Maharashtra State Board 12th Maths Solutions Chapter 2 Applications of Derivatives Ex 2.1

Question 1.
Find the equations of tangents and normals to the curve at the point on it.
(i) $y = x^2 + 2e^x + 2$ at (0, 4)
Solution:

$$y = x^2 + 2e^x + 2$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^2 + 2e^x + 2)$$

$$= 2x + 2 \times e^x + 0 = 2x + 2e^x$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at }(0,\,4)} = 2(0) + 2e^0 = 2$$

$$= \text{slope of the tangent at } (0, 4)$$

$\therefore$ the equation of the tangent at (0, 4) is

$$y - 4 = 2(x - 0)$$

$$\therefore y - 4 = 2x$$

$$\therefore 2x - y + 4 = 0$$

The slope of the normal at (0, 4)

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at}(0,4)}} = -\frac{1}{2}$$

$\therefore$ the equation of the normal at (0, 4) is

$$y - 4 = -\frac{1}{2}(x - 0)$$

$$\therefore 2y - 8 = -x \qquad \therefore x + 2y - 8 = 0$$

Hence, the equations of tangent and normal are $2x - y + 4 = 0$ and $x + 2y - 8 = 0$ respectively.

(ii) $x^3 + y^3 - 9xy = 0$ at (2, 4)
Solution:
$x^3 + y^3 - 9xy = 0$
Differentiating both sides w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left[x\frac{dy}{dx} + y \cdot \frac{d}{dx}(x)\right] = 0$$

$$\therefore 3x^2 + 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} - 9y \times 1 = 0$$

$$\therefore (3y^2 - 9x)\frac{dy}{dx} = 9y - 3x^2$$

$$\therefore \frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at}(2,4)} = \frac{9(4) - 3(2)^2}{3(4)^2 - 9(2)}$$

$$= \frac{36 - 12}{48 - 18} = \frac{24}{30} = \frac{4}{5}$$

$$= \text{slope of the tangent at } (2, 4)$$

$\therefore$ the equation of the tangent at (2, 4) is

$$y - 4 = \frac{4}{5}(x - 2)$$

$\therefore$ $5y - 20 = 4x - 8$

$\therefore$ $4x - 5y + 12 = 0$

The slope of normal at (2, 4) $= \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{\text{at }(2,4)}} = -\dfrac{5}{4}$

$\therefore$ the equation of the normal at (2, 4) is

$$y - 4 = -\frac{5}{4}(x - 2)$$

$\therefore$ $4y - 16 = -5x + 10$

$\therefore$ $5x + 4y - 26 = 0$

Hence, the equations of tangent and normal are 4x – 5y + 12 = 0 and 5x + 4y – 26 = 0 respectively.

(iii) $x^2 - \sqrt{3}xy + 2y^2 = 5$ at (√3, 2)
Solution:
$x^2 - \sqrt{3}xy + 2y^2 = 5$
Differentiating both sides w.r.t. x, we get

$$2x - \sqrt{3}\left[x\frac{dy}{dx} + y \cdot \frac{d}{dx}(x)\right] + 2 \times 2y\frac{dy}{dx} = 0$$

$$\therefore\ 2x - \sqrt{3}x\frac{dy}{dx} - \sqrt{3}y \times 1 + 4y\frac{dy}{dx} = 0$$

$$\therefore\ (4y - \sqrt{3}x)\frac{dy}{dx} = \sqrt{3}y - 2x$$

$$\therefore\ \frac{dy}{dx} = \frac{\sqrt{3}y - 2x}{4y - \sqrt{3}x} = \frac{2x - \sqrt{3}x}{\sqrt{3}x - 4y}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }(\sqrt{3},2)} = \frac{2\sqrt{3} - \sqrt{3}(2)}{\sqrt{3}(\sqrt{3}) - 4(2)} = 0$$

$$= \text{slope of the tangent at } (\sqrt{3}, 2)$$

$\therefore$ the equation of the tangent at $(\sqrt{3}, 2)$ is

$$y - 2 = 0(x - \sqrt{3})$$

$\therefore$ $y - 2 = 0$ $\quad\therefore$ $y = 2$

The slope of normal at $(\sqrt{3}, 2)$

$$= \frac{-1}{\left(\dfrac{dy}{dx}\right)_{\text{at }(\sqrt{3},2)}} \quad \text{where } \left(\frac{dy}{dx}\right)_{\text{at }(\sqrt{3},2)} = 0$$

the slope of normal at (√3, 2) does not exist.
normal is parallel to Y-axis.
equation of the normal is of the form x = k
Since, it passes through the point (√3, 2), k = √3
equation of the normal is x = √3.
Hence, the equations of tangent and normal are y = 2 and x = √3 respectively.

(iv) 2xy + π sin y = 2π at $(1, \frac{\pi}{2})$
Solution:
2xy + π sin y = 2π
Differentiating both sides w.r.t. x, we get

$$2\left[x\frac{dy}{dx}+y\cdot\frac{d}{dx}(x)\right]+\pi\cos y\frac{dy}{dx}=0$$

$$\therefore\ 2x\frac{dy}{dx}+2y\times 1+\pi\cos y\frac{dy}{dx}=0$$

$$\therefore\ (2x+\pi\cos y)\frac{dy}{dx}=-2y$$

$$\therefore\ \frac{dy}{dx}=\frac{-2y}{2x+\pi\cos y}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at}\left(1,\frac{\pi}{2}\right)}=\frac{-2\left(\frac{\pi}{2}\right)}{2(1)+\pi\cos\frac{\pi}{2}}$$

$$=\frac{-\pi}{2+\pi(0)}=-\frac{\pi}{2}$$

$$=\text{slope of the tangent at }\left(1,\frac{\pi}{2}\right)$$

$\therefore$ the equation of the tangent at $\left(1,\frac{\pi}{2}\right)$ is

$$y-\frac{\pi}{2}=-\frac{\pi}{2}(x-1)$$

$$\therefore\ 2y-\pi=-\pi x+\pi$$

$$\therefore\ \pi x+2y-2\pi=0$$

The slope of normal at $\left(1,\frac{\pi}{2}\right)$

$$=\frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at}\left(1,\frac{\pi}{2}\right)}}=\frac{-1}{\left(-\frac{\pi}{2}\right)}=\frac{2}{\pi}$$

$\therefore$ the equation of the normal at $\left(1,\frac{\pi}{2}\right)$ is

$$y-\frac{\pi}{2}=\frac{2}{\pi}(x-1)$$

$$\therefore\ \pi y-\frac{\pi^2}{2}=2x-2$$

$$\therefore\ 2\pi y-\pi^2=4x-4$$

$$\therefore\ 4x-2\pi y+\pi^2-4=0$$

Hence, the equations of tangent and normal are $\pi x + 2y - 2\pi = 0$ and $4x - 2\pi y + \pi^2 - 4 = 0$ respectively.

(v) x sin 2y = y cos 2x at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
Solution:
x sin 2y = y cos 2x
Differentiating both sides w.r.t. x, we get

$$x\frac{d}{dx}(\sin 2y)+\sin 2y\cdot\frac{d}{dx}(x)=y\cdot\frac{d}{dx}(\cos 2x)+\cos 2x\cdot\frac{dy}{dx}$$

$$\therefore\ x\cdot\cos 2y\cdot\frac{d}{dx}(2y)+\sin 2y\times 1$$

$$=y\cdot(-\sin 2x)\cdot\frac{d}{dx}(2x)+\cos 2x\cdot\frac{dy}{dx}$$

$$\therefore\ x\cos 2y\times 2\frac{dy}{dx}+\sin 2y=-y\sin 2x\times 2+\cos 2x\cdot\frac{dy}{dx}$$

$$\therefore\ (2x\cos 2y-\cos 2x)\frac{dy}{dx}=-2y\sin 2x-\sin 2y$$

$$\therefore\ \frac{dy}{dx}=\frac{-2y\sin 2x-\sin 2y}{2x\cos 2y-\cos 2x}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at}\left(\frac{\pi}{4},\frac{\pi}{2}\right)}=\frac{-2\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2}-\sin\pi}{2\left(\frac{\pi}{4}\right)\cos\pi-\cos\frac{\pi}{2}}$$

$$=\frac{-\pi(1)-0}{\frac{\pi}{2}(-1)-0}=\frac{-\pi}{\left(\frac{-\pi}{2}\right)}=2$$

$$=\text{slope of the tangent at }\left(\frac{\pi}{4},\frac{\pi}{2}\right)$$

$\therefore$ the equation of the tangent at $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ is

$$y-\frac{\pi}{2}=2\left(x-\frac{\pi}{4}\right)$$

$$\therefore\ y-\frac{\pi}{2}=2x-\frac{\pi}{2}$$

$$\therefore\ 2x-y=0$$

The slope of normal at $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

$$=\frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at}\left(\frac{\pi}{4},\frac{\pi}{2}\right)}}=\frac{-1}{2}$$

$\therefore$ the equation of the normal at $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ is

$$y-\frac{\pi}{2}=-\frac{1}{2}\left(x-\frac{\pi}{4}\right)$$

$$\therefore\ 2y-\pi=-x+\frac{\pi}{4}$$

$$\therefore\ 8y-4\pi=-4x+\pi$$

$$\therefore\ 4x+8y-5\pi=0$$

Hence, the equations of the tangent and normal are 2x – y = 0 and 4x + 8y – 5π = 0 respectively.

(vi) x = sin θ and y = cos 2θ at θ = π6
Solution:
When θ = π6, x = sinπ6 and y = cosπ3
∴ x = 12 and y = 12
Hence, the point at which we want to find the equations of tangent and normal is (12, 12)
Now, x = sin θ, y = cos 2θ
Differentiating x and y w.r.t. θ, we get

$$\frac{dx}{d\theta}=\frac{d}{d\theta}(\sin\theta)=\cos\theta$$

and $\dfrac{dy}{d\theta}=\dfrac{d}{d\theta}(\cos 2\theta)=-\sin 2\theta\cdot\dfrac{d}{d\theta}(2\theta)$

$$=-\sin 2\theta\times 2=-2\sin 2\theta$$

$$\therefore\ \frac{dy}{dx}=\frac{(dy/d\theta)}{(dx/d\theta)}=\frac{-2\sin 2\theta}{\cos\theta}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }\theta=\frac{\pi}{6}}=\frac{-2\sin\frac{\pi}{3}}{\cos\frac{\pi}{6}}=\frac{-2\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}=-2$$

$$=\text{slope of the tangent at }\theta=\frac{\pi}{6}$$

$\therefore$ the equation of the tangent at $\theta=\dfrac{\pi}{6}$ i.e. at $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$ is

$$y-\frac{1}{2}=-2\left(x-\frac{1}{2}\right)$$

$$\therefore\ y-\frac{1}{2}=-2x+1$$

$$\therefore\ 2y-1=-4x+2$$

$$\therefore\ 4x+2y-3=0$$

The slope of normal at $\theta=\dfrac{\pi}{6}$

$$=-\frac{1}{\left(\frac{dy}{dx}\right)_{\text{at }\theta=\frac{\pi}{6}}}=\frac{-1}{-2}=\frac{1}{2}$$

$\therefore$ equation of the normal at $\theta=\dfrac{\pi}{6}$, i.e. at $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$ is

$$y-\frac{1}{2}=\frac{1}{2}\left(x-\frac{1}{2}\right)$$

2y – 1 = x – 12
4y – 2 = 2x – 1
2x – 4y + 1 = 0
Hence, equations of the tangent and normal are 4x + 2y – 3 = 0 and 2x – 4y + 1 = 0 respectively.

(vii) x = √t, y = t – 1t√, at t = 4.
Solution:
When t = 4, x = √4 and y = 4 – 14√
∴ x = 2 and y = 4 – 12 = 72
Hence, the point at which we want to find the equations of tangent and normal is (2, 72).
Now, x = √t, y = t – 1t√
Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}(\sqrt{t}) = \frac{1}{2\sqrt{t}}$$

and $\frac{dy}{dt} = \frac{d}{dt}\left(t - \frac{1}{\sqrt{t}}\right) = 1 - \left(-\frac{1}{2}\right)t^{-\frac{3}{2}} = 1 + \frac{1}{2t^{\frac{3}{2}}}$

$$= \frac{2t^{\frac{3}{2}} + 1}{2t^{\frac{3}{2}}}$$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{\left(\frac{2t^{\frac{3}{2}}+1}{2t^{\frac{3}{2}}}\right)}{\left(\frac{1}{2\sqrt{t}}\right)}$$

$$= \frac{2t^{\frac{3}{2}}+1}{2t^{\frac{3}{2}}} \times 2\sqrt{t} = \frac{2t^{\frac{3}{2}}+1}{t}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } t=4} = \frac{2(4)^{\frac{3}{2}}+1}{4} = \frac{2 \times 8 + 1}{4} = \frac{17}{4}$$

$=$ slope of the tangent at $t = 4$

$\therefore$ the equation of the tangent at $t = 4$, i.e. at $\left(2, \frac{7}{2}\right)$ is

$$y - \frac{7}{2} = \frac{17}{4}(x - 2)$$

$\therefore 4y - 14 = 17x - 34$

$\therefore 17x - 4y - 20 = 0$

The slope of normal at $t = 4$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at } t=4}} = \frac{-1}{\left(\frac{17}{4}\right)} = -\frac{4}{17}$$

$\therefore$ the equation of the normal at $t = 4$, i.e. at $\left(2, \frac{7}{2}\right)$ is

$$y - \frac{7}{2} = -\frac{4}{17}(x - 2)$$

$\therefore 34y - 119 = -8x + 16$

$\therefore 8x + 34y - 135 = 0$

Hence, the equations of tangent and normal are 17x – 4y – 20 = 0 and 8x + 34y – 135 = 0 respectively.

Question 2.
Find the point of the curve y = $\sqrt{x-3}$ where the tangent is perpendicular to the line 6x + 3y – 5 = 0.
Solution:
Let the required point on the curve y = $\sqrt{x-3}$ be P(x_1, y_1).

Differentiating y = $\sqrt{x-3}$ w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\sqrt{x-3})=\frac{1}{2\sqrt{x-3}}\cdot\frac{d}{dx}(x-3)$$

$$=\frac{1}{2\sqrt{x-3}}\times(1-0)=\frac{1}{2\sqrt{x-3}}$$

$\therefore$ slope of the tangent at (x_1, y_1)

$$=\left(\frac{dy}{dx}\right)_{\text{at }(x_1, y_1)}=\frac{1}{2\sqrt{x_1-3}}$$

Since, this tangent is perpendicular to $6x+3y-5=0$

whose slope is $\frac{-6}{3}=-2$,

slope of the tangent $=\frac{-1}{-2}=\frac{1}{2}$

$$\therefore \frac{1}{2\sqrt{x_1-3}}=\frac{1}{2}$$

$$\therefore \sqrt{x_1-3}=1$$

$$\therefore x_1-3=1 \qquad \therefore x_1=4$$

Since, (x_1, y_1) lies on $y=\sqrt{x-3}$, $y_1=\sqrt{x_1-3}$

When $x_1=4$, $y_1=\sqrt{4-3}=\pm 1$

Hence, the required points are (4, 1) and (4, -1).

Question 3.
Find the points on the curve $y = x^3 - 2x^2 - x$ where the tangents are parallel to $3x - y + 1 = 0$.
Solution:
Let the required point on the curve $y = x^3 - 2x^2 - x$ be $P(x_1, y_1)$.

Differentiating $y = x^3 - 2x^2 - x$ w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 2x^2 - x)$$

$$= 3x^2 - 2 \times 2x - 1 = 3x^2 - 4x - 1$$

$\therefore$ slope of the tangent at (x_1, y_1)

$$= \left(\frac{dy}{dx}\right)_{\text{at }(x_1, y_1)} = 3x_1^2 - 4x_1 - 1$$

Since this tangent is parallel to $3x - y + 1 = 0$ whose slope is $\frac{-3}{-1} = 3$,

slope of the tangent $= 3$

$\therefore \; 3x_1^2 - 4x_1 - 1 = 3$

$\therefore \; 3x_1^2 - 4x_1 - 4 = 0$

$\therefore \; 3x_1^2 - 6x_1 + 2x_1 - 4 = 0$

$\therefore \; 3x_1(x_1 - 2) + 2(x_1 - 2) = 0$

$\therefore \; (x_1 - 2)(3x_1 + 2) = 0$

$\therefore \; x_1 - 2 = 0 \quad \text{or} \quad 3x_1 + 2 = 0$

$\therefore \; x_1 = 2 \qquad \text{or} \quad x_1 = -\frac{2}{3}$

Since, (x_1, y_1) lies on $y = x^3 - 2x^2 - x$, $y_1 = x_1^3 - 2x_1^2 - x_1$

When $x_1 = 2$, $y_1 = (2)^3 - 2(2)^2 - 2 = 8 - 8 - 2 = -2$

When $x_1 = -\frac{2}{3}$, $y_1 = \left(\frac{-2}{3}\right)^3 - 2\left(\frac{-2}{3}\right)^2 + \frac{2}{3}$

$$= \frac{-8}{27} - \frac{8}{9} + \frac{2}{3} = \frac{-14}{27}$$

Hence, the required points are $(2, -2)$ and $\left(\frac{-2}{3}, \frac{-14}{27}\right)$.

Question 4.
Find the equations of the tangents to the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ which are parallel to the X-axis.
Solution:
Let P (x_1, y_1) be the point on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ where the tangent is parallel to X-axis.
Differentiating $x^2 + y^2 - 2x - 4y + 1 = 0$ w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2 \times 1 - 4\frac{dy}{dx} + 0 = 0$$

$$\therefore \; (2y - 4)\frac{dy}{dx} = 2 - 2x$$

$$\therefore \; \frac{dy}{dx} = \frac{2 - 2x}{2y - 4} = \frac{1 - x}{y - 2}$$

$$\therefore \; \left(\frac{dy}{dx}\right)_{\text{at }(x_1, y_1)} = \frac{1 - x_1}{y_1 - 2}$$

$= $ slope of the tangent at (x_1, y_1)

Since, the tangent is parallel to X-axis, slope of the tangent $= 0$.

$\therefore \dfrac{1-x_1}{y_1-2} = 0$

$\therefore 1 - x_1 = 0 \qquad \therefore x_1 = 1$

Since, (x_1, y_1) lies on $x^2 + y^2 - 2x - 4y + 1 = 0$,

$x_1^2 + y_1^2 - 2x_1 - 4y_1 + 1 = 0$

When $x_1 = 1$, $(1)^2 + y_1^2 - 2(1) - 4y_1 + 1 = 0$

$\therefore 1 + y_1^2 - 2 - 4y_1 + 1 = 0$

$\therefore y_1^2 - 4y_1 = 0$

$\therefore y_1(y_1 - 4) = 0$

$\therefore y_1 = 0$ or $y_1 = 4$

the coordinates of the points are (1, 0) or (1, 4)
Since the tangents are parallel to X-axis, their equations are of the form y = k
If it passes through the point (1, 0), k = 0, and if it passes through the point (1, 4), k = 4
Hence, the equations of the tangents are y = 0 and y = 4.

Question 5.
Find the equations of the normals to the curve $3x^2 - y^2 = 8$, which are parallel to the line x + 3y = 4.
Solution:
Let $P(x_1, y_1)$ be the foot of the required normal to the curve $3x^2 - y^2 = 8$.
Differentiating $3x^2 - y^2 = 8$ w.r.t. x, we get

$3 \times 2x - 2y\dfrac{dy}{dx} = 0$

$\therefore -2y\dfrac{dy}{dx} = -6x$

$\therefore \dfrac{dy}{dx} = \dfrac{3x}{y}$

$\therefore \left(\dfrac{dy}{dx}\right)_{\text{at }(x_1, y_1)} = \dfrac{3x_1}{y_1}$

$= \text{slope of the tangent at } (x_1, y_1)$

$\therefore$ slope of the normal at $P(x_1, y_1)$

$= m_1 = \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{\text{at }(x_1, y_1)}} \qquad = -\dfrac{y_1}{3x_1}$

The slope of line $x + 3y = 4$ is $m_2 = \dfrac{-1}{3}$

Since, the normal at $P(x_1, y_1)$ is parallel to the line $x + 3y = 4$, $m_1 = m_2$

$\therefore -\dfrac{y_1}{3x_1} = -\dfrac{1}{3} \qquad \therefore y_1 = x_1$

Since, (x_1, y_1) lies on the curve $3x^2 - y^2 = 8$,

$3x_1^2 - y_1^2 = 8$

$\therefore\ 3x_1^2 - x_1^2 = 8 \qquad \ldots [\because\ y_1 = x_1]$

$\therefore\ 2x_1^2 = 8 \qquad \therefore\ x_1^2 = 4$

$\therefore\ x_1 = \pm 2$

When $x_1 = 2$, $y_1 = 2$

When $x_1 = -2$, $y_1 = -2$

$\therefore$ the coordinates of the point P are (2, 2) or $(-2, -2)$

and the slope of the normal is $m_1 = m_2 = -\frac{1}{3}$

$\therefore$ the equation of the normal at (2, 2) is

$$y - 2 = -\frac{1}{3}(x - 2)$$

$\therefore\ 3y - 6 = -x + 2$

$\therefore\ x + 3y - 8 = 0$

and the equation of the normal at $(-2, -2)$ is

$$y + 2 = -\frac{1}{3}(x + 2)$$

$\therefore\ 3y - 6 = -x + 2$

$\therefore\ x + 3y + 8 = 0$

Hence, the equations of the normals are x + 3y – 8 = 0 and x + 3y + 8 = 0.

Question 6.
If the line y = 4x – 5 touches the curve $y^2 = ax^3 + b$ at the point (2, 3), find a and b.
Solution:
$y^2 = ax^3 + b$
Differentiating both sides w.r.t. x, we get

$$2y\frac{dy}{dx} = a \times 3x^2 + 0$$

$$\therefore\ \frac{dy}{dx} = \frac{3ax^2}{2y}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }(2,\,3)} = \frac{3a(2)^2}{2(3)} = 2a$$

= slope of the tangent at (2, 3)
Since, the line y = 4x – 5 touches the curve at the point (2, 3), slope of the tangent at (2, 3) is 4.
2a = 4 ⇒ a = 2
Since (2, 3) lies on the curve $y^2 = ax^3 + b$
$(3)^2 = a(2)^3 + b$
9 = 8a + b
9 = 8(2) + b …… [∵ a = 2]
b = -7
Hence, a = 2 and b = -7.

Question 7.
A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which y-coordinate is changing 8 times as fast as the x-coordinate.
Solution:

Let $P(x_1, y_1)$ be the point on the curve $6y = x^3 + 2$ whose y-coordinate is changing 8 times as fast as the x-coordinate.

Then $\left(\frac{dy}{dt}\right)_{\text{at }(x_1, y_1)} = 8\left(\frac{dx}{dt}\right)_{\text{at }(x_1, y_1)}$ ... (1)

Differentiating $6y = x^3 + 2$ w.r.t. t, we get

$$6\frac{dy}{dt} = \frac{d}{dt}(x^3 + 2) = 3x^2\frac{dx}{dt} + 0$$

$$\therefore\ 2\frac{dy}{dt} = x^2\frac{dx}{dt}$$

$$\therefore\ 2\left(\frac{dy}{dt}\right)_{\text{at }(x_1, y_1)} = x_1^2 \cdot \left(\frac{dx}{dt}\right)_{\text{at }(x_1, y_1)}$$

$$\therefore\ 2 \times 8\left(\frac{dx}{dt}\right)_{\text{at }(x_1, y_1)} = x_1^2 \cdot \left(\frac{dx}{dt}\right)_{\text{at }(x_1, y_1)} \quad \text{... [By (1)]}$$

$\therefore\ x_1^2 = 16 \qquad \therefore\ x_1 = \pm 4$

Now, (x_1, y_1) lies on the curve $6y = x^3 + 2$.

$\therefore\ 6y_1 = x_1^3 + 2$

When $x_1 = 4$, $6y_1 = (4)^3 + 2 = 66 \quad \therefore\ y_1 = 11$

When $x_1 = -4$, $6y_1 = (-4)^3 + 2 = -62 \quad \therefore\ y_1 = -\frac{31}{3}$

Hence, the required points on the curve are (4, 11) and $\left(-4, \frac{-31}{3}\right)$.

Question 8.
A spherical soap bubble is expanding so that its radius is increasing at the rate of 0.02 cm/sec. At what rate is the surface area increasing, when its radius is 5 cm?
Solution:
Let r be the radius and S be the surface area of the soap bubble at any time t.
Then $S = 4\pi r^2$
Differentiating w.r.t. t, we get

$$\frac{dS}{dt} = 4\pi \times 2r\frac{dr}{dt}$$

$$\therefore\ \frac{dS}{dt} = 8\pi r\frac{dr}{dt} \quad \text{........(1)}$$

Now, $\frac{dr}{dt} = 0.02$ cm/sec and $r = 5$ cm

$$\therefore\ \text{(1) gives, } \frac{dS}{dt} = 8\pi(5)(0.02) = 0.8\pi$$

Hence, the surface area of the soap bubble is increasing at the rate of 0.87c cm^2 / sec.

Question 9.
The surface area of a spherical balloon is increasing at the rate of 2 cm^2/sec. At what rate is the volume of the balloon is increasing, when the radius of the balloon is 6 cm?
Solution:
Let r be the radius, S be the surface area and V be the volume of the spherical balloon at any time t.
Then $S = 4\pi r^2$ and V = 43πr3
Differentiating w.r.t. t, we get

$$\frac{dS}{dt} = 4\pi \times 2r\frac{dr}{dt} = 8\pi r\frac{dr}{dt} \qquad \dots (1)$$

and $\dfrac{dV}{dt} = \dfrac{4}{3}\pi \times 3r^2\dfrac{dr}{dt} = 4\pi r^2\dfrac{dr}{dt}$

From (1), $\dfrac{dr}{dt} = \dfrac{1}{8\pi r}\cdot\dfrac{dS}{dt}$

$\therefore\ \dfrac{dV}{dt} = 4\pi r^2 \times \dfrac{1}{8\pi r}\dfrac{dS}{dt}$

$\therefore\ \dfrac{dV}{dt} = \dfrac{r}{2}\cdot\dfrac{dS}{dt} \qquad \dots (2)$

Now, $\dfrac{dS}{dt} = 2\ \text{cm}^2/\text{sec}$ and $r = 6$ cm

$\therefore$ (2) gives, $\dfrac{dV}{dt} = \dfrac{6}{2} \times 2 = 6$

Hence, the volume of the spherical balloon is increasing at the rate of 6 cm^3 / sec.

