Maharashtra State Board 12th Commerce Maths Solutions Chapter 4 Applications of Derivatives Ex 4.1

Question 1.
Find the equations of tangent and normal to the following curves at the given point on it:
(i) $y = 3x^2 - x + 1$ at (1, 3)
Solution:
$y = 3x^2 - x + 1$
$\therefore \frac{dy}{dx} = \frac{d}{dx}(3x^2 - x + 1)$
$= 3 \times 2x - 1 + 0$
$= 6x - 1$
$\therefore \left(\frac{dy}{dx}\right)_{at\ (1,3)} = 6(1) - 1$
$= 5$
= slope of the tangent at (1, 3).
∴ the equation of the tangent at (1, 3) is
$y - 3 = 5(x - 1)$
$\therefore y - 3 = 5x - 5$
$\therefore 5x - y - 2 = 0$.
The slope of the normal at (1, 3) $= \frac{-1}{\left(\frac{dy}{dx}\right)_{at\ (1,3)}} = -\frac{1}{5}$
∴ the equation of the normal at (1, 3) is
$y - 3 = -\frac{1}{5}(x - 1)$
$\therefore 5y - 15 = -x + 1$
$\therefore x + 5y - 16 = 0$
Hence, the equations of the tangent and normal are $5x - y - 2 = 0$ and $x + 5y - 16 = 0$ respectively.

(ii) $2x^2 + 3y^2 = 5$ at (1, 1)
Solution:
$2x^2 + 3y^2 = 5$
Differentiating both sides w.r.t. x, we get

$$2 \times 2x + 3 \times 2y\frac{dy}{dx} = 0$$

$$\therefore\ 6y\frac{dy}{dx} = -4x \qquad \therefore\ \frac{dy}{dx} = -\frac{2x}{3y}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{at\ (1,\ 1)} = \frac{-2(1)}{3(1)} = -\frac{2}{3}$$

= slope of the tangent at (1, 1)
∴ the equation of the tangent at (1, 1) is
$y - 1 = \frac{-2}{3}(x - 1)$
$\therefore 3y - 3 = -2x + 2$
$\therefore 2x + 3y - 5 = 0$.
The slope of normal at (1, 1) $= \frac{-1}{\left(\frac{dy}{dx}\right)_{at\ (1,1)}} = \frac{-1}{\left(\frac{-2}{3}\right)} = \frac{3}{2}$
∴ the equation of the normal at (1, 1) is
$y - 1 = \frac{3}{2}(x - 1)$
$\therefore 2y - 2 = 3x - 3$
$\therefore 3x - 2y - 1 = 0$
Hence, the equations of the tangent and normal are $2x + 3y - 5 = 0$ and $3x - 2y - 1 = 0$ respectively.

(iii) $x^2 + y^2 + xy = 3$ at (1, 1)
Solution:
$x^2 + y^2 + xy = 3$
Differentiating both sides w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} + x\frac{dy}{dx} + y\frac{d}{dx}(x) = 0$$

$$\therefore\ 2x + 2y\frac{dy}{dx} + x\frac{dy}{dx} + y \times 1 = 0$$

$$\therefore\ (x + 2y)\frac{dy}{dx} = -2x - y$$

$$\therefore\ \frac{dy}{dx} = \frac{-2x - y}{x + 2y} = -\left(\frac{2x + y}{x + 2y}\right)$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }(1,\,1)} = -\left[\frac{2(1) + 1}{1 + 2(1)}\right] = -\frac{3}{3} = -1$$

= slope of the tangent at (1, 1)
the equation of the tangent at (1, 1) is
y – 1= -1(x – 1)
∴ y – 1 = -x + 1
∴ x + y = 2
The slope of the normal at (1, 1) = $\frac{-1}{\left(\frac{dy}{dx}\right)_{at(1,1)}}$
= $\frac{-1}{-1}$
= 1
∴ the equation of the normal at (1, 1) is y – 1 = 1(x – 1)
∴ y – 1 = x – 1
∴ x – y = 0
Hence, the equations of tangent and normal are x + y = 2 and x – y = 0 respectively.

Question 2.
Find the equations of the tangent and normal to the curve $y = x^2 + 5$ where the tangent is parallel to the line 4x – y + 1 = 0.
Solution:
Let $P(x_1, y_1)$ be the point on the curve $y = x^2 + 5$ where the tangent is parallel to the line 4x – y + 1 = 0.
Differentiating $y = x^2 + 5$ w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 5) = 2x + 0 = 2x$
$\left(\frac{dy}{dx}\right)_{at\ (x_1, y_1)} = 2x_1$
= slope of the tangent at (x_1, y_1)
Let $m_1 = 2x_1$
The slope of the line 4x – y + 1 = 0 is
$m_2 = \frac{-4}{-1} = 4$
Since, the tangent at $P(x_1, y_1)$ is parallel to the line 4x – y + 1 = 0,
$m_1 = m_2$
∴ $2x_1 = 4$
∴ $x_1 = 2$
Since, (x_1, y_1) lies on the curve $y = x^2 + 5$, $y_1 = x_1^2 + 5$
∴ $y_1 = (2)^2 + 5 = 9$ ……$[x_1 = 2]$
∴ the coordinates of the point are (2, 9) and the slope of the tangent = $m_1 = m_2 = 4$.
∴ the equation of the tangent at (2, 9) is
y – 9 = 4(x – 2)
∴ y – 9 = 4x – 8
∴ 4x – y + 1 = 0
Slope of the normal = $\frac{-1}{m_1} = -\frac{1}{4}$
∴ the equation of the normal at (2, 9) is
y – 9 = $-\frac{1}{4}$(x – 2)
∴ 4y – 36 = -x + 2
∴ x + 4y – 38 = 0
Hence, the equations of tangent and normal are 4x – y + 1 = 0 and x + 4y – 38 = 0 respectively.

Question 3.
Find the equations of the tangent and normal to the curve $y = 3x^2 – 3x – 5$ where the tangent is parallel to the line 3x – y + 1 = 0.
Solution:
Let $P(x_1, y_1)$ be the point on the curve $y = 3x^2 – 3x – 5$ where the tangent is parallel to the line 3x – y + 1 = 0.
Differentiating $y = 3x^2 – 3x – 5$ w.r.t. x, we get
$\frac{dy}{dx} = \frac{d}{dx}(3x^2 – 3x – 5)$
= 3 × 2x – 3 × 1 – 0
= 6x – 3

$\therefore \left(\frac{dy}{dx}\right)_{at\ (x_1, y_1)} = 6x_1 - 3$

= slope of the tangent at (x_1, y_1)

Let $m_1 = 6x_1 - 3$

The slope of the line $3x - y + 1 = 0$

$m_2 = \frac{-3}{-1} = 3$

Since, the tangent at $P(x_1, y_1)$ is parallel to the line $3x - y + 1 = 0$,

$m_1 = m_2$

$\therefore 6x_1 - 3 = 3$

$\therefore 6x_1 = 6$

$\therefore x_1 = 1$

Since, (x_1, y_1) lies on the curve $y = 3x^2 - 3x - 5$,

$y_1 = 3x_1^2 - 3x_1 - 5$, where $x_1 = 1$

$= 3(1)^2 - 3(1) - 5$

$= 3 - 3 - 5$

$= -5$

$\therefore$ the coordinates of the point are (1, -5) and the slope of the tangent $= m_1 = m_2 = 3$.

