\frac{\sin 2x}{5x} - a\right) = 4$

$\therefore \quad \lim_{x\to 0^+}\frac{\sin 2x}{5x} - \lim_{x\to 0^+} a = 4$

$\therefore \quad \frac{1}{5}\lim_{x\to 0^+}\frac{\sin 2x}{2x}\times(2) - \lim_{x\to 0^+} a = 4$

$\therefore \quad \frac{1}{5}(1)(2) - a = 4 \qquad \ldots\left[\begin{array}{l}\because x\to 0, 2x\to 0\\ \text{and } \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1\end{array}\right]$

$\therefore \quad \frac{2}{5} - a = 4$

$\therefore \quad \frac{2}{5} - 4 = a$

$\therefore \quad a = -\frac{18}{5}$

Also, $\lim_{x\to 0^-} f(x) = f(0)$

$\therefore \quad \lim_{x\to 0^-}(x^2 + b - 3) = 4$

$\therefore \quad b - 3 = 4$

$\therefore \quad b = 7$

$\therefore \quad a = \frac{-18}{5}$ and $b = 7$

(iv) For what values of a and b is the function
f(x) = ax + 2b + 18, for x ≤ 0
= x^2 + 3a – b, for 0 < x ≤ 2 = 8x – 2, for x > 2,
continuous for every x?
Solution:
f(x) is continuous for every x …..(given)
∴ f(x) is continuous at x = 0 and x = 2.
As f(x) is continuous at x = 0,

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

$\therefore \quad \lim_{x\to 0^-}(ax + 2b + 18) = \lim_{x\to 0^+}(x^2 + 3a - b)$

$\therefore \quad a(0) + 2b + 18 = (0)^2 + 3a - b$

$\therefore \quad 3a - 3b = 18$

$\therefore \quad a - b = 6 \qquad \ldots(1)$

As f(x) is continuous at x = 2,

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$

$\therefore \quad \lim_{x\to 2^-}(x^2 + 3a - b) = \lim_{x\to 2^+}(8x - 2)$

∴ $(2)^2$ + 3a – b = 8(2) – 2
∴ 4 + 3a – b = 14
∴ 3a – b = 10 …….(ii)
Subtracting (i) from (ii), we get
2a = 4
∴ a = 2
Substituting a = 2 in (i), we get
2 – b = 6
∴ b = -4
∴ a = 2 and b = -4

(v) For what values of a and b is the function
f(x) = x^2-4x-2, for x < 2
= ax^2 – bx + 3, for 2 ≤ x < 3
= 2x – a + b, for x ≥ 3
continuous for every x on R?
Solution:
f(x) is continuous for every x on R …..(given)
∴ f(x) is continuous at x = 2 and x = 3.

As f(x) is continuous at x = 2,

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$$\therefore \quad \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^+} (ax^2 - bx + 3)$$

$$\therefore \quad \lim_{x\to 2^-} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2^+} (ax^2 - bx + 3)$$

$$\therefore \quad \lim_{x\to 2^-} (x+2) = \lim_{x\to 2^+} (ax^2 - bx + 3) \ \ldots \begin{bmatrix} \because x \to 2, x \neq 2 \\ \therefore \ x-2 \neq 0 \end{bmatrix}$$

$$\therefore \quad 2 + 2 = a(2)^2 - b(2) + 3$$

$$\therefore \quad 4a - 2b + 3 = 4$$

$$\therefore \quad 4a - 2b = 1 \qquad \ldots(i)$$

As f(x) is continuous at x = 3,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

$$\therefore \quad \lim_{x\to 3^-} (ax^2 - bx + 3) = \lim_{x\to 3^+} (2x - a + b)$$

∴ a(3)2 – b(3) + 3 = 2(3) – a + b
∴ 9a – 3b + 3 = 6 – a + b
∴ 10a – 4b = 3 …..(ii)
Multiplying (i) by 2, we get
8a – 4b = 2 ….(iii)
Subtracting (ii) from (iii), we get
-2a = -1
∴ a = 12
Substituting a = 12 in (i), we get
4(12) – 2b = 1
∴ 2 – 2b = 1
∴ 1 = 2b
∴ b = 12
∴ a = 12 and b = 12

Question 12.
Discuss the continuity of f on its domain, where
f(x) = |x + 1|, for -3 ≤ x ≤ 2
= |x – 5|, for 2 < x ≤ 7
Solution:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} |x+1|$$
$$= |2 + 1|$$
$$= |3|$$
$$= 3$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} |x-5|$$
$$= |2 - 5|$$
$$= |-3|$$
$$= 3$$

$$f(2) = |2 + 1|$$
$$= 3$$

$$\therefore \quad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

∴ f(x) is continuous at x = 2.

Question 13.
Discuss the continuity of f(x) at x = π4 where,
f(x) = (sinx+cosx)3−22√sin2x−1, for x ≠ π4
= 32√, for x = π4
Solution:

$$f\left(\frac{\pi}{4}\right)=\frac{3}{\sqrt{2}}$$

$$\lim_{x\to\frac{\pi}{4}} f(x)=\lim_{x\to\frac{\pi}{4}} \frac{(\sin x+\cos x)^3-2\sqrt{2}}{\sin 2x-1}$$

$$(\sin x+\cos x)^3 = \left[(\sin x+\cos x)^2\right]^{\frac{3}{2}}$$

$$= (1+\sin 2x)^{\frac{3}{2}}$$

$$\therefore \quad \lim_{x\to\frac{\pi}{4}} f(x)=\lim_{x\to\frac{\pi}{4}} \frac{(1+\sin 2x)^{\frac{3}{2}}-2^{\frac{3}{2}}}{\sin 2x-1}$$

Put $1 + \sin 2x = t$

$\therefore \quad \sin 2x = t - 1$

As $x \to \frac{\pi}{4}$, $t \to 1 + \sin 2\left(\frac{\pi}{4}\right)$

i.e., $t \to 1 + \sin\frac{\pi}{2}$

i.e., $t \to 1 + 1$

i.e., $t \to 2$

$$\therefore \quad \lim_{x\to\frac{\pi}{4}} f(x) = \lim_{t\to 2} \frac{t^{\frac{3}{2}}-2^{\frac{3}{2}}}{t-1-1}$$

$$= \lim_{t\to 2} \frac{t^{\frac{3}{2}}-2^{\frac{3}{2}}}{t-2}$$

$$= \frac{3}{2}(2)^{\frac{1}{2}} \quad \ldots\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}\right]$$

$$= \frac{3\sqrt{2}}{2} = \frac{3}{\sqrt{2}}$$

$$\therefore \quad \lim_{x\to\frac{\pi}{4}} f(x)=f\left(\frac{\pi}{4}\right)$$

$\therefore$ **f(x) is continuous at $x = \frac{\pi}{4}$.**

Question 14.
Determine the values of p and q such that the following function is continuous on the entire real number line.
f(x) = x + 1, for 1 < x < 3
= x2 + px + q, for |x – 2| ≥ 1.
Solution:
|x – 2| ≥ 1
∴ x – 2 ≥ 1 or x – 2 ≤ -1
∴ x ≥ 3 or x ≤ 1
∴ f(x) = x2 + px + q for x ≥ 3 as well as x ≤ 1
Thus, f(x) = x2 + px + q; x ≤ 1
= x + 1; 1 < x < 3 = x2 + px + q; x > 3
f(x) is continuous for all x ∈ R.
∴ f(x) is continuous at x = 1 and x = 3.
As f(x) is continuous at x = 1,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$$