Question 10.
If each side of an equilateral triangle increases at the rate of √2 cm/sec, find the rate of increase of its area when its side of length is 3 cm.
Solution:
If x cm is the side of the equilateral triangle and A is its area, then A=3√4x2
Differentiating w.r.t. f, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2x\frac{dx}{dt} = \frac{\sqrt{3}}{2}\cdot x\frac{dx}{dt} \qquad \dots (1)$$

Now, $\dfrac{dx}{dt} = \sqrt{2}$ cm/sec and $x = 3$ cm

$\therefore$ (1) gives, $\dfrac{dA}{dt} = \dfrac{\sqrt{3}}{2} \times 3 \times \sqrt{2}$

$$= \frac{3\sqrt{6}}{2}\ \text{cm}^2/\text{sec}$$

Hence, rate of increase of the area of equilateral triangle = 36√2 cm^2 / sec.

Question 11.
The volume of a sphere increases at the rate of 20 cm^3/sec. Find the rate of change of its surface area, when its radius is 5 cm.
Solution:
Let r be the radius, S be the surface area and V be the volume of the sphere at any time t.
Then S = $4\pi r^2$ and V = 43πr3
Differentiating w.r.t. t, we get

$$\frac{dS}{dt} = 4\pi \times 2r\frac{dr}{dt} = 8\pi r\frac{dr}{dt}$$

and $\dfrac{dV}{dt} = \dfrac{4\pi}{3} \times 3r^2\dfrac{dr}{dt} = 4\pi r^2\dfrac{dr}{dt} \qquad \dots (1)$

From (1), $\dfrac{dr}{dt} = \dfrac{1}{4\pi r^2}\dfrac{dV}{dt}$

$\therefore\ \dfrac{dS}{dt} = 8\pi r \times \dfrac{1}{4\pi r^2}\dfrac{dV}{dt}$

$\therefore\ \dfrac{dS}{dt} = \dfrac{2}{r}\cdot\dfrac{dV}{dt} \qquad .. (2)$

Now, $\dfrac{dV}{dt} = 20\ \text{cm}^3/\text{sec}$ and $r = 5$ cm

$\therefore$ (2) gives, $\dfrac{dS}{dt} = \dfrac{2}{5} \times 20 = 8$

Hence, the surface area of the sphere is changing at the rate of 8 cm^2/sec.

Question 12.
The edge of a cube is decreasing at the rate of 0.6 cm/sec. Find the rate at which its volume is decreasing, when the edge of the cube is 2 cm.
Solution:
Let x be the edge of the cube and V be its volume at any time t.
Then V = x^3
Differentiating both sides w.r.t. t, we get

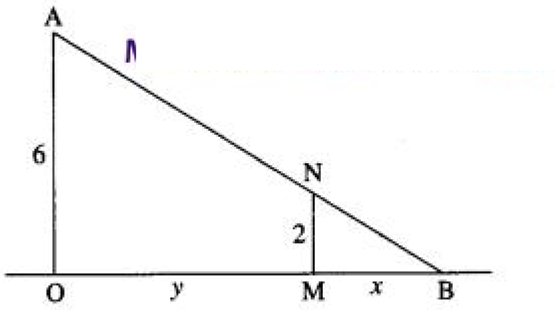
$$\frac{dV}{dt}=3x^2\frac{dx}{dt}$$

Now, $\frac{dx}{dt}=0.6$ cm/sec and $x=2$ cm

$$\therefore\ \frac{dV}{dt}=3(2)^2(0.6)$$
$$=7.2$$

Hence, the volume of the cube is decreasing at the rate of 7.2 cm^3/sec.

Question 13.
A man of height 2 meters walks at a uniform speed of 6 km/hr away from a lamp post of 6 meters high. Find the rate at which the length of the shadow is increasing.
Solution:
Let OA be the lamp post, MN the man, MB = x, his shadow, and OM = y, the distance of the man from the lamp post at time t.



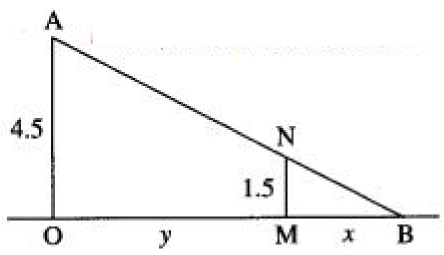
Then $\frac{dy}{dt}$ = 6 km/hr is the rate at which the man is moving at away from the lamp post.
$\frac{dx}{dt}$ is the rate at which his shadow is increasing.
From the figure,

$\frac{x}{2}=\frac{x+y}{6}$
6x = 2x + 2y
4x = 2y
x = $\frac{1}{2}$ y

$\frac{dx}{dt}=\frac{1}{2}\frac{dy}{dt}=\frac{1}{2}\times 6=3$ km/hr

Hence, the length of the shadow is increasing at the rate of 3 km/hr.

Question 14.
A man of height 1.5 meters walks towards a lamp post of height 4.5 meters, at the rate of ($\frac{3}{4}$) meter/sec.
Find the rate at which
(i) his shadow is shortening
(ii) the tip of the shadow is moving.
Solution:



Let OA be the lamp post, MN the man, MB = x his shadow and OM = y the distance of the man from lamp post at time t.
Then $\frac{dy}{dt}=\frac{3}{4}$ is the rate at which the man is moving towards the lamp post.
$\frac{dx}{dt}$ is the rate at which his shadow is shortening.
B is the tip of the shadow and it is at a distance of x + y from the post.
$\frac{d}{dt}(x+y)=\frac{dx}{dt}+\frac{dy}{dt}$ is the rate at which the tip of the shadow is moving.
From the figure,

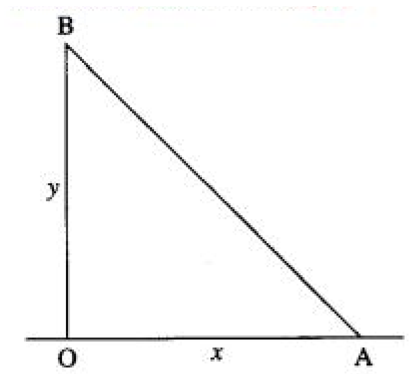
$\frac{x}{1.5}=\frac{x+y}{4.5}$
45x = 15x + 15y
30x = 15y
x = $\frac{1}{2}$y

$\frac{dx}{dt}=\frac{1}{2}\cdot\frac{dy}{dt}=\frac{1}{2}\left(\frac{3}{4}\right)=\left(\frac{3}{8}\right)$ metre/sec and $\frac{dx}{dt}+\frac{dy}{dt}=\frac{3}{8}+\frac{3}{4}=\left(\frac{9}{8}\right)$ metres /sec

Hence (i) the shadow is shortening at the rate of $\left(\frac{3}{8}\right)$ metre/sec, and

(ii) the tip of shadow is moving at the rate of $\left(\frac{9}{8}\right)$ metres/sec.

Question 15.
A ladder 10 metres long is leaning against a vertical wall. If the bottom of the ladder is pulled horizontally away from the wall at the rate of 1.2 metres per second, find how fast the top of the ladder is sliding down the wall, when the bottom is 6 metres away from the wall.
Solution:



Let AB be the ladder, where AB = 10 metres.
Let at time t seconds, the end A of the ladder be x metres from the wall and the end B be y metres from the ground.
Since, OAB is a right angled triangle, by Pythagoras' theorem
$x^2 + y^2 = 10^2$ i.e. $y^2 = 100 - x^2$
Differentiating w.r.t. t, we get
$2y \frac{dy}{dt} = 0 - 2x \frac{dx}{dt}$

$\therefore \frac{dy}{dt}=-\frac{x}{y}\cdot\frac{dx}{dt}$ ....(1)

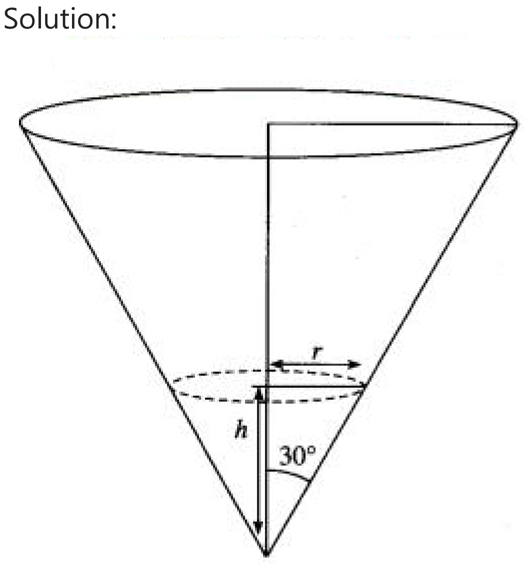
Now, $\frac{dx}{dt}$ = 1.2 metres/sec is the rate at which the bottom at of the ladder is pulled horizontally and $\frac{dy}{dt}$ is the rate at which the top of ladder B is sliding.
When x = 6, $y^2 = 100 - 36 = 64$
y = 8

(1) gives $\frac{dy}{dt}=-\frac{6}{8}(1.2)=-\frac{6}{8}\times\frac{12}{10}$

$=-\frac{9}{10}=-0.9$

Hence, the top of the ladder is sliding down the wall, at the rate of 0.9 metre/sec.

Question 16.
If water is poured into an inverted hollow cone whose semi-vertical angle is 30° so that its depth (measured along the axis) increases at the rate of 1 cm/sec. Find the rate at which the volume of water increases when the depth is 2 cm.
Solution:



Let r be the radius, h be the height, θ be the semi-vertical angle and V be the volume of the water at any time t.

Given : $\frac{dh}{dt} = 1$ cm/sec, $\theta = 30°$

Now, $V = \frac{1}{3}\pi r^2 h$

But, $\tan 30° = \frac{r}{h}$

$\therefore \frac{1}{\sqrt{3}} = \frac{r}{h} \quad \therefore r = \frac{h}{\sqrt{3}}$

$\therefore V = \frac{1}{3}\pi\left(\frac{h}{\sqrt{3}}\right)^2 h = \frac{\pi}{9}h^3$

Differentiating w.r.t. t, we get,

$\frac{dV}{dt} = \frac{\pi}{9} \times 3h^2 \frac{dh}{dt} = \frac{\pi}{3}h^2 \frac{dh}{dt}$

When $h = 2$ cm, then

$\frac{dV}{dt} = \frac{\pi}{3} \times (2)^2 \times 1 = \frac{4\pi}{3}$

Hence, the volume of water is increasing at the rate of $(4\pi 3)$ cm3/sec.

Maharashtra State Board 12th Maths Solutions Chapter 2 Applications of Derivatives Ex 2.2

Question 1.
Find the approximate value of given functions, at required points.
(i) √8.95
Solution:

Let $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x}}$.

Take $a = 9$ and $h = -0.05$. Then $f(a) = f(9) = \sqrt{9} = 3$ and

$f'(a) = f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$.

The formula for approximation is

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$$\begin{aligned} \therefore \sqrt{8 \cdot 95} &= f(9 - 0 \cdot 05) \\ &\doteqdot f(9) - (0 \cdot 05) f'(9) \\ &\doteqdot 3 - 0 \cdot 05 \times \frac{1}{6} \\ &\doteqdot 3 - 0 \cdot 0083 = 2 \cdot 9917 \end{aligned}$$

√8.95 = 2.9917

(ii) $\sqrt[3]{28}$

Solution:

Let $f(x) = \sqrt[3]{x}$

Then $f'(x) = \frac{d}{dx}(x^{\frac{1}{3}}) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$

Take $a = 27$ and $h = 1$.

Then $f(a) = f(27) = \sqrt[3]{27} = 3$

and $f'(a) = f'(27) = \frac{1}{3(27)^{\frac{2}{3}}} = \frac{1}{3 \times 9} = \frac{1}{27} = 0.03704$

The formula for approximation is

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$\therefore \sqrt[3]{28} = f(28) = f(27+1)$

$\doteqdot f(27) - 1 \cdot f'(27)$

$\doteqdot 3 + 1 \times 0.03704 = 3.03704$

$\therefore \sqrt[3]{28} \doteqdot 3.03704.$

(iii) $\sqrt[5]{31.98}$

Solution:

Let $f(x) = \sqrt[5]{x}$

Then $f'(x) = \frac{d}{dx}(x^{\frac{1}{5}}) = \frac{1}{5}x^{-\frac{4}{5}} = \frac{1}{5x^{\frac{4}{5}}}$

Take $a = 32$ and $h = -0.02$.

Then $f(a) = f(32) = \sqrt[5]{32} = 2$

$f'(a) = f'(32) = \frac{1}{5(32)^{\frac{4}{5}}} = \frac{1}{5 \times 16} = \frac{1}{80} = 0.0125$

The formula for approximation is

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$\therefore \sqrt[5]{31.98} = f(31.98) = f(32 - 0.02)$

$\doteqdot f(32) - 0.02 \cdot f'(32)$

$\doteqdot 2 - 0.02 \times 0.0125$

$\doteqdot 2 - 0.000250 = 1.99975$

$\therefore \sqrt[5]{31.98} \doteqdot 1.99975.$

(iv) $(3.97)^4$

Solution:

Let $f(x) = x^4$

Then $f'(x) = \frac{d}{dx}(x^4) = 4x^3$

Take $a = 4$ and $h = -0.03$.

Then $f(a) = f(4) = (4)^4 = 256$ and

$f'(a) = f'(4) = 4(4)^3 = 256$

The formula for approximation is

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$\therefore (3.97)^4 = f(3.97) = f(4 - 0.03)$

$\doteqdot f(4) - (0.03) f'(4)$

$\doteqdot 256 - 0.03 \times 256$

$\doteqdot 256 - 7.68 = 248.32 \quad \therefore (3.97)^4 \doteqdot 248.32.$

(v) $(4.01)^3$
Solution:

Let $f(x) = x^3$. Then, $f'(x) = 3x^2$
Take $a = 4$ and $h = 0.01$. Then
$f(a) = f(4) = 4^3 = 64$ and
$f'(a) = f'(4) = 3 \times 4^2 = 48$.
The formula for approximation is
$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$$\therefore \ (4.01)^3 = f(4 + 0.01)$$
$$\doteqdot f(4) + (0.01) f'(4)$$
$$\doteqdot 64 + 0.01 \times 48$$
$$\doteqdot 64 + 0.48 = 64.48 \qquad \therefore \ (4.01)^3 \doteqdot 64.48.$$

Question 2.
Find the approximate values of:
(i) sin 61°, given that 1° = 0.0174^c, √3 = 1.732.
Solution:

Let $f(x) = \sin x$

Then $f'(x) = \dfrac{d}{dx}(\sin x) = \cos x$

Take $a = 60° = \dfrac{\pi}{3}$ and $h = 1° = 0.0174^c$

Then $f(a) = f\left(\dfrac{\pi}{3}\right) = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2} = \dfrac{1.732}{2} = 0.866$

and $f'(a) = f'\left(\dfrac{\pi}{3}\right) = \cos\dfrac{\pi}{3} = \dfrac{1}{2} = 0.5$

The formula for approximation is
$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$$\therefore \ \sin 61° = f(61°) = f\left(\frac{\pi}{3} + 0.0174\right)$$
$$\doteqdot f\left(\frac{\pi}{3}\right) + 0.0174 \cdot f'\left(\frac{\pi}{3}\right)$$
$$\doteqdot 0.866 + 0.0174 \times 0.5$$
$$\doteqdot 0.866 + 0.00870 = 0.8747$$
$$\therefore \ \sin 61° \doteqdot 0.8747.$$

(ii) sin(29° 30'), given that 1° = 0.0175^c, √3 = 1.732.
Solution:

$$\text{Let } f(x) = \sin x$$

$$\text{Then } f'(x) = \frac{d}{dx}(\sin x) = \cos x$$

$$\text{Take } a = 30° = \frac{\pi}{6} \text{ and}$$

$$h = -30' = -\left(\frac{1}{2}\right)^{\circ} = -\frac{1}{2} \times 0.0175 = -0.00875$$

$$\text{Then } f(a) = f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2} = 0.5$$

$$\text{and } f'(a) = f'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} = \frac{1.732}{2} = 0.866$$

The formula for approximation is

$$f(a+h) \doteqdot f(a) + h \cdot f'(a)$$

$$\therefore \sin(29° 30') = f(29° 30') = f\left(\frac{\pi}{6} + 0.00875\right)$$

$$\doteqdot f\left(\frac{\pi}{6}\right) - (0.00875) \cdot f'\left(\frac{\pi}{6}\right)$$

$$\doteqdot 0.5 - 0.00875 \times 0.866$$

$$\doteqdot 0.5 - 0.0075775 = 0.4924$$

$$\therefore \sin(29° 30') \doteqdot 0.4924.$$

(iii) cos(60° 30'), given that 1° = 0.0175^c, √3 = 1.732.
Solution:

$$\text{Let } f(x) = \cos x$$

$$\text{Then } f'(x) = \frac{d}{dx}(\cos x) = -\sin x$$

$$\text{Take } a = 60° = \frac{\pi}{3} \text{ and}$$

$$h = 30' = \left(\frac{1}{2}\right)^{\circ} = \left(\frac{1}{2} \times 0.0175\right)^{c} = 0.00875^{c}$$

$$\text{Then } f(a) = f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2} = 0.5$$

$$f'(a) = f'\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2} = -\frac{1.732}{2} = -0.866$$

The formula for approximation is

$$f(a+h) \doteqdot f(a) + h \cdot f'(a)$$

$$\therefore \cos(60° 30') = f(60° 30') = f\left(\frac{\pi}{3} + 0.00875\right)$$

$$\doteqdot f\left(\frac{\pi}{3}\right) + 0.00875 \cdot f'\left(\frac{\pi}{3}\right)$$

$$\doteqdot 0.5 + (0.00875)(-0.8660)$$

$$\doteqdot 0.5 - 0.0075775 = 0.4924225$$

$$\therefore \cos(60° 30') \doteqdot 0.4924.$$

(iv) tan (45° 40'), given that 1° = 0.0175^c.
Solution:

$$\text{Let } f(x) = \tan x$$

$$\text{Then } f'(x) = \frac{d}{dx}(\tan x) = \sec^2 x$$

$$\text{Take } a = 45° = \frac{\pi}{4} \text{ and } h = 40' = \left(\frac{40}{60} \times 0.0175\right)^c = 0.01167^c$$

$$\text{Then } f(a) = f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$$

$$\text{and } f'(a) = f'\left(\frac{\pi}{4}\right) = \sec^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$$

The formula for approximation is

$$f(a+h) \doteqdot f(a) + h \cdot f'(a)$$

$$\therefore \tan(45°40') = f(45°40') = f\left(\frac{\pi}{4} + 0.01167\right)$$

$$\doteqdot f\left(\frac{\pi}{4}\right) + (0.01167) \cdot f'\left(\frac{\pi}{4}\right)$$

$$\doteqdot 1 + 0.01167 \times 2 = 1 + 0.02334 = 1.02334$$

$$\therefore \tan(45°40') \doteqdot 1.02334.$$

Question 3.
Find the approximate values of
(i) $\tan^{-1}(0.999)$.
Solution:

$$\text{Let } f(x) = \tan^{-1} x$$

$$\text{Then } f'(x) = \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\text{Take } a = 1 \text{ and } h = -0.001$$

$$\text{Then } f(a) = f(1) = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\text{and } f'(a) = f'(1) = \frac{1}{1+1^2} = \frac{1}{2}$$

The formula for approximation is

$$f(a+h) \doteqdot f(a) + hf'(a)$$

$$\therefore \tan^{-1}(0.999) = f(0.999) = f(1 - 0.001)$$

$$\doteqdot f(1) - (0.001) \cdot f'(1)$$

$$\doteqdot \frac{\pi}{4} - 0.001 \times \frac{1}{2} = \frac{\pi}{4} - 0.0005$$

$$\therefore \tan^{-1}(0.999) \doteqdot \frac{\pi}{4} - 0.0005.$$

Remark : The answer can also be given as :

$$\tan^{-1}(0.999) \doteqdot \frac{3.1416}{4} - 0.0005$$

$$\doteqdot 0.7854 - 0.0005 = 0.7849.$$

(ii) $\cot^{-1}(0.999)$.
Solution:

$$\text{Let } f(x) = \cot^{-1} x$$

$$\therefore\ f'(x) = \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

Take $a = 1$ and $h = -0.001$

Then $f(a) = f(1) = \cot^{-1} 1 = \frac{\pi}{4}$

and $f'(a) = f'(1) = \frac{-1}{1+1^2} = \frac{-1}{2}$

The formula for approximation is

$$f(a+h) \doteqdot f(a) + h \cdot f'(a)$$

$$\begin{aligned}\therefore\ \cot^{-1}(0.999) &= f(0.999) = f(1-0.001)\\ &\doteqdot f(1) - (0.001)\cdot f'(1)\\ &\doteqdot \frac{\pi}{4} - (0.001)\cdot\left(\frac{-1}{2}\right) = \frac{\pi}{4} + 0.0005\end{aligned}$$

$$\therefore\ \cot^{-1}(0.999) \doteqdot \frac{\pi}{4} + 0.0005.$$

Remark : The answer can also be given as :

$$\begin{aligned}\cot^{-1}(0.999) &\doteqdot \frac{3.1416}{4} + 0.0005\\ &\doteqdot 0.7854 + 0.0005 = 0.7859.\end{aligned}$$

(iii) $\tan^{-1}(1.001)$.
Solution:

$$\text{Let } f(x) = \tan^{-1} x$$

$$\therefore\ f'(x) = \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Take $a = 1$ and $h = 0.001$

Then $f(a) = f(1) = \tan^{-1} 1 = \frac{\pi}{4}$

and $f'(a) = f'(1) = \frac{1}{1+1^2} = \frac{1}{2}$

The formula for approximation is

$$f(a+h) \doteqdot f(a) + h \cdot f'(a)$$

$$\begin{aligned}\therefore\ \tan^{-1}(1.001) &= f(1.001) = f(1+0.001)\\ &\doteqdot f(1) + (0.001)\cdot f'(1)\\ &\doteqdot \frac{\pi}{4} + (0.001) \times \frac{1}{2} = \frac{\pi}{4} + 0.0005\end{aligned}$$

$$\therefore\ \tan^{-1}(1.001) \doteqdot \frac{\pi}{4} + 0.0005.$$

Remark : The answer can also be given as :

$$\begin{aligned}\tan^{-1}(1.001) &\doteqdot f(1) + (0.001)\cdot f'(1)\\ &\doteqdot \frac{\pi}{4} + (0.001) \times \frac{1}{2}\\ &\doteqdot \frac{3.1416}{4} + 0.0005\\ &\doteqdot 0.7854 + 0.0005 = 0.7859.\end{aligned}$$

Question 4.
Find the approximate values of:
(i) $e^{0.995}$, given that e = 2.7183.
Solution:

Let $f(x) = e^x$.

Then $f'(x) = \frac{d}{dx}(e^x) = e^x$

Take $a = 1$ and $h = -0.005$.

Then $f(a) = f(1) = e = 2.7183$

and $f'(a) = f'(1) = e = 2.7183$

The formula for approximation is

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$\therefore \; e^{0.995} = f(0.995) = f(1 - 0.005)$

$\doteqdot f(1) - (0.005) \cdot f'(1)$

$\doteqdot 2.7183 - 0.005 \times 2.7183$

$\doteqdot 2.7183 - 0.01359 = 2.70471$

$\therefore \; e^{0.995} \doteqdot 2.70471$

(ii) $e^{2.1}$, given that $e^2 = 7.389$.
Solution:

Let $f(x) = e^x$

Then $f'(x) = \frac{d}{dx}(e^x) = e^x$

Take $a = 2$ and $h = 0.1$

Then $f(a) = f(2) = e^2 = 7.389$

$f'(a) = f'(2) = e^2 = 7.389$

The formula for approximation is

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$\therefore \; e^{2.1} = f(2.1) = f(2 + 0.1)$

$\doteqdot f(2) + (0.1) \cdot f'(2)$

$\doteqdot 7.389 + 0.1 \times 7.389$

$\doteqdot 7.389 + 0.7389 = 8.1279$

$\therefore \; e^{2.1} \doteqdot 8.1279.$

(iii) $3^{2.01}$, given that log 3 = 1.0986.
Solution:

Let $f(x) = 3^x$

Then $f'(x) = \frac{d}{dx}(3^x) = 3^x \cdot \log 3$

Take $a = 2$ and $h = 0.01$

Then $f(a) = f(2) = 3^2 = 9$

and $f'(a) = f'(2) = 3^2 \cdot \log 3 = 9 \times 1.0986 = 9.8874$

The formula for approximation is

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$\therefore \; 3^{2.01} = f(2.01) = f(2 + 0.01)$

$\doteqdot f(2) + (0.01) \cdot f'(2)$

$\doteqdot 9 + 0.01 \times 9.8874$

$\doteqdot 9 + 0.098874 = 9.098874$

$\therefore \; 3^{2.01} \doteqdot 9.098874.$

Question 5.
Find the approximate values of:
(i) $\log_e(101)$, given that $\log_e 10 = 2.3026$.
Solution:

Let $f(x) = \log_e x$. Then $f'(x) = \frac{1}{x}$.

Take $a = 100$ and $h = 1$. Then

$f(a) = f(100) = \log_e 100 = 2\log_e 10$

$= 2 \times 2{\cdot}3026 = 4{\cdot}6052$

$f'(a) = f'(100) = \frac{1}{100} = 0{\cdot}01$

The formula for approximation is

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$\therefore \log_e 101 = f(101) = f(100+1)$

$\doteqdot f(100) + 1 \cdot f'(100)$

$\doteqdot 4{\cdot}6052 + 1 \times 0{\cdot}01 = 4{\cdot}6152$

$\log_e(101) \doteqdot 4{\cdot}6152$.