$\therefore$ the equation of the tangent at (1, -5) is

$y - (-5) = 3(x - 1)$

$\therefore y + 5 = 3x - 3$

$\therefore 3x - y - 8 = 0$

Slope of the normal $= \frac{-1}{m_1} = \frac{-1}{3}$

$\therefore$ the equation of the normal at (1, -5) is

$y - (-5) = \frac{-1}{3}(x - 1)$

$\therefore 3y + 15 = -x + 1$

$\therefore x + 3y + 14 = 0$

Hence, the equations of tangent and normal are $3x - y - 8 = 0$ and $x + 3y + 14 = 0$ respectively.

Maharashtra State Board 12th Commerce Maths Solutions Chapter 4 Applications of Derivatives Ex 4.2

Question 1.

Test whether the following functions are increasing and decreasing:

(i) $f(x) = x^3 - 6x^2 + 12x - 16$, $x \in R$

Solution:

$f(x) = x^3 - 6x^2 + 12x - 16$

$\therefore f'(x) = \frac{d}{dx}(x^3 - 6x^2 + 12x - 16)$

$= 3x^2 - 6 \times 2x + 12 \times 1 - 0$

$= 3x^2 - 12x + 12$

$= 3(x^2 - 4x + 4)$

$= 3(x - 2)^2 > 0$ for all $x \in R$, $x \neq 2$

$\therefore f'(x) > 0$ for all $x \in R - \{2\}$

$\therefore$ f is increasing for all $x \in R - \{2\}$.

(ii) $f(x) = x - \frac{1}{x}$, $x \in R$, $x \neq 0$

Solution:

$f(x) = x - \frac{1}{x}$

$\therefore f'(x) = \frac{d}{dx}\left(x - \frac{1}{x}\right)$

$= 1 - \left(-\frac{1}{x^2}\right)$

$= 1 + \frac{1}{x^2} > 0$ for all $x \in R$, $x \neq 0$

$\therefore f'(x) > 0$ for all $x \in R$, where $x \neq 0$

$\therefore$ f is increasing for all $x > R$, where $x \neq 0$.

(iii) $f(x) = \frac{7}{x} - 3$, $x \in R$, $x \neq 0$

Solution:

$f(x) = \frac{7}{x} - 3$
$\therefore f'(x) = \frac{d}{dx}\left(\frac{7}{x} - 3\right) = 7\left(\frac{-1}{x^2}\right) - 0$
$= \frac{-7}{x^2} < 0$ for all $x \in R, x \neq 0$
$\therefore f'(x) < 0$ for all $x \in R$, where $x \neq 0$.
$\therefore$ f is decreasing for all $x \in R$, where $x \neq 0$.

Question 2.
Find the values of x, such that f(x) is increasing function:
(i) $f(x) = 2x^3 - 15x^2 + 36x + 1$
Solution:
$f(x) = 2x^3 - 15x^2 + 36x + 1$
$\therefore f'(x) = \frac{d}{dx}(2x^3 - 15x^2 + 36x + 1)$
$= 2 \times 3x^2 - 15 \times 2x + 36 \times 1 + 0$
$= 6x^2 - 30x + 36$
$= 6(x^2 - 5x + 6)$
f is increasing, if $f'(x) > 0$
i.e. if $6(x^2 - 5x + 6) > 0$
i.e. if $x^2 - 5x + 6 > 0$
i.e. if $x^2 - 5x > -6$
i.e. if $x - 5x + \frac{25}{4} > -6 + \frac{25}{4}$
i.e. if $\left(x - \frac{5}{2}\right)^2 > \frac{1}{4}$
i.e. if $x - \frac{5}{2} > \frac{1}{2}$ or $x - \frac{5}{2} < -\frac{1}{2}$
i.e. if $x > 3$ or $x < 2$
i.e. if $x \in (-\infty, 2) \cup (3, \infty)$
$\therefore$ f is increasing, if $x \in (-\infty, 2) \cup (3, \infty)$.

(ii) $f(x) = x^2 + 2x - 5$
Solution:
$f(x) = x^2 + 2x - 5$
$\therefore f'(x) = \frac{d}{dx}(x^2 + 2x - 5)$
$= 2x + 2 \times 1 - 0$
$= 2x + 2$
f is increasing, if $f'(x) > 0$
i.e. if $2x + 2 > 0$
i.e. if $2x > -2$
i.e. if $x > -1$, i.e. $x \in (-1, \infty)$
$\therefore$ f is increasing, if $x > -1$, i.e. $x \in (-1, \infty)$

(iii) $f(x) = 2x^3 - 15x^2 - 144x - 7$
Solution:
$f(x) = 2x^3 - 15x^2 - 144x - 7$
$\therefore f'(x) = \frac{d}{dx}(2x^3 - 15x^2 - 144x - 7)$
$= 2 \times 3x^2 - 15 \times 2x - 144 \times 1 - 0$
$= 6x^2 - 30x - 144$
$= 6(x^2 - 5x - 24)$
f is increasing if, $f'(x) > 0$
i.e. if $6(x^2 - 5x - 24) > 0$
i.e. if $x^2 - 5x - 24 > 0$
i.e. if $x^2 - 5x > 24$
i.e. if $x^2 - 5x + \frac{25}{4} > 24 + \frac{25}{4}$
i.e. if $\left(x - \frac{5}{2}\right)^2 > \frac{121}{4}$
i.e. if $x - \frac{5}{2} > \frac{11}{2}$ or $x - \frac{5}{2} < -\frac{11}{2}$
i.e. if $x > 8$ or $x < -3$
i.e. if $x \in (-\infty, -3) \cup (8, \infty)$
$\therefore$ f is increasing, if $x \in (-\infty, -3) \cup (8, \infty)$.