$$\therefore \quad \lim_{x\to 1^-} (x^2+px+q) = \lim_{x\to 1^+} (x+1)$$

$$\therefore \quad (1)^2+p(1)+q = 1+1$$

$$\therefore \quad 1+p+q=2$$

$$\therefore \quad p+q=1 \quad \ldots(i)$$

As f(x) is continuous at $x = 3$,

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

$$\therefore \quad \lim_{x\to 3^-} (x+1) = \lim_{x\to 3^+} (x^2+px+q)$$

$$\therefore \quad 3+1=(3)^2+3p+q$$

$$\therefore \quad 3p+q+9=4$$

$$\therefore \quad 3p+q=-5 \quad \ldots(ii)$$

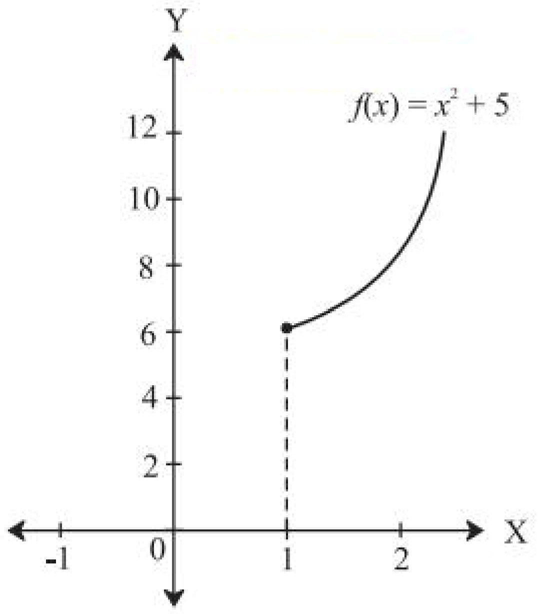
Subtracting (i) from (ii), we get
2p = -6

∴ $p = -3$
Substituting $p = -3$ in (i), we get
$-3 + q = 1$
∴ $q = 4$
∴ $p = -3$ and $q = 4$

Question 15.
Show that there is a root for the equation $2x^3 - x - 16 = 0$ between 2 and 3.
Solution:
Let $f(x) = 2x^3 - x - 16$
f(x) is a polynomial function and hence it is continuous for all $x \in R$.
A root of f(x) exists, if f(x) = 0 for at least one value of x.
$f(2) = 2(2)^3 - 2 - 16 = -2 < 0$
$f(3) = 2(3)^3 - 3 - 16 = 35 > 0$
∴ $f(2) < 0$ and $f(3) > 0$
∴ By intermediate value theorem,
there has to be point 'c' between 2 and 3 such that f(c) = 0.
∴ There is a root of the given equation between 2 and 3.

Question 16.
Show that there is a root for the equation $x^3 - 3x = 0$ between 1 and 2.
Solution:
Let $f(x) = x^3 - 3x$
f(x) is a polynomial function and hence it is continuous for all $x \in R$.
A root of f(x) exists, if f(x) = 0 for at least one value of x.
$f(1) = (1)^3 - 3(1) = -2 < 0$
$f(2) = (2)^3 - 3(2) = 2 > 0$
∴ $f(1) < 0$ and $f(2) > 0$
∴ By intermediate value theorem,
there has to be point 'c' between 1 and 2 such that f(c) = 0.
There is a root of the given equation between 1 and 2.

Question 17.
Let $f(x) = ax + b$ (where a and b are unknown)
$= x^2 + 5$ for $x \in R$
Find the values of a and b, so that f(x) is continuous at x = 1.



Solution:
$f(x) = x^2 + 5, x \in R$
∴ $f(1) = 1 + 5 = 6$
If $f(x) = ax + b$ is continuous at x = 1, then
$f(1) = \lim_{x \to 1}(ax+b) = a + b$
∴ $6 = a + b$ where, $a, b \in R$
∴ There are infinitely many values of a and b.

Question 18.
Activity: Suppose $f(x) = px + 3$ for $a \le x \le b$
$= 5x^2 - q$ for $b < x \le c$
Find the condition on p, q, so that f(x) is continuous on [a, c], by filling in the boxes.

$f(b) = \boxed{}$

$\lim\limits_{x \to b^+} f(x) = \boxed{}$

$\therefore\ pb + 3 = \boxed{} - q$

$\therefore\ p = \frac{\boxed{}}{b}$ is the required condition.

Solution:

$f(b) = \boxed{pb + 3}$

$\lim\limits_{x \to b^+} f(x) = \boxed{5b^2} - q$

$\therefore\ pb + 3 = \boxed{5b^2} - q$

$\therefore\ p = \frac{\boxed{5b^2 - q - 3}}{b}$ is the required condition.

Maharashtra State Board 11th Maths Solutions Chapter 8 Continuity Miscellaneous Exercise 8

(I) Select the correct answer from the given alternatives.

Question 1.
f(x) = $\frac{2^{\cot x} - 1}{\pi - 2x}$, for x ≠ $\frac{\pi}{2}$
= log √2, for x = $\frac{\pi}{2}$
(A) f is continuous at x = $\frac{\pi}{2}$
(B) f has a jump discontinuity at x = $\frac{\pi}{2}$
(C) f has a removable discontinuity
(D) $\lim_{x \to \frac{\pi}{2}} f(x) = 2 \log 3$
Answer:
(A) f is continuous at x = $\frac{\pi}{2}$

Hint:

$$f\left(\frac{\pi}{2}\right) = \log\sqrt{2}$$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{2^{\cot x}-1}{\pi-2x} = \lim_{x\to\frac{\pi}{2}} \frac{2^{\tan\left(\frac{\pi}{2}-x\right)}-1}{2\left(\frac{\pi}{2}-x\right)}$$

Put $\frac{\pi}{2} - x = h$

As $x \to \frac{\pi}{2}$, $h \to 0$

$$\therefore \quad \lim_{x\to\frac{\pi}{2}} f(x) = \lim_{h\to 0} \frac{2^{\tan h}-1}{2h}$$

$$= \frac{1}{2}\lim_{h\to 0}\left(\frac{2^{\tan h}-1}{\tan h} \times \frac{\tan h}{h}\right)$$

$$\ldots(\because h \to 0, \tan h \to 0, \tan h \neq 0)$$

$$= \frac{1}{2}\lim_{h\to 0}\frac{2^{\tan h}-1}{\tan h} \times \lim_{h\to 0}\frac{\tan h}{h}$$

$$= \frac{1}{2}\cdot\log 2\cdot(1)$$

$$= \log\sqrt{2} = f\left(\frac{\pi}{2}\right)$$

$\therefore \quad f(x)$ is continuous at $x = \frac{\pi}{2}$

Question 2.
If f(x) = 1−2√sinxπ−4x, for x ≠ π4 is continuous at x = π4, then f(π4) =
(A) 12√
(B) −12√
(C) −14
(D) 14
Answer:
(D) 14
Hint:

f(x) is continuous at $x = \frac{\pi}{4}$

$$\therefore \quad f\left(\frac{\pi}{4}\right) = \lim_{x \to \frac{\pi}{4}} f(x)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{1-\sqrt{2}\sin x}{\pi - 4x}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}\left(\sin x - \frac{1}{\sqrt{2}}\right)}{4\left(x - \frac{\pi}{4}\right)}$$

$$= \frac{\sqrt{2}}{4} \lim_{x \to \frac{\pi}{4}} \frac{\sin x - \sin\frac{\pi}{4}}{x - \frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{4} \lim_{x \to \frac{\pi}{4}} \frac{2\cos\left(\frac{x+\frac{\pi}{4}}{2}\right) \cdot \sin\left(\frac{x-\frac{\pi}{4}}{2}\right)}{x - \frac{\pi}{4}}$$

$$= \frac{\sqrt{2}}{4} \cdot \lim_{x \to \frac{\pi}{4}} \cos\left(\frac{x}{2} + \frac{\pi}{8}\right) \cdot \lim_{x \to \frac{\pi}{4}} \frac{\sin\left(\frac{x-\frac{\pi}{4}}{2}\right)}{\frac{x-\frac{\pi}{4}}{2}}$$

$$= \frac{\sqrt{2}}{4} \cdot \cos\left(\frac{x}{8} + \frac{\pi}{8}\right) \times 1$$

$$\ldots \left[\begin{array}{l} \because x \to \frac{\pi}{4}, x - \frac{\pi}{4} \to 0, \frac{x-\frac{\pi}{4}}{2} \to 0 \\ \text{and } \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \end{array}\right]$$

$$= \frac{\sqrt{2}}{4} \times \cos\frac{\pi}{4}$$

$$= \frac{1}{4}$$

Question 3.
If f(x) = $(sin2x)tan5x(e_{2x}-1)_2$, for x ≠ 0 is continuous at x = 0, then f(0) is
(A) $10e_2$
(B) $10e_4$
(C) 54
(D) 52
Answer:
(D) 52