(ii) $\log_e(9.01)$, given that log 3 = 1.0986.
Solution:

Let f(x) - $\log_e x$.

Then f'(x) = $\frac{1}{x}$

Take a = 100 and h = 1. then

f(a) = f(100)

= $\log_e 100$

= 3 $\log_e 10$

= 3 x 1.0986

= 3.2958

f'(a) = f'(100)

= $\frac{1}{100}$

= 0.01

The formula for a approximation is

f(a + h) ≑ f(a) + h.f'(a)

∴ $\log_e$ 9.01 = f(9.01)

= f(900 + 1)

≑ f(100) + 1.f'(100)

≑ 3.2958 + 1 x 0.01

= 2.1983

$\log_e(901)$ ≑ 2.1983.

(iii) $\log_{10}(1016)$, given that $\log_{10}e = 0.4343$.
Solution:

$$\text{Let } f(x) = \log_{10}x = \frac{\log_e x}{\log_e 10}$$

$$= (\log_{10}e)(\log x) = (0.4343)\log x$$

$$\text{Then } f'(x) = (0.4343).\frac{d}{dx}(\log x) = \frac{0.4343}{x}$$

Take $a = 1000$ and $h = 16$. Then

$$f(a) = f(1000) = \log_{10}1000 = \log_{10}10^3 = 3$$

$$f'(a) = f'(1000) = \frac{0.4343}{1000}$$

The formula for approximation is

$$f(a+h) \doteqdot f(a) + hf'(a)$$

$$\therefore\ \log_{10}1016 = f(1016) = f(1000+16)$$

$$\doteqdot f(1000) + 16 \cdot f'(1000)$$

$$\doteqdot 3 + 16 \times \frac{0.4343}{1000}$$

$$\doteqdot 3 + 0.0069488 \doteqdot 3.006949$$

$$\therefore\ \log_{10}1016 \doteqdot 3.006949.$$

Question 6.
Find the approximate values of:
(i) $f(x) = x^3 - 3x + 5$ at $x = 1.99$
Solution:

$$f(x) = x^3 - 3x + 5$$

$$\therefore\ f'(x) = \frac{d}{dx}(x^3 - 3x + 5)$$

$$= 3x^2 - 3 \times 1 + 0 = 3x^2 - 3$$

Take $a = 2$, $h = -0.01$

$$\text{Then } f(a) = f(2) = (2)^3 - 3(2) + 5$$

$$= 8 - 6 + 5 = 7$$

$$f'(a) = f'(2) = 3(2)^2 - 3 = 12 - 3 = 9$$

The formula for approximation is

$$f(a+h) \doteqdot f(a) + h \cdot f'(a)$$

$$\therefore\ f(1.99) = f(2 - 0.01) \doteqdot f(2) - (0.01) \cdot f'(2)$$

$$\doteqdot 7 - 0.01 \times 9 = 7 - 0.09 = 6.91$$

$$\therefore\ f(1.99) \doteqdot 6.91.$$

(ii) $f(x) = x^3 + 5x^2 – 7x + 10$ at $x = 1.12$
Solution:

$$f(x) = x^3 + 5x^2 - 7x + 10$$

$$\therefore\ f'(x) = \frac{d}{dx}(x^3 + 5x^2 - 7x + 10)$$

$$= 3x^2 + 5 \times 2x - 7 \times 1 + 0 = 3x^2 + 10x - 7$$

Take $a = 1$, $h = 0.12$

Then $f(a) = f(1) = (1)^3 + 5(1)^2 - 7(1) + 10$

$$= 1 + 5 - 7 + 10 = 9$$

and $f'(a) = f'(1) = 3(1)^2 + 10(1) - 7 = 3 + 10 - 7 = 6$

The formula for approximation is

$$f(a + h) \doteqdot f(a) + h \cdot f'(a)$$

$$\therefore\ f(1.12) = f(1 + 0.12) \doteqdot f(1) + (0.12) \cdot f'(1)$$

$$\doteqdot 9 + 0.12 \times 6 \doteqdot 9 + 0.72 = 9.72$$

$\therefore\ f(1.12) \doteqdot 9.72.$

Maharashtra State Board 12th Maths Solutions Chapter 2 Applications of Derivatives Ex 2.3

Question 1.
Check the validity of the Rolle's theorem for the following functions.
(i) $f(x) = x^2 – 4x + 3$, $x \in [1, 3]$
Solution:
The function f given as $f(x) = x^2 – 4x + 3$ is polynomial function.
Hence, it is continuous on [1, 3] and differentiable on (1, 3).
Now, $f(1) = 1^2 – 4(1) + 3 = 1 – 4 + 3 = 0$
and $f(3) = 3^2 – 4(3) + 3 = 9 – 12 + 3 = 0$
$\therefore f(1) = f(3)$
Thus, the function f satisfies all the conditions of Rolle's theorem.

(ii) $f(x) = e^{-x} \sin x$, $x \in [0, \pi]$.
Solution:
The functions e^{-x} and $\sin x$ are continuous and differentiable on their domains.
$\therefore f(x) = e^{-x} \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.
Now, $f(0) = e^0 \sin 0 = 1 \times 0 = 0$
and $f(\pi) = e^{-\pi} \sin \pi = e^{-\pi} \times 0 = 0$
$\therefore f(0) = f(\pi)$
Thus, the function f satisfies all the conditions of the Rolle's theorem.

(iii) $f(x) = 2x^2 – 5x + 3$, $x \in [1, 3]$.
Solution:
The function f given as $f(x) = 2x^2 – 5x + 3$ is a polynomial function.
Hence, it is continuous on [1, 3] and differentiable on (1, 3).
Now, $f(1) = 2(1)^2 – 5(1) + 3 = 2 – 5 + 3 = 0$
and $f(3) = 2(3)^2 – 5(3) + 3 = 18 – 15 + 3 = 6$

$\therefore$ f(1) ≠ f(3)
Hence, the conditions of Rolle's theorem are not satisfied.

(iv) f(x) = sin x – cos x + 3, x ∈ [0, 2π].
Solution:
The functions sin x, cos x and 3 are continuous and differentiable on their domains.
$\therefore$ f(x) = sin x – cos x + 3 is continuous on [0, 2π] and differentiable on (0, 2π).
Now, f(0) = sin 0 – cos 0 + 3 = 0 – 1 + 3 = 2
and f(2π) = sin 2π – cos 2π + 3 = 0 – 1 + 3 = 2
$\therefore$ f(0) = f(2π)
Thus, the function f satisfies all the conditions of the Rolle's theorem.

(v) f(x) = x^2, if 0 ≤ x ≤ 2
= 6 – x, if 2 < x ≤ 6.
Solution:
f(x) = x^2, if 0 ≤ x ≤ 2
= 6 – x, if 2 < x ≤ 6
$\therefore$ f(x) = $\frac{d}{dx}(x^2)$ = 2x, if 0 ≤ x ≤ 2
= $\frac{d}{dx}(6-x)$ = -1, if 2 < x ≤ 6
$\therefore$ Lf'(2) = 2(2) = 4 and Rf'(2) = -1
$\therefore$ Lf'(2) ≠ Rf'(2)
$\therefore$ f is not differentiable at x = 2 and 2 ∈ (0, 6).
$\therefore$ f is not differentiable at all the points on (0, 6).
Hence, the conditions of Rolle's theorem are not satisfied.

(vi) f(x) = $x^{\frac{2}{3}}$, x ∈ [-1, 1].
Solution:
f(x) = $x^{\frac{2}{3}}$
$\therefore f'(x)=\frac{d}{dx}\left(x^{\frac{2}{3}}\right)=\frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$
This does not exist at x = 0 and 0 ∈ (-1, 1)
$\therefore$ f is not differentiable on the interval (-1, 1).
Hence, the conditions of Rolle's theorem are not satisfied.

Question 2.
Given an interval [a, b] that satisfies hypothesis of Rolle's theorem for the function f(x) = $x^4 + x^2 - 2$. It is known that a = -1. Find the value of b.
Solution:
f(x) = $x^4 + x^2 - 2$
Since the hypothesis of Rolle's theorem are satisfied by f in the interval [a, b], we have
f(a) = f(b), where a = -1
Now, f(a) = f(-1) = $(-1)^4 + (-1)^2 - 2$ = 1 + 1 – 2 = 0
and f(b) = $b^4 + b^2 - 2$
$\therefore$ f(a) = f(b) gives
0 = $b^4 + b^2 - 2$ i.e. $b^4 + b^2 - 2 = 0$.
Since, b = 1 satisfies this equation, b = 1 is one of the roots of this equation.
Hence, b = 1.

Question 3.
Verify Rolle's theorem for the following functions.
(i) f(x) = sin x + cos x + 7, x ∈ [0, 2π]
Solution:
The functions sin x, cos x and 7 are continuous and differentiable on their domains.
$\therefore$ f(x) = sin x + cos x + 7 is continuous on [0, 2π] and differentiable on (0, 2π)
Now, f(0) = sin 0 + cos 0 + 7 = 0 + 1 + 7 = 8
and f(2π) = sin 2π + cos 2π + 7 = 0 + 1 + 7 = 8
$\therefore$ f(0) = f(2π)
Thus, the function f satisfies all the conditions of Rolle's theorem.
$\therefore$ there exists c ∈ (0, 2π) such that f'(c) = 0.
Now, f(x) = sin x + cos x + 7
$\therefore$ f'(x) = $\frac{d}{dx}$ (sin x + cos x + 7)
= cos x – sin x + 0
= cos x – sin x
$\therefore$ f'(c) = cos c – sin c
$\therefore$ f'(c) = 0 gives, cos c – sin c = 0
$\therefore$ cos c = sin c

∴ c = $\frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$

But $\frac{\pi}{4}, \frac{5\pi}{4}$ ∈ (0, 2π)

∴ c = $\frac{\pi}{4}$ or $\frac{5\pi}{4}$

Hence, the Rolle's theorem is verified.

(ii) f(x) = sin($\frac{x}{2}$), x ∈ [0, 2π]
Solution:
The function f(x) = sin($\frac{x}{2}$) is continuous on [0, 2π] and differentiable on (0, 2π).
Now, f(0) = sin 0 = 0
and f(2π) = sin π = 0
∴ f(0) = f(2π)
Thus, the function f satisfies all the conditions of Rolle's theorem.
∴ there exists c ∈ (0, 2π) such that f'(c) = 0.

Now, $f(x) = \sin\left(\frac{x}{2}\right)$

$$\therefore f'(x) = \frac{d}{dx}\left[\sin\left(\frac{x}{2}\right)\right]$$

$$= \cos\left(\frac{x}{2}\right) \cdot \frac{d}{dx}\left(\frac{x}{2}\right)$$

$$= \cos\left(\frac{x}{2}\right) \cdot \frac{1}{2} = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$\therefore f'(c) = \frac{1}{2}\cos\left(\frac{c}{2}\right)$$

$$\therefore f'(c) = 0 \text{ gives } \frac{1}{2}\cos\left(\frac{c}{2}\right) = 0$$

$$\therefore \cos\left(\frac{c}{2}\right) = 0$$

$$\therefore \cos\frac{c}{2} = \cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = \cos\frac{5\pi}{2} = \ldots$$

$$\therefore \frac{c}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\therefore c = \pi, 3\pi, 5\pi, \ldots$$

But $\pi \in (0, 2\pi)$

$$\therefore c = \pi$$

Hence, Rolle's theorem is verified.

(iii) f(x) = x^2 – 5x + 9, x ∈ [1, 4].
Solution:
The function f given as f(x) = x^2 – 5x + 9 is a polynomial function.
Hence it is continuous on [1, 4] and differentiable on (1, 4).
Now, f(1) = 1^2 – 5(1) + 9 = 1 – 5 + 9 = 5
and f(4) = 4^2 – 5(4) + 9 = 16 – 20+ 9 = 5
∴ f(1) = f(4)
Thus, the function f satisfies all the conditions of the Rolle's theorem.
∴ there exists c ∈ (1, 4) such that f'(c) = 0.
Now, f(x) = x2 – 5x + 9
∴ f'(x) = $\frac{d}{dx}$ (x^2 – 5x + 9)
= 2x – 5 × 1 + 0
= 2x – 5
∴ f'(c) = 2c – 5
∴ f'(c) = 0 gives, 2c – 5 = 0
∴ c = 5/2 ∈ (1, 4)
Hence, the Rolle's theorem is verified.

Question 4.
If Rolle's theorem holds for the function f(x) = x^3 + px^2 + qx + 5, x ∈ [1, 3] with c = 2 + $\frac{1}{\sqrt{3}}$, find the values of p and q.
Solution:
The Rolle's theorem holds for the function f(x) = x^3 + px^2 + qx + 5, x ∈ [1, 3]

$\therefore$ f(1) = f(3)
$\therefore$ $1^3 + p(1)^2 + q(1) + 5 = 3^3 + p(3)^2 + q(3) + 5$
$\therefore$ 1 + p + q + 5 = 27 + 9p + 3q + 5
$\therefore$ 8p + 2q = -26
$\therefore$ 4p + q = -13 ..... (1)
Also, there exists at least one point c ∈ (1, 3) such that f'(c) = 0.
Now, f'(x) = $\frac{d}{dx}(x^3 + px^2 + qx + 5)$
= $3x^2$ + p × 2x + q × 1 + 0
= $3x^2 + 2px + q$

$$\therefore\ f'(c) = 3c^2 + 2pc + q, \text{ where } c = 2 + \frac{1}{\sqrt{3}}$$

$$\therefore\ f'(c) = 3\left(2 + \frac{1}{\sqrt{3}}\right)^2 + 2p\left(2 + \frac{1}{\sqrt{3}}\right) + q$$

$$= 3\left(4 + \frac{4}{\sqrt{3}} + \frac{1}{3}\right) + 4p + \frac{2p}{\sqrt{3}} + q$$

$$= 12 + \frac{12}{\sqrt{3}} + 1 + 4p + \frac{2p}{\sqrt{3}} + q$$

$$= 4p + \frac{2p}{\sqrt{3}} + q + 13 + \frac{12}{\sqrt{3}}$$

But f'(c) = 0
$\therefore$ $4p + \frac{2p}{\sqrt{3}} + q + 13 + \frac{12}{\sqrt{3}} = 0$
$\therefore$ (4√3 + 2)p + √3q + (13√3 + 12) = 0
$\therefore$ (4√3 + 2)p + √3q = -13√3 – 12 .... (2)
Multiplying equation (1) by √3, we get
4√3p + √3q= -13√3
Subtracting this equation from (2), we get
2p = -12 ⇒ p= -6
$\therefore$ from (1), 4(-6) + q = -13 ⇒ q = 11
Hence, p = -6 and q = 11.

Question 5.
If Rolle's theorem holds for the function f(x) = (x – 2) log x, x ∈ [1, 2], show that the equation x log x = 2 – x is satisfied by at least one value of x in (1, 2).
Solution:
The Rolle's theorem holds for the function f(x) = (x – 2) log x, x ∈ [1, 2].
$\therefore$ there exists at least one real number c ∈ (1, 2) such that f'(c) = 0.
Now, f(x) = (x – 2) log x

$$\therefore\ f'(x) = \frac{d}{dx}[(x-2)\log x]$$

$$= (x-2)\cdot\frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x-2)$$

$$= (x-2) \times \frac{1}{x} + (\log x)(1-0)$$

$$= 1 - \frac{2}{x} + \log x$$

$$\therefore\ f'(c) = 1 - \frac{2}{c} + \log c$$

$\therefore$ f'(c) = 0 gives 1 – $\frac{2}{c}$ + log c = 0
$\therefore$ c – 2 + c log c = 0
$\therefore$ c log c = 2 – c, where c ∈ (1, 2)
$\therefore$ c satisfies the equation x log x = 2 – x, c ∈ (1, 2).
Hence, the equation x log x = 2 – x is satisfied by at least one value of x in (1, 2).

Question 6.
The function f(x) = $x(x+3)e^{-\frac{x}{2}}$ satisfies all the conditions of Rolle's theorem on [-3, 0]. Find the value of c such that f'(c) = 0.
Solution:
The function f(x) satisfies all the conditions of Rolle's theorem, therefore there exist c ∈ (-3, 0) such that f'(c) = 0.

Now, $f(x) = x(x+3)e^{-\frac{x}{2}} = (x^2+3x)e^{-\frac{x}{2}}$

$$\therefore\ f'(x) = \frac{d}{dx}[(x^2+3x)e^{-\frac{x}{2}}]$$

$$= (x^2+3x)\cdot\frac{d}{dx}(e^{-\frac{x}{2}}) + e^{-\frac{x}{2}}\cdot\frac{d}{dx}(x^2+3x)$$

$$= (x^2+3x)\cdot e^{-\frac{x}{2}}\cdot\frac{d}{dx}\left(-\frac{x}{2}\right) + e^{-\frac{x}{2}} \times (2x+3\times 1)$$

$$= (x^2+3x)\cdot e^{-\frac{x}{2}} \times -\frac{1}{2} + e^{-\frac{x}{2}}(2x+3)$$

$$= e^{-\frac{x}{2}}\left[(2x+3) - \frac{x^2+3x}{2}\right]$$

$$= e^{-\frac{x}{2}}\left[\frac{4x+6-x^2-3x}{2}\right]$$

$$= \frac{e^{-\frac{x}{2}}}{2}(6+x-x^2)$$

$$= \frac{e^{-\frac{x}{2}}}{2}(3-x)(2+x)$$

$$\therefore\ f'(c) = \frac{e^{-\frac{c}{2}}}{2}(3-c)(2+c)$$

$$\therefore\ f'(c) = 0 \text{ gives } \frac{e^{-\frac{c}{2}}}{2}(3-c)(2+c) = 0$$

$$\therefore\ (3-c)(2+c) = 0 \qquad \ldots\left[\because\ \frac{e^{-\frac{c}{2}}}{2} \neq 0\right]$$

$\therefore\ (3-c) = 0$ or $(2+c) = 0$

$\therefore\ c = 3$ or $c = -2$

But $3 \notin (-3, 0)$ $\quad\therefore\ c \neq 3$

Hence, $c = -2$.

Question 7.
Verify Lagrange's mean value theorem for the following functions:
(i) f(x) = log x on [1, e].
Solution:
The function f given as f(x) = log x is a logarithmic function that is continuous for all positive real numbers.
Hence, it is continuous on [1, e] and differentiable on (1, e).
Thus, the function f satisfies the conditions of Lagrange's mean value theorem.
∴ there exists c ∈ (1, e) such that

$$f'(c) = \frac{f(e)-f(1)}{e-1} \qquad \text{.........(1)}$$

Now, $f(x) = \log x$

$\therefore\ f(1) = \log 1 = 0$ and $f(e) = \log e = 1$

Also, $f'(x) = \frac{d}{dx}(\log x) = \frac{1}{x}$

$$\therefore\ f'(c) = \frac{1}{c}$$

$$\therefore\ \text{from (1)},\ \frac{1}{c} = \frac{1-0}{e-1} = \frac{1}{e-1}$$

$\therefore\ c = e-1 \in (1, e)$

Hence, Lagrange's mean value theorem is verified.

(ii) f(x) = (x – 1)(x – 2)(x – 3) on [0, 4].
Solution:

The function f given as
f(x) = (x – 1)(x – 2)(x – 3)
= (x – 1)(x^2 – 5x + 6)
= x^3 – 5x^2 + 6x – x^2 + 5x – 6
= x^3 – 6x^2 + 11x – 6 is a polynomial function.
Hence, it is continuous on [0, 4] and differentiable on (0, 4).
Thus, the function f satisfies the conditions of Lagrange's, mean value theorem.
∴ there exists c ∈ (0, 4) such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0} \qquad \text{.......(1)}$$

Now, $f(x) = (x-1)(x-2)(x-3)$

$\therefore\ f(0) = (0-1)(0-2)(0-3) = (-1)(-2)(-3) = -6$

and $f(4) = (4-1)(4-2)(4-3) = (3)(2)(1) = 6$

Also, $f'(x) = \frac{d}{dx}(x^3 - 6x^2 + 11x - 6)$

$$= 3x^2 - 6 \times 2x + 11 \times 1 - 0$$

$$= 3x^2 - 12x + 11$$

$\therefore\ f'(c) = 3c^2 - 12c + 11$

$\therefore$ from (1), $3c^2 - 12c + 11 = \frac{6-(-6)}{4}$

$\therefore\ 3c^2 - 12c + 11 = 3$

$\therefore\ 3c^2 - 12c + 8 = 0.$

$$\therefore\ c = \frac{12 \pm \sqrt{144 - 4(3)(8)}}{2(3)}$$

$$\therefore\ c = \frac{12 \pm \sqrt{48}}{6} = \frac{12 \pm 4\sqrt{3}}{6}$$

$$\therefore\ c = 2 \pm \frac{2}{\sqrt{3}} \in (0, 4)$$

Hence, Lagrange's mean value theorem is verified.

(iii) f(x) = x^2 – 3x – 1, x ∈ [-117,137]
Solution:
The function f given as f(x) = x^2 – 3x – 1 is a polynomial function.
Hence, it is continuous on [-117,137] and differentiable on (-117,137).
Thus, the function f satisfies the conditions of LMVT.
∴ there exists c ∈ (-117,137) such that

$$f'(c) = \frac{f\left(\frac{13}{7}\right) - f\left(\frac{-11}{7}\right)}{\frac{13}{7} - \left(\frac{-11}{7}\right)} \qquad \text{.........(1)}$$

Now, $f(x) = x^2 - 3x - 1$

$$\therefore\ f\left(\frac{-11}{7}\right) = \left(\frac{-11}{7}\right)^2 - 3\left(\frac{-11}{7}\right) - 1$$

$$= \frac{121}{49} + \frac{33}{7} - 1$$

$$= \frac{121 + 231 - 49}{49} = \frac{303}{49}$$

and $f\left(\frac{13}{7}\right) = \left(\frac{13}{7}\right)^2 - 3\left(\frac{13}{7}\right) - 1 = \frac{169}{49} - \frac{39}{7} - 1$

$$= \frac{169 - 273 - 49}{49} = \frac{-153}{49}$$

Also, $f'(x) = \frac{d}{dx}(x^2 - 3x - 1) = 2x - 3 \times 1 - 0$

$$= 2x - 3$$

$\therefore\ f'(c) = 2c - 3$

$$\therefore\ \text{from (1)},\ 2c - 3 = \frac{\frac{-153}{49} - \frac{303}{49}}{\frac{13}{7} + \frac{11}{7}}$$

$$\therefore\ 2c - 3 = -\frac{456}{49} \times \frac{7}{24} = \frac{-57}{21}$$

$$\therefore\ 2c = \frac{-57}{21} + 3 = \frac{-57 + 63}{21} = \frac{6}{21} = \frac{2}{7}$$

$$\therefore\ c = \frac{1}{7} \in \left(\frac{-11}{7}, \frac{13}{7}\right)$$

Hence, Lagrange’s mean value theorem is verified.

(iv) $f(x) = 2x - x^2$, $x \in [0, 1]$.
Solution:
The function f given as $f(x) = 2x - x^2$ is a polynomial function.
Hence, it is continuous on [0, 1] and differentiable on (0, 1).
Thus, the function f satisfies the conditions of Lagrange’s mean value theorem.
∴ there exists $c \in (0, 1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} \qquad \text{.........(1)}$$

Now, $f(x) = 2x - x^2$

$\therefore\ f(0) = 0 - 0 = 0$ and $f(1) = 2(1) - 1^2 = 1$

Also, $f'(x) = \frac{d}{dx}(2x - x^2) = 2 \times 1 - 2x$

$$= 2 - 2x$$

$\therefore\ f'(c) = 2 - 2c$

$$\therefore\ \text{from (1)},\ 2 - 2c = \frac{1 - 0}{1} = 1$$

$$\therefore\ 2c = 1 \qquad \therefore\ c = \frac{1}{2} \in (0, 1)$$

Hence, Lagrange’s mean value theorem is verified.

(v) f(x) = $\frac{x-1}{x-3}$ on [4, 5].
Solution:
The function f given as
f(x) = $\frac{x-1}{x-3}$ is a rational function which is continuous except at x = 3.
But 3 ∉ [4, 5]
Hence, it is continuous on [4, 5] and differentiable on (4, 5).
Thus, the function f satisfies the conditions of Lagrange's mean value theorem.
∴ there exists c ∈ (4, 5) such that
f'(c) = $\frac{f(5)-f(4)}{5-4}$ ....(1)

$$\text{Now, } f(x)=\frac{x-1}{x-3}$$

$$\therefore\ f(4)=\frac{4-1}{4-3}=\frac{3}{1}=3$$

$$\text{and } f(5)=\frac{5-1}{5-3}=\frac{4}{2}=2$$

$$\text{Also, } f'(x)=\frac{d}{dx}\left(\frac{x-1}{x-3}\right)$$

$$=\frac{(x-3)\cdot\frac{d}{dx}(x-1)-(x-1)\cdot\frac{d}{dx}(x-3)}{(x-3)^2}$$

$$=\frac{(x-3)\times(1-0)-(x-1)\times(1-0)}{(x-3)^2}$$

$$=\frac{x-3-x+1}{(x-3)^2}=\frac{-2}{(x-3)^2}$$

$$\therefore\ f'(c)=\frac{-2}{(c-3)^2}$$

$$\therefore\ \text{from (1), } \frac{-2}{(c-3)^2}=\frac{2-3}{1}=-1$$

$$\therefore\ (c-3)^2=2$$

$$\therefore\ c-3=\pm\sqrt{2} \qquad \therefore\ c=3\pm\sqrt{2}$$

$$\text{But } (3-\sqrt{2})\notin(4,5) \qquad \therefore\ c\neq 3-\sqrt{2}$$

$$\therefore\ c=3+\sqrt{2}\in(4,5)$$

Hence, Lagrange's mean value theorem is verified.