Question 3.
Find the values of x such that f(x) is decreasing function:
(i) $f(x) = 2x^3 - 15x^2 - 144x - 7$
Solution:
$f(x) = 2x^3 - 15x^2 - 144x - 7$
$\therefore f'(x) = \frac{d}{dx}(2x^3 - 15x^2 - 144x - 7)$
$= 2 \times 3x^2 - 15 \times 2x - 144 \times 1 - 0$
$= 6x^2 - 30x - 144$
$= 6(x^2 - 5x - 24)$

f is decreasing, if f'(x) < 0
i.e. if $6(x^2 – 5x – 24) < 0$
i.e. if $x^2 – 5x – 24 < 0$
i.e. if $x^2 – 5x < 24$
i.e. if $x^2 – 5x + 254 < 1214$
i.e. if $(x−52)2<1214$
i.e. if $−112<x−52<112$
i.e. if $−112+52<x−52+52<112+52$
i.e. if -3 < x < 8
∴ f is decreasing, if -3 < x < 8.

(ii) $f(x) = x^4 – 2x^3 + 1$
Solution:
$f(x) = x^4 – 2x^3 + 1$
∴ $f'(x) = ddx(x^4 – 2x^3 + 1)$
$= 4x^3 – 2 \times 3x^2 + 0$
$= 4x^3 – 6x^2$
f is decreasing, if f'(x) < 0
i.e. if $4x^3 – 6x^2 < 0$
i.e. if $x^2(4x – 6) < 0$
i.e. if 4x – 6 < 0 …….[∵ $x^2 > 0$]
i.e. if x < 32
i.e. -∞ < x < 32
∴ f is decreasing, if -∞ < x < 32.

(iii) $f(x) = 2x^3 – 15x^2 – 84x – 7$
Solution:
$f(x) = 2x^3 – 15x^2 – 84x – 7$
∴ $f'(x) = ddx(2x^3 – 15x^2 – 84x – 7)$
$= 2 \times 3x^2 – 15 \times 2x – 84 \times 1 – 0$
$= 6x^2 – 30x – 84$
$= 6(x^2 – 5x – 14)$
f is decreasing, if f'(x) < 0
i.e. if $6(x^2 – 5x – 14) < 0$
i.e. if $x^2 – 5x – 14 < 0$
i.e. if $x^2 – 5x < 14$
i.e. if x – 5x + 254 < 14 + 254
i.e. if $(x−52)2<814$
i.e. if $−92<x−52<92$
i.e. if $−92+52<x−52+52<92+52$
i.e. if -2 < x < 7
∴ f is decreasing, if -2 < x < 7.

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Question 1.
Determine the maximum and minimum values of the following functions:
(i) $f(x) = 2x^3 – 21x^2 + 36x – 20$
Solution:
$f(x) = 2x^3 – 21x^2 + 36x – 20$
∴ $f'(x) = ddx(2x^3 – 21x^2 + 36x – 20)$
$= 2 \times 3x^2 – 21 \times 2x + 36 \times 1 – 0$
$= 6x^2 – 42x + 36$
and $f''(x) = ddx(6x^2 – 42x + 36)$

= 6 × 2x – 42 × 1 + 0
= 12x – 42
f'(x) = 0 gives $6x^2 – 42x + 36 = 0$.
∴ $x^2 – 7x + 6 = 0$
∴ (x – 1)(x – 6) = 0
∴ the roots of f'(x) = 0 are $x_1 = 1$ and $x_2 = 6$.
For x = 1, f''(1) = 12(1) – 42 = -30 < 0
∴ by the second derivative test,
f has maximum at x = 1 and maximum value of f at x = 1
$f(1) = 2(1)^3 – 21(1)^2 + 36(1) – 20$
= 2 – 21 + 36 – 20
= -3
For x = 6, f''(6) = 12(6) – 42 = 30 > 0
∴ by the second derivative test,
f has minimum at x = 6 and minimum value of f at x = 6
$f(6) = 2(6)^3 – 21(6)^2 + 36(6) – 20$
= 432 – 756 + 216 – 20
= -128
Hence, the function f has maximum value -3 at x = 1 and minimum value -128 at x = 6.

(ii) f(x) = x . log x
Solution:
f(x) = x . log x
f'(x) = $\frac{d}{dx}$(x.log x)
= x.$\frac{d}{dx}$(log x) + log x.$\frac{d}{dx}$(x)
= x × $\frac{1}{x}$ + (logx) × 1
= 1 + log x
and f''(x) = $\frac{d}{dx}$(1 + logx)
= 0 + $\frac{1}{x}$
= $\frac{1}{x}$
Now, f'(x) = 0, if 1 + log x = 0
i.e. if log x = -1 = -log e
i.e. if log x = log(e^{-1}) = log $\frac{1}{e}$
i.e. if x = $\frac{1}{e}$
When x = $\frac{1}{e}$, f''(x) = $\frac{1}{(1/e)}$ = e > 0
∴ by the second derivative test,
f is minimum at x = $\frac{1}{e}$
Minimum value of f at x = $\frac{1}{e}$
= $\frac{1}{e}$ log($\frac{1}{e}$)
= $\frac{1}{e}$ log(e^{-1})
= $\frac{1}{e}$ (-1) log e
= $-\frac{1}{e}$ ....[∵ log e = 1]
Hence, the function f has minimum at x = $\frac{1}{e}$ and minimum value is $-\frac{1}{e}$.

(iii) f(x) = $x^2 + \frac{16}{x}$
Solution:

$$f(x) = x^2 + \frac{16}{x}$$

$$\therefore\ f'(x) = \frac{d}{dx}\left(x^2 + \frac{16}{x}\right)$$

$$= 2x + 16(-1)x^{-2} = 2x - \frac{16}{x^2}$$

$$\text{and } f''(x) = \frac{d}{dx}\left(2x - \frac{16}{x^2}\right)$$

$$= 2 \times 1 - 16(-2)x^{-3} = 2 + \frac{32}{x^3}$$

f'(x) = 0 gives 2x – $\frac{16}{x^2}$ = 0
∴ $2x^3 – 16 = 0$
∴ $x^3 = 8$
∴ x = 2
For x = 2, f''(2) = 2 + $\frac{32}{(2)^3}$ = 6 > 0
∴ by the second derivative test, f has minimum at x = 2 and minimum value of f at x = 2
f(2) = $(2)^2 + \frac{16}{2}$

= 4 + 8
= 12
Hence, the function f has a minimum at x = 2 and a minimum value is 12.

Question 2.
Divide the number 20 into two parts such that their product is maximum.
Solution:
Let the first part of 20 be x.
Then the second part is 20 – x.
∴ their product = $x(20 – x) = 20x – x^2 = f(x)$ …..(Say)
∴ $f'(x) = \frac{d}{dx}(20x – x^2)$
= 20 × 1 – 2x
= 20 – 2x
and $f''(x) = \frac{d}{dx}(20 – 2x)$
= 0 – 2 × 1
= -2
The root of the equation f'(x) = 0
i.e. 20 – 2x = 0 is x = 10
and f''(10) = -2 < 0
∴ by the second derivative test, f is maximum at x = 10.
Hence, the required parts of 20 are 10 and 10.