Hint:

$f(x)$ is continuous at $x = 0$

$$f(0) = \lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0}\frac{(\sin 2x)(\tan 5x)}{\left(e^{2x}-1\right)^2}$$

$$= \frac{\lim_{x\to 0}\frac{\sin 2x}{2x}\times\lim_{x\to 0}\frac{\tan 5x}{5x}\times 2\times 5}{\left(\lim_{x\to 0}\frac{e^{2x}-1}{2x}\right)^2\times(2)^2}$$

$$= \frac{1\times 1\times 2\times 5}{(1)^2\times 4}$$

$$\ldots\left[\because x\to 0,\ 2x\to 0,\ 5x\to 0 \text{ and } \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1,\ \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right]$$

$$= \frac{5}{2}$$

Question 4.
f(x) = $\frac{x^2-7x+10}{x^2+2x-8}$, for x ∈ [-6, -3]
(A) f is discontinuous at x = 2
(B) f is discontinuous at x = -4
(C) f is discontinuous at x = 0
(D) f is discontinuous at x = 2 and x = -4
Answer:
(B) f is discontinuous at x = -4
Hint:
f(x) = $\frac{x^2-7x+10}{x^2+2x-8}$, for x ∈ [-6, -3]
= $\frac{x^2-7x+10}{(x+4)(x-2)}$
Here f(x) is a rational function and is continuous everywhere except at the points Where denominator becomes zero.
Here, denominator becomes zero when x = -4 or x = 2
But x = 2 does not lie in the given interval.
∴ x = -4 is the point of discontinuity.

Question 5.
If f(x) = ax^2 + bx + 1, for |x – 1| ≥ 3 and
= 4x + 5, for -2 < x < 4
is continuous everywhere then,
(A) a = $\frac{1}{2}$, b = 3
(B) a = $-\frac{1}{2}$, b = -3
(C) a = $-\frac{1}{2}$, b = 3
(D) a = $\frac{1}{2}$, b = -3
Answer:
(A) a = $\frac{1}{2}$, b = 3
Hint:
f(x) = ax^2 + bx + 1, |x – 1| ≥ 3
= 4x + 5; -2 < x < 4
The first interval is
|x – 1| ≥ 3
∴ x – 1 ≥ 3 or x – 1 ≤ -3
∴ x ≥ 4 or x ≤ -2
∴ f(x) is same for x ≤ -2 as well as x ≥ 4.
∴ f(x) is defined as:
f(x) = ax^2 + bx + 1; x ≤ -2
= 4x + 5; -2 < x < 4
= ax^2 + bx + 1; x ≥ 4
f(x) is continuous everywhere.
∴ f(x) is continuous at x = -2 and x = 4.
As f(x) is continuous at x = -2,
$\lim_{x\to -2^-} f(x)=\lim_{x\to -2^+} f(x)$
∴ $\lim_{x\to -2}(ax^2+bx+1)=\lim_{x\to -2}(4x+5)$
∴ $a(-2)^2$ + b(-2) + 1 = 4(-2) + 5
∴ 4a – 2b + 1 = -3
∴ 4a – 2b = -4
∴ 2a – b = -2 .....(i)

∵ f(x) is continuous at x = 4,

$lim_{x\to4^-}f(x)=lim_{x\to4^+}f(x)$

∴ $lim_{x\to4}(4x+5)=lim_{x\to4}(ax^2+bx+1)$

4(4) + 5 = $a(4)^2$ + b(4) + 1

16a + 4b + 1 = 21

16a + 4b = 20

4a + b = 5 …..(ii)

Adding (i) and (ii), we get

6a = 3

∴ a = 12

Substituting a = 12 in (ii), we get

4(12) + b = 5

∴ 2 + b = 5

∴ b = 3

∴ a = 12, b = 3

Question 6.

f(x) = $(16^x-1)(9^x-1)(27^x-1)(32^x-1)$, for x ≠ 0

= k, for x = 0

is continuous at x = 0, then 'k' =

(A) 83

(B) 815

(C) −815

(D) 203

Answer:

(B) 815

Hint:

f(x) is continuous at x = 0

$$f(0) = \lim_{x\to0} f(x)$$

$$k = \lim_{x\to0}\frac{(16^x-1)(9^x-1)}{(27^x-1)(32^x-1)}$$

$$= \frac{\lim_{x\to0}\left(\frac{16^x-1}{x}\right)\times\lim_{x\to0}\left(\frac{9^x-1}{x}\right)}{\lim_{x\to0}\left(\frac{27^x-1}{x}\right)\times\lim_{x\to0}\left(\frac{32^x-1}{x}\right)}$$

$$= \frac{\log16\times\log9}{\log27\times\log32} \quad \ldots\left[\because \lim_{x\to0}\frac{a^x-1}{x}=\log a\right]$$

$$= \frac{4\log2\times2\log3}{3\log3\times5\log2}$$

$$= \frac{8}{15}$$

Question 7.

f(x) = $32^x-8^x-4^x+14^x-2^{x+1}+1$, for x ≠ 0

= k, for x = 0,

is continuous at x = 0, then value of 'k' is

(A) 6

(B) 4

(C) (log 2) (log 4)

(D) 3 log 4

Answer:

(A) 6

Hint:

f(x) is continuous at $x = 0$. …(given)

$$f(0) = \lim_{x\to 0} f(x)$$

$$k = \lim_{x\to 0}\frac{32^x - 8^x - 4^x + 1}{4^x - 2^{x+1} + 1}$$

$$= \lim_{x\to 0}\frac{\left(4^x-1\right)\left(8^x-1\right)}{\left(2^x-1\right)^2}$$

$$= \frac{\lim\limits_{x\to 0}\left(\frac{4^x-1}{x}\right)\left(\frac{8^x-1}{x}\right)}{\lim\limits_{x\to 0}\left(\frac{2^x-1}{x}\right)^2}$$

$$= \frac{\lim\limits_{x\to 0}\left(\frac{4^x-1}{x}\right)\cdot\lim\limits_{x\to 0}\left(\frac{8^x-1}{x}\right)}{\left(\lim\limits_{x\to 0}\frac{2^x-1}{x}\right)^2}$$

$$= \frac{\log 4 \times \log 8}{(\log 2)}$$

$$= \frac{2\log 2 \times 3\log 2}{(\log 2)^2} = 6$$

Question 8.
If f(x) = $\frac{12^x-4^x-3^x+1}{1-\cos 2x}$, for x ≠ 0 is continuous at x = 0 then the value of f(0) is
(A) $\frac{\log 12}{2}$
(B) log 2 . log 3
(C) $\log 2 \cdot \frac{\log 3}{2}$
(D) None of these
Answer:
(B) log 2 . log 3
Hint:

f(x) is continuous at $x = 0$. …(given)

$$f(0) = \lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0}\frac{12^x - 4^x - 3^x + 1}{1-\cos 2x}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{4^x\left(3^x-1\right)\left(3^x-1\right)}{\sin^2 x}$$