Maharashtra State Board 12th Maths Solutions Chapter 2 Applications of Derivatives Ex 2.3

Question 1.
Check the validity of the Rolle's theorem for the following functions.
(i) f(x) = x^2 – 4x + 3, x ∈ [1, 3]
Solution:
The function f given as f(x) = x^2 – 4x + 3 is polynomial function.
Hence, it is continuous on [1, 3] and differentiable on (1, 3).
Now, f(1) = 1^2 – 4(1) + 3 = 1 – 4 + 3 = 0
and f(3) = 3^2 – 4(3) + 3 = 9 – 12 + 3 = 0
∴ f(1) = f(3)
Thus, the function f satisfies all the conditions of Rolle's theorem.

(ii) $f(x) = e^{-x} \sin x$, $x \in [0, \pi]$.
Solution:
The functions e^{-x} and $\sin x$ are continuous and differentiable on their domains.
∴ $f(x) = e^{-x} \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.
Now, $f(0) = e^{0} \sin 0 = 1 \times 0 = 0$
and $f(\pi) = e^{-\pi} \sin \pi = e^{-\pi} \times 0 = 0$
∴ $f(0) = f(\pi)$
Thus, the function f satisfies all the conditions of the Rolle's theorem.

(iii) $f(x) = 2x^2 - 5x + 3$, $x \in [1, 3]$.
Solution:
The function f given as $f(x) = 2x^2 - 5x + 3$ is a polynomial function.
Hence, it is continuous on $[1, 3]$ and differentiable on $(1, 3)$.
Now, $f(1) = 2(1)^2 - 5(1) + 3 = 2 - 5 + 3 = 0$
and $f(3) = 2(3)^2 - 5(3) + 3 = 18 - 15 + 3 = 6$
∴ $f(1) \neq f(3)$
Hence, the conditions of Rolle's theorem are not satisfied.

(iv) $f(x) = \sin x - \cos x + 3$, $x \in [0, 2\pi]$.
Solution:
The functions $\sin x$, $\cos x$ and 3 are continuous and differentiable on their domains.
∴ $f(x) = \sin x - \cos x + 3$ is continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$.
Now, $f(0) = \sin 0 - \cos 0 + 3 = 0 - 1 + 3 = 2$
and $f(2\pi) = \sin 2\pi - \cos 2\pi + 3 = 0 - 1 + 3 = 2$
∴ $f(0) = f(2\pi)$
Thus, the function f satisfies all the conditions of the Rolle's theorem.

(v) $f(x) = x^2$, if $0 \leq x \leq 2$
$= 6 - x$, if $2 < x \leq 6$.
Solution:
$f(x) = x^2$, if $0 \leq x \leq 2$
$= 6 - x$, if $2 < x \leq 6$
∴ $f(x) = \frac{d}{dx}(x^2) = 2x$, if $0 \leq x \leq 2$
$= \frac{d}{dx}(6 - x) = -1$, if $2 < x \leq 6$
∴ $Lf'(2) = 2(2) = 4$ and $Rf'(2) = -1$
∴ $Lf'(2) \neq Rf'(2)$
∴ f is not differentiable at $x = 2$ and $2 \in (0, 6)$.
∴ f is not differentiable at all the points on $(0, 6)$.
Hence, the conditions of Rolle's theorem are not satisfied.

(vi) $f(x) = x^{\frac{2}{3}}$, $x \in [-1, 1]$.
Solution:
$f(x) = x^{\frac{2}{3}}$
∴ $f'(x) = \frac{d}{dx}\left(x^{\frac{2}{3}}\right) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$
This does not exist at $x = 0$ and $0 \in (-1, 1)$
∴ f is not differentiable on the interval $(-1, 1)$.
Hence, the conditions of Rolle's theorem are not satisfied.

Question 2.
Given an interval $[a, b]$ that satisfies hypothesis of Rolle's theorem for the function $f(x) = x^4 + x^2 - 2$. It is known that $a = -1$. Find the value of b.
Solution:
$f(x) = x^4 + x^2 - 2$
Since the hypothesis of Rolle's theorem are satisfied by f in the interval $[a, b]$, we have
$f(a) = f(b)$, where $a = -1$
Now, $f(a) = f(-1) = (-1)^4 + (-1)^2 - 2 = 1 + 1 - 2 = 0$
and $f(b) = b^4 + b^2 - 2$
∴ $f(a) = f(b)$ gives
$0 = b^4 + b^2 - 2$ i.e. $b^4 + b^2 - 2 = 0$.
Since, $b = 1$ satisfies this equation, $b = 1$ is one of the roots of this equation.
Hence, $b = 1$.

Question 3.
Verify Rolle's theorem for the following functions.

(i) $f(x) = \sin x + \cos x + 7$, $x \in [0, 2\pi]$
Solution:
The functions sin x, cos x and 7 are continuous and differentiable on their domains.
∴ $f(x) = \sin x + \cos x + 7$ is continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$
Now, $f(0) = \sin 0 + \cos 0 + 7 = 0 + 1 + 7 = 8$
and $f(2\pi) = \sin 2\pi + \cos 2\pi + 7 = 0 + 1 + 7 = 8$
∴ $f(0) = f(2\pi)$
Thus, the function f satisfies all the conditions of Rolle's theorem.
∴ there exists $c \in (0, 2\pi)$ such that $f'(c) = 0$.
Now, $f(x) = \sin x + \cos x + 7$
∴ $f'(x) = \frac{d}{dx}(\sin x + \cos x + 7)$
$= \cos x - \sin x + 0$
$= \cos x - \sin x$
∴ $f'(c) = \cos c - \sin c$
∴ $f'(c) = 0$ gives, $\cos c - \sin c = 0$
∴ $\cos c = \sin c$
∴ $c = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots$
But $\frac{\pi}{4}, \frac{5\pi}{4} \in (0, 2\pi)$
∴ $c = \frac{\pi}{4}$ or $\frac{5\pi}{4}$
Hence, the Rolle's theorem is verified.

(ii) $f(x) = \sin\left(\frac{x}{2}\right)$, $x \in [0, 2\pi]$
Solution:
The function $f(x) = \sin\left(\frac{x}{2}\right)$ is continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$.
Now, $f(0) = \sin 0 = 0$
and $f(2\pi) = \sin \pi = 0$
∴ $f(0) = f(2\pi)$
Thus, the function f satisfies all the conditions of Rolle's theorem.
∴ there exists $c \in (0, 2\pi)$ such that $f'(c) = 0$.

Now, $f(x) = \sin\left(\frac{x}{2}\right)$

$$\therefore f'(x) = \frac{d}{dx}\left[\sin\left(\frac{x}{2}\right)\right]$$

$$= \cos\left(\frac{x}{2}\right)\cdot\frac{d}{dx}\left(\frac{x}{2}\right)$$

$$= \cos\left(\frac{x}{2}\right)\cdot\frac{1}{2} = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$\therefore f'(c) = \frac{1}{2}\cos\left(\frac{c}{2}\right)$$

$$\therefore f'(c) = 0 \text{ gives } \frac{1}{2}\cos\left(\frac{c}{2}\right) = 0$$

$$\therefore \cos\left(\frac{c}{2}\right) = 0$$

$$\therefore \cos\frac{c}{2} = \cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = \cos\frac{5\pi}{2} = \ldots$$

$$\therefore \frac{c}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\therefore c = \pi, 3\pi, 5\pi, \ldots$$

But $\pi \in (0, 2\pi)$

$$\therefore c = \pi$$

Hence, Rolle's theorem is verified.

(iii) $f(x) = x^2 - 5x + 9$, $x \in [1, 4]$.
Solution:
The function f given as $f(x) = x^2 - 5x + 9$ is a polynomial function.
Hence it is continuous on $[1, 4]$ and differentiable on $(1, 4)$.
Now, $f(1) = 1^2 - 5(1) + 9 = 1 - 5 + 9 = 5$
and $f(4) = 4^2 - 5(4) + 9 = 16 - 20 + 9 = 5$

$\therefore$ f(1) = f(4)
Thus, the function f satisfies all the conditions of the Rolle's theorem.
$\therefore$ there exists c $\in$ (1, 4) such that f'(c) = 0.
Now, f(x) = x2 – 5x + 9
$\therefore$ f'(x) = $\frac{d}{dx}$ (x^2 – 5x + 9)
= 2x – 5 × 1 + 0
= 2x – 5
$\therefore$ f'(c) = 2c – 5
$\therefore$ f'(c) = 0 gives, 2c – 5 = 0
$\therefore$ c = 5/2 $\in$ (1, 4)
Hence, the Rolle's theorem is verified.

Question 4.
If Rolle's theorem holds for the function f(x) = x^3 + px^2 + qx + 5, x $\in$ [1, 3] with c = 2 + $\frac{1}{\sqrt{3}}$, find the values of p and q.
Solution:
The Rolle's theorem holds for the function f(x) = x^3 + px^2 + qx + 5, x $\in$ [1, 3]
$\therefore$ f(1) = f(3)
$\therefore$ 1^3 + p$(1)^2$ + q(1) + 5 = 3^3 + p $(3)^2$ + q(3) + 5
$\therefore$ 1 + p + q + 5 = 27 + 9p + 3q + 5
$\therefore$ 8p + 2q = -26
$\therefore$ 4p + q = -13 ..... (1)
Also, there exists at least one point c $\in$ (1, 3) such that f'(c) = 0.
Now, f'(x) = $\frac{d}{dx}$ (x^3 + px^2 + qx + 5)
= 3x^2 + p × 2x + q × 1 + 0
= 3x^2 + 2px + q

$$\therefore\ f'(c) = 3c^2 + 2pc + q, \text{ where } c = 2 + \frac{1}{\sqrt{3}}$$

$$\therefore\ f'(c) = 3\left(2 + \frac{1}{\sqrt{3}}\right)^2 + 2p\left(2 + \frac{1}{\sqrt{3}}\right) + q$$

$$= 3\left(4 + \frac{4}{\sqrt{3}} + \frac{1}{3}\right) + 4p + \frac{2p}{\sqrt{3}} + q$$

$$= 12 + \frac{12}{\sqrt{3}} + 1 + 4p + \frac{2p}{\sqrt{3}} + q$$

$$= 4p + \frac{2p}{\sqrt{3}} + q + 13 + \frac{12}{\sqrt{3}}$$

But f'(c) = 0
$\therefore$ $4p + \frac{2p}{\sqrt{3}} + q + 13 + \frac{12}{\sqrt{3}} = 0$
$\therefore$ (4√3 + 2)p + √3q + (13√3 + 12) = 0
$\therefore$ (4√3 + 2)p + √3q = -13√3 – 12 .... (2)
Multiplying equation (1) by √3, we get
4√3p + √3q= -13√3
Subtracting this equation from (2), we get
2p = -12 ⇒ p= -6
$\therefore$ from (1), 4(-6) + q = -13 ⇒ q = 11
Hence, p = -6 and q = 11.

Question 5.
If Rolle's theorem holds for the function f(x) = (x – 2) log x, x $\in$ [1, 2], show that the equation x log x = 2 – x is satisfied by at least one value of x in (1, 2).
Solution:
The Rolle's theorem holds for the function f(x) = (x – 2) log x, x $\in$ [1, 2].
$\therefore$ there exists at least one real number c $\in$ (1, 2) such that f'(c) = 0.
Now, f(x) = (x – 2) log x

$$\therefore\ f'(x) = \frac{d}{dx}[(x-2)\log x]$$

$$= (x-2)\cdot\frac{d}{dx}(\log x) + \log x\cdot\frac{d}{dx}(x-2)$$

$$= (x-2)\times\frac{1}{x} + (\log x)(1-0)$$

$$= 1 - \frac{2}{x} + \log x$$

$$\therefore\ f'(c) = 1 - \frac{2}{c} + \log c$$

∴ f'(c) = 0 gives $1 - \frac{2}{c} + \log c = 0$
∴ c – 2 + c log c = 0
∴ c log c = 2 – c, where c ∈ (1, 2)
∴ c satisfies the equation x log x = 2 – x, c ∈ (1, 2).
Hence, the equation x log x = 2 – x is satisfied by at least one value of x in (1, 2).

Question 6.
The function f(x) = $x(x+3)e^{-\frac{x}{2}}$ satisfies all the conditions of Rolle's theorem on [-3, 0]. Find the value of c such that f'(c) = 0.
Solution:
The function f(x) satisfies all the conditions of Rolle's theorem, therefore there exist c ∈ (-3, 0) such that f'(c) = 0.

$$\text{Now, } f(x) = x(x+3)e^{-\frac{x}{2}} = (x^2+3x)e^{-\frac{x}{2}}$$

$$\therefore\ f'(x) = \frac{d}{dx}[(x^2+3x)e^{-\frac{x}{2}}]$$

$$= (x^2+3x)\cdot\frac{d}{dx}(e^{-\frac{x}{2}}) + e^{-\frac{x}{2}}\cdot\frac{d}{dx}(x^2+3x)$$

$$= (x^2+3x)\cdot e^{-\frac{x}{2}}\cdot\frac{d}{dx}\left(-\frac{x}{2}\right) + e^{-\frac{x}{2}}\times(2x+3\times 1)$$

$$= (x^2+3x)\cdot e^{-\frac{x}{2}}\times -\frac{1}{2} + e^{-\frac{x}{2}}(2x+3)$$

$$= e^{-\frac{x}{2}}\left[(2x+3) - \frac{x^2+3x}{2}\right]$$

$$= e^{-\frac{x}{2}}\left[\frac{4x+6-x^2-3x}{2}\right]$$

$$= \frac{e^{-\frac{x}{2}}}{2}(6+x-x^2)$$

$$= \frac{e^{-\frac{x}{2}}}{2}(3-x)(2+x)$$

$$\therefore\ f'(c) = \frac{e^{-\frac{c}{2}}}{2}(3-c)(2+c)$$

$$\therefore\ f'(c) = 0 \text{ gives } \frac{e^{-\frac{c}{2}}}{2}(3-c)(2+c) = 0$$

$$\therefore\ (3-c)(2+c) = 0 \qquad \ldots\left[\because\ \frac{e^{-\frac{c}{2}}}{2} \neq 0\right]$$

$$\therefore\ (3-c) = 0 \text{ or } (2+c) = 0$$

$$\therefore\ c = 3 \text{ or } c = -2$$

But $3 \notin (-3, 0)$ $\quad\therefore\ c \neq 3$

Hence, $c = -2$.

Question 7.
Verify Lagrange's mean value theorem for the following functions:
(i) f(x) = log x on [1, e].
Solution:
The function f given as f(x) = log x is a logarithmic function that is continuous for all positive real numbers.

Hence, it is continuous on [1, e] and differentiable on (1, e).
Thus, the function f satisfies the conditions of Lagrange's mean value theorem.
∴ there exists c ∈ (1, e) such that

$$f'(c) = \frac{f(e) - f(1)}{e - 1} \quad \text{.........(1)}$$

Now, $f(x) = \log x$

$\therefore\ f(1) = \log 1 = 0$ and $f(e) = \log e = 1$

Also, $f'(x) = \dfrac{d}{dx}(\log x) = \dfrac{1}{x}$

$\therefore\ f'(c) = \dfrac{1}{c}$

$\therefore$ from (1), $\dfrac{1}{c} = \dfrac{1-0}{e-1} = \dfrac{1}{e-1}$

$\therefore\ c = e - 1 \in (1, e)$

Hence, Lagrange's mean value theorem is verified.

(ii) f(x) = (x – 1)(x – 2)(x – 3) on [0, 4].
Solution:
The function f given as
f(x) = (x – 1)(x – 2)(x – 3)
= (x – 1)(x^2 – 5x + 6)
= x^3 – 5x^2 + 6x – x^2 + 5x – 6
= x^3 – 6x^2 + 11x – 6 is a polynomial function.
Hence, it is continuous on [0, 4] and differentiable on (0, 4).
Thus, the function f satisfies the conditions of Lagrange's, mean value theorem.
∴ there exists c ∈ (0, 4) such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0} \quad \text{.......(1)}$$

Now, $f(x) = (x-1)(x-2)(x-3)$

$\therefore\ f(0) = (0-1)(0-2)(0-3) = (-1)(-2)(-3) = -6$

and $f(4) = (4-1)(4-2)(4-3) = (3)(2)(1) = 6$

Also, $f'(x) = \dfrac{d}{dx}(x^3 - 6x^2 + 11x - 6)$

$= 3x^2 - 6 \times 2x + 11 \times 1 - 0$

$= 3x^2 - 12x + 11$

$\therefore\ f'(c) = 3c^2 - 12c + 11$

$\therefore$ from (1), $3c^2 - 12c + 11 = \dfrac{6-(-6)}{4}$

$\therefore\ 3c^2 - 12c + 11 = 3$

$\therefore\ 3c^2 - 12c + 8 = 0.$

$\therefore\ c = \dfrac{12 \pm \sqrt{144 - 4(3)(8)}}{2(3)}$

$\therefore\ c = \dfrac{12 \pm \sqrt{48}}{6} = \dfrac{12 \pm 4\sqrt{3}}{6}$

$\therefore\ c = 2 \pm \dfrac{2}{\sqrt{3}} \in (0, 4)$

Hence, Lagrange's mean value theorem is verified.

(iii) f(x) = x^2 – 3x – 1, x ∈ [-117,137]
Solution:
The function f given as f(x) = x^2 – 3x – 1 is a polynomial function.
Hence, it is continuous on [-117,137] and differentiable on (-117,137).
Thus, the function f satisfies the conditions of LMVT.
∴ there exists c ∈ (-117,137) such that

$$f'(c)=\frac{f\left(\frac{13}{7}\right)-f\left(\frac{-11}{7}\right)}{\frac{13}{7}-\left(\frac{-11}{7}\right)} \qquad \text{.........(1)}$$

Now, $f(x)=x^2-3x-1$

$$\therefore\ f\left(\frac{-11}{7}\right)=\left(\frac{-11}{7}\right)^2-3\left(\frac{-11}{7}\right)-1$$

$$=\frac{121}{49}+\frac{33}{7}-1$$

$$=\frac{121+231-49}{49}=\frac{303}{49}$$

and $f\left(\frac{13}{7}\right)=\left(\frac{13}{7}\right)^2-3\left(\frac{13}{7}\right)-1=\frac{169}{49}-\frac{39}{7}-1$

$$=\frac{169-273-49}{49}=\frac{-153}{49}$$

Also, $f'(x)=\frac{d}{dx}(x^2-3x-1)=2x-3\times 1-0$

$$=2x-3$$

$\therefore\ f'(c)=2c-3$

$$\therefore\ \text{from (1)},\ 2c-3=\frac{\frac{-153}{49}-\frac{303}{49}}{\frac{13}{7}+\frac{11}{7}}$$

$$\therefore\ 2c-3=-\frac{456}{49}\times\frac{7}{24}=\frac{-57}{21}$$

$$\therefore\ 2c=\frac{-57}{21}+3=\frac{-57+63}{21}=\frac{6}{21}=\frac{2}{7}$$

$$\therefore\ c=\frac{1}{7}\in\left(\frac{-11}{7},\frac{13}{7}\right)$$

Hence, Lagrange's mean value theorem is verified.

(iv) $f(x) = 2x - x^2$, $x \in [0, 1]$.
Solution:
The function f given as $f(x) = 2x - x^2$ is a polynomial function.
Hence, it is continuous on [0, 1] and differentiable on (0, 1).
Thus, the function f satisfies the conditions of Lagrange's mean value theorem.
$\therefore$ there exists $c \in (0, 1)$ such that

$$f'(c)=\frac{f(1)-f(0)}{1-0} \qquad \text{.........(1)}$$

Now, $f(x)=2x-x^2$

$\therefore\ f(0)=0-0=0$ and $f(1)=2(1)-1^2=1$

Also, $f'(x)=\frac{d}{dx}(2x-x^2)=2\times 1-2x$

$$=2-2x$$

$\therefore\ f'(c)=2-2c$

$$\therefore\ \text{from (1)},\ 2-2c=\frac{1-0}{1}=1$$

$\therefore\ 2c=1 \qquad \therefore\ c=\frac{1}{2}\in(0,1)$

Hence, Lagrange's mean value theorem is verified.

(v) f(x) = $\frac{x-1}{x-3}$ on [4, 5].

Solution:

The function f given as

f(x) = $\frac{x-1}{x-3}$ is a rational function which is continuous except at x = 3.

But 3 ∉ [4, 5]

Hence, it is continuous on [4, 5] and differentiable on (4, 5).

Thus, the function f satisfies the conditions of Lagrange's mean value theorem.

∴ there exists c ∈ (4, 5) such that

f'(c) = $\frac{f(5)-f(4)}{5-4}$ ....(1)

Now, $f(x)=\frac{x-1}{x-3}$

$\therefore\ f(4)=\frac{4-1}{4-3}=\frac{3}{1}=3$

and $f(5)=\frac{5-1}{5-3}=\frac{4}{2}=2$

Also, $f'(x)=\frac{d}{dx}\left(\frac{x-1}{x-3}\right)$

$$=\frac{(x-3)\cdot\frac{d}{dx}(x-1)-(x-1)\cdot\frac{d}{dx}(x-3)}{(x-3)^2}$$

$$=\frac{(x-3)\times(1-0)-(x-1)\times(1-0)}{(x-3)^2}$$

$$=\frac{x-3-x+1}{(x-3)^2}=\frac{-2}{(x-3)^2}$$

$\therefore\ f'(c)=\frac{-2}{(c-3)^2}$

$\therefore$ from (1), $\frac{-2}{(c-3)^2}=\frac{2-3}{1}=-1$

$\therefore\ (c-3)^2=2$

$\therefore\ c-3=\pm\sqrt{2}$ $\qquad\therefore\ c=3\pm\sqrt{2}$

But $(3-\sqrt{2})\notin(4,5)$ $\qquad\therefore\ c\neq 3-\sqrt{2}$

$\therefore\ c=3+\sqrt{2}\in(4,5)$

Hence, Lagrange's mean value theorem is verified.

Maharashtra State Board 12th Maths Solutions Chapter 2 Applications of Derivatives Ex 2.4

Question 1.

Test whether the following functions are increasing or decreasing.

(i) f(x) = $x^3 - 6x^2 + 12x - 16$, x ∈ R.

Solution:

f(x) = $x^3 - 6x^2 + 12x - 16$

∴ f'(x) = $\frac{d}{dx}$ ($x^3 - 6x^2 + 12x - 16$)

= $3x^2 - 6 \times 2x + 12 \times 1 - 0$

= $3x^2 - 12x + 12$

= $3(x^2 - 4x + 4)$

= $3(x - 2)^2 \geq 0$ for all x ∈ R

∴ f(x) ≥ 0 for all x ∈ R

∴ f is increasing for all x ∈ R.

(ii) $f(x) = 2 - 3x + 3x^2 - x^3$, $x \in R$.
Solution:
$f(x) = 2 - 3x + 3x^2 - x^3$
$\therefore f'(x) = \frac{d}{dx}(2 - 3x + 3x^2 - x^3)$
$= 0 - 3 \times 1 + 3 \times 2x - 3x^2$
$= -3 + 6x - 3x^2$
$= -3(x^2 - 2x + 1)$
$= -3(x - 1)^2 \le 0$ for all $x \in R$
$\therefore f'(x) \le 0$ for all $x \in R$
$\therefore$ f is decreasing for all $x \in R$.

(iii) $f(x) = x - \frac{1}{x}$, $x \in R$, $x \neq 0$.
Solution:
$f(x) = x - \frac{1}{x}$
$f'(x) = \frac{d}{dx}\left(x - \frac{1}{x}\right) = 1 - \left(-\frac{1}{x^2}\right)$
$= 1 + \frac{1}{x^2} > 0$ for all $x \in R$, $x \neq 0$
$\therefore f'(x) > 0$ for all $x \in R$, where $x \neq 0$
$\therefore$ f is increasing for all $x \in R$, where $x \neq 0$.

Question 2.
Find the values of x for which the following functions are strictly increasing:
(i) $f(x) = 2x^3 - 3x^2 - 12x + 6$
Solution:
$f(x) = 2x^3 - 3x^2 - 12x + 6$
$\therefore f'(x) = \frac{d}{dx}(2x^3 - 3x^2 - 12x + 6)$
$= 2 \times 3x^2 - 3 \times 2x - 12 \times 1 + 0$
$= 6x^2 - 6x - 12$
$= 6(x^2 - x - 2)$
f is strictly increasing if $f'(x) > 0$
i.e. if $6(x^2 - x - 2) > 0$
i.e. if $x^2 - x - 2 > 0$
i.e. if $x^2 - x > 2$
i.e. if $x^2 - x + \frac{1}{4} > 2 + \frac{1}{4}$
i.e. if $\left(x - \frac{1}{2}\right)^2 > \frac{9}{4}$
i.e. if $x - \frac{1}{2} > \frac{3}{2}$ or $x - \frac{1}{2} < -\frac{3}{2}$ i.e. if $x > 2$ or $x < -1$
$\therefore$ f is strictly increasing if $x < -1$ or $x > 2$.

(ii) $f(x) = 3 + 3x - 3x^2 + x^3$
Solution:
$f(x) = 3 + 3x - 3x^2 + x^3$
$\therefore f'(x) = \frac{d}{dx}(3 + 3x - 3x^2 + x^3)$
$= 0 + 3 \times 1 - 3 \times 2x + 3x^2$
$= 3 - 6x + 3x^2$
$= 3(x^2 - 2x + 1)$
f is strictly increasing if $f'(x) > 0$
i.e. if $3(x^2 - 2x + 1) > 0$
i.e. if $x^2 - 2x + 1 > 0$
i.e. if $(x - 1)^2 > 0$
This is possible if $x \in R$ and $x \neq 1$
i.e. $x \in R - \{1\}$
$\therefore$ f is strictly increasing if $x \in R - \{1\}$.