Question 3.
A metal wire of 36 cm long is bent to form a rectangle. Find its dimensions where its area is maximum.
Solution:
Let x cm and y cm be the length and breadth of the rectangle.
Then its perimeter is 2(x + y) = 36
∴ x + y = 18
∴ y = 18 – x
Area of the rectangle = xy = x(18 – x)
Let $f(x) = x(18 – x) = 18x – x^2$
Then $f'(x) = \frac{d}{dx}(18x – x^2)$
= 18 × 1 – 2x
= 18 – 2x
and $f''(x) = \frac{d}{dx}(18 – 2x)$
= 0 – 2 × 1
= -2
Now, f(x) = 0, if 18 – 2x = 0
i.e. if x = 9
and f''(9) = -2 < 0
∴ by the second derivative test, f has maximum value at x = 9
When x = 9, y = 18 – 9 = 9
Hence, the rectangle is a square of side 9 cm.

Question 4.
The total cost of producing x units is ₹$(x^2 + 60x + 50)$ and the price is ₹(180 – x) per unit. For what units is the profit maximum?
Solution:
Let the number of units sold be x.
Then profit = S.P. – C.P.
∴ $P(x) = (180 – x)x – (x^2 + 60x + 50)$
∴ $P(x) = 180x – x^2 – x^2 – 60x – 50$
∴ $P(x) = 120x – 2x^2 – 50$
$P'(x) = \frac{d}{dx}(120x – 2x^2 – 50)$
= 120 × 1 – 2 × 2x – 0
= 120 – 4x
and $P''(x) = \frac{d}{dx}(120 – 4x)$
= 0 – 4 × 1
= -4
P'(x) = 0 if 120 – 4x = 0
i.e. if x = 30 and P''(30) = -4 < 0
∴ by the second derivative test, P(x) is maximum when x = 30.
Hence, the number of units sold for maximum profit is 30.

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Question 1.
The demand function of a commodity at price P is given as D = 40 – $\frac{5P}{8}$. Check whether it is an increasing or decreasing function.
Solution:
D = 40 – $\frac{5P}{8}$
∴ $\frac{dD}{dP} = \frac{d}{dP}\left(40 - \frac{5P}{8}\right)$
= 0 – $\frac{5}{8}$ × 1
= $-\frac{5}{8}$
Hence, the given function is decreasing function.

Question 2.
The price P for demand D is given as P = 183 + 120D – $3D^2$, find D for which price is increasing.
Solution:
P = 183 + 120D – $3D^2$
∴ $\frac{dP}{dD} = \frac{d}{dD}$(183 + 120D – $3D^2$)
= 0 + 120 × 1 – 3 × 2D
= 120 – 6D
If price P is increasing, then $\frac{dP}{dD}$ > 0
∴ 120 – 6D > 0
∴ 120 > 6D
∴ D < 20
Hence, the price is increasing when D < 20.

Question 3.
The total cost function for production of x articles is given as C = 100 + 600x – $3x^2$. Find the values of x for which the total cost is decreasing.
Solution:
The cost function is given as
C = 100 + 600x – $3x^2$
∴ $\frac{dC}{dD} = \frac{d}{dD}$(100 + 600x – $3x^2$)
= 0 + 600 × 1 – 3 × 2x
= 600 – 6x
If the total cost is decreasing, then $\frac{dC}{dx}$ < 0
∴ 600 – 6x < 0
∴ 600 < 6x
∴ x > 100
Hence, the total cost is decreasing for x > 100.

Question 4.
The manufacturing company produces x items at the total cost of ₹(180 + 4x). The demand function for this product is P = (240 – x). Find x for which
(i) revenue is increasing
(ii) profit is increasing.
Solution:
(i) Let R be the total revenue.
Then R = P.x = (240 – x)x
∴ R = 240x – x^2
∴ $\frac{dR}{dD} = \frac{d}{dD}$(240x – x^2)
= 240 × 1 – 2x
= 240 – 2x
R is increasing, if $\frac{dR}{dx}$ > 0
i.e. if 240 – 2x > 0
i.e. if 240 > 2x
i.e. if x < 120
Hence, the revenue is increasing, if x < 120.

(ii) Profit π = R – C
∴ π = (240x – x^2) – (180 + 4x)
= 240x – x^2 – 180 – 4x
= 236x – x^2 – 180
∴ $\frac{d\pi}{dx} = \frac{d}{dx}$(236x – x^2 – 180)

= 236 × 1 – 2x – 0
= 236 – 2x
Profit is increasing, if $\frac{d\pi}{dx} > 0$
i.e. if 236 – 2x > 0
i.e. if 236 > 2x
i.e. if x < 118
Hence, the profit is increasing, if x < 118.

Question 5.
For manufacturing x units, labour cost is 150 – 54x and processing cost is x^2. Price of each unit is p = 10800 – $4x^2$. Find the values of x for which
(i) total cost is decreasing
(ii) revenue is increasing.
Solution:
(i) Total cost C = labour cost + processing cost
∴ C = 150 – 54x + x^2
∴ $\frac{dC}{dx} = \frac{d}{dx}(150 - 54x + x^2)$
= 0 – 54 × 1 + 2x
= -54 + 2x
The total cost is decreasing, if $\frac{dC}{dx} < 0$
i.e. if -54 + 2x < 0
i.e. if 2x < 54
i.e. if x < 27
Hence, the total cost is decreasing, if x < 27.

(ii) The total revenue R is given as
R = p.x
R = (10800 – $4x^2$) x
R = 10800x – $4x^3$
∴ $\frac{dR}{dx} = \frac{d}{dx}(10800x - 4x^3)$
= 10800 × 1 – 4 × $3x^2$
= 10800 – $12x^2$
The revenue is increasing, if $\frac{dR}{dx} > 0$
i.e. if 10800 – $12x^2$ > 0
i.e. if 10800 > $12x^2$
i.e. if x^2 < 900
i.e. if x < 30 ……[∵ x > 0]
Hence, the revenue is increasing, if x < 30.