$$= \frac{1}{2}\lim_{x\to 0}\frac{\left(3^x-1\right)\left(4^x-1\right)}{\sin^2 x}$$

$$= \frac{1}{2}\,\frac{\lim\limits_{x\to 0}\left(\frac{3^x-1}{x}\right)\cdot\lim\limits_{x\to 0}\left(\frac{4^x-1}{x}\right)}{\left(\lim\limits_{x\to 0}\frac{\sin x}{x}\right)^2}$$

$$= \frac{1}{2}\times\frac{(\log 3)\times(\log 4)}{(1)^2}$$

$$= \frac{1}{2}\times\log 3\times\log(2)^2$$

$$= \log 3.\ \log 2$$

Question 9.
If f(x) = $\left(\frac{4+5x}{4-7x}\right)^{\frac{4}{x}}$, for x ≠ 0 and f(0) = k, is continuous at x = 0, then k is
(A) e^7
(B) e^3
(C) e^{12}
(D) $e^{\frac{3}{4}}$
Answer:
(C) e^{12}
Hint:

f(x) is continuous at $x = 0$

$$f(0) = \lim_{x\to 0} f(x)$$

$$= \lim_{x\to 0}\left(\frac{4+5x}{4-7x}\right)^{\frac{4}{x}}$$

$$= \lim_{x\to 0}\left[\frac{4\left(1+\frac{5x}{4}\right)}{4\left(1-\frac{7x}{4}\right)}\right]^{\frac{4}{x}}$$

$$= \frac{\lim_{x\to 0}\left[\left(1+\frac{5x}{4}\right)^{\frac{4}{5x}}\right]^{5}}{\lim_{x\to 0}\left[\left(1-\frac{7x}{4}\right)^{\frac{-4}{7x}}\right]^{-7}}$$

$$= \frac{e^{5}}{e^{-7}} \quad \ldots \left[\begin{array}{l}\because x\to 0, \frac{5x}{4}\to 0, \frac{-7x}{4}\to 0 \\ \text{and } \lim_{x\to 0}(1+x)^{\frac{1}{x}} = e\end{array}\right]$$

$$= e^{12}$$

Question 10.

If $f(x) = \lfloor x \rfloor$ for $x \in (-1, 2)$, then f is discontinuous at

(A) x = -1, 0, 1, 2
(B) x = -1, 0, 1
(C) x = 0, 1
(D) x = 2

Answer:
(C) x = 0, 1

Hint:

$f(x) = \lfloor x \rfloor$, $x \in (-1, 2)$

This function is discontinuous at all integer values of x between -1 and 2.
∴ f(x) is discontinuous at x = 0 and x = 1.

II. Discuss the continuity of the following functions at the point(s) or on the interval indicated against them.

Question 1.

$f(x) = \frac{x^2-3x-10}{x-5}$, for $3 \le x \le 6$, $x \ne 5$
= 10, for x = 5
$= \frac{x^2-3x-10}{x-5}$, for $6 < x \le 9$

Solution:

$\frac{x^2-3x-10}{x-5}$ is not defined at $x = 5$.

$\therefore \quad f(x) = \frac{x^2-3x-10}{x-5}$, where $x \in [3, 5) \cup (5, 6]$

We can write f(x) explicitly, as follows:

$$
\begin{aligned}
f(x) &= \frac{x^2-3x-10}{x-5}, \quad 3 \le x < 5 \\
&= 10 \quad\quad , x = 5 \\
&= \frac{x^2-3x-10}{x-5}, \quad 5 < x \le 6 \\
&= \frac{x^2-3x-10}{x-5}, \quad 6 < x \le 9
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad f(x) &= \frac{(x-5)(x+2)}{(x-5)}, \quad 3 \le x < 5 \\
&= 10, \quad \text{for } x = 5 \\
&= \frac{(x-5)(x+2)}{(x-5)}, \quad \text{for } 5 < x \le 9
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad f(x) &= x + 2, \quad 3 \le x < 5 \\
&= 10, \quad x = 5 \\
&= x + 2, \quad 5 < x \le 9
\end{aligned}
$$

$f(5) = 10$

$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (x + 2) = 5 + 2 = 7$

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (x + 2) = 5 + 2 = 7$

$\therefore \quad f(5) \ne \lim_{x \to 5} f(x)$

$\therefore$ f(x) is continuous on its domain except at $x = 5$.

Question 2.
$f(x) = 2x^2 - 2x + 5$, for $0 \le x \le 2$
$= \frac{1-3x-x^2}{1-x}$, for $2 < x < 4$
$= \frac{x^2-25}{x-5}$, for $4 \le x \le 7$ and $x \ne 5$
$= 7$, for $x = 5$
Solution:
The domain of f(x) is [0, 7].
(i) For $0 \le x \le 2$
$f(x) = 2x^2 - 2x + 5$
It is a polynomial function and is Continuous at all point in [0, 2).

(ii) For $2 < x < 4$
$f(x) = \frac{1-3x-x^2}{1-x}$
It is a rational function and is continuous everwhere except at points where its denominator becomes zero.
Denominator becomes zero at x = 1
But x = 1 does not lie in the interval.
f(x) is continuous at all points in (2, 4).

(iii) For $4 \le x \le 7$, $x \le 5$
$f(x) = \frac{x^2-25}{x-5}$
It is a rational function and is continuous everywhere except at points where its denominator becomes zero.
Denominator becomes zero at x = 5
But x = 5 does not lie in the interval.
$\therefore$ f(x) is continuous at all points in (4, 7] – {5}.

(iv) For continuity at x = 2:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} f(x)\ (2x^2 - 2x + 5)$$
$$= 2(2)^2 - 2(2) + 5$$
$$= 8 - 4 + 5$$
$$= 9$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{1-3x-x^2}{1-x}$$
$$= \frac{\lim_{x\to 2^+}\left(1-3x-x^2\right)}{\lim_{x\to 2^+}(1-x)}$$
$$= \frac{1-3(2)-(2)^2}{1-2}$$
$$= \frac{1-6-4}{-1}$$
$$= \frac{-9}{-1}$$
$$= 9$$

Also $f(2) = 2(2)^2 - 2(2) + 5$
$$= 8 - 4 + 5$$
$$= 9$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

∴ f(x) is continuous at x = 2.

(v) For continuity at x = 4:

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{1-3x-x^2}{1-x}$$
$$= \frac{\lim_{x\to 4^-}\left(1-3x-x^2\right)}{\lim_{x\to 4^-}(1-x)}$$
$$= \frac{1-3(4)-(4)^2}{1-4}$$
$$= \frac{1-12-16}{1-4} = \frac{-27}{-3} = 9$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{x^2-25}{x-5}$$
$$= \frac{\lim_{x\to 4^+}\left(x^2-25\right)}{\lim_{x\to 4^+}(x-5)}$$
$$= \frac{(4)^2-25}{4-5}$$
$$= \frac{16-25}{-1} = 9$$

Also $f(4) = \frac{(4)^2-25}{4-5}$
$$= \frac{16-25}{-1} = 9$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^+} f(x) = f(4)$$

∴ f(x) is continuous at x = 4.

(vi) For continuity at x = 5.
f(5) = 7

$$\lim_{x\to5} f(x) = \lim_{x\to5}\frac{x^2-25}{x-5}$$
$$= \lim_{x\to5}\frac{(x-5)(x+5)}{x-5}$$
$$= \lim_{x\to5}(x+5) \quad \ldots\begin{bmatrix}\because x\to5, x\neq5\\ \therefore\ x-5\neq0\end{bmatrix}$$
$$= 5+5$$
$$= 10$$
$$\lim_{x\to5} f(x) \neq f(5)$$

∴ f(x) is discontinuous at x = 5.
Thus, f(x) is continuous at all points on its domain except at x = 5.