(iii) $f(x) = x^3 - 6x^2 - 36x + 7$
Solution:
$f(x) = x^3 - 6x^2 - 36x + 7$
$\therefore f'(x) = \frac{d}{dx}(x^3 - 6x^2 - 36x + 7)$
$= 3x^2 - 6 \times 2x - 36 \times 1 + 0$
$= 3x^2 - 12x - 36$
$= 3(x^2 - 4x - 12)$
f is strictly increasing if $f'(x) > 0$
i.e. if $3(x^2 - 4x - 12) > 0$
i.e. if $x^2 - 4x - 12 > 0$
i.e. if $x^2 - 4x > 12$
i.e. if $x^2 - 4x + 4 > 12 + 4$
i.e. if $(x - 2)^2 > 16$

i.e. if $x - 2 > 4$ or $x - 2 < -4$ i.e. if $x > 6$ or $x < -2$
∴ f is strictly increasing if $x < -2$ or $x > 6$.

Question 3.
Find the values of x for which the following functions are strictly decreasiong:
(i) $f(x) = 2x^3 - 3x^2 - 12x + 6$
Solution:
$f(x) = 2x^3 - 3x^2 - 12x + 6$
∴ $f'(x) = \frac{d}{dx}(2x^3 - 3x^2 - 12x + 6)$
$= 2 \times 3x^2 - 3 \times 2x - 12 \times 1 + 0$
$= 6x^2 - 6x - 12$
$= 6(x^2 - x - 2)$
f is strictly decreasing if $f'(x) < 0$
i.e. if $6(x^2 - x - 2) < 0$
i.e. if $x^2 - x - 2 < 0$
i.e. if $x^2 - x < 2$
i.e. if $x^2 - x + \frac{1}{4} < 2 + \frac{1}{4}$
i.e. if $\left(x - \frac{1}{2}\right)^2 < \frac{9}{4}$
i.e. if $-\frac{3}{2} < x - \frac{1}{2} < \frac{3}{2}$
i.e. if $-\frac{3}{2} + \frac{1}{2} < x - \frac{1}{2} + \frac{1}{2} < \frac{3}{2} + \frac{1}{2}$
i.e. if $-1 < x < 2$
∴ f is strictly decreasing if $-1 < x < 2$.

(ii) $f(x) = x + \frac{25}{x}$
Solution:
$f(x) = x + \frac{25}{x}$, $x \neq 0$
∴ $f'(x) = \frac{d}{dx}\left(x + \frac{25}{x}\right)$
$= 1 + 25(-1)\, x^{-2}$
$= 1 - \frac{25}{x^2}$
f is is strictly decreasing if $f'(x) < 0$
i.e. if $1 - \frac{25}{x^2} < 0$
i.e. if $1 < \frac{25}{x^2}$
i.e. if $x^2 < 25$
i.e. if $-5 < x < 5$, $x \neq 0$
i.e. if $x \in (-5, 5) - \{0\}$
∴ f is strictly decreasing if $x \in (-5, 5) - \{0\}$.

(iii) $f(x) = x^3 - 9x^2 + 24x + 12$
Solution:
$f(x) = x^3 - 9x^2 + 24x + 12$
∴ $f'(x) = \frac{d}{dx}(x^3 - 9x^2 + 24x + 12)$
$= 3x^2 - 9 \times 2x + 24 \times 1 + 0$
$= 3x^2 - 18x + 24$
$= 3(x^2 - 6x + 8)$
f is strictly decreasing if $f'(x) < 0$
i.e. if $3(x^2 - 6x + 8) < 0$
i.e. if $x^2 - 6x + 8 < 0$
i.e. if $x^2 - 6x < -8$
i.e. if $x^2 - 6x + 9 < -8 + 9$
i.e. if $(x - 3)^2 < 1$
i.e. if $-1 < x - 3 < 1$
i.e. if $-1 + 3 < x - 3 + 3 < 1 + 3$
i.e. if $2 < x < 4$
i.e., if $x \in (2, 4)$
∴ f is strictly decreasing if $x \in (2, 4)$

Question 4.
Find the values of x for which the function $f(x) = x^3 - 12x^2 - 144x + 13$
(a) increasing
(b) decreasing.
Solution:
$f(x) = x^3 - 12x^2 - 144x + 13$
∴ $f'(x) = \frac{d}{dx}(x^3 - 12x^2 - 144x + 13)$

$= 3x^2 - 12 \times 2x - 144 \times 1 + 0$
$= 3x^2 - 24x - 144$
$= 3(x^2 - 8x - 48)$

(a) f is increasing if $f'(x) \geq 0$
i.e. if $3(x^2 - 8x - 48) \geq 0$
i.e. if $x^2 - 8x - 48 \geq 0$
i.e. if $x^2 - 8x \geq 48$
i.e. if $x^2 - 8x + 16 \geq 48 + 16$
i.e. if $(x - 4)^2 \geq 64$
i.e. if $x - 4 \geq 8$ or $x - 4 \leq -8$
i.e. if $x > 12$ or $x \leq -4$
∴ f is increasing if $x \leq -4$ or $x \geq 12$,
i.e. $x \in (-\infty, -4] \cup [12, \infty)$.

(b) f is decreasing if $f'(x) \leq 0$
i.e. if $3(x^2 - 8x - 48) \leq 0$
i.e. if $x^2 - 8x - 48 \leq 0$
i.e. if $x^2 - 8x \leq 48$
i.e. if $x^2 - 8x + 16 \leq 48 + 16$
i.e. if $(x - 4)^2 \leq 64$
i.e. if $-8 \leq x - 4 \leq 8$
i.e. if $-4 \leq x \leq 12$
∴ f is decreasing if $-4 \leq x \leq 12$, i.e. $x \in [-4, 12]$.

Question 5.
Find the values of x for which $f(x) = 2x^3 - 15x^2 - 144x - 7$ is
(a) strictly increasing
(b) strictly decreasing.
Solution:
$f(x) = 2x^3 - 15x^2 - 144x - 7$
$f'(x) = \frac{d}{dx}(2x^3 - 15x^2 - 144x - 7)$
$= 2 \times 3x^2 - 15 \times 2x - 144 \times 1 - 0$
$= 6x^2 - 30x - 144$
$= 6(x^2 - 5x - 24)$
(a) f is strictly increasing if $f'(x) > 0$
i.e. if $6(x^2 - 5x - 24) > 0$
i.e. if $x^2 - 5x - 24 > 0$
i.e. if $x^2 - 5x > 24$
i.e. if $x^2 - 5x + \frac{25}{4} > 24 + \frac{25}{4}$
i.e. if $\left(x - \frac{5}{2}\right)^2 > \frac{121}{4}$
i.e. if $x - \frac{5}{2} > \frac{11}{2}$ or $x - \frac{5}{2} < -\frac{11}{2}$ i.e. if $x > 8$ or $x < -3$
∴ f is strictly increasing, if $x < -3$ or $x > 8$.

(b) f is strictly decreasing if $f'(x) < 0$
i.e. if $6(x^2 - 5x - 24) < 0$
i.e. if $x^2 - 5x - 24 < 0$
i.e. if $x^2 - 5x < 24$
i.e. if $x^2 - 5x + \frac{25}{4} < 24 + \frac{25}{4}$
i.e. if $\left(x - \frac{5}{2}\right)^2 < \frac{121}{4}$
i.e. if $-\frac{11}{2} < x - \frac{5}{2} < \frac{11}{2}$
i.e. if $-\frac{11}{2} + \frac{5}{2} < x - \frac{5}{2} + \frac{5}{2} < \frac{11}{2} + \frac{5}{2}$
i.e. if $-3 < x < 8$
∴ f is strictly decreasing, if $-3 < x < 8$.

Question 6.
Find the values of x for which $f(x) = \frac{x}{x^2+1}$ is
(a) strictly increasing
(b) strictly decreasing.
Solution:
$f(x) = \frac{x}{x^2+1}$

$$\therefore\ f'(x) = \frac{d}{dx}\left(\frac{x}{x^2+1}\right)$$

$$= \frac{(x^2+1)\cdot\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(1) - x(2x+0)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

(a) f is strictly increasing if f'(x) > 0
i.e. if $\frac{1-x^2}{(x^2+1)^2} > 0$
i.e. if $1 - x^2 > 0$ ....[$\because (x^2 + 1)^2 > 0$]
i.e. if $1 > x^2$
i.e. if $x^2 < 1$
i.e. if $-1 < x < 1$
$\therefore$ f is strictly increasing if $-1 < x < 1$

(b) f is strictly decreasing if f'(x) < 0
i.e. if $\frac{1-x^2}{(x^2+1)^2} < 0$
i.e. if $1 - x^2 < 0$ ....[$\because (x^2 + 1)^2 > 0$]
i.e. if $1 < x^2$ i.e. if $x^2 > 1$
i.e. if $x > 1$ or $x < -1$
$\therefore$ f is strictly decreasing if $x < -1$ or $x > 1$
i.e. $x \in (-\infty, -1) \cup (1, \infty)$.

Question 7.
Show that $f(x) = 3x + \frac{1}{3x}$ is increasing in $\left(\frac{1}{3}, 1\right)$ and decreasing in $\left(\frac{1}{9}, \frac{1}{3}\right)$
Solution:
$f(x) = 3x + \frac{1}{3x}$

$$\therefore\ f'(x) = 3\frac{d}{dx}(x) + \frac{1}{3}\frac{d}{dx}(x^{-1})$$

$$= 3 \times 1 + \frac{1}{3}(-1)x^{-2} = 3 - \frac{1}{3x^2}$$

Now, f is increasing if $f'(x) > 0$ and is decreasing if $f'(x) < 0$.

Let $x \in \left(\frac{1}{3}, 1\right)$. Then $\frac{1}{3} < x < 1$

$$\therefore\ \frac{1}{9} < x^2 < 1$$

$$\therefore\ \frac{1}{3} < 3x^2 < 3$$

$$\therefore\ 3 > \frac{1}{3x^2} > \frac{1}{3}$$

$$\therefore\ -3 < -\frac{1}{3x^2} < -\frac{1}{3}$$

$$\therefore\ 3 - 3 < 3 - \frac{1}{3x^2} < 3 - \frac{1}{3}$$

$$\therefore\ 0 < f'(x) < \frac{8}{3}$$

$\therefore\ f'(x) > 0$ for all $x \in \left(\frac{1}{3}, 1\right)$

$\therefore\ f$ is increasing in the interval $\left(\frac{1}{3}, 1\right)$.

Let $x \in \left(\frac{1}{9}, \frac{1}{3}\right)$. Then $\frac{1}{9} < x < \frac{1}{3}$

$$\therefore\ \frac{1}{81} < x^2 < \frac{1}{9}$$

$$\therefore\ \frac{1}{27} < 3x^2 < \frac{1}{3} \quad \therefore\ 27 > \frac{1}{3x^2} > 3$$

$$\therefore\ -27 < -\frac{1}{3x^2} < -3$$

$$\therefore\ 3 - 27 < 3 - \frac{1}{3x^2} < 3 - 3 \quad \therefore\ -24 < f'(x) < 0$$

$\therefore\ f'(x) < 0$ for all $x \in \left(\frac{1}{9}, \frac{1}{3}\right)$

$\therefore\ f$ is decreasing in the interval $\left(\frac{1}{9}, \frac{1}{3}\right)$.

Question 8.
Show that f(x) = x – cos x is increasing for all x.
Solution:
f(x) = x – cos x
∴ f'(x) = $\frac{d}{dx}$ (x – cos x)
= 1 – (-sin x)
= 1 + sin x
Now, -1 ≤ sin x ≤ 1 for all x ∈ R
∴ -1 + 1 ≤ 1 + sin x ≤ 1 for all x ∈ R
∴ 0 ≤ f'(x) ≤ 1 for all x ∈ R
∴ f'(x) ≥ 0 for all x ∈ R
∴ f is increasing for all x.

Question 9.
Find the maximum and minimum of the following functions:
(i) y = $5x^3 + 2x^2 - 3x$
Solution:

$y = 5x^3 + 2x^2 - 3x$

$\therefore \dfrac{dy}{dx} = \dfrac{d}{dx}(5x^3 + 2x^2 - 3x)$

$= 5 \times 3x^2 + 2 \times 2x - 3 \times 1$

$= 15x^2 + 4x - 3$

and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(15x^2 + 4x - 3)$

$= 15 \times 2x + 4 \times 1 - 0 = 30x + 4$

$\dfrac{dy}{dx} = 0$ gives $15x^2 + 4x - 3 = 0$

$\therefore\ 15x^2 + 9x - 5x - 3 = 0$

$\therefore\ 3x(5x + 3) - 1(5x + 3) = 0$

$\therefore\ (5x + 3)(3x - 1) = 0$

$\therefore\ x = -\dfrac{3}{5}$ or $x = \dfrac{1}{3}$

$\therefore$ the roots of $\dfrac{dy}{dx} = 0$ are $x_1 = -\dfrac{3}{5}$ and $x_2 = \dfrac{1}{3}$.

Method 1 (Second Derivative Test) :

(a) $\left(\dfrac{d^2y}{dx^2}\right)_{\text{at } x = -\frac{3}{5}} = 30\left(-\dfrac{3}{5}\right) + 4 = -14 < 0$

$\therefore$ by the second derivative test, y is maximum at $x = -\dfrac{3}{5}$ and maximum value of y at $x = -\dfrac{3}{5}$

$= 5\left(-\dfrac{3}{5}\right)^3 + 2\left(-\dfrac{3}{5}\right)^2 - 3\left(-\dfrac{3}{5}\right)$

$= \dfrac{-27}{25} + \dfrac{18}{25} + \dfrac{9}{5} = \dfrac{36}{25}$

(b) $\left(\dfrac{d^2y}{dx^2}\right)_{\text{at } x = \frac{1}{3}} = 30\left(\dfrac{1}{3}\right) + 4 = 14 > 0$

$\therefore$ by the second derivative test, y is minimum at $x = \dfrac{1}{3}$ and minimum value of y at $x = \dfrac{1}{3}$

$= 5\left(\dfrac{1}{3}\right)^3 + 2\left(\dfrac{1}{3}\right)^2 - 3\left(\dfrac{1}{3}\right)$

$= \dfrac{5}{27} + \dfrac{2}{9} - 1 = -\dfrac{16}{27}$

Hence, the function has maximum value $\dfrac{36}{25}$ at $x = -\dfrac{3}{5}$ and minimum value $-\dfrac{16}{27}$ at $x = \dfrac{1}{3}$.

Method 2 (First Derivative Test) :

(a) $\dfrac{dy}{dx} = 15x^2 + 4x - 3 = (5x + 3)(3x - 1)$

Consider $x = -\dfrac{3}{5}$

Let h be a small positive number. Then

$$\left(\frac{dy}{dx}\right)_{\text{at } x=-\frac{3}{5}-h} = \left[5\left(-\frac{3}{5}-h\right)+3\right]\left[3\left(-\frac{3}{5}-h\right)-1\right]$$

$$= (-3-5h+3)\left(-\frac{9}{5}-3h-1\right)$$

$$= -5h\left(-\frac{14}{5}-3h\right)$$

$$= 5h\left(\frac{14}{5}+3h\right) > 0$$

and $$\left(\frac{dy}{dx}\right)_{\text{at } x=-\frac{3}{5}+h} = \left[5\left(-\frac{3}{5}+h\right)+3\right]\left[3\left(-\frac{3}{5}+h\right)-1\right]$$

$$= (-3+5h+3)\left(-\frac{9}{5}+3h-1\right)$$

$$= 5h\left(3h-\frac{14}{5}\right) < 0,$$

as h is small positive number.

$\therefore$ by the first deritative test, y is maximum at $x = -\frac{3}{5}$ and maximum value of y at $x = -\frac{3}{5}$

$$= 5\left(-\frac{3}{5}\right)^3 + 2\left(-\frac{3}{5}\right)^2 - 3\left(-\frac{3}{5}\right)$$

$$= -\frac{27}{25} + \frac{18}{25} + \frac{9}{5} = \frac{36}{25}$$

(b) $\frac{dy}{dx} = 15x^2 + 4x - 3 = (5x+3)(3x-1)$

Consider $x = \frac{1}{3}$

Let h be a small positive number. Then

$$\left(\frac{dy}{dx}\right)_{\text{at } x=\frac{1}{3}-h} = \left[5\left(\frac{1}{3}-h\right)+3\right]\left[3\left(\frac{1}{3}-h\right)-1\right]$$

$$= \left(\frac{5}{3} - 5h + 3\right)(1 - 3h - 1)$$

$$= \left(\frac{14}{5} - 5h\right)(-3h) < 0, \text{ as } h \text{ is small positive number}$$

and $$\left(\frac{dy}{dx}\right)_{\text{at } x=\frac{1}{3}+h} = \left[5\left(\frac{1}{3}+h\right)+3\right]\left[3\left(\frac{1}{3}+h\right)-1\right]$$

$$= \left(\frac{5}{3} + 5h + 3\right)(1 + 3h - 1)$$

$$= \left(\frac{14}{3} + 5h\right)(3h) > 0$$

$\therefore$ by the first derivative test, y is minimum at $x = \frac{1}{3}$

and minimum value of y at $x = \frac{1}{3}$

$$= 5\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right)$$

$$= \frac{5}{27} + \frac{2}{9} - 1 = \frac{-16}{27}$$

Hence, the function has maximum value $\frac{36}{25}$ at $x = -\frac{3}{5}$ and minimum value $-\frac{16}{27}$ at $x = \frac{1}{3}$.

(ii) $f(x) = 2x^3 - 21x^2 + 36x - 20$
Solution:
$f(x) = 2x^3 - 21x^2 + 36x - 20$
$\therefore f'(x) = \frac{d}{dx}(2x^3 - 21x^2 + 36x - 20)$
$= 2 \times 3x^2 - 21 \times 2x + 36 \times 1 - 0$
$= 6x^2 - 42x + 36$
and $f''(x) = \frac{d}{dx}(6x^2 - 42x + 36)$
$= 6 \times 2x - 42 \times 1 + 0$
$= 12x - 42$
$f'(x) = 0$ gives $6x^2 - 42x + 36 = 0$
$\therefore x^2 - 7x + 6 = 0$
$\therefore (x - 1)(x - 6) = 0$
the roots of $f'(x) = 0$ are $x_1 = 1$ and $x_2 = 6$.

Method 1 (Second Derivative Test):
(a) $f''(1) = 12(1) - 42 = -30 < 0$
$\therefore$ by the second derivative test, f has maximum at $x = 1$
and maximum value of f at $x = 1$
$f(1) = 2(1)^3 - 21(1)^2 + 36(1) - 20$
$= 2 - 21 + 36 - 20$
$= -3$

(b) $f''(6) = 12(6) - 42 = 30 > 0$
$\therefore$ by the second derivative test, f has minimum at $x = 6$
and minimum value of f at $x = 6$
$f(6) = 2(6)^3 - 21(6)^2 + 36(6) - 20$
$= 432 - 756 + 216 - 20$
$= -128$.
Hence, the function f has maximum value -3 at $x = 1$ and minimum value -128 at $x = 6$.

Method 2 (First Derivative Test):
(a) $f'(x) = 6(x - 1)(x - 6)$
Consider $x = 1$
Let h be a small positive number. Then
$f'(1 - h) = 6(1 - h - 1)(1 - h - 6)$
$= 6(-h)(-5 - h)$
$= 6h(5 + h) > 0$
and $f'(1 + h) = 6(1 + h - 1)(1 + h - 6)$
$= 6h(h - 5) < 0$, as h is small positive number.
∴ by the first derivative test, f has maximum at x = 1 and maximum value of f at x = 1
$f(1) = 2(1)^3 - 21(1)^2 + 36(1) - 20$
$= 2 - 21 + 36 - 20$
$= -3$

(b) $f'(x) = 6(x - 1)(x - 6)$
Consider $x = 6$
Let h be a small positive number. Then
$f'(6 - h) = 6(6 - h - 1)(6 - h - 6)$
$= 6(5 - h)(-h)$
$= -6h(5 - h) < 0$, as h is small positive number
and $f'(6 + h) = 6(6 + h - 1)(6 + h - 6) = 6(5 + h)(h) > 0$
∴ by the first derivative test, f has minimum at x = 6
and minimum value of f at x = 6
$f(6) = 2(6)^3 - 21(6)^2 + 36(6) - 20$
$= 432 - 756 + 216 - 20$
$= -128$
Hence, the function f has maximum value -3 at x = 1
and minimum value -128 at x = 6.

(iii) $f(x) = x^3 - 9x^2 + 24x$
Solution:
$f(x) = x^3 - 9x^2 + 24x$
∴ $f'(x) = \frac{d}{dx}(x^3 - 9x^2 + 24x)$
$= 3x^2 - 9 \times 2x + 24 \times 1$
$= 3x^2 - 18x + 24$
and $f''(x) = \frac{d}{dx}(3x^2 - 18x + 24)$
$= 3 \times 2x - 18 \times 1 + 0$
$= 6x - 18$
$f'(x) = 0$ gives $3x^2 - 18x + 24 = 0$
∴ $x^2 - 6x + 8 = 0$
∴ $(x - 2)(x - 4) = 0$
∴ the roots of $f'(x) = 0$ are $x_1 = 2$ and $x_2 = 4$.

(a) $f''(2) = 6(2) - 18 = -6 < 0$
∴ by the second derivative test, f has maximum at x = 2
and maximum value of f at x = 2
$f(2) = (2)^3 - 9(2)^2 + 24(2)$
$= 8 - 36 + 48$
$= 20$

(b) $f''(4) = 6(4) - 18 = 6 > 0$
∴ by the second derivative test, f has minimum at x = 4
and minimum value of f at x = 4
$f(4) = (4)^3 - 9(4)^2 + 24(4)$
$= 64 - 144 + 96$
$= 16$
Hence, the function f has maximum value 20 at x = 2 and minimum value 16 at x = 4.

(iv) $f(x) = x^2 + \frac{16}{x^2}$
Solution:

$f(x) = x^2 + \frac{16}{x^2}$

$\therefore\ f'(x) = \frac{d}{dx}(x^2) + 16\frac{d}{dx}(x^{-2})$

$= 2x + 16(-2)x^{-3} = 2x - \frac{32}{x^3}$

and $f''(x) = \frac{d}{dx}(2x) - 32\frac{d}{dx}(x^{-3})$

$= 2 \times 1 - 32(-3)x^{-4} = 2 + \frac{96}{x^4}$

$f'(x) = 0$ gives $2x - \frac{32}{x^3} = 0$

$\therefore\ 2x^4 - 32 = 0 \quad \therefore\ x^4 = 16$

$\therefore\ x = \pm 2$

$\therefore$ the roots of $f'(x) = 0$ are $x_1 = 2$ and $x_2 = -2$.

(a) $f''(2) = 2 + \frac{96}{(2)^4} = 8 > 0$

$\therefore$ by the second derivative test, f has minimum at $x = 2$ and minimum value of f at $x = 2$

$= f(2) = (2)^2 + \frac{16}{(2)^2} = 4 + 4 = 8$

(b) $f''(-2) = 2 + \frac{96}{(-2)^4} = 8 > 0$

$\therefore$ by the second derivative test, f has minimum at $x = -2$ and minimum value of f at $x = -2$

$= f(-2) = (-2)^2 + \frac{16}{(-2)^2} = 4 + 4 = 8$

Hence, the function f has minimum value 8 at $x = \pm 2$.

(v) f(x) = x log x
Solution:

$f(x) = x \log x$

$\therefore\ f'(x) = \frac{d}{dx}(x \log x)$

$= x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x)$

$= x \times \frac{1}{x} + (\log x) \times 1 = 1 + \log x$

and $f''(x) = \frac{d}{dx}(1 + \log x) = 0 + \frac{1}{x} = \frac{1}{x}$

Now, $f'(x) = 0$, if $1 + \log x = 0$

i.e. if $\log x = -1 = -\log e$

i.e. if $\log x = \log(e^{-1}) = \log \frac{1}{e}$

i.e. if $x = \frac{1}{e}$

When $x = \frac{1}{e}$, $f''(x) = \frac{1}{(1/e)} = e > 0$

$\therefore$ by the second derivative test, f is minimum at $x = \frac{1}{e}$.

Minimum value of f at $x = \frac{1}{e}$

$= \frac{1}{e} \log\left(\frac{1}{e}\right) = \frac{1}{e} \cdot \log(e^{-1})$

$= \frac{1}{e} \cdot (-1) \log e$

$= -\frac{1}{e}$ ... $[\because\ \log e = 1]$

(vi) f(x) = *logxx*

Solution:

$$f(x) = \frac{\log x}{x}$$

$$\therefore f'(x) = \frac{d}{dx}\left(\frac{\log x}{x}\right)$$

$$= \frac{x\frac{d}{dx}(\log x) - \log x\frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\left(\frac{1}{x}\right) - (\log x)(1)}{x^2} = \frac{1 - \log x}{x^2}$$

$$\text{and } f''(x) = \frac{d}{dx}\left(\frac{1 - \log x}{x^2}\right)$$

$$= \frac{x^2\frac{d}{dx}(1 - \log x) - (1 - \log x)\frac{d}{dx}(x^2)}{x^4}$$

$$= \frac{x^2\left(0 - \frac{1}{x}\right) - (1 - \log x) \times 2x}{x^4}$$

$$= \frac{-x - 2x + 2x\log x}{x^4} = \frac{x(2\log x - 3)}{x^4}$$

$$\therefore f''(x) = \frac{2\log x - 3}{x^3}$$

Now, $f'(x) = 0$, if $\frac{1 - \log x}{x^2} = 0$

i.e. if $1 - \log x = 0$, i.e. if $\log x = 1 = \log e$

i.e. if $x = e$

and $f''(e) = \frac{2\log e - 3}{e^3} = \frac{-1}{e^3} < 0$ ... $[\because \log e = 1]$

$\therefore$ by the second derivative test, $f(x)$ is maximum at $x = e$.

Maximum value of f at $x = e$

$= \frac{\log e}{e} = \frac{1}{e}$ ... $[\because \log e = 1]$

Question 10.
Divide the number 30 into two parts such that their product is maximum.
Solution:
Let the first part of 30 be x.
Then the second part is 30 – x.
∴ their product = x(30 – x) = 30x – x^2 = f(x) ......(Say)
∴ f'(x) = $\frac{d}{dx}$ (30x – x^2)
= 30 × 1 – 2x
= 30 – 2x
and f''(x) = $\frac{d}{dx}$ (30 – 2x)
= 0 – 2 × 1
= -2
The root of the equation f(x) = 0,
i.e. 30 – 2x = 0 is x = 15 and f''(15) = -2 < 0
∴ by the second derivative test, f is maximum at x = 15.
Hence, the required parts of 30 are 15 and 15.