Question 6.
The total cost of manufacturing x articles is C = 47x + $300x^2$ – x^4. Find x, for which average cost is
(i) increasing
(ii) decreasing.
Solution:
The total cost is given as C = 47x + $300x^2$ – x^4
∴ the average cost is given by

$$C_A = \frac{C}{x} = \frac{47x + 300x^2 - x^4}{x}$$

$$\therefore\ C_A = 47 + 300x - x^3$$

$$\therefore\ \frac{dC_A}{dx} = \frac{d}{dx}(47 + 300x - x^3)$$

$$= 0 + 300 \times 1 - 3x^2 = 300 - 3x^2$$

(i) C_A is increasing, if $\frac{dC_A}{dx} > 0$
i.e. if 300 – $3x^2$ > 0
i.e. if 300 > $3x^2$
i.e. if x^2 < 100
i.e. if x < 10 …..[∵ x > 0]
Hence, the average cost is increasing, if x < 10.

(ii) C_A is decreasing, if $\frac{dC_A}{dx} < 0$
i.e. if 300 – $3x^2$ < 0
i.e. if 300 < $3x^2$
i.e. if x^2 > 100
i.e. if x > 10 ……[∵ x > 0]
Hence, the average cost is decreasing, if x > 10.

Question 7.
(i) Find the marginal revenue, if the average revenue is 45 and the elasticity of demand is 5.
Solution:
Given $R_A = 45$ and $\eta = 5$
Now, $R_m = R_A\left(1-\frac{1}{\eta}\right)$
$= 45\left(1 - \frac{1}{5}\right)$
$= 45\left(\frac{4}{5}\right)$
$= 36$
Hence, the marginal revenue = 36.

(ii) Find the price, if the marginal revenue is 28 and elasticity of demand is 3.
Solution:
Given $R_m = 28$ and $\eta = 3$

$$\text{Now, } R_m = R_A\left(1-\frac{1}{\eta}\right)$$

$$\therefore\ 28 = R_A\left(1-\frac{1}{3}\right) = \frac{2}{3}R_A$$

$$\therefore\ R_A = \frac{28 \times 3}{2} = 42$$

Hence, the price = 42.

(iii) Find the elasticity of demand, if the marginal revenue is 50 and price is ₹ 75.
Solution:
Given $R_m = 50$ and $R_A = 75$

$$\text{Now, } R_m = R_A\left(1-\frac{1}{\eta}\right)$$

$$\therefore\ 50 = 75\left(1-\frac{1}{\eta}\right) \qquad \therefore\ 1-\frac{1}{\eta} = \frac{50}{75} = \frac{2}{3}$$

$$\therefore\ \frac{1}{\eta} = 1 - \frac{2}{3} = \frac{1}{3} \qquad \therefore\ \eta = 3$$

Hence, the elasticity of demand = 3.

Question 8.
If the demand function is $D = \frac{p+6}{p-3}$, find the elasticity of demand at p = 4.
Solution:
The demand function is

$$D = \frac{p+6}{p-3}$$

$$\therefore\ \frac{dD}{dp} = \frac{d}{dp}\left(\frac{p+6}{p-3}\right)$$

$$= \frac{(p-3)\frac{d}{dp}(p+6) - (p+6)\frac{d}{dp}(p-3)}{(p-3)^2}$$

$$= \frac{(p-3)(1+0) - (p+6)(1-0)}{(p-3)^2}$$

$$= \frac{p-3-p-6}{(p-3)^2} = \frac{-9}{(p-3)^2}$$

Elasticity of demand is given by

$$\eta = \frac{-p}{D} \cdot \frac{dD}{dp}$$

$$= \frac{-p}{\left(\frac{p+6}{p-3}\right)} \times \frac{-9}{(p-3)^2}$$

$$= \frac{9p}{(p+6)(p-3)}$$

When $p = 4$, then

$$\eta = \frac{9(4)}{(4+6)(4-3)} = \frac{36}{10 \times 1} = 3.6.$$

Hence, the elasticity of demand at p = 4 is 3.6

Question 9.
Find the price for the demand function D = $\frac{2p+3}{3p-1}$, when elasticity of demand is $\frac{11}{14}$.
Solution:
The demand function is

$$D = \frac{2p+3}{3p-1}$$

$$\therefore \frac{dD}{dp} = \frac{d}{dp}\left(\frac{2p+3}{3p-1}\right)$$

$$= \frac{(3p-1)\frac{d}{dp}(2p+3) - (2p+3)\frac{d}{dp}(3p-1)}{(3p-1)^2}$$

$$= \frac{(3p-1)(2 \times 1 + 0) - (2p+3)(3 \times 1 - 0)}{(3p-1)^2}$$

$$= \frac{6p - 2 - 6p - 9}{(3p-1)^2} = \frac{-11}{(3p-1)^2}$$

Elasticity of demand is given by

$$\eta = -\frac{p}{D} \cdot \frac{dD}{dp}$$

$$= \frac{-p}{\left(\frac{2p+3}{3p-1}\right)} \times \frac{-11}{(3p-1)^2}$$

$$= \frac{11p}{(2p+3)(3p-1)} = \frac{11p}{6p^2 + 7p - 3}$$

If $\eta = \frac{11}{14}$, then

$$\frac{11}{14} = \frac{11p}{6p^2 + 7p - 3}$$

$$\therefore 66p^2 + 77p - 33 = 154p$$

$$\therefore 66p^2 - 77p - 33 = 0$$

$$\therefore 6p^2 - 7p - 3 = 0$$

$$\therefore (2p-3)(3p+1) = 0$$

$$\therefore 2p - 3 = 0 \quad \ldots [\because p \geqslant 0]$$

$$\therefore p = \frac{3}{2}.$$

Question 10.
If the demand function is D = 50 – 3p – p^2 elasticity of demand at (i) p = 5 (ii) p = 2. Comment on the result.
Solution:
The demand function is D = 50 – 3p – p^2

∴ $\frac{dD}{dp}=\frac{d}{dp}(50 - 3p - p^2)$
= 0 – 3 × 1 – 2p
= -3 – 2p
Elasticity of demand is given by

$$\eta = -\frac{p}{D}\cdot\frac{dD}{dp}$$

$$= \frac{-p}{(50-3p-p^2)} \times (-3-2p)$$

$$= \frac{p(3+2p)}{50-3p-p^2}$$

(i) When p = 5, then

$$\eta = \frac{5(3+2\times 5)}{50-3(5)-(5)^2} = \frac{5\times 13}{50-15-25}$$

$$= \frac{65}{10} = 6.5.$$

Since, η >1, the demand is elastic.
(ii) When p = 2, then

$$\eta = \frac{2(3+2\times 2)}{50-3(2)-(2)^2} = \frac{2\times 7}{50-6-4}$$

$$= \frac{14}{40} = \frac{7}{20}.$$

Since, 0 < η < 1, the demand is inelastic.