Question 3.
f(x) = $\frac{\cos 4x-\cos 9x}{1-\cos x}$, for x ≠ 0

f(0) = $\frac{68}{15}$, at x = 0 on $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Solution:

The domain of f(x) is $[-\frac{\pi}{2}, \frac{\pi}{2}]$

(i) For $[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$:

f(x) = $\frac{\cos 4x-\cos 9x}{1-\cos x}$

It is a rational function and is continuous everywhere except at points where its denominator becomes zero.
Denominator becomes zero when cos x = 1,
i.e., x = 0
But x = 0 does not lie in the interval.
∴ f(x) is continuous at all points in $[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$

(ii) For continuity at x = 0:

$$f(0) = \frac{68}{15}$$
$$\lim_{x\to0} f(x) = \lim_{x\to0}\frac{\cos 4x-\cos 9x}{1-\cos x}$$
$$= \lim_{x\to0}\frac{2\sin\left(\frac{4x+9x}{2}\right)\cdot\sin\left(\frac{9x-4x}{2}\right)}{2\sin^2\frac{x}{2}}$$
$$= \lim_{x\to0}\frac{\sin\left(\frac{13x}{2}\right)\cdot\sin\left(\frac{5x}{2}\right)}{\left(\sin\frac{x}{2}\right)^2}$$
$$= \lim_{x\to0}\left[\frac{\frac{\sin\left(\frac{13x}{2}\right)\cdot\sin\left(\frac{5x}{2}\right)}{x^2}}{\frac{\left(\sin\frac{x}{2}\right)^2}{x^2}}\right]$$
$$\ldots[\because x\to0, x\neq0, x^2\neq0]$$

$$= \frac{\lim_{x\to0} \frac{\sin\left(\frac{13x}{2}\right)}{x} \cdot \frac{\sin\left(\frac{5x}{2}\right)}{x}}{\lim_{x\to0}\left(\frac{\sin\frac{x}{2}}{x}\right)^2}$$

$$= \frac{\lim_{x\to0} \frac{\sin\left(\frac{13x}{2}\right)}{\frac{13x}{2}} \times \frac{13}{2} \cdot \lim_{x\to0} \frac{\sin\left(\frac{5x}{2}\right)}{\frac{5x}{2}} \times \frac{5}{2}}{\lim_{x\to0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 \times \frac{1}{4}}$$

$$= \frac{1 \times \frac{13}{2} \times 1 \times \frac{5}{2}}{(1)^2 \times \frac{1}{4}} \ldots \left[\because x\to0, \frac{13x}{2}\to0, \frac{5x}{2}\to0 \text{ and } \lim_{\theta\to0}\frac{\sin\theta}{\theta} = 1\right]$$

$$= 65$$

$\therefore \lim_{x\to0} f(x) \neq f(0)$
∴ f(x) is discontinuous at x = 0.

Question 4.
f(x) = $\frac{\sin^2\pi x}{3(1-x)^2}$, for x ≠ 1
= $\frac{\pi^2\sin^2\left(\frac{\pi x}{2}\right)}{3+4\cos^2\left(\frac{\pi x}{2}\right)}$, for x = 1, at x = 1
Solution:

$$f(1) = \frac{\pi^2\sin\left(\frac{\pi}{2}\right)}{3+4\cos^2\left(\frac{\pi}{2}\right)}$$

$$= \frac{\pi^2 \times 1}{3+4(0)^2} = \frac{\pi^2}{3}$$

$$\lim_{x\to1} f(x) = \lim_{x\to1}\frac{\sin^2\pi x}{3(1-x)^2}$$

Put $1 - x = h$

$\therefore$ $x = 1 - h$

As $x \to 1$, $h \to 0$

$$\therefore \lim_{x\to1} f(x) = \lim_{h\to0}\frac{\sin^2\pi(1-h)}{3[1-(1-h)]^2}$$

$$= \lim_{h\to0}\frac{[\sin(\pi-\pi h)]^2}{3h^2}$$

$$= \frac{1}{3}\lim_{h\to0}\frac{(\sin\pi h)^2}{h^2} = \frac{1}{3}\lim_{h\to0}\left(\frac{\sin\pi h}{h}\right)^2$$

$$= \frac{1}{3}\lim_{h\to0}\left(\frac{\sin\pi h}{\pi h}\right)^2 \times \pi^2$$

$$= \frac{1}{3} \times (1)^2 \times \pi^2 \ldots \left[\because h\to0, \pi h\to0 \text{ and } \lim_{\theta\to0}\frac{\sin\theta}{\theta} = 1\right]$$

$$= \frac{\pi^2}{3}$$

$\therefore$ $\lim_{x\to1} f(x) = f(1)$

$\therefore$ **f(x) is continuous at x = 1.**

Question 5.
f(x) = $\frac{|x+1|}{2x^2+x-1}$, for x ≠ -1
= 0, for x = -1, at x = -1
Solution:

$|x+1| = x+1 \quad , x \geq -1$

$= -(x+1) \quad , x < -1$

$f(x) = \frac{-(x+1)}{2x^2+x-1} \quad , x < -1$

$= 0 \quad , x = -1$

$= \frac{x+1}{2x^2+x-1} \quad , x > -1$

$f(-1) = 0$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{-(x+1)}{2x^2+x-1}$$

$$= \lim_{x \to -1^-} \frac{-(x+1)}{(x+1)(2x-1)}$$

$$= \lim_{x \to -1^-} \frac{-1}{2x-1} \quad \ldots \begin{bmatrix} \text{As } x \to -1, x \neq -1 \\ \therefore x+1 \neq 0 \end{bmatrix}$$

$$= \frac{-1}{2(-1)-1}$$

$$= \frac{1}{3}$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{x+1}{2x^2+x-1}$$

$$= \lim_{x \to -1^+} \frac{x+1}{(x+1)(2x-1)}$$

$$= \lim_{x \to -1^+} \frac{-1}{2x-1} \quad \ldots \begin{bmatrix} \text{As } x \to -1, x \neq -1 \\ \therefore x+1 \neq 0 \end{bmatrix}$$

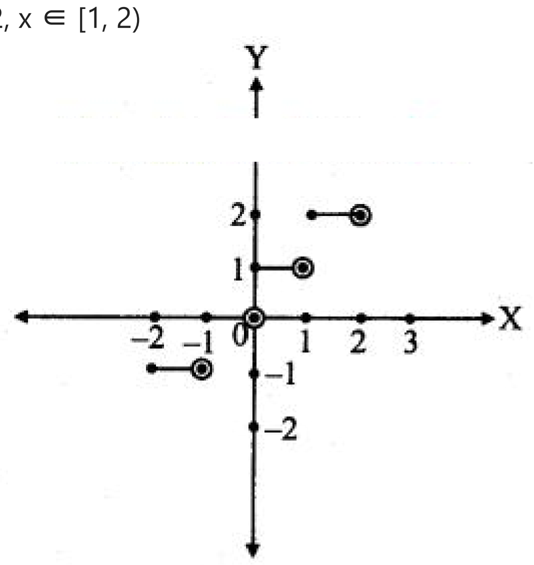
$$= \frac{1}{2(-1)-1}$$

$$= \frac{-1}{3}$$

$\therefore \quad \lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$

$\therefore \quad$ **f(x) is discontinuous at $x = -1$.**

Question 6.
f(x) = [x + 1] for x ∈ [-2, 2)
Where [*] is greatest integer function.
Solution:
f(x) = [x + 1], x ∈ [-2, 2)
∴ f(x) = -1, x ∈ [-2, -1)
= 0, x ∈ [-1, 0)
= 1, x ∈ [0, 1)
= 2, x ∈ [1, 2)



For continuity at $x = -1$

$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} [x+1]$

$= -1$

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} [x+1]$

$= 0$

$\therefore \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$

∴ f(x) is discontinuous at x = -1.
Similarly, f(x) is discontinuous at the points x = 0 and x = 1.