Question 11.
Divide the number 20 into two parts such that the sum of their squares is minimum.
Solution:
Let the first part of 20 be x.

Then the second part is 20 – x.
∴ sum of their squares = $x^2 + (20 - x)^2 = f(x)$ …… (Say)
∴ $f'(x) = \frac{d}{dx} [x^2 + (20 - x)^2]$
$= 2x + 2(20 - x) \cdot \frac{d}{dx} (20 - x)$
$= 2x + 2(20 - x) \times (0 - 1)$
$= 2x - 40 + 2x$
$= 4x - 40$
and $f''(x) = \frac{d}{dx} (4x - 40)$
$= 4 \times 1 - 0$
$= 4$
The root of the equation $f'(x) = 0$,
i.e. $4x - 40 = 0$ is $x = 10$ and $f''(10) = 4 > 0$
∴ by the second derivative test, f is minimum at x = 10.
Hence, the required parts of 20 are 10 and 10.

Question 12.
A wire of length 36 meters is bent in the form of a rectangle. Find its dimensions if the area of the rectangle is maximum.
Solution:
Let x metres and y metres be the length and breadth of the rectangle.
Then its perimeter is $2(x + y) = 36$
$x + y = 18$
$y = 18 - x$
Area of the rectangle = $xy = x (18 - x)$
Let $f(x) = x(18 - x) = 18x - x^2$
∴ $f'(x) = \frac{d}{dx} (18x - x^2) = 18 - 2x$
and $f''(x) = \frac{d}{dx} (18 - 2x) = 0 - 2 \times 1 = -2$
Now, $f'(x) = 0$, if $18 - 2x = 0$
i.e. if x = 9
and $f''(9) = -2 < 0$
∴ by the second derivative test, f has maximum value at x = 9.
When x = 9, y = 18 – 9 = 9
∴ x = 9 cm, y = 9 cm
∴ the rectangle is a square of side 9 metres.

Question 13.
A ball is thrown in the air. Its height at any time t is given by $h = 3 + 14t - 5t^2$. Find the maximum height it can reach.
Solution:
The height h at any t is given by $h = 3 + 14t - 5t^2$

$$\therefore \frac{dh}{dt} = \frac{d}{dt}(3 + 14t - 5t^2)$$
$$= 0 + 14 \times 1 - 5 \times 2t = 14 - 10t$$
$$\text{and } \frac{d^2h}{dt^2} = \frac{d}{dt}(14 - 10t) = 0 - 10 \times 1 = -10$$

The root of $\frac{dh}{dt} = 0$, i.e. $14 - 10t = 0$ is $t = \frac{14}{10} = \frac{7}{5}$

and $\left(\frac{d^2h}{dt^2}\right)_{\text{at } t = \frac{7}{5}} = -10 < 0$

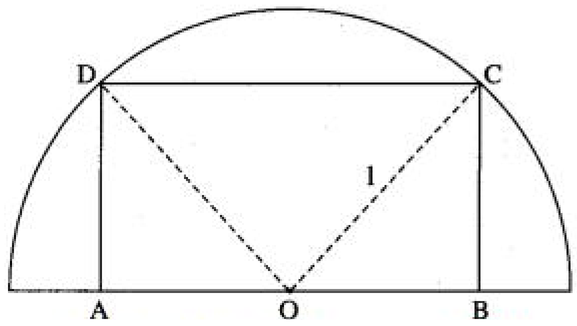
∴ by the second derivative test, h is maximum at $t = \frac{7}{5}$.

∴ maximum height $= (3 + 14t - 5t^2)_{\text{at } t = \frac{7}{5}}$

$$= 3 + 14\left(\frac{7}{5}\right) - 5\left(\frac{7}{5}\right)^2$$
$$= 3 + \frac{98}{5} - \frac{245}{25}$$
$$= \frac{75 + 490 - 245}{25}$$
$$= \frac{320}{25} = 12.8$$

Hence, the maximum height the ball can reach = 12.8 units.

Question 14.
Find the largest size of a rectangle that can be inscribed in a semicircle of radius 1 unit, so that two vertices lie on the diameter.
Solution:



Let ABCD be the rectangle inscribed in a semicircle of radius 1 unit such that the vertices A and B lie on the diameter.
Let AB = DC = x and BC = AD = y.
Let O be the centre of the semicircle.
Join OC and OD. Then OC = OD = radius = 1.
Also, AD = BC and m∠A = m∠B = 90°.
∴ OA = OB
∴ OB = $\frac{1}{2}$ AB = $\frac{x}{2}$
In right angled triangle OBC,
$OB^2 + BC^2 = OC^2$

$$\therefore \left(\frac{x}{2}\right)^2 + y^2 = 1^2$$

$$\therefore y^2 = 1 - \frac{x^2}{4} = \frac{1}{4}(4 - x^2)$$

$$\therefore y = \frac{1}{2}\sqrt{4 - x^2} \quad \ldots [\because y > 0]$$

$$\text{Area of the rectangle} = xy = x \cdot \frac{1}{2}\sqrt{4 - x^2}$$

$$\text{Let } f(x) = \frac{1}{2}x\sqrt{4 - x^2} = \frac{1}{2}\sqrt{4x^2 - x^4}$$

$$\therefore f'(x) = \frac{1}{2}\frac{d}{dx}(\sqrt{4x^2 - x^4})$$

$$= \frac{1}{2} \times \frac{1}{2\sqrt{4x^2 - x^4}} \cdot \frac{d}{dx}(4x^2 - x^4)$$

$$= \frac{1}{4\sqrt{4x^2 - x^4}} \times (4 \times 2x - 4x^3)$$

$$= \frac{4x(2 - x^2)}{4x\sqrt{4 - x^2}} = \frac{2 - x^2}{\sqrt{4 - x^2}} \quad \ldots [\because x \neq 0]$$

$$\text{and } f''(x) = \frac{d}{dx}\left(\frac{2 - x^2}{\sqrt{4 - x^2}}\right) = \frac{d}{dx}\left[\frac{(4 - x^2) - 2}{\sqrt{4 - x^2}}\right]$$

$$= \frac{d}{dx}\left[\sqrt{4 - x^2} - \frac{2}{\sqrt{4 - x^2}}\right]$$

$$= \frac{d}{dx}(\sqrt{4 - x^2}) - 2\frac{d}{dx}(4 - x^2)^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{4-x^2}} \cdot \frac{d}{dx}(4-x^2) - 2\left(-\frac{1}{2}\right)(4-x^2)^{-\frac{3}{2}} \cdot \frac{d}{dx}(4-x^2)$$

$$= \frac{1}{2\sqrt{4-x^2}} \times (0-2x) + \frac{1}{(4-x^2)^{\frac{3}{2}}} \times (0-2x)$$

$$= \frac{-x}{\sqrt{4-x^2}} - \frac{2x}{(4-x^2)^{\frac{3}{2}}}$$

$$= \frac{-x(4-x^2)-2x}{(4-x^2)^{\frac{3}{2}}}$$

$$= \frac{-4x+x^3-2x}{(4-x^2)^{\frac{3}{2}}} = \frac{x^3-6x}{(4-x^2)^{\frac{3}{2}}}$$

For maximum value of $f(x)$, $f'(x) = 0$

$$\therefore \frac{2-x^2}{\sqrt{4-x^2}} = 0 \qquad \therefore 2-x^2 = 0$$

$$\therefore x^2 = 2 \qquad \therefore x = \sqrt{2} \qquad \ldots [\because x > 0]$$

$$\text{Now, } f''(\sqrt{2}) = \frac{(\sqrt{2})^3 - 6\sqrt{2}}{[4-(\sqrt{2})^2]^{\frac{3}{2}}} = \frac{-4\sqrt{2}}{2\sqrt{2}} = -2 < 0$$

$\therefore$ by the second derivative test, f is maximum when $x = \sqrt{2}$

When $x = \sqrt{2}$, $y = \frac{1}{2}\sqrt{4-x^2} = \frac{1}{2}\sqrt{4-2}$

$$= \frac{1}{2} \times \sqrt{2} = \frac{1}{\sqrt{2}}$$

$\therefore x = \sqrt{2}$ and $y = \frac{1}{\sqrt{2}}$

Hence, the area of the rectangle is maximum (i.e. rectangle has the largest size) when its length is $\sqrt{2}$ units and breadth is $\frac{1}{\sqrt{2}}$ unit.

Question 15.
An open cylindrical tank whose base is a circle is to be constructed of metal sheet so as to contain a volume of πa^3 cu cm of water. Find the dimensions so that the quantity of the metal sheet required is minimum.
Solution:
Let x be the radius of the base, h be the height, V be the volume and S be the total surface area of the cylindrical tank.
Then V = πa^3 … (Given)
$\therefore \pi x^2 h = \pi a^3$
$\therefore h = \frac{a^3}{x^2}$ ……..(1)
Now, S = $2\pi x h + \pi x^2$

$$= 2\pi x\left(\frac{a^3}{x^2}\right) + \pi x^2 \qquad \ldots \text{ [By (1)]}$$

$$= \frac{2\pi a^3}{x} + \pi x^2$$

$$\therefore \frac{dS}{dx} = \frac{d}{dx}\left(\frac{2\pi a^3}{x} + \pi x^2\right)$$

$$= 2\pi a^3(-1)x^{-2} + \pi \times 2x = \frac{-2\pi a^3}{x^2} + 2\pi x$$

$$\text{and } \frac{d^2S}{dx^2} = \frac{d}{dx}\left(\frac{-2\pi a^3}{x^2} + 2\pi x\right)$$

$$= -2\pi a^3(-2)x^{-3} + 2\pi \times 1$$

$$= \frac{4\pi a^3}{x^3} + 2\pi$$

$$\text{Now, } \frac{dS}{dx} = 0 \text{ gives } \frac{-2\pi a^3}{x^2} + 2\pi x = 0$$

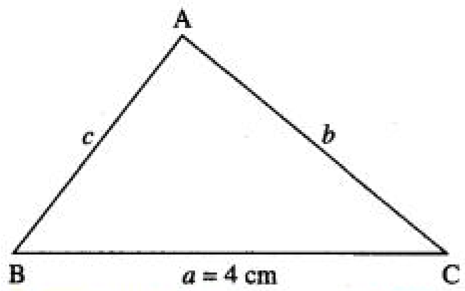
$$\therefore -2\pi a^3 + 2\pi x^3 = 0$$

$$\therefore 2\pi x^3 = 2\pi a^3 \qquad \therefore x = a$$

$$\text{and } \left(\frac{d^2S}{dx^2}\right)_{\text{at } x=a} = \frac{4\pi a^3}{a^3} + 2\pi = 6\pi > 0$$

∴ by the second derivative test, S is minimum when x = a
When x = a, from (1)
h = $\frac{a^3}{a^2}$ = a
Hence, the quantity of metal sheet is minimum when radius height = a cm.

Question 16.
The perimeter of a triangle is 10 cm. If one of the sides is 4 cm. What are the other two sides of the triangle for its maximum area?
Solution:



Let ABC be the triangle such that the side BC = a = 4 cm.
Also, the perimeter of the triangle is 10 cm.
i.e. a + b + c = 10
∴ 2s = 10
∴ s = 5
Also, 4 + b + c = 10
∴ b + c = 6
∴ b = 6 – c
Let Δ be the area of the triangle.

$$\text{Then } \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{5(5-4)(5-6+c)(5-c)}$$

$$= \sqrt{5(c-1)(5-c)}$$

$$\therefore \Delta^2 = 5(c-1)(5-c) = 5(5c - c^2 - 5 + c)$$

$$\therefore \Delta^2 = 5(-c^2 + 6c - 5)$$

Differentiating both sides w.r.t. c, we get

$$2\Delta\frac{d\Delta}{dc}=5\frac{d}{dc}(-c^2+6c-5)$$

$$=5(-2c+6\times1-0)=5(-2c+6)$$

$$\therefore\ \frac{d\Delta}{dc}=\frac{5(-c+3)}{\Delta}$$

and $\frac{d^2\Delta}{dc^2}=5\frac{d}{dc}\left(\frac{-c+3}{\Delta}\right)$

$$=5\cdot\frac{\Delta\frac{d}{dc}(-c+3)-(-c+3)\frac{d\Delta}{dc}}{\Delta^2}$$

$$=5\cdot\frac{\Delta(-1+0)-(c+3)\frac{d\Delta}{dc}}{\Delta^2}$$

$$=\frac{5}{\Delta^2}\left(-\Delta-(c+3)\frac{d\Delta}{dc}\right)$$

$$=\frac{-5}{\Delta^2}\left[\Delta+(c+3)\frac{d\Delta}{dc}\right]$$

For maximumΔ, $\frac{d\Delta}{dc}=0$

$\therefore\ \frac{5(-c+3)}{\Delta}=0$

$\therefore\ -c+3=0$ ... $[\because\ \Delta\neq0]$

$\therefore\ c=3$

If $c=3$, $\Delta=\sqrt{5(3-1)(5-3)}=2\sqrt{5}$

$$\therefore\ \left(\frac{d^2\Delta}{dc^2}\right)_{\text{at } c=3}=\frac{-5}{4\times5}[2\sqrt{5}+(3+3)(0)]$$

$$=-\frac{\sqrt{5}}{2}<0$$

∴ by the second derivative test, Δ is maximum when c = 3.
When c = 3, b = 6 – c = 6 – 3 = 3
Hence, the area of the triangle is maximum when the other two sides are 3 cm and 3 cm.

Question 17.
A box with a square base is to have an open top. The surface area of the box is 192 sq cm. What should be its dimensions in order that the volume is largest?
Solution:
Let x cm be the side of square base and h cm be its height.
Then $x^2 + 4xh = 192$
∴ $h = \frac{192-x^2}{4x}$ ...... (1)
Let V be the volume of the box.

Then $V = x^2 h = x^2\left(\frac{192 - x^2}{4x}\right)$ ... [By (1)]

$\therefore\ V = \frac{1}{4}(192x - x^3)$

$\therefore\ \frac{dV}{dx} = \frac{1}{4}\frac{d}{dx}(192x - x^3)$

$= \frac{1}{4}(192 \times 1 - 3x^2) = \frac{3}{4}(64 - x^2)$

and $\frac{d^2V}{dx^2} = \frac{3}{4}\frac{d}{dx}(64 - x^2)$

$= \frac{3}{4}(0 - 2x) = -\frac{3}{2}x$

For maximum V, $\frac{dV}{dx} = 0$

$\therefore\ \frac{3}{4}(64 - x^2) = 0$

$\therefore\ x^2 = 64 \quad \therefore\ x = 8$ ... $[\because\ x > 0]$

and $\left(\frac{d^2V}{dx^2}\right)_{\text{at } x=8} = -\frac{3}{2} \times 8 = -12 < 0$

∴ by the second derivative test, V is maximum at x = 8.

If x = 8, $h = \frac{192-64}{4(8)} = \frac{128}{32} = 4$

Hence, the volume of the box is largest, when the side of square base is 8 cm and its height is 4 cm.

Question 18.
The profit function P (x) of a firm, selling x items per day is given by P(x) = (150 – x)x – 1625. Find the number of items the firm should manufacture to get maximum profit. Find the maximum profit.
Solution:
Profit function P (x) is given by
$P(x) = (150 - x)x - 1625 = 150x - x^2 - 1625$
$\therefore\ P'(x) = \frac{d}{dx}(150x - x^2 - 1625)$
$= 150 \times 1 - 2x - 0$
$= 150 - 2x$
and $P''(x) = \frac{d}{dx}(150 - 2x)$
$= 0 - 2 \times 1$
$= -2$
Now, P'(x) = 0 gives, 150 – 2x = 0
∴ x = 75
and P"(75) = -2 < 0
∴ by the second derivative test, P(x) is maximum when x = 75
Maximum profit = P(75)
= (150 – 75)75 – 1625
= 75 × 75 – 1625
= 4000
Hence, the profit will be maximum, if the manufacturer manufactures 75 items and the maximum profit is 4000.

Question 19.
Find two numbers whose sum is 15 and when the square of one multiplied by the cube of the other is maximum.
Solution:
Let the two numbers be x and y.
Then x + y = 15
∴ y = 15 – x
Let P is the product of square of y and cube of x.
Then $P = x^3y^2$
$= x^3(15 - x)^2$
$= x^3(225 - 30x + x^2)$
$= x^5 - 30x^4 + 225x^3$
$\therefore\ \frac{dP}{dx} = \frac{d}{dx}(x^5 - 30x^4 + 225x^3)$
$= 5x^4 - 30 \times 4x^3 + 225 \times 3x^2$
$= 5x^4 - 120x^3 + 675x^2$
and $\frac{d^2P}{dx^2} = \frac{d}{dx}(5x^4 - 120x^3 + 675x^2)$
$= 5 \times 4x^3 - 120 \times 3x^2 + 675 \times 2x$

$= 20x^3 – 360x^2 + 1350x$
$= 10x(2x^2 – 36x + 135)$
Now, $\frac{dP}{dx} = 0$ gives $5x^4 – 120x^3 + 675x^2 = 0$
$\therefore 5x^2(x^2 – 24x + 135) = 0$
$\therefore 5x^2(x^2 – 15x – 9x + 135) = 0$
$\therefore 5x^2[x(x – 15) – 9(x – 35)] = 0$
$\therefore 5x^2(x – 15)(x – 9) = 0$
$\therefore$ the roots of $\frac{dP}{dx} = 0$ are $x_1 = 0$, $x_2 = 15$ and $x_3 = 9$
If x = 0, then y = 15 – 0 = 15
If x = 15, then y = 15 – 15 = 0
In both cases, product x^3y^2 is zero, which is not maximum.
$\therefore x \neq 0$ and $x \neq 15$
$\therefore x = 6$
Now, $\left(\frac{d^2P}{dx^2}\right)_{at\ x=6} = 10(6)[2(6)^2 – 36 \times 6 + 135]$
$= 60[72 – 216 + 135]$
$= 60(-9)$
$= -540 < 0$
$\therefore$ P is maximum when x = 6
If x = 6, then y = 15 – 6 = 9
Hence, the required numbers are 6 and 9.

Question 20.
Show that among rectangles of given area, the square has least perimeter.
Solution:
Let x be the length and y be the breadth of the rectangle whose area is A sq units (which is given as constant).
Then xy = A
$\therefore y = \frac{A}{x}$ ....(1)
Let P be the perimeter of the rectangle.

$$\text{Then } P = 2(x+y) = 2\left(x + \frac{A}{x}\right) \quad \ldots \text{ [By (1)]}$$

$$\therefore \frac{dP}{dx} = 2 \cdot \frac{d}{dx}\left(x + \frac{A}{x}\right) = 2[1 + A(-1)x^{-2}]$$

$$= 2\left(1 - \frac{A}{x^2}\right)$$

$$\text{and } \frac{d^2P}{dx^2} = 2\frac{d}{dx}\left(1 - \frac{A}{x^2}\right) = 2[0 - A(-2)x^{-3}]$$

$$= \frac{4A}{x^3}$$

$$\text{Now, } \frac{dP}{dx} = 0, \text{ gives } 2\left(1 - \frac{A}{x^2}\right) = 0$$

$$\therefore x^2 - A = 0 \qquad \therefore x^2 = A$$

$$\therefore x = \sqrt{A} \quad \ldots [\because x > 0]$$

$$\text{and } \left(\frac{d^2P}{dx^2}\right)_{\text{at } x = \sqrt{A}} = \frac{4A}{(\sqrt{A})^3} > 0$$

$$\therefore P \text{ is minimum when } x = \sqrt{A}$$

$$\text{If } x = \sqrt{A}, \text{ then } y = \frac{A}{x} = \frac{A}{\sqrt{A}} = \sqrt{A}$$

x = y
$\therefore$ rectangle is a square.
Hence, among rectangles of given area, the square has least perimeter.

Question 21.
Show that the height of a closed right circular cylinder of given volume and least surface area is equal to its diameter.
Solution:
Let x be the radius of base, h be the height and S be the surface area of the closed right circular cylinder whose volume is V which is given to be constant.
Then $\pi r^2 h = V$
$\therefore h = \frac{V}{\pi r^2} = \frac{A}{x^2}$ ....(1)

where A = V/π, which is constant.

Now, $S = 2\pi xh + 2\pi x^2$

$$= 2\pi x\left(\frac{A}{x^2}\right) + 2\pi x^2 \qquad \ldots \text{ [By (1)]}$$

$$= \frac{2\pi A}{x} + 2\pi x^2$$

$$\therefore \frac{dS}{dx} = \frac{d}{dx}\left(\frac{2\pi A}{x} + 2\pi x^2\right)$$

$$= 2\pi A(-1)x^{-2} + 2\pi \times 2x$$

$$= \frac{-2\pi A}{x^2} + 4\pi x$$

and $\frac{d^2S}{dx^2} = \frac{d}{dx}\left(\frac{-2\pi A}{x^2} + 4\pi x\right)$

$$= -2\pi A(-2)x^{-3} + 4\pi \times 1$$

$$= \frac{4\pi A}{x^3} + 4\pi$$

Now, $\frac{dS}{dx} = 0$ gives $\frac{-2\pi A}{x^2} + 4\pi x = 0$

$\therefore -2\pi A + 4\pi x^3 = 0$

$\therefore 4\pi x^3 = 2\pi A$

$\therefore x^3 = \frac{A}{2}$

$\therefore x = \left(\frac{A}{2}\right)^{\frac{1}{3}}$

and $\left(\frac{d^2S}{dx^2}\right)_{\text{at } x = \left(\frac{A}{2}\right)^{\frac{1}{3}}} = \frac{4\pi A}{\left(\frac{A}{2}\right)} + 4\pi = 12\pi > 0$

$\therefore$ by the second derivative test, S is minimum when

$x = \left(\frac{A}{2}\right)^{\frac{1}{3}}$

When $x = \left(\frac{A}{2}\right)^{\frac{1}{3}}$, from (1),

$$h = \frac{A}{\left(\frac{A}{2}\right)^{\frac{2}{3}}} = 2^{\frac{2}{3}} \cdot A^{\frac{1}{3}} = 2 \cdot \left(\frac{A}{2}\right)^{\frac{1}{3}}$$

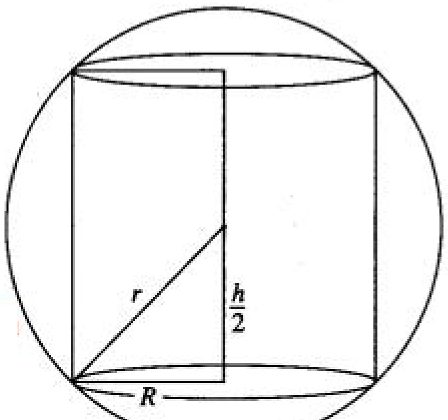
$\therefore h = 2x$

Hence, the surface area is least when height of the closed right circular cylinder is equal to its diameter.

Question 22.
Find the volume of the largest cylinder that can be inscribed in a sphere of radius ‘r’ cm.
Solution:
Let R be the radius and h be the height of the cylinder which is inscribed in a sphere of radius r cm.
Then from the figure,

$R^2 + \left(\frac{h}{2}\right)^2 = r^2$

$\therefore\ R^2 = r^2 - \frac{h^2}{4}$ ... (1)



Let V be the volume of the cylinder.

Then $V = \pi R^2 h$

$= \pi\left(r^2 - \frac{h^2}{4}\right)h$ ... [By (1)]

$= \pi\left(r^2 h - \frac{h^3}{4}\right)$

$\therefore\ \frac{dV}{dh} = \pi \frac{d}{dh}\left(r^2 h - \frac{h^3}{4}\right)$

$= \pi\left(r^2 \times 1 - \frac{1}{4} \times 3h^2\right) = \pi\left(r^2 - \frac{3}{4}h^2\right)$

and $\frac{d^2V}{dh^2} = \pi \frac{d}{dh}\left(r^2 - \frac{3}{4}h^2\right)$

$= \pi\left(0 - \frac{3}{4} \times 2h\right) = -\frac{3}{2}\pi h$

Now, $\frac{dV}{dh} = 0$ gives, $\pi\left(r^2 - \frac{3}{4}h^2\right) = 0$

$\therefore\ r^2 - \frac{3}{4}h^2 = 0$

$\therefore\ \frac{3}{4}h^2 = r^2$ $\therefore\ h^2 = \frac{4r^2}{3}$

$\therefore\ h = \frac{2r}{\sqrt{3}}$. $[\because\ h > 0]$

and $\left(\frac{d^2V}{dh^2}\right)_{\text{at } h = \frac{2r}{\sqrt{3}}} = -\frac{3}{2}\pi \times \frac{2r}{\sqrt{3}} < 0$

$\therefore\ V$ is maximum at $h = \frac{2r}{\sqrt{3}}$

If $h = \frac{2r}{\sqrt{3}}$, then from (1)

$R^2 = r^2 - \frac{1}{4} \times \frac{4r^2}{3} = \frac{2r^2}{3}$

$\therefore$ volume of the largest cylinder

$= \pi \times \frac{2r^2}{3} \times \frac{2r}{\sqrt{3}} = \frac{4\pi r^3}{3\sqrt{3}}$ cu cm.

Hence, the volume of the largest cylinder inscribed in a sphere of radius 'r' cm = $\frac{4\pi r^3}{3\sqrt{3}}$ cu cm.