Question 11.
For the demand function D = 100 – $\frac{p^2}{2}$, find the elasticity of demand at (i) p = 10 (ii) p = 6 and comment on the results.
Solution:
The demand function is

$$D = 100 - \frac{p^2}{2}$$

$$\therefore \frac{dD}{dp} = \frac{d}{dp}\left(100 - \frac{p^2}{2}\right)$$

$$= 0 - \frac{1}{2}\times 2p = -p$$

The elasticity of demand is given by

$$\eta = \frac{-p}{D}\cdot\frac{dD}{dp}$$

$$= \frac{-p}{\left(100-\frac{p^2}{2}\right)} \times (-p) = \frac{p^2}{\left(100-\frac{p^2}{2}\right)}$$

(i) When p = 10, then

$$\eta = \frac{(10)^2}{100-\frac{(10)^2}{2}} = \frac{100}{100-50}$$

$$= \frac{100}{50} = 2$$

Since, η > 1, the demand is elastic.
(ii) When p = 6, then

$$\eta = \frac{(6)^2}{100-\frac{(6)^2}{2}} = \frac{36}{100-18}$$

$$= \frac{36}{82} = \frac{18}{41}$$

Since, 0 < η < 1, the demand is inelastic.

Question 12.
A manufacturing company produces, x items at a total cost of ₹(40 + 2x). Their price is given as p = 120 – x. Find the value of x for which

(i) revenue is increasing
(ii) profit is increasing
(iii) Also find an elasticity of demand for price 80.
Solution:
(i) The total revenue R is given by
R = p.x = (120 – x)x
∴ $R = 120x - x^2$
∴ $\frac{dR}{dx} = \frac{d}{dx}(120x - x^2)$
= 120 × 1 – 2x
= 120 – 2x
If the revenue is increasing, then $\frac{dR}{dx} > 0$
∴ 120 – 2x > 0
∴ 120 > 2x
∴ x < 60
Hence, the revenue is increasing when x < 60.

(ii) Profit π = R – C
$= (120x - x^2) - (40 + 2x)$
$= 120x - x^2 - 40 - 2x$
$= 118x - x^2 - 40$
∴ $\frac{d\pi}{dx} = \frac{d}{dx}(118x - x^2 - 40)$
= 118 × 1 – 2x – 0
= 118 – 2x
If the profit is increasing, then $\frac{d\pi}{dx} > 0$
∴ 118 – 2x > 0
∴ 118 > 2x
∴ x < 59
Hence, the profit is increasing when x < 59.

(iii) p = 120 – x
∴ x = 120 – p
∴ $\frac{dx}{dp} = \frac{d}{dp}(120 - p)$
= 0 – 1
= -1
Elasticity of demand is given by

$$\eta = \frac{-p}{x} \cdot \frac{dx}{dp}$$

$$= \frac{-p}{120 - p} \times (-1) = \frac{p}{120 - p}$$

When $p = 80$, then

$$\eta = \frac{80}{120 - 80} = \frac{80}{40} = 2.$$

Question 13.
Find MPC, MPS, APC and APS, if the expenditure E_c of a person with income I is given as
$E_c = (0.0003)I^2 + (0.075)I$, when I = 1000.
Solution:
$E_c = (0.0003)I^2 + (0.075)I$
MPC = $\frac{dE_c}{dI} = \frac{d}{dI}[(0.0003)I^2 + (0.075)I]$
= (0.0003)(2I) + (0.075)(1)
= (0.0006)I + 0.075
When I = 1000, then
MPC = (0.0006)(1000) + 0.075
= 0.6 + 0.075
= 0.675.
∴ MPC + MPS = 1
∴ 0.675 + MPS = 1
∴ MPS = 1 – 0.675 = 0.325
Now, APC = $\frac{E_c}{I} = \frac{(0.0003)I^2 + (0.075)I}{I}$
= (0.0003)I + (0.075)
When I = 1000, then
APC = (0.0003)(1000) + 0.075
= 0.3 + 0.075
= 0.375

∵ APC + APS = 1
∴ 0.375 + APS = 1
∴ APS = 1 – 0.375 = 0.625
Hence, MPC = 0.675, MPS = 0.325, APC = 0.375, APS = 0.625.

Maharashtra State Board 12th Commerce Maths Solutions Chapter 4 Applications of Derivatives Miscellaneous Exercise 4

(I) Choose the correct alternative:

Question 1.
The equation of tangent to the curve $y = x^2 + 4x + 1$ at (-1, -2) is
(a) $2x – y = 0$
(b) $2x + y – 5 = 0$
(c) $2x – y – 1 = 0$
(d) $x + y – 1 = 0$
Answer:
(a) $2x – y = 0$

Question 2.
The equation of tangent to the curve $x^2 + y^2 = 5$, where the tangent is parallel to the line $2x – y + 1 = 0$ are
(a) $2x – y + 5 = 0$; $2x – y – 5 = 0$
(b) $2x + y + 5 = 0$; $2x + y – 5 = 0$
(c) $x – 2y + 5 = 0$; $x – 2y – 5 = 0$
(d) $x + 2y + 5$; $x + 2y – 5 = 0$
Answer:
(a) $2x – y + 5 = 0$; $2x – y – 5 = 0$

Question 3.
If the elasticity of demand η = 1, then demand is
(a) constant
(b) inelastic
(c) unitary elastic
(d) elastic
Answer:
(c) unitary elastic

Question 4.
If 0 < η < 1, then the demand is
(a) constant
(b) inelastic
(c) unitary elastic
(d) elastic
Answer:
(b) inelastic

Question 5.
The function $f(x) = x^3 – 3x^2 + 3x – 100$, $x \in R$ is
(a) increasing for all $x \in R$, $x \neq 1$
(b) decreasing
(c) neither increasing nor decreasing
(d) decreasing for all $x \in R$, $x \neq 1$
Answer:
(a) increasing for all $x \in R$, $x \neq 1$

Question 6.
If $f(x) = 3x^3 – 9x^2 – 27x + 15$, then
(a) f has maximum value 66

(b) f has minimum value 30
(c) f has maxima at x = -1
(d) f has minima at x = -1
Answer:
(c) f has maxima at x = -1

(II) Fill in the blanks:

Question 1.
The slope of tangent at any point (a, b) is called as __________
Answer:
gradient

Question 2.
If f(x) = $x^3 – 3x^2 + 3x – 100$, x ∈ R, then f"(x) is __________
Answer:
6x – 6 = 6(x – 1)

Question 3.
If f(x) = $7x$ – 3, x ∈ R, x ≠ 0, then f"(x) is __________
Answer:
$14x^{-3}$

Question 4.
A rod of 108 m in length is bent to form a rectangle. If area j at the rectangle is maximum, then its dimensions are __________
Answer:
27 and 27

Question 5.
If f(x) = x . log x, then its maximum value is __________
Answer:
$-1e$

(III) State whether each of the following is True or False:

Question 1.
The equation of tangent to the curve y = $4xe^x$ at (-1, $-4e$) is y.e + 4 = 0.
Answer:
True

Question 2.
x + 10y + 21 = 0 is the equation of normal to the curve y = $3x^2 + 4x – 5$ at (1, 2).
Answer:
False

Question 3.
An absolute maximum must occur at a critical point or at an endpoint.
Answer:
True

Question 4.
The function f(x) = $x.e^{x(1-x)}$ is increasing on (-12, 1).
Answer:
True.
Hint:

$$f(x) = xe^{x(1-x)} = xe^{x-x^2}$$

$$\therefore\ f'(x) = x \cdot \frac{d}{dx}(e^{x-x^2}) + e^{x-x^2} \cdot \frac{d}{dx}(x)$$

$$= x \cdot e^{x-x^2} \cdot \frac{d}{dx}(x - x^2) + e^{x-x^2} \times 1$$

$$= x \cdot e^{x-x^2} \times (1 - 2x) + e^{x-x^2}$$

$$= e^{x-x^2}(x - 2x^2 + 1)$$

$$= -2e^{x-x^2}\left(x^2 - \frac{1}{2}x - \frac{1}{2}\right)$$

$$= -2e^{x-x^2}\left[\left(x^2 - \frac{1}{2}x + \frac{1}{16}\right) - \frac{1}{2} - \frac{1}{16}\right]$$

$$\therefore\ f'(x) = -2e^{x-x^2}\left[\left(x - \frac{1}{4}\right)^2 - \frac{9}{16}\right] \quad \ldots (1)$$

Now, $x \in \left(-\frac{1}{2}, 1\right)$ $\quad \therefore\ -\frac{1}{2} < x < 1$

$$\therefore\ -\frac{1}{2} - \frac{1}{4} < x - \frac{1}{4} < 1 - \frac{1}{4} = \frac{3}{4}$$

$$\therefore\ 0 < \left(x - \frac{1}{4}\right)^2 < \frac{9}{16}$$

$$\therefore\ \left(x - \frac{1}{4}\right)^2 - \frac{9}{16} < 0 \quad \ldots (2)$$

Also, $2e^{x-x^2} > 0$ for $x \in \left(-\frac{1}{2}, 1\right)$

$$\therefore\ -2e^{x-x^2} < 0 \quad \ldots (3)$$

From (2) and (3),

$$-2e^{x-x^2}\left[\left(x - \frac{1}{4}\right)^2 - \frac{9}{16}\right] > 0$$

$$\therefore\ f'(x) > 0 \quad \ldots [\text{By } (1)]$$

Hence, function f(x) is increasing on (-12, 1).

(IV) Solve the following:

Question 1.
Find the equations of tangent and normal to the following curves:
(i) xy = c2 at (ct, ct), where t is a parameter.
Solution:
xy = c2
Differentiating both sides w.r.t. x, we get

$$x\frac{dy}{dx} + y \cdot \frac{d}{dx}(x) = 0$$

$$\therefore\ x\frac{dy}{dx} + y \times 1 = 0$$

$$\therefore\ x\frac{dy}{dx} = -y \qquad \therefore\ \frac{dy}{dx} = -\frac{y}{x}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at}\left(ct, \frac{c}{t}\right)} = -\frac{\left(\frac{c}{t}\right)}{ct} = -\frac{1}{t^2}$$

$$= \text{slope of the tangent at } \left(ct, \frac{c}{t}\right)$$

$\therefore$ the equation of the tangent at $\left(ct, \frac{c}{t}\right)$ is

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$\therefore\ t^2y - ct = -x + ct$

$\therefore\ x + t^2y - 2ct = 0$

The slope of the normal at $\left(ct, \frac{c}{t}\right)$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at } \left(ct, \frac{c}{t}\right)}} = \frac{-1}{\left(-\frac{1}{t^2}\right)} = t^2$$

$\therefore$ the equation of the normat at $\left(ct, \frac{c}{t}\right)$ is

$$y - \frac{c}{t} = t^2(x - ct)$$

$\therefore\ ty - c = t^3x - ct^4$

$\therefore\ t^3x - ty - c(t^4 - 1) = 0$

Hence, equations of tangent and normal are $x + t^2y - 2ct = 0$ and $t^3x - ty - c(t^4 + 1) = 0$ respectively.

(ii) $y = x^2 + 4x$ at the point whose ordinate is -3.
Solution:
Let $P(x_1, y_1)$ be the point on the curve
$y = x^2 + 4x$, where $y_1 = -3$

$\therefore\ y_1 = x_1{}^2 + 4x_1$

$\therefore\ x_1{}^2 + 4x_1 = -3$

$\therefore\ x_1{}^2 + 4x_1 + 3 = 0$

$\therefore\ (x_1 + 3)(x_1 + 1) = 0$

$\therefore\ x_1 = -3$ or $x_1 = -1$

$\therefore$ coordinates of the points are $(-3, -3)$ and $(-1, -3)$.

Differentiating $y = x^2 + 4x$ w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 4x) = 2x + 4$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at } (-3,\ -3)} = 2(-3) + 4 = -2$$

$= $ slope of the tangent at $(-3, -3)$

$\therefore$ equation of the tangent at $(-3, -3)$ is

$y - (-3) = -2[x - (-3)]$

$\therefore\ y + 3 = -2x - 6$

$\therefore\ 2x + y + 9 = 0$

The slope of the normal at $(-3, -3)$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at } (-3,\ -3)}} = \frac{-1}{-2} = \frac{1}{2}.$$

$\therefore$ the equation of the normal at $(-3, -3)$ is

$$y - (-3) = \frac{1}{2}[x - (-3)]$$

$\therefore\ 2y + 6 = x + 3$

$\therefore\ x - 2y - 3 = 0$

$$\left(\frac{dy}{dx}\right)_{\text{at } (-1,\ -3)} = 2(-1) + 4 = 2$$

$= \text{slope of the tangent at } (-1, -3)$

$\therefore$ the equation of the tangent at $(-1, -3)$ is

$$y - (-3) = 2[x - (-1)]$$

$\therefore\ y + 3 = 2x + 2$

$\therefore\ 2x - y - 1 = 0$

The slope of the normal at $(-1, -3)$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at } (-1,\ -3)}} = -\frac{1}{2}$$

The equation of the normal at $(-1, -3)$ is

$$y - (-3) = -\frac{1}{2}[x - (-1)]$$

$\therefore\ 2y + 6 = -x - 1$

$\therefore\ x + 2y + 7 = 0$

Hence, the equations of tangent and normal at
(i) (-3, -3) are 2x + y + 9 = 0 and x – 2y – 3 = 0
(ii) (-1, -3) are 2x – y – 1 = 0 and x + 2y + 7 = 0