Question 7.
$f(x) = 2x^2 + x + 1$, for $|x – 3| \geq 2$
$= x^2 + 3$, for $1 < x < 5$
Solution:
$|x – 3| \geq 2$
∴ $x – 3 \geq 2$ or $x – 3 \leq -2$
∴ $x \geq 5$ or $x \leq 1$
∴ $f(x) = 2x^2 + x + 1, x \leq 1$
$= x^2 + 3, 1 < x < 5$
$= 2x^2 + x + 1, x \geq 5$
Consider the intervals
x < 1 , i.e., (-∞, 1)
1 < x < 5, i.e., (1, 5) x > 5, i.e., (5, ∞)
In all these intervals, f(x) is a polynomial function and hence is continuous at all points.
For continuity at x = 1:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x^2 + x + 1)$$
$$= 2(1)^2 + 1 + 1$$
$$= 4$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 3)$$
$$= (1)^2 + 3$$
$$= 4$$
Also, $f(1) = 2(1)^2 + 1 + 1$.
$$= 4$$
$$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$
∴ f(x) is continuous at $x = 1$
For continuity at $x = 5$:
$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (x^2 + 3)$$
$$= (5)^2 + 3$$
$$= 28$$
$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (2x^2 + x + 1)$$
$$= 2(5)^2 + 5 + 1$$
$$= 56$$
$$\therefore \quad \lim_{x \to 5^-} f(x) \neq \lim_{x \to 5^+} f(x)$$

∴ f(x) is discontinuous at x = 5.
∴ f(x) is continuous for all x ∈ R, except at x = 5.

III. Identify discontinuities if any for the following functions as either a jump or a removable discontinuity on their respective domains.

Question 1.
$f(x) = x^2 + x – 3$, for $x \in [-5, -2)$
$= x^2 – 5$, for $x \in (-2, 5]$
Solution:
f(-2) has not been defined.
$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (x^2 + x - 3)$
$= (-2)^2 + (-2) – 3$
= 4 – 2 – 3
= -1
$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (x^2 - 5)$
$= (-2)^2 – 5$
= 4 – 5
= -1
∴ $\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x)$
∴ $\lim_{x \to -2} f(x)$ exists.
But f(-2) has not been defined.
∴ f(x) has a removable discontinuity at x = -2.

Question 2.
f(x) = $x^2 + 5x + 1$, for $0 \le x \le 3$
= $x^3 + x + 5$, for $3 < x \le 6$
Solution:
$\lim_{x\to3^-} f(x) = \lim_{x\to3^-}(x^2+5x+1)$
= $(3)^2 + 5(3) + 1$
= 9 + 15 + 1
= 25
$\lim_{x\to3^+} f(x) = \lim_{x\to3^+}(x^3+x+5)$
= $(3)^3 + 3 + 5$
= 27 + 3 + 5
= 35
$\therefore \lim_{x\to3^-} f(x) \ne \lim_{x\to3^+} f(x)$
$\therefore \lim_{x\to3} f(x)$ does not exist.
∴ f(x) is discontinuous at x = 3.
∴ f(x) has a jump discontinuity at x = 3.

Question 3.
f(x) = $\frac{x^2+x+1}{x+1}$, for $x \in [0, 3)$
= $\frac{3x+4}{x^2-5}$, for $x \in [3, 6]$
Solution:

$$\begin{aligned}\lim_{x\to3^-} f(x) &= \lim_{x\to3^-}\frac{x^2+x+1}{x+1}\\ &= \frac{\lim_{x\to3^-}(x^2+x+1)}{\lim_{x\to3^-}(x+1)}\\ &= \frac{(3)^2+3+1}{3+1}\\ &= \frac{13}{4}\end{aligned}$$

$$\begin{aligned}\lim_{x\to3^+} f(x) &= \lim_{x\to3^+}\frac{3x+4}{x^2-5}\\ &= \frac{\lim_{x\to3^+}(3x+4)}{\lim_{x\to3^+}(x^2-5)}\\ &= \frac{3(3)+4}{(3)^2-5}\\ &= \frac{13}{4}\end{aligned}$$

$$\begin{aligned}\text{Also, } f(3) &= \frac{3(3)+4}{(3)^2-5}\\ &= \frac{13}{4}\end{aligned}$$

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^+} f(x) = f(3)$$

∴ f(x) is continuous at x = 3.

IV. Discuss the continuity of the following functions at the point or on the interval indicated against them. If the function is discontinuous, identify the type of discontinuity and state whether the discontinuity is removable. If it has a removable discontinuity, redefine the function so that it becomes continuous.

Question 1.
f(x) = $\frac{(x+3)(x^2-6x+8)}{x^2-x-12}$
Solution:
f(x) = $\frac{(x+3)(x^2-6x+8)}{x^2-x-12}$
= $\frac{(x+3)(x-2)(x-4)}{(x-4)(x+3)}$
∴ f(x) is not defined at x = 4 and x = -3.
∴ The domain of function f = R – {-3, 4}.
For x ≠ -3 and 4,

$$f(x) = \frac{(x+3)(x-2)(x-4)}{(x-4)(x+3)}$$

$$\therefore \quad \lim_{x \to -3} f(x) = \lim_{x \to -3} \frac{(x+3)(x-2)(x-4)}{(x-4)(x+3)}$$

$$= \lim_{x \to -3} (x-2) \qquad \ldots \left[\begin{array}{l} \because x \to -3, x \neq 3 \\ \therefore x+3 \neq 0 \end{array}\right]$$

$$= -5$$

$$\text{Also, } \lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{(x+3)(x-2)(x-4)}{(x-4)(x+3)}$$

$$= \lim_{x \to 4} (x-2) \qquad \ldots \left[\begin{array}{l} \because x \to 4, x \neq 4 \\ \therefore x-4 \neq 0 \end{array}\right]$$

$$= 2$$

f(x) is discontinuous at x = 4 and x = -3.
This discontinuity is removable.
∴ f(x) can be redefined as
f(x) = $\frac{(x+3)(x^2-6x+8)}{x^2-x-12}$, for x ≠ 4, x ≠ -3
= -5, for x ∈ R – {-3, 4}, x = -3
= 2, for x ∈ R – {-3, 4}, x = 4

Question 2.
f(x) = $x^2 + 2x + 5$, for x ≤ 3
= $x^3 - 2x^2 - 5$, for x > 3
Solution:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \left(x^2 + 2x + 5\right)$$

$$= (3)^2 + 2(3) + 5$$

$$= 9 + 6 + 5$$

$$= 20$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \left(x^3 - 2x^2 - 5\right)$$

$$= (3)^3 - 2(3)^2 - 5$$

$$= 27 - 18 - 5$$

$$= 4$$

$$\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$$

$\lim_{x \to 3} f(x)$ does not exist.

∴ f(x) is discontinuous at x = 3.
This discontinuity is irremovable.