Question 23.
Show that $y = \log(1 + x) - \frac{2x}{2+x}$, $x > -1$ is an increasing function on its domain.
Solution:
$y = \log(1 + x) - \frac{2x}{2+x}$, $x > -1$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left[\log(1+x)-\frac{2x}{2+x}\right]$$

$$=\frac{1}{1+x}\cdot\frac{d}{dx}(1+x)-\frac{(2+x)\cdot\frac{d}{dx}(2x)-2x\cdot\frac{d}{dx}(2+x)}{(2+x)^2}$$

$$=\frac{1}{1+x}\times(0+1)-\frac{(2+x)\times 2-2x(0+1)}{(2+x)^2}$$

$$=\frac{1}{1+x}-\frac{4+2x-2x}{(2+x)^2}$$

$$=\frac{1}{1+x}-\frac{4}{(2+x)^2}$$

$$=\frac{(2+x)^2-4(1+x)}{(1+x)(2+x)^2}$$

$$=\frac{4+4x+x^2-4-4x}{(1+x)(2+x)^2}$$

$$=\frac{x^2}{(1+x)(2+x)^2}>0 \text{ for all } x>-1$$

Hence, the given function is increasing function on its domain.

Question 24.
Prove that y = $4sin\theta 2+cos\theta$ – θ is an increasing function if θ ∈ [0, $\pi 2$]
Solution:
y = $4sin\theta 2+cos\theta$ – θ

$$\therefore \frac{dy}{d\theta} = \frac{d}{d\theta}\left[\frac{4\sin\theta}{2+\cos\theta} - \theta\right]$$

$$= \frac{d}{d\theta}\left(\frac{4\sin\theta}{2+\cos\theta}\right) - \frac{d}{d\theta}(\theta)$$

$$= \frac{(2+\cos\theta)\cdot\frac{d}{d\theta}(4\sin\theta) - 4\sin\theta\cdot\frac{d}{d\theta}(2+\cos\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{(2+\cos\theta)(4\cos\theta) - (4\sin\theta)(0-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4(\cos^2\theta + \sin^2\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4}{(2+\cos\theta)^2} - 1$$

$$= \frac{(8\cos\theta + 4) - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4 - 4 - 4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

Since, $\theta \in \left[0, \frac{\pi}{2}\right]$, $\cos\theta \geqslant 0$, Also, $\cos\theta < 4$

$\therefore\ 4 - \cos\theta > 0$

$\therefore\ \cos\theta(4-\cos\theta) \geqslant 0$

$$\therefore\ \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \geqslant 0$$

$$\therefore\ \frac{dy}{d\theta} \geqslant 0 \text{ for all } \theta \in \left[0, \frac{\pi}{2}\right]$$

Hence, y is an increasing function if $\theta \in \left[0, \frac{\pi}{2}\right]$.

Maharashtra State Board 12th Maths Solutions Chapter 2 Applications of Derivatives Miscellaneous Exercise 2

I. Choose the correct option from the given alternatives:

Question 1.
If the function f(x) = $ax^3 + bx^2 + 11x - 6$ satisfies conditions of Rolle's theorem in [1, 3] and f'(2 + 13√) = 0, then values of a and b are respectively.
(a) 1, -6

(b) -2, 1
(c) -1, -6
(d) -1, 6
Answer:
(a) 1, -6

Hint: $f(x) = ax^3 + bx^2 + 11x – 6$ satisfies the conditions of Rolle's theorem in [1, 3]
$\therefore f(1) = f(3)$
$a(1)^3 + b(1)^2 + 11(1) – 6 = a(3)^3 + b(3)^2 + 11(3) – 6$
$a + b + 11 = 27a + 9b + 33$
$26a + 8b = -22$
$13a + 4b = -11$
Only a = 1, b = -6 satisfy this equation.

Question 2.
If $f(x) = \frac{x^2-1}{x^2+1}$, for every real x, then the minimum value of f is
(a) 1
(b) 0
(c) -1
(d) 2
Answer:
(c) -1

Question 3.
A ladder 5 m in length is resting against a vertical wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 1.5 m/sec. The length of the higher point of the ladder when the foot of the ladder is 4.0 m away from the wall decreases at the rate of
(a) 1
(b) 2
(c) 2.5
(d) 3
Answer:
(b) 2

Question 4.
Let f(x) and g(x) be differentiable for 0 < x < 1 such that f(0) = 0, g(0) = 0, f(1) = 6. Let there exist a real number c in (0, 1) such that f'(c) = 2g'(c), then the value of g(1) must be
(a) 1
(b) 3
(c) 2.5
(d) -1
Answer:
(b) 3

Hint: f(x) and g(x) both satisfies the conditions of LMVT in (0, 1).
$\therefore f'(c) = \frac{f(1)-f(0)}{1-0} = \frac{6-0}{1} = 6$
and $g'(c) = \frac{g(1)-g(0)}{1-0} = \frac{g(1)-0}{1} = g(1)$
But f'(c) = 2g'(c)
6 = 2g(1)
$\therefore$ g(1) = 3

Question 5.
Let $f(x) = x^3 – 6x^2 + 9x + 18$, then f(x) is strictly decreasing in
(a) (-∞, 1)
(b) [3, ∞)
(c) (-∞, 1] ∪ [3, ∞)
(d) (1, 3)
Answer:
(d) (1, 3)

Question 6.
If x = -1 and x = 2 are the extreme points of $y = \alpha \log x + \beta x^2 + x$, then
(a) $\alpha = -6, \beta = \frac{1}{2}$
(b) $\alpha = -6, \beta = -\frac{1}{2}$
(c) $\alpha = 2, \beta = -\frac{1}{2}$
(d) $\alpha = 2, \beta = \frac{1}{2}$

Answer:
(c) α = 2, β = $-\frac{1}{2}$

Hint: y = α log x + βx^2 + x
∴ $\frac{dy}{dx} = \frac{\alpha}{x} + \beta \times 2x + 1 = \frac{\alpha}{x} + 2\beta x + 1$
f(x) has extreme values at x = -1 and x = 2
∴ f'(-1) = 0 and f(2) = 0
α + 2β = 1
and $\frac{\alpha}{2}$ + 4β = -1
By solving these two equations, we get
α = 2, β = $-\frac{1}{2}$

Question 7.
The normal to the curve x^2 + 2xy – 3y^2 = 0 at (1, 1)
(a) meets the curve again in the second quadrant
(b) does not meet the curve again
(c) meets the curve again in the third quadrant
(d) meets the curve again in the fourth quadrant
Answer:
(d) meets the curve again in fourth quadrant

Hint: x^2 + 2xy – 3y^2 = 0

$$\therefore\ 2x + 2\left(x\frac{dy}{dx} + y \times 1\right) - 3 \times 2y\frac{dy}{dx} = 0$$

$$\therefore\ (2x - 6y)\frac{dy}{dx} = -2x - 2y$$

$$\therefore\ \frac{dy}{dx} = \frac{-(x+y)}{x-3y}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }(1,1)} = \frac{-(1+1)}{1-3} = 1$$

= slope of the tangent at (1, 1)
∴ equation of the tangent at (1, 1) is -1
∴ equation of the normal is
y – 1= -1 (x – 1) = -x + 1
∴ x + y = 2
∴ y = 2 – x
Substituting y = 2 – x in x^2 + 2xy – 3y^2 = 0, we get
x^2 + 2x(2 – x) – 3 $(2 – x)^2$ = 0
⇒ x^2 + 4x – 2x^2 – 3(4 – 4x + x^2) = 0
⇒ x^2 – 4x + 3 = 0
⇒ (x – 1)(x – 3) = 0
⇒ x = 1, x = 3
When x = 1, y = 2 – 1 = 1
When x = 3, y = 2 – 3 = -1
∴ the normal at (1, 1) meets the curve at (3, -1) which is in the fourth quadrant.

Question 8.
The equation of the tangent to the curve y = 1 – $e^{\frac{x}{2}}$ at the point of intersection with Y-axis is
(a) x + 2y = 0
(b) 2x + y = 0
(c) x – y = 2
(d) x + y = 2
Answer:
(a) x + 2y = 0
Hint: The point of intersection of the curve with the Y-axis is the origin (0, 0).

Question 9.
If the tangent at (1, 1) on y^2 = x$(2 – x)^2$ meets the curve again at P, then P is
(a) (4, 4)
(b) (-1, 2)
(c) (3, 6)
(d) $\left(\frac{9}{4}, \frac{3}{8}\right)$
Answer:

(d) (94,38)
Hint: $y^2 = x(2 - x)^2$
$= x(4 - 4x + x^2)$
$= x^3 - 4x^2 + 4x$

$$\therefore\ 2y\frac{dy}{dx} = 3x^2 - 8x + 4$$

$$\therefore\ \frac{dy}{dx} = \frac{3x^2 - 8x + 4}{2y}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }(1,1)} = \frac{3(1)^2 - 8(1) + 4}{2(1)} = -\frac{1}{2}$$

= slope of the tangent at (1, 1)
∴ equation of the tangent at (1, 1) is
y – 1 = −12 (x – 1)
∴ 2y – 2 = -x + 1
∴ x + 2y = 3
Only the coordinates (94,38) satisfy both the equations $y^2 = x(2 - x)^2$ and x + 2y = 3
∴ P is (94,38)

Question 10.
The approximate value of tan (44° 30′), given that 1° = 0.0175, is
(a) 0.8952
(b) 0.9528
(c) 0.9285
(d) 0.9825
Answer:
(d) 0.9825

II. Solve the following:

Question 1.
If the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$, intersect orthogonally, then prove that 1a−1b=1a′−1b′
Solution:
Let $P(x_1, y_1)$ be the point of intersection of the curves.

$\therefore\ ax_1^2 + by_1^2 = 1$ and $a'x_1^2 + b'y_1^2 = 1$

$\therefore\ ax_1^2 + by_1^2 = a'x_1^2 + b'y_1^2$

$\therefore\ (a - a')x_1^2 = (b' - b)y_1^2 \quad \ldots (1)$

Differentiating $ax^2 + by^2 = 1$ w.r.t. x, we get

$$a \times 2x + b \times 2y\frac{dy}{dx} = 0$$

$$\therefore\ \frac{dy}{dx} = \frac{-ax}{by}$$

$\therefore$ slope of the tangent at $(x_1, y_1) = m_1$

$$= \left(\frac{dy}{dx}\right)_{\text{at }(x_1, y_1)} = \frac{-ax_1}{by_1}$$

Differentiating $a'x^2 + b'y^2 = 1$ w.r.t. x, we get

$$a' \times 2x + b' \times 2y\frac{dy}{dx} = 0$$

$$\therefore\ \frac{dy}{dx} = \frac{-a'x}{b'y}$$

$\therefore$ slope of the tangent at $(x_1, y_1) = m_2$

$$= \left(\frac{dy}{dx}\right)_{\text{at }(x_1, y_1)} = \frac{-a'x_1}{b'y_1}$$

Since, curves intersect each other orthogonally,

$$m_1 m_2 = -1$$

$$\therefore\ \left(\frac{-ax_1}{by_1}\right)\left(\frac{-a'x_1}{b'y_1}\right) = -1$$

$$\therefore\ \frac{aa'x_1^2}{bb'y_1^2} = -1 \qquad \therefore\ \frac{aa'}{bb'} = \frac{-y_1^2}{x_1^2}$$

$$\therefore\ \frac{aa'}{bb'} = -\left(\frac{a - a'}{b' - b}\right) \quad \ldots [\text{By (1)}]$$

$$\therefore\ \frac{aa'}{bb'} = \frac{a - a'}{b - b'}$$

$$\therefore\ \frac{a - a'}{aa'} = \frac{b - b'}{bb'}$$

$$\therefore\ \frac{1}{a'} - \frac{1}{a} = \frac{1}{b'} - \frac{1}{b}$$

$$\therefore\ \frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}.$$

Question 2.
Determine the area of the triangle formed by the tangent to the graph of the function $y = 3 – x^2$ drawn at the point (1, 2) and the coordinate axes.
Solution:
$y = 3 – x^2$

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}(3 - x^2) = 0 - 2x = -2x$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }(1, 2)} = -2(1) = -2$$

= slope of the tangent at (1, 2)
∴ equation of the tangent at (1, 2) is
$y – 2= -2(x – 1)$
⇒ $y – 2= -2x + 2$
⇒ $2x + y = 4$
Let this tangent cuts the coordinate axes at A(a, 0) and B(0, b).
∴ $2a + 0 = 4$ and $2(0) + b = 4$
∴ $a = 2$ and $b = 4$
∴ area of required triangle = $\frac{1}{2}$ × l(OA) × l(OB)
= $\frac{1}{2}$ ab

$= \frac{1}{2}(2)(4)$
= 4 sq units.

Question 3.
Find the equation of the tangent and normal drawn to the curve $y^4 - 4x^4 - 6xy = 0$ at the point M (1, 2).
Solution:
$y^4 - 4x^4 - 6xy = 0$
Differentiating both sides w.r.t. x, we get

$$4y^3\frac{dy}{dx} - 4 \times 4x^3 - 6\left[x\frac{dy}{dx} + y \cdot \frac{d}{dx}(x)\right] = 0$$

$$\therefore\ 4y^3\frac{dy}{dx} - 16x^3 - 6x\frac{dy}{dx} - 6y \times 1 = 0$$

$$\therefore\ (4y^3 - 6x)\frac{dy}{dx} = 16x^3 + 6y$$

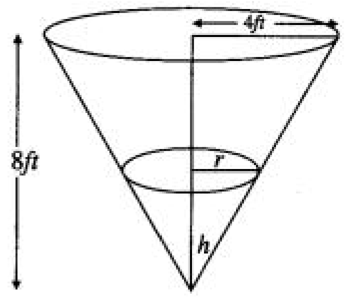
$$\therefore\ \frac{dy}{dx} = \frac{16x^3 + 6y}{4y^3 - 6x} = \frac{8x^3 + 3y}{2y^3 - 3x}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }(1,2)} = \frac{8(1)^3 + 3(2)}{2(2)^3 - 3(1)}$$

$$= \frac{8+6}{16-3} = \frac{14}{13}$$

= slope of the tangent at (1, 2)
∴ the equation of normal at M (1, 2) is
$y - 2 = \frac{14}{13}(x - 1)$
∴ 13y – 26 = 14x – 14
∴ 14x – 13y + 12 = 0
The slope of normal at (1, 2)
$= \frac{-1}{\left(\frac{dy}{dx}\right)_{at(1,2)}} = \frac{-1}{\left(\frac{14}{13}\right)} = -\frac{13}{14}$
∴ the equation of normal at M (1, 2) is
$y - 2 = -\frac{13}{14}(x - 1)$
14y – 28 = -13x + 13
13x + 14y – 41 = 0
Hence, the equations of tangent and normal are 14x – 13y + 12 = 0 and 13x + 14y – 41 = 0 respectively.

Question 4.
A water tank in the form of an inverted cone is being emptied at the rate of 2 cubic feet per second. The height of the cone is 8 feet and the radius is 4 feet. Find the rate of change of the water level when the depth is 6 feet.
Solution:



Let r be the radius of the base, h be the height and V be the volume of the water level at any time t.
Since, the height of the cone is 8 feet and the radius is 4 feet,
$\frac{r}{h} = \frac{4}{8} = \frac{1}{2}$
$r = \frac{h}{2}$ ....(1)

Given : $\frac{dV}{dt} = 2$ cu feet/sec.

Now, $V = \frac{1}{3} r^2 h$

$$= \frac{1}{3} \pi \left(\frac{h}{2}\right)^2 h \qquad \text{........[By (1)]}$$

$\therefore \ V = \frac{\pi}{12} h^3$

Differentiating w.r.t. t, we get

$$\frac{dV}{dt} = \frac{\pi}{12} \times 3h^2 \frac{dh}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

$$\therefore \ \frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt}$$

When $h = 6$, then

$$\frac{dh}{dt} = \frac{4}{\pi (6)^2} \times 2 = \left(\frac{2}{9\pi}\right) \text{ft/sec}$$

Hence, the rate of change of water level is (29π) ft/sec.

Question 5.
Find all points on the ellipse $9x^2 + 16y^2 = 400$, at which the y-coordinate is decreasing and the x-coordinate is increasing at the same rate.
Solution:
Let $P(x_1, y_1)$ be the point on the ellipse $9x^2 + 16y^2 = 400$ whose y-coordinate decreasing x-coordinate is increasing at the same rate.

Then $-\left(\frac{dy}{dt}\right)_{\text{at }(x_1,y_1)} = \left(\frac{dx}{dt}\right)_{\text{at }(x_1,y_1)}$ ... (1)

Differentiating $9x^2 + 16y^2 = 400$ w.r.t. t, we get

$$9 \times 2x\frac{dx}{dt} + 16 \times 2y\frac{dy}{dt} = 0$$

$$\therefore\ 9x\frac{dx}{dt} + 16y\frac{dy}{dt} = 0$$

$$\therefore\ 9x_1\left(\frac{dx}{dt}\right)_{\text{at }(x_1,y_1)} + 16y_1\left(\frac{dy}{dt}\right)_{\text{at }(x_1,y_1)} = 0$$

$$\therefore\ 9x_1\left(\frac{dx}{dt}\right)_{\text{at }(x_1,y_1)} - 16y_1\left(\frac{dx}{dt}\right)_{\text{at }(x_1,y_1)} = 0 \quad \text{... [By (1)]}$$

$$\therefore\ 9x_1 - 16y_1 = 0 \quad \text{... (2)}$$

Now, (x_1, y_1) lies on the ellipse $9x^2 + 16y^2 = 400$

$$\therefore\ 9x_1{}^2 + 16y_1{}^2 = 400 \quad \text{... (3)}$$

From (2), $x_1 = \frac{16y_1}{9}$

Substitute $x_1 = \frac{16y_1}{9}$ in (3), we get

$$\therefore\ 9\left(\frac{16y_1}{9}\right)^2 + 16y_1{}^2 = 400$$

$$\therefore\ 16y_1^2 + 9y_1^2 = 225$$

$$\therefore\ 25y_1^2 = 225 \qquad \therefore\ y_1^2 = 9$$

$$\therefore\ y_1 = \pm 3$$

When $y_1 = 3$, $x_1 = \frac{16(3)}{9} = \frac{16}{3}$

When $y_1 = -3$, $x_1 = \frac{16(-3)}{9} = -\frac{16}{3}$

Hence, the required points on the ellipse are

$\left(\frac{16}{3}, 3\right)$ and $\left(-\frac{16}{3}, -3\right)$.

Question 6.
Verify Rolle's theorem for the function f(x) = $2e^{x}+e^{-x}$ on [-1, 1].
Solution:
The functions e^x, e^{-x}, and 2 are continuous and differentiable in their respective domains.
∴ f(x) = $2e^{x}+e^{-x}$ is continuous on [-1, 1] and differentiable on (-1, 1), because $e^x + e^{-x} \neq 0$ for all x ∈ [-1, 1].

Now, f(-1) = $2e^{-1}+e$=$2e+e^{-1}$ and f(1) = $2e+e^{-1}$

∴ f(-1) = f(1)
Thus, the function f satisfies all the conditions of the Rolle's theorem.
∴ there exist c ∈ (-1, 1) such that f'(c) = 0
Now, f(x) = $2e^{x}+e^{-x}$

$$\therefore\ f'(x) = \frac{d}{dx}\left(\frac{2}{e^x + e^{-x}}\right) = 2\frac{d}{dx}(e^x + e^{-x})^{-1}$$

$$= 2(-1)(e^x + e^{-x})^{-2} \cdot \frac{d}{dx}(e^x + e^{-x})$$

$$= \frac{-2}{(e^x + e^{-x})^2} \times [e^x + e^{-x} - (-1)]$$

$$= \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$\therefore\ f'(c) = \frac{-2(e^c - e^{-c})}{(e^c + e^{-c})^2}$$

$$\therefore\ f'(c) = 0 \text{ gives, } \frac{-2(e^c - e^{-c})}{(e^c + e^{-c})^2} = 0$$

$$\therefore\ e^c - e^{-c} = 0$$

$$\therefore\ e^c = e^{-c} = \frac{1}{e^c}$$

$$\therefore\ e^{2c} = 1 = e^0$$

$$\therefore\ 2c = 0 \qquad \therefore\ c = 0 \in (-1, 1)$$

Hence, Rolle's theorem is verified.

Question 7.
The position of a particle is given by the function $s(t) = 2t^2 + 3t - 4$. Find the time t = c in the interval $0 \le f \le 4$ when the instantaneous velocity of the particle is equal to its average velocity in this interval.
Solution:
$s(t) = 2t^2 + 3t - 4$
$\therefore s(0) = 2(0)^2 + 3(0) - 4 = -4$
and $s(4) = 2(4)^2 + 3(4) - 4 = 32 + 12 - 4 = 40$
$\therefore$ average velocity = $\frac{s(4)-s(0)}{4-0}$
$= \frac{40-(-4)}{4}$
= 11
Also, instantaneous velocity = $\frac{ds}{dt}$
$= \frac{d}{dt}(2t^2 + 3t - 4)$
$= 2 \times 2t + 3 \times 1 - 0$
$= 4t + 3$
$\therefore$ instantaneous velocity at t = c is $\left(\frac{ds}{dt}\right)_{t=c} = 4c + 3$
When instantaneous velocity at t = c equal to its average velocity, we get
$4c + 3 = 11$
$4c = 8$
$\therefore c = 2 \in [0, 4]$
Hence, t = c = 2.

Question 8.
Find the approximate value of the function $f(x) = \sqrt{x^2+3x}$ at x = 1.02.
Solution:

f(x) = $\sqrt{x^2+3x}$

$$\therefore\ f'(x) = \frac{d}{dx}(\sqrt{x^2+3x})$$

$$= \frac{1}{2\sqrt{x^2+3x}} \cdot \frac{d}{dx}(x^2+3x)$$

$$= \frac{1}{2\sqrt{x^2+3x}} \times (2x+3\times 1) = \frac{2x+3}{2\sqrt{x^2+3x}}$$

Take $a = 1$ and $h = 0.02$.

Then $f(a) = f(1) = \sqrt{1^2+3(1)} = 2$

and $f'(a) = f'(1) = \dfrac{2(1)+3}{2\sqrt{1^2+3(1)}} = \dfrac{5}{2\times 2} = \dfrac{5}{4}$

The formula for approximation is

$f(a+h) \doteqdot f(a) + h \cdot f'(a)$

$$\therefore\ f(1.02) = f(1+0.02)$$

$$\doteqdot f(1) + (0.02)f'(1)$$

$$\doteqdot 2 + 0.02 \times \frac{5}{4}$$

$$\doteqdot \frac{8+0.1}{4} = \frac{8.1}{4} = 2.025$$

$\therefore\ f(1.02) \doteqdot 2.025.$

Question 9.
Find the approximate value of $\cos^{-1}(0.51)$, given π = 3.1416, $\frac{2}{\sqrt{3}}$ = 1.1547.
Solution:
Let f(x) = $\cos^{-1} x$

Then $f'(x) = \dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$

Take $a = 0.5$ and $h = 0.01$.

Then $f(a) = f(0.5) = \cos^{-1}(0.5) = \cos^{-1}\left(\cos\dfrac{\pi}{3}\right) = \dfrac{\pi}{3}$

and $f'(a) = f'(0.5) = -\dfrac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = -\dfrac{2}{\sqrt{3}} = -1.1547$

The formula for approximation is f(a + h)= f(a) + h . f'(a)
∴ $\cos^{-1}$ (0.51) = f(0.51)
= f(0.5 + 0.01)
= f(0.5) + (0.01) f'(0.5)
= $\frac{\pi}{3}$ + 0.01 × (-1.1547)
= $\frac{3.1416}{3}$ – 0.011547
= 1.0472 – 0.011547
= 1.035653
∴ $\cos^{-1}$ (0.51) = 1.035653.

Question 10.
Find the intervals on which the function y = x^x, (x > 0) is increasing and decreasing.
Solution:
y = x^x
∴ log y = log x^x = x log x
Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(x\log x)$$

$$=x\cdot\frac{d}{dx}(\log x)+(\log x)\cdot\frac{d}{dx}(x)$$

$$=x\times\frac{1}{x}+(\log x)\times 1$$

$$\therefore\ \frac{dy}{dx}=y(1+\log x)=x^x(1+\log x)$$

y is increasing if $\frac{dy}{dx} \geq 0$
i.e. if $x^x(1 + \log x) \geq 0$
i.e. if $1 + \log x \geq 0$ ......[$\because$ x > 0]
i.e. if $\log x \geq -1$
i.e. if $\log x \geq -\log e$ ....[$\because$ log e = 1]
i.e. if $\log x \geq \log \frac{1}{e}$
i.e. if $x \geq \frac{1}{e}$
$\therefore$ y is increasing in $\left[\frac{1}{e},\infty\right)$
y is decreasing if $\frac{dy}{dx} \leq 0$
i.e. if $x^x(1 + \log x) \leq 0$
i.e. if $1 + \log x \leq 0$ ......[$\because$ x > 0]
i.e. if $\log x \leq -1$
i.e. if $\log x \leq -\log e$
i.e. if $\log x \leq \log \frac{1}{e}$
i.e. if $x \leq \frac{1}{e}$ where x > 0
$\therefore$ y is decreasing is $\left(0,\frac{1}{e}\right]$
Hence, the given function is increasing in $\left[\frac{1}{e},\infty\right)$ and decreasing in $\left(0,\frac{1}{e}\right]$

Question 11.
Find the intervals on which the function f(x) = $\frac{x}{\log x}$ is increasing and decreasing.
Solution:
f(x) = $\frac{x}{\log x}$

$$\therefore\ f'(x)=\frac{d}{dx}\left(\frac{x}{\log x}\right)$$

$$=\frac{(\log x)\cdot\frac{d}{dx}(x)-x\cdot\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$=\frac{(\log x)\times 1-x\times\frac{1}{x}}{(\log x)^2}$$

$$=\frac{\log x-1}{(\log x)^2}$$

f is increasing if f'(x) ≥ 0
i.e. if $\frac{\log x-1}{(\log x)^2} \geq 0$
i.e. if $\log x - 1 \geq 0$ ....[$\because$ $(\log x)^2 > 0$]
i.e. if $\log x \geq 1$
i.e. if $\log x \geq \log e$ ....[$\because$ log e = 1]
i.e. if $x \geq e$
$\therefore$ f is increasing on $[e, \infty)$
f is decreasing if f'(x) ≤ 0
i.e. if $\frac{\log x-1}{(\log x)^2} \leq 0$
i.e. if $\log x - 1 \leq 0$ ....[$\because$ $(\log x)^2 > 0$]
i.e. if $\log x \leq 1$
i.e. if $\log x \leq \log e$
i.e. if $x \leq e$
Also, x > 0 and x ≠ 1 because f(x) = $\frac{x}{\log x}$ is not defined at x = 1.
$\therefore$ f is decreasing in (0, e] – {1}
Hence, f is increasing in [e, ∞) and decreasing in (0, e] – {1}.

Question 12.
An open box with a square base is to be made out of the given quantity of sheet of area a^2. Show that the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$.
Solution:
Let x be the side of square base and h be the height of the box.
Then $x^2 + 4xh = a^2$
$\therefore h = \frac{a^2-x^2}{4x}$ ....(1)
Let V be the volume of the box.
Then $V = x^2h$

$$\therefore \ V = x^2\left(\frac{a^2-x^2}{4x}\right) \quad \ldots \text{[By (1)]}$$

$$\therefore \ V = \frac{1}{4}(a^2x - x^3) \quad \ldots (2)$$

$$\therefore \ \frac{dV}{dx} = \frac{1}{4}(a^2x - x^3) = \frac{1}{4}(a^2 \times 1 - 3x^2) = \frac{1}{4}(a^2 - 3x^2)$$

$$\text{and } \frac{d^2V}{dx^2} = \frac{1}{4}\cdot\frac{d}{dx}(a^2 - 3x^2) = \frac{1}{4}(0 - 3 \times 2x) = -\frac{3}{2}x$$

$$\text{Now, } \frac{dV}{dx} = 0 \text{ gives } \frac{1}{4}(a^2 - 3x^2) = 0$$

$$\therefore \ a^2 - 3x^2 = 0 \qquad \therefore \ 3x^2 = a^2$$

$$\therefore \ x^2 = \frac{a^2}{3} \qquad \therefore \ x = \frac{a}{\sqrt{3}} \quad \ldots [\because \ x > 0]$$

$$\text{and } \left(\frac{d^2V}{dx^2}\right)_{\text{at } x = \frac{a}{\sqrt{3}}} = -\frac{3}{2} \times \frac{a}{\sqrt{3}} = -\frac{\sqrt{3}}{2}a < 0$$

$$\therefore \ V \text{ is maximum when } x = \frac{a}{\sqrt{3}}$$

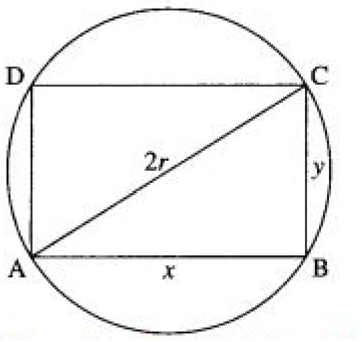
$$\text{From (2), maximum volume} = \left[\frac{1}{4}(a^2x - x^3)\right]_{\text{at } x = \frac{a}{\sqrt{3}}}$$

$$= \frac{1}{4}\left(a^2 \times \frac{a}{\sqrt{3}} - \frac{a^3}{3\sqrt{3}}\right)$$

$$= \frac{1}{4}\left(\frac{2a^3}{3\sqrt{3}}\right) = \frac{a^3}{6\sqrt{3}}$$

Hence, the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$ cu units.

Question 13.
Show that of all rectangles inscribed in a given circle, the square has the maximum area.
Solution:



Let ABCD be a rectangle inscribed in a circle of radius r.
Let AB = x and BC = y.
Then $x^2 + y^2 = 4r^2$ ....(1)
Area of rectangle = xy
$= x\sqrt{4r^2 - x^2}$ ......[By (1)]
Let $f(x) = x^2(4r^2 - x^2)$
$= 4r^2x^2 - x^4$
$\therefore f'(x) = \frac{d}{dx}(4r^2x^2 - x^4)$

$= 4r^2 \times 2x - 4x^3$
$= 8r^2x - 4x^3$
and $f''(x) = \frac{d}{dx}(8r^2x - 4x^3)$
$= 8r^2 \times 1 - 4 \times 3x^2$
$= 8r^2 - 12x^2$
For maximum area, $f'(x) = 0$
$\Rightarrow 8r^2x - 4x^3 = 0$
$\Rightarrow 4x^3 = 8r^2x$
$\Rightarrow x^2 = 2r^2$ ....[$\because x \neq 0$]
$\Rightarrow x = \sqrt{2}r$ .....[$x > 0$]
and $f''(\sqrt{2}r) = 8r^2 - 12(\sqrt{2}r)^2 = -16r^2 < 0$
$\therefore f(x)$ is maximum when $x = \sqrt{2}r$
If $x = \sqrt{2}r$, then from (1),
$(\sqrt{2}r)^2 + y^2 = 4r^2$
$\Rightarrow y^2 = 4r^2 - 2r^2 = 2r^2$
$\Rightarrow y = \sqrt{2}r$ ......[$\because y > 0$]
$\Rightarrow x = y$
$\therefore$ rectangle is a square.
Hence, amongst all rectangles inscribed in a circle, the square has maximum area.

Question 14.
Show that a closed right circular cylinder of a given surface area has maximum volume if its height equals the diameter of its base.
Solution:
Let r be the radius of the base, h be the height and V be the volume of the closed right circular cylinder, whose surface area is a^2 sq units (which is given).
$2\pi rh + 2\pi r^2 = a^2$
$\Rightarrow 2\pi r(h + r) = a^2$
$\Rightarrow h = \frac{a^2}{2\pi r} - r$ ......(1)

$$\text{Now, } V = \pi r^2 h = \pi r^2\left(\frac{a^2}{2\pi r} - r\right) \quad \ldots \text{ [By (1)]}$$
$$= \frac{1}{2}a^2 r - \pi r^3$$
$$\therefore \frac{dV}{dr} = \frac{d}{dr}\left(\frac{1}{2}a^2 r - \pi r^3\right)$$
$$= \frac{1}{2}a^2 \times 1 - \pi \times 3r^2 = \frac{a^2}{2} - 3\pi r^2$$
$$\text{and } \frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{a^2}{2} - 3\pi r^2\right)$$
$$= 0 - 3\pi \times 2r = -6\pi r$$
For maximum volume, $\frac{dV}{dr} = 0$
$$\therefore \frac{a^2}{2} - 3\pi r^2 = 0$$
$$\therefore 3\pi r^2 = \frac{a^2}{2} \qquad \therefore r^2 = \frac{a^2}{6\pi}$$
$$\therefore r = \frac{a}{\sqrt{6\pi}} \quad \ldots [\because r > 0]$$

and $\left(\frac{d^2V}{dr^2}\right)_{\text{at } x=\frac{a}{\sqrt{6\pi}}} = -6\pi\left(\frac{a}{\sqrt{6\pi}}\right) < 0$

$\therefore$ V is maximum when $r = \frac{a}{\sqrt{6\pi}}$

When $r = \frac{a}{\sqrt{6\pi}}$, then from (1),

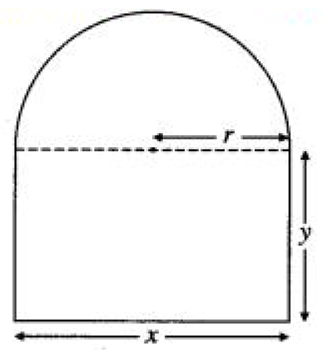
$$h = \frac{a^2}{2\pi \times \frac{a}{\sqrt{6\pi}}} - \frac{a}{\sqrt{6\pi}} = \frac{\sqrt{6\pi}a}{2\pi} - \frac{a}{\sqrt{6\pi}}$$

$$= \frac{6\pi a - 2\pi a}{2\pi\sqrt{6\pi}} = \frac{4\pi a}{2\pi\sqrt{6\pi}} = \frac{2a}{\sqrt{6\pi}}$$

$\therefore$ $h = 2r$

Hence, the volume of the cylinder is maximum if its height is equal to the diameter of the base.

Question 15.
A window is in the form of a rectangle surmounted by a semicircle. If the perimeter is 30 m, find the dimensions so that the greatest possible amount of light may be admitted.
Solution:



Let x be the length, y be the breadth of the rectangle and r be the radius of the semicircle.
Then perimeter of the window = x + 2y + πr, where x = 2r
This is given to be 30 m
⇒ 2r + 2y + πr = 30
⇒ 2y = 30 – (π + 2)r
⇒ y = 15 – $\frac{(\pi+2)r}{2}$ ……..(1)
The greatest possible amount of light may be admitted if the area of the window is maximized.
Let A be the area of the window.

$$\text{Then } A = xy + \frac{\pi r^2}{2}$$

$$= 2yr + \frac{\pi r^2}{2} \qquad \ldots [\because\ x = 2r]$$

$$= 2r\left[15 - \frac{(\pi+2)r}{2}\right] + \frac{\pi r^2}{2} \qquad \ldots [\text{By (1)}]$$

$$= 30r - (\pi+2)r^2 + \frac{\pi}{2}r^2$$

$$= 30r - \left(\pi + 2 - \frac{\pi}{2}\right)r^2$$

$$\therefore\ A = 30r - \left(\frac{\pi+4}{2}\right)r^2$$

$$\therefore\ \frac{dA}{dr} = \frac{d}{dr}\left[30r - \left(\frac{\pi+4}{2}\right)r^2\right]$$

$$= 30 \times 1 - \left(\frac{\pi+4}{2}\right) \times 2r = 30 - (\pi+4)r$$

and $\frac{d^2A}{dr^2} = \frac{d}{dr}[30-(\pi+4)r]$

$= 0-(\pi+4)\times 1 = -(\pi+4)$

For maximum A, $\frac{dA}{dr} = 0$

$\therefore\ 30-(\pi+4)r = 0$

$\therefore\ r = \frac{30}{\pi+4}$

and $\left(\frac{d^2A}{dr^2}\right)_{\text{at } r=\frac{30}{\pi+4}} = -(\pi+4) < 0$

$\therefore\ A$ is maximum when $r = \frac{30}{\pi+4}$

When $r = \frac{30}{\pi+4}$, $x = 2r = \frac{60}{\pi+4}$

and $y = 15 - \frac{(\pi+2)}{2}\times\frac{30}{\pi+4}$ ... [By (1)]

$= \frac{30\pi+120-30\pi-60}{2(\pi+4)} = \frac{30}{\pi+4}$

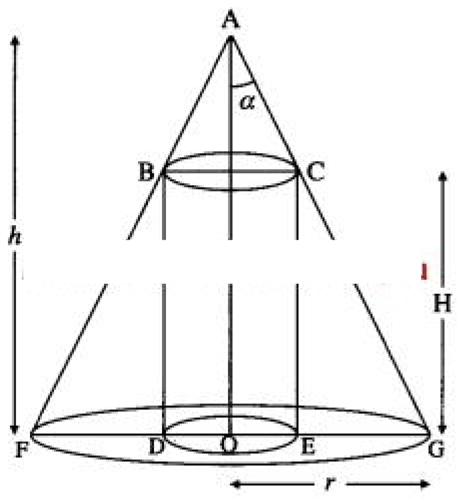
Hence, the required dimensions of the window are as follows:

Length of rectangle = $\left(\frac{60}{\pi+4}\right)$ metres

breadth of rectangle = $\left(\frac{30}{\pi+4}\right)$ metres

radius of the semicircle = $\left(\frac{30}{\pi+4}\right)$ metres

Question 16.
Show that the height of a right circular cylinder of greatest volume that can be inscribed in a right circular cone is one-third of that of the cone.
Solution:



Given the right circular cone of fixed height h and semi-vertical angle a.
Let R be the radius of the base and H be the height of the right circular cylinder that can be inscribed in the right circular cone.
In the figure, ∠GAO = α, OG = r, OA = h, OE = R, CE = H.
We have, $\frac{r}{h}$ = tan α
∴ r = h tan α ......(1)
Since ΔAOG and ΔCEG are similar.

$$\therefore \frac{AO}{OG} = \frac{CE}{EG} = \frac{CE}{OG - OE}$$

$$\therefore \frac{h}{r} = \frac{H}{r - R}$$

$$\therefore H = \frac{h}{r}(r - R) = \frac{h}{h\tan\alpha}(h\tan\alpha - R) \quad \ldots \text{[By (1)]}$$

$$\therefore H = \frac{1}{\tan\alpha}(h\tan\alpha - R) \quad \ldots (2)$$

Let V be the volume of the cylinder.

Then $V = \pi R^2 H = \frac{\pi R^2}{\tan\alpha}(h\tan\alpha - R)$

$$\therefore V = \pi R^2 h - \frac{\pi R^3}{\tan\alpha}$$

$$\therefore \frac{dV}{dR} = \frac{d}{dR}\left(\pi R^2 h - \frac{\pi R^3}{\tan\alpha}\right)$$

$$= \pi h \times 2R - \frac{\pi}{\tan\alpha} \times 3R^2$$

$$= 2\pi Rh - \frac{3\pi R^2}{\tan\alpha}$$

and $\frac{d^2V}{dR^2} = \frac{d}{dR}\left(2\pi Rh - \frac{3\pi R^2}{\tan\alpha}\right)$

$$= 2\pi h \times 1 - \frac{3\pi}{\tan\alpha} \times 2R$$

$$= 2\pi h - \frac{6\pi R}{\tan\alpha}$$

For maximum volume, $\frac{dV}{dR} = 0$

$$\therefore 2\pi Rh - \frac{3\pi R^2}{\tan\alpha} = 0$$

$$\therefore \frac{3\pi R^2}{\tan\alpha} = 2\pi Rh$$

$$\therefore R = \frac{2h}{3}\tan\alpha \quad \ldots\ldots[\because R \neq 0]$$

and $\left(\frac{d^2V}{dR^2}\right)_{\text{at } R = \frac{2h}{3}\tan\alpha} = 2\pi h - \frac{6\pi}{\tan\alpha} \times \frac{2h}{3}\tan\alpha$

$$= 2\pi h - 4\pi h = -2\pi h < 0$$

$\therefore$ V is maximum when $R = \frac{2h}{3}\tan\alpha$

When $R = \frac{2h}{3}\tan\alpha$, then from (2), we get

$$H = \frac{1}{\tan\alpha}\left(h\tan\alpha - \frac{2h}{3}\tan\alpha\right) = \frac{h}{3}$$

Hence, the height of the right circular cylinder is one-third of that of the cone.

Question 17.
A wire of length l is cut into two parts. One part is bent into a circle and the other into a square. Show that the sum of the areas of the circle and the square is the least if the radius of the circle is half of the side of the square.
Solution:
Let r be the radius of the circle and x be the length of the side of the square. Then
(circumference of the circle) + (perimeter of the square) = l
$\therefore 2\pi r + 4x = l$

∴ $r = \frac{l-4x}{2\pi}$
A = (area of the circle) + (area of the square)
= $\pi r^2 + x^2$

$$= \pi\left(\frac{l-4x}{2\pi}\right)^2 + x^2 = x^2 + \frac{1}{4\pi}(l-4x)^2 = f(x) \quad \dots \text{(Say)}$$

Then $f'(x) = 2x + \frac{1}{4\pi} \times 2(l-4x)(-4)$

$$= 2x - \frac{2}{\pi}(l-4x)$$

and $f''(x) = 2 - \frac{2}{\pi}(-4) = 2 + \frac{8}{\pi}$

Now, $f'(x) = 0$ when $2x - \frac{2}{\pi}(l-4x) = 0$

i.e. when $2\pi x - 2l + 8x = 0$

i.e. when $2(\pi+4)x = 2l$

i.e. when $x = \frac{l}{\pi+4}$

and $f''\left(\frac{l}{\pi+4}\right) = 2 + \frac{8}{\pi} > 0$

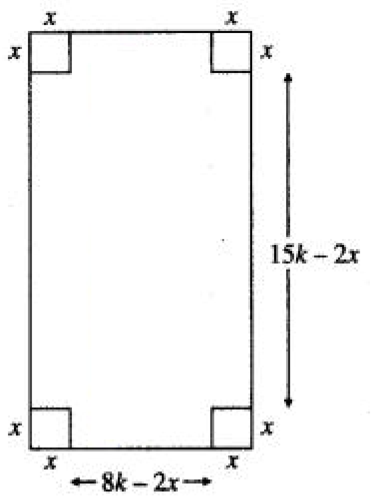
∴ by the second derivative test, f has a minimum,

when $x = \frac{l}{\pi+4}$. For this value of x,

$$r = \frac{l - 4\left(\frac{l}{\pi+4}\right)}{2\pi} = \frac{\pi l + 4l - 4l}{2\pi(\pi+4)} = \frac{l}{2(\pi+4)} = \frac{x}{2}$$

This shows that the sum of the areas of circle and square is least when the radius of the circle = ($\frac{1}{2}$) side of the square.

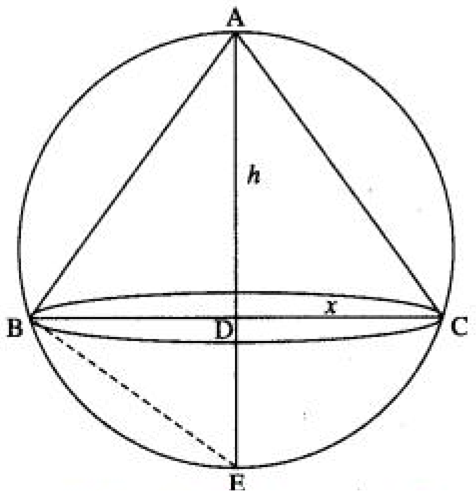
Question 18.
A rectangular sheet of paper of fixed perimeter with the sides having their lengths in the ratio 8 : 15 converted into an open rectangular box by folding after removing the squares of the equal area from all comers. If the total area of the removed squares is 100, the resulting box has maximum volume. Find the lengths of the rectangular sheet of paper.
Solution:



The sides of the rectangular sheet of paper are in the ratio 8 : 15.
Let the sides of the rectangular sheet of paper be 8k and 15k respectively.
Let x be the side of the square which is removed from the comers of the sheet of paper.
The total area of removed squares is $4x^2$, which is given to be 100.
$4x^2 = 100$
⇒ $x^2 = 25$
⇒ $x = 5$ ……[x > 0]
Now, the length, breadth, and height of the rectangular box are 15k – 2x, 8k – 2x, and x respectively.
Let V be the volume of the box.
Then V = (15k – 2x) (8k – 2x) . x
⇒ V = $(120k^2 - 16kx - 30kx + 4x^2) . x$
⇒ V = $4x^3 - 46kx^2 + 120k^2x$
$\frac{dV}{dx} = \frac{d}{dx}(4x^3 - 46kx^2 + 120k^2x)$
= $4 \times 3x^2 - 46k \times 2x + 120k^2 \times 1$
= $12x^2 - 92kx + 120k^2$
Since, volume is maximum when the square of side x = 5 is removed from the corners,

$\left(\frac{dV}{dx}\right)_{at\ x=5} = 0$
⇒ $12(5)^2 - 92k(5) + 120k^2 = 0$
⇒ $60 - 92k + 24k^2 = 0$
⇒ $6k^2 - 23k + 15 = 0$
⇒ $6k^2 - 18k - 5k + 15 = 0$
⇒ $6k(k - 3) - 5(k - 3) = 0$
⇒ $(k - 3)(6k - 5) = 0$
⇒ $k = 3$ or $k = \frac{5}{6}$
If $k = \frac{5}{6}$, then
$8k - 2x = 8k - 10 < 0$
$\therefore k \neq \frac{5}{6}$
$\therefore k = 3$
$\therefore 8k = 8 \times 3 = 24$ and $15k = 15 \times 3 = 45$
Hence, the lengths of the rectangular sheet are 24 and 45.

Question 19.
Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.
Solution:



Let x be the radius of the base and h be the height of the cone which is inscribed in a sphere of radius r.
In the figure, AD = h and CD = x = BD
Since, ΔABD and ΔBDE are similar,
$\frac{AD}{BD} = \frac{BD}{DE}$
$BD^2 = AD \cdot DE = AD(AE - AD)$
$x^2 = h(2r - h)$ ...... (1)
Let V be the volume of the cone.

Then $V = \frac{1}{3}\pi x^2 h = \frac{\pi}{3} h(2r - h)h$ .....[By (1)]

$\therefore\ V = \frac{\pi}{3}(2rh^2 - h^3)$

$$\therefore\ \frac{dV}{dh} = \frac{\pi}{3}\frac{d}{dh}(2rh^2 - h^3)$$

$$= \frac{\pi}{3}(2r \times 2h - 3h^2) = \frac{\pi}{3}(4rh - 3h^2)$$

and $\frac{d^2V}{dh^2} = \frac{\pi}{3}\frac{d}{dh}(4rh - 3h^2)$

$$= \frac{\pi}{3}(4r \times 1 - 3 \times 2h) = \frac{\pi}{3}(4r - 6h)$$

For maximum volume, $\frac{dV}{dh} = 0$

$$\therefore \frac{\pi}{3}(4rh - 3h^2) = 0$$

$$\therefore 4rh = 3h^2$$

$$\therefore h = \frac{4r}{3} \qquad [\because h \neq 0]$$

$$\text{and} \left(\frac{d^2V}{dh^2}\right)_{\text{at } h=\frac{4r}{3}} = \frac{\pi}{3}\left(4r - 6 \times \frac{4r}{3}\right)$$

$$= \frac{\pi}{3}(4r - 8r) = -\frac{4\pi r}{3} < 0$$

∴ V is maximum when h = $\frac{4r}{3}$
Hence, the altitude (i.e. height) of the right circular cone of maximum volume = $\frac{4r}{3}$.

Question 20.
Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also, find the maximum Volume.
Solution:
Let R be the radius and h be the height of the cylinder which is inscribed in a sphere of radius r cm.
Then from the figure,
$R^2+\left(\frac{h}{2}\right)^2=r^2$
∴ $R^2 = r^2 - \frac{h^2}{4}$ ....(1)
Let V be the volume of the cylinder.
Then V = $\pi R^2 h$

$$= \pi\left(r^2 - \frac{h^2}{4}\right)h \quad \text{...[By (1)]}$$

$$= \pi\left(r^2 - \frac{h^3}{4}\right)$$

$$\therefore \frac{dV}{dh} = \pi\frac{d}{dh}\left(r^2h - \frac{h^3}{4}\right)$$

$$= \pi\left(r^2 \times 1 - \frac{1}{4} \times 3h^2\right)$$

$$= \pi\left(r^2 - \frac{3}{4}h^2\right)$$

and

$$\frac{d^2V}{dh^2} = \pi\frac{d}{dh}\left(r^2 - \frac{3}{4}h^2\right)$$

$$= \pi\left(0 - \frac{3}{4} \times 2h\right)$$

$$= -\frac{3}{2}\pi h$$

Now, $\frac{dV}{dh} = 0$ gives, $\pi\left(r^2 - \frac{3}{4}h^2\right) = 0$

$\therefore r^2 - \frac{3}{4}h^2 = 0$

$\therefore \frac{3}{4}h^2 = r^2$

$\therefore h^2 = \frac{4r^2}{3}$

$\therefore h = \frac{2r}{\sqrt{3}}$ ...[$\because h > 0$]

and

$\left(\frac{d^2V}{dh^2}\right)_{\text{at } h=\frac{2r}{\sqrt{3}}}$

$= -\frac{3}{2}\pi \times \frac{2r}{\sqrt{3}} < 0$

$\therefore$ V is maximum at h = $\frac{2r}{\sqrt{3}}$

If h = $\frac{2r}{\sqrt{3}}$, then from (1)

$R^2 = r^2 - \frac{1}{4} \times \frac{4r^2}{3} = \frac{2r^2}{3}$

$\therefore$ volumeof the largest cylinder

$= \pi \times \frac{2r^2}{3} \times \frac{2r}{\sqrt{3}} = \frac{4\pi r^3}{3\sqrt{3}}$ cu cm.

Hence, the volume of the largest cylinder inscribed in a sphere of radius 'r' cm = $\frac{4\pi r^3}{3\sqrt{3}}$ cu cm.

Question 21.
Find the maximum and minimum values of the function f(x) = $\cos^2 x$ + sin x.
Solution:
f(x) = $\cos^2 x$ + sin x
$\therefore$ f'(x) = $\frac{d}{dx}$ ($\cos^2 x$ + sin x)
= 2 cos x . $\frac{d}{dx}$ (cos x) + cos x
= 2 cos x(-sin x) + cos x
= -sin 2x + cos x
and f''(x) = $\frac{d}{dx}$ (-sin 2x + cos x)
= -cos 2x . $\frac{d}{dx}$ (2x) – sin x
= -cos 2x × 2 – sin x
= -2 cos 2x – sin x
For extreme values of f(x), f'(x) = 0
-sin 2x + cos x = 0
-2 sin x cos x + cos x = 0
cos x (-2 sin x + 1) = 0
cos x = 0 or -2 sin x + 1 = 0
cos x = cos $\frac{\pi}{2}$ or sin x = $\frac{1}{2}$ = sin $\frac{\pi}{6}$
$\therefore$ x = $\frac{\pi}{2}$ or x = $\frac{\pi}{6}$

(i) f''($\frac{\pi}{2}$) = -2 cos π – sin $\frac{\pi}{2}$
= -2(-1) – 1
= 1 > 0
$\therefore$ by the second derivative test, f is minimum at x = $\frac{\pi}{2}$ and minimum value of f at x = $\frac{\pi}{2}$
= f($\frac{\pi}{2}$)
= $\cos^2\frac{\pi}{2} + \sin\frac{\pi}{2}$
= 0 + 1
= 1

(ii) $f''(\pi6) = -2\cos\pi3 - \sin\pi6$

$= -2(12) - 12$

$= -32 < 0$

∴ by the second derivative test, f is maximum at x = $\pi6$ and maximum value of f at x = $\pi6$

$= f(\pi6)$

$= \cos2\pi6 + \sin\pi6$

$= (3\sqrt{2})2 + 12$

$= 54$

Hence, the maximum and minimum values of the function f(x) are 54 and 1 respectively.