V. Find k if the following functions are continuous at the points indicated against them.

Question 1.
f(x) = $\left(\frac{5x-8}{8-3x}\right)^{\frac{3}{2x-4}}$, for x ≠ 2
= k, for x = 2 at x = 2.
Solution:
f(x) is continuous at x = 2. …..(given)

$$\therefore \quad f(2) = \lim_{x \to 2} f(x)$$

$$\therefore \quad k = \lim_{x \to 2} \left(\frac{5x-8}{8-3x}\right)^{\frac{3}{2x-4}}$$

Put $x - 2 = h$

$\therefore \quad x = 2 + h$

As $x \to 2, h \to 0$

$$\therefore \quad k = \lim_{h \to 0} \left[\frac{5(2+h)-8}{8-3(2+h)}\right]^{\frac{3}{2(2+h)-4}}$$

$$= \lim_{h \to 0} \left(\frac{10+5h-8}{8-6-3h}\right)^{\frac{3}{4+2h-4}}$$

$$= \lim_{h \to 0} \left(\frac{2+5h}{2-3h}\right)^{\frac{3}{2h}}$$

$$= \lim_{h \to 0} \left[\frac{2\left(1+\frac{5h}{2}\right)}{2\left(1-\frac{3h}{2}\right)}\right]^{\frac{3}{2h}}$$

$$= \lim_{h \to 0} \frac{\left(1+\frac{5h}{2}\right)^{\frac{3}{2h}}}{\left(1-\frac{3h}{2}\right)^{\frac{3}{2h}}}$$

$$= \frac{\lim_{h \to 0} \left[\left(1+\frac{5h}{2}\right)^{\frac{2}{5h}}\right]^{\frac{5}{2} \times \frac{3}{2}}}{\lim_{h \to 0} \left[\left(1-\frac{3h}{2}\right)^{\frac{-2}{3h}}\right]^{\frac{-3}{2} \times \frac{3}{2}}}$$

$$= \frac{e^{\frac{15}{4}}}{e^{\frac{-9}{4}}} \quad \ldots \left[\begin{array}{l} \because h \to 0, \frac{5h}{2} \to 0, \frac{-3h}{2} \to 0 \\ \text{and } \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e \end{array}\right]$$

$$= e^{\frac{24}{4}}$$

$$= e^6$$

Question 2.

f(x) = $45^x - 9^x - 5^x + 1 (k^x - 1)(3^x - 1)$, for x ≠ 0

= 23, for x = 0, at x = 0

Solution:

f(x) is continuous at x = 0 …..(given)

$\therefore \quad \lim_{x\to 0} f(x) = f(0)$

$$\therefore \quad \lim_{x\to 0}\frac{(45)^x - 9^x - 5^x + 1}{(k^x-1)(3^x-1)} = \frac{2}{3}$$

$$\therefore \quad \lim_{x\to 0}\frac{9^x \cdot 5^x - 9^x - 5^x + 1}{(k^x-1)(3^x-1)} = \frac{2}{3}$$

$$\therefore \quad \lim_{x\to 0}\frac{9^x(5^x-1) - 1(5^x-1)}{(k^x-1)(3^x-1)} = \frac{2}{3}$$

$$\therefore \quad \lim_{x\to 0}\frac{(5^x-1)(9^x-1)}{(k^x-1)(3^x-1)} = \frac{2}{3}$$

$$\therefore \quad \lim_{x\to 0}\frac{\frac{(5^x-1)(9^x-1)}{x^2}}{\frac{(k^x-1)(3^x-1)}{x^2}} = \frac{2}{3}$$

$$\ldots [\because x\to 0, x\neq 0, x^2 \neq 0]$$

$$\therefore \quad \frac{\lim_{x\to 0}\left(\frac{5^x-1}{x}\right)\left(\frac{9^x-1}{x}\right)}{\lim_{x\to 0}\left(\frac{k^x-1}{x}\right)\left(\frac{3^x-1}{x}\right)} = \frac{2}{3}$$

$$\therefore \quad \frac{\left(\lim_{x\to 0}\frac{5^x-1}{x}\right)\cdot\left(\lim_{x\to 0}\frac{9^x-1}{x}\right)}{\left(\lim_{x\to 0}\frac{k^x-1}{x}\right)\cdot\left(\lim_{x\to 0}\frac{3^x-1}{x}\right)} = \frac{2}{3}$$

$$\therefore \quad \frac{\log 5 \cdot \log 9}{\log k \cdot \log 3} = \frac{2}{3} \qquad \ldots\left[\because \lim_{x\to 0}\frac{a^x-1}{x} = \log a\right]$$

$$\therefore \quad \frac{\log 5 \cdot \log (3)^2}{\log k \cdot \log 3} = \frac{2}{3}$$

$$\therefore \quad \frac{\log 5 \times \log 3}{\log k \times \log 3} = \frac{1}{3}$$

$\therefore \quad 3\log 5 = \log k$

$\therefore \quad \log (5)^3 = \log k$

$\therefore \quad (5)^3 = k$

$\therefore \quad k = 125$

VI. Find a and b if the following functions are continuous at the points or on the interval indicated against them.

Question 1.
f(x) = 4tanx+5sinxax−1, for x < 0
= 9log2, for x = 0
= 11x+7x· cosxbx−1, for x < 0
Solution:

f(x) is continuous at $x = 0$. ...(given)

$\therefore \lim_{x\to 0^-} f(x) = f(0)$

$\therefore \lim_{x\to 0^-}\left(\frac{4\tan x+5\sin x}{a^x-1}\right) = \frac{9}{\log 2}$

$\therefore \lim_{x\to 0^-}\left(\frac{\frac{4\tan x+5\sin x}{x}}{\frac{a^x-1}{x}}\right) = \frac{9}{\log 2}$ ...[$\because x \to 0, x \neq 0$]

$\therefore \frac{\lim_{x\to 0}\left(\frac{4\tan x}{x}+\frac{5\sin x}{x}\right)}{\lim_{x\to 0}\frac{a^x-1}{x}} = \frac{9}{\log 2}$

$\therefore \frac{4\lim_{x\to 0}\frac{\tan x}{x}+5\lim_{x\to 0}\frac{\sin x}{x}}{\lim_{x\to 0}\frac{a^x-1}{x}} = \frac{9}{\log 2}$

$\therefore \frac{4(1)+5(1)}{\log a} = \frac{9}{\log 2}$ ...$\left[\because \lim_{x\to 0}\frac{a^x-1}{x}=\log a\right]$

$\therefore \frac{9}{\log a} = \frac{9}{\log 2}$

$\therefore \log a = \log 2$

$\therefore a = 2$

Also, $\lim_{x\to 0^+} f(x) = f(0)$

$\therefore \lim_{x\to 0^+}\frac{11x + 7x\cdot\cos x}{b^x-1} = \frac{9}{\log 2}$

$\therefore \lim_{x\to 0^+}\frac{\frac{11x + 7x\cos x}{x}}{\frac{b^x-1}{x}} = \frac{9}{\log 2}$

...[$\because x \to 0, x \neq 0$]

$\therefore \frac{\lim_{x\to 0^+}(11+7\cos x)}{\lim_{x\to 0^+}\left(\frac{b^x-1}{x}\right)} = \frac{9}{\log 2}$

$\therefore \frac{11+7\cos 0}{\log b} = \frac{9}{\log 2}$ ...$\left[\because \lim_{x\to 0}\frac{a^x-1}{x}=\log a\right]$

$\therefore \frac{11+7(1)}{\log b} = \frac{9}{\log 2}$

$\therefore 9\log b = 18\log 2$

$\therefore \log b = 2\log 2$

$= \log(2)^2$

$\therefore \log b = \log$

$\therefore b = 4$

$\therefore a = 2$ and $b = 4$

Question 2.
f(x) = ax2 + bx + 1, for |2x – 3| ≥ 2
= 3x + 2, for 12 < x < 52
Solution:

$|2x-3| \geq 2$

$\therefore \quad 2x-3 \geq 2 \quad$ or $\quad 2x-3 \leq -2$

$\therefore \quad 2x \geq 5 \quad$ or $\quad 2x \leq 1$

$\therefore \quad x \geq \frac{5}{2} \quad$ or $\quad x \leq \frac{1}{2}$

$\therefore$ f(x) is redefined as

$$f(x) = ax^2 + bx + 1 \ ; \quad x \leq \frac{1}{2}$$

$$= 3x + 2 \ ; \quad \frac{1}{2} < x < \frac{5}{2}$$

$$= ax^2 + bx + 1 \ ; \quad x \geq \frac{5}{2}$$

f(x) is continuous everywhere on its domain. ...(given)

$\therefore$ f(x) is continuous at $x = \frac{1}{2}$ and $x = \frac{5}{2}$.

As f(x) is continuous at $x = \frac{1}{2}$,

$$\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^+} f(x)$$

$$\therefore \quad \lim_{x \to \frac{1}{2}^-} \left(ax^2 + bx + 1\right) = \lim_{x \to \frac{1}{2}^+} (3x + 2)$$

$$\therefore \quad a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + 1 = 3\left(\frac{1}{2}\right) + 2$$

$$\therefore \quad \frac{a}{4} + \frac{b}{2} + 1 = \frac{7}{2}$$

$\therefore \quad a + 2b + 4 = 14 \quad$ ...[Multiplying by 4]

$\therefore \quad a + 2b = 10 \quad$ ...(i)

Also, f(x) is continuous at $x = \frac{5}{2}$.

$$\therefore \quad \lim_{x \to \frac{5}{2}^-} f(x) = \lim_{x \to \frac{5}{2}^+} f(x)$$

$$\therefore \quad \lim_{x \to \frac{5}{2}^-} (3x + 2) = \lim_{x \to \frac{5}{2}^+} (ax^2 + bx + 1)$$

$$\therefore \quad 3\left(\frac{5}{2}\right) + 2 = a\left(\frac{5}{2}\right)^2 + b\left(\frac{5}{2}\right) + 1$$

$$\therefore \quad \frac{15}{2} + 2 = \frac{25a}{4} + \frac{5b}{2} + 1$$

$\therefore \quad 30 + 8 = 25a + 10b + 4 \quad$ ...[Multiplying by 4]

$\therefore \quad 25a + 10b = 34 \quad$ ...(ii)

Multiplyinig (i) by 5, we get

$5a + 10b = 50 \quad$ ...(iii)

Subtracting (iii) from (ii), we get

$20a = -16$

$$\therefore \quad a = \frac{-16}{20} = \frac{-4}{5}$$

Substituting $a = \frac{-4}{5}$ in (iii), we get

$$5\left(\frac{-4}{5}\right) + 10b = 50$$

$\therefore \quad -4 + 10b = 50$

$\therefore \quad 10b = 54$

$$\therefore \quad b = \frac{54}{10} = \frac{27}{5}$$

$$\therefore \quad a = \frac{-4}{5}, b = \frac{27}{5}$$

VII. Find f(a), if f is continuous at x = a where,

Question 1.
f(x) = $\frac{1+\cos(\pi x)}{\pi(1-x)^2}$, for x ≠ 1 and at a = 1.
Solution:

f(x) is continuous at $x = 1$. ...(given)

$\therefore \quad f(1) = \lim_{x \to 1} f(x)$

$\therefore \quad f(1) = \lim_{x \to 1} \frac{1+\cos \pi x}{\pi(1-x)^2}$

Put $1 - x = h$

$\therefore \quad x = 1 - h$

As $x \to 1, h \to 0$

$$\therefore \quad f(1) = \lim_{h \to 0} \frac{1+\cos[\pi(1-h)]}{\pi h^2}$$

$$= \lim_{h \to 0} \frac{1+\cos(\pi - \pi h)}{\pi h^2}$$

$$= \lim_{h \to 0} \frac{1-\cos \pi h}{\pi h^2}$$

$$= \lim_{h \to 0} \frac{1-\cos \pi h}{\pi h^2} \times \frac{1+\cos \pi h}{1+\cos \pi h}$$

$$= \lim_{h \to 0} \frac{1-\cos^2 \pi h}{\pi h^2 (1+\cos \pi h)}$$

$$= \frac{1}{\pi} \lim_{h \to 0} \frac{\sin^2 \pi h}{h^2 (1+\cos \pi h)}$$

$$= \frac{1}{\pi} \lim_{h \to 0} \left(\frac{\sin \pi h}{h}\right)^2 \times \frac{1}{1+\cos \pi h}$$

$$= \frac{1}{\pi} \lim_{h \to 0} \left(\frac{\sin \pi h}{\pi h}\right)^2 \times \pi^2 \times \frac{1}{\lim_{h \to 0}(1+\cos \pi h)}$$

$$= \frac{1}{\pi} \times (1)^2 \times \pi^2 \times \frac{1}{1+1}$$

$$\ldots \begin{bmatrix} \because h \to 0, \pi h \to 0 \\ \text{and } \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \end{bmatrix}$$

$$= \frac{\pi}{2}$$

Question 2.
f(x) = $\frac{1-\cos[7(x-\pi)]}{5(x-\pi)^2}$, for x ≠ π and at a = π.

Solution:

$f(x)$ is continuous at $x = \pi$. ...(given)

$$\therefore \quad f(\pi) = \lim_{x\to\pi} f(x) = \lim_{x\to\pi} \frac{1-\cos[7(x-\pi)]}{5(x-\pi)^2}$$

Put $x - \pi = h$

As $x \to \pi, h \to 0$

$$\therefore \quad f(\pi) = \lim_{h\to 0} \frac{1-\cos 7h}{5h^2}$$

$$= \lim_{h\to 0} \frac{2\sin^2\left(\frac{7h}{2}\right)}{5h^2}$$

$$= \frac{2}{5} \lim_{h\to 0} \frac{\sin^2\left(\frac{7h}{2}\right)}{\left(\frac{7h}{2}\right)^2} \times \left(\frac{7}{2}\right)^2$$

$$= \frac{2}{5}\left[\lim_{h\to 0} \frac{\sin\left(\frac{7h}{2}\right)}{\left(\frac{7h}{2}\right)}\right]^2 \times \frac{49}{4}$$

$$= \frac{2}{5} \times (1)^2 \times \frac{49}{4} \quad \ldots\left[\because \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1\right]$$

$$\therefore \quad f(\pi) = \frac{49}{10}$$

VIII. Solve using intermediate value theorem.

Question 1.
Show that $5^x - 6x = 0$ has a root in [1, 2].
Solution:
Let $f(x) = 5^x - 6x$
5^x and $6x$ are continuous functions for all $x \in R$.
$\therefore 5^x - 6x$ is also continuous for all $x \in R$.
i.e., f(x) is continuous for all $x \in R$.
A root of f(x) exists, if f(x) = 0 for at least one value of x.
$f(1) = 5^1 - 6(1) = -1 < 0$
$f(2) = (5)^2 - 6(2) = 13 > 0$
$\therefore f(1) < 0$ and $f(2) > 0$
∴ By intermediate value theorem, there has to be a point ‘c’ between 1 and 2 such that f(c) = 0.
∴ There is a root of the given equation in [1, 2].

Question 2.
Show that $x^3 - 5x^2 + 3x + 6 = 0$ has at least two real roots between x = 1 and x = 5.
Solution:
Let $f(x) = x^3 - 5x^2 + 3x + 6$
f(x) is a polynomial function and hence it is continuous for all $x \in R$.
A root of f(x) exists, if f(x) = 0 for at least one value of x.
Here, we have been asked to show that f(x) has at least two roots between x = 1 and x = 5.
$f(1) = (1)^3 - 5(1)^2 + 3(1) + 6$
$= 5 > 0$
$f(2) = (2)^3 - 5(2)^2 + 3(2) + 6$
$= 8 - 20 + 6 + 6$
$= 0$
∴ x = 2 is a root of f(x).
Also, $f(3) = (3)^3 - 5(3)^2 + 3(3) + 6$
$= 27 - 45 + 9 + 6$
$= -3 < 0$
$f(4) = (4)^3 - 5(4)^2 + 3(4) + 6$
$= 64 - 80 + 12 + 6$
$= 2 > 0$
$\therefore f(3) < 0$ and $f(4) > 0$
∴ By intermediate value theorem, there has to be a point ‘c’ between 3 and 4 such that f(c) = 0.
∴ There are two roots, x = 2 and a root between x = 3 and x = 4.
Thus, there are at least two roots of the given equation between x = 1 and x = 5.