(iii) x = 1t, y = t – 1t, at t = 2.
Solution:
When t = 2, x = 12 and y = 2 – 12 = 32
Hence, the point P at which we want to find the equations of tangent and normal is (12, 32)

Now, $x = \frac{1}{t}$, $y = t - \frac{1}{t}$

Differentiating x and y w.r.t. t, we get

$$\frac{dx}{dt} = \frac{d}{dt}\left(\frac{1}{t}\right) = -\frac{1}{t^2}$$

and $\frac{dy}{dt} = \frac{d}{dt}\left(t - \frac{1}{t}\right) = 1 - \left(-\frac{1}{t^2}\right) = 1 + \frac{1}{t^2} = \frac{t^2+1}{t^2}$

$$\therefore \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = = \frac{\left(\frac{t^2+1}{t^2}\right)}{\left(-\frac{1}{t^2}\right)} = -(t^2+1)$$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } t=2} = -(4+1) = -5$$

$= \text{slope of the tangent at } t = 2$

$\therefore$ the equation of the tangent at $\left(\frac{1}{2}, \frac{3}{2}\right)$ is

$$y - \frac{3}{2} = -5\left(x - \frac{1}{2}\right)$$

$$\therefore y - \frac{3}{2} = -5x + \frac{5}{2}$$

$$\therefore 5x + y - 4 = 0$$

The slope of the normal at $t = 2$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at } t=2}} = \frac{-1}{-5} = \frac{1}{5}$$

$\therefore$ the equation of the normal at $\left(\frac{1}{2}, \frac{3}{2}\right)$ is

$$y - \frac{3}{2} = \frac{1}{5}\left(x - \frac{1}{2}\right)$$

$$\therefore 5y - \frac{15}{2} = x - \frac{1}{2}$$

$$\therefore x - 5y + 7 = 0$$

Hence, the equations of tangent and normal are $5x + y - 4 = 0$ and $x - 5y + 7 = 0$ respectively.

(iv) $y = x^3 - x^2 - 1$ at the point whose abscissa is -2.
Solution:
$y = x^3 - x^2 - 1$
$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^3 - x^2 - 1)$
$= 3x^2 - 2x - 0$
$= 3x^2 - 2x$
$\therefore \left(\frac{dy}{dx}\right)_{\text{at } x=-2} = 3(-2)^2 - 2(-2) = 16$
= slope of the tangent at $x = -2$
When $x = -2$, $y = (-2)^3 - (-2)^2 - 1 = -13$
$\therefore$ the point P is $(-2, -13)$
$\therefore$ the equation of the tangent at $(-2, -13)$ is
$y - (-13) = 16[x - (-2)]$
$\therefore y + 13 = 16x + 32$
$\therefore 16x - y + 19 = 0$
The slope of the normal at $x = -2$
$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at } x=-2}} = \frac{-1}{16}$
$\therefore$ the equation of the normal at $(-2, -13)$ is
$y - (-13) = \frac{-1}{16}[x - (-2)]$
$\therefore 16y + 208 = -x - 2$
$\therefore x + 16y + 210 = 0$
Hence, equations of tangent and normal are $16x - y + 19 = 0$ and $x + 16y + 210 = 0$ respectively.

Question 2.
Find the equation of the normal to the curve y = x−3−−−−−√ which is perpendicular to the line 6x + 3y – 4 = 0.
Solution:
Let P(x_1, y_1) be the foot of the required normal to the curve y = x−3−−−−−√
Differentiating y = x−3−−−−−√ w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}(\sqrt{x}-3)=\frac{1}{2\sqrt{x-3}}\cdot\frac{d}{dx}(x-3)$$

$$=\frac{1}{2\sqrt{x-3}}\times(1-0)=\frac{1}{2\sqrt{x-3}}$$

$$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at }(x_1,\ y_1)}=\frac{1}{2\sqrt{x_1-3}}$$

= slope of the tangent at $P(x_1, y_1)$

$\therefore$ slope of the normal at $P(x_1, y_1)$

$$=m_1=\frac{-1}{\left(\frac{dy}{dx}\right)_{\text{at }(x_1,\ y_1)}}=-2\sqrt{x_1-3}$$

The slope of the line $6x+3y-4=0$ is

$$m_2=\frac{-6}{3}=-2$$

Since, the normal at $P(x_1, y_1)$ is perpendicular to the line

$6x+3y-4=0$, $m_1\cdot m_2=-1$, i.e. $m_1=\frac{-1}{m_2}$

$$\therefore\ -2\sqrt{x_1-3}=\frac{-1}{-2}=\frac{1}{2}$$

$$\therefore\ x_1-3=\frac{1}{16}\qquad\therefore\ x_1=\frac{49}{16}$$

Since, (x_1, y_1) lies on the curve $y=\sqrt{x-3}$

$$\therefore\ y_1=\sqrt{x_1-3}$$

$$\therefore\ y_1=\sqrt{\frac{49}{16}-3}=\pm\frac{1}{4}$$

$\therefore$ the coordinates of the point P are $\left(\frac{49}{16},\frac{1}{4}\right)$ or $\left(\frac{49}{16},-\frac{1}{4}\right)$

and the slopes of the normal is $m_1=-\frac{1}{m_2}=\frac{1}{2}$.

$\therefore$ the equation of the normal at $\left(\frac{49}{16},\frac{1}{4}\right)$ is

$$y-\frac{1}{4}=\frac{1}{2}\left(x-\frac{49}{16}\right)$$

$$\therefore\ 2y-\frac{1}{2}=x-\frac{49}{16}$$

$$\therefore\ x-2y-\frac{41}{16}=0$$

$$\therefore\ 16x-32y-41=0$$

and the equation of the normal at $\left(\frac{49}{16},-\frac{1}{4}\right)$ is

$$y-\left(-\frac{1}{4}\right)=\frac{1}{2}\left(x-\frac{49}{16}\right)$$

$$\therefore\ 2y+\frac{1}{2}=x-\frac{49}{16}$$

∴ $x - 2y - \frac{57}{16} = 0$
i.e. $16x - 32y - 57 = 0$
Hence, the equation of the normals are $16x - 32y - 41 = 0$ and $16x - 32y - 57 = 0$.

Question 3.
Show that the function $f(x) = \frac{x-2}{x+1}$, $x \neq -1$ is increasing.
Solution:
$f(x) = \frac{x-2}{x+1}$

$$\therefore f'(x) = \frac{d}{dx}\left(\frac{x-2}{x+1}\right) = \frac{(x+1)\cdot\frac{d}{dx}(x-2) - (x-2)\cdot\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$= \frac{(x+1)\cdot(1-0) - (x-2)\cdot(1+0)}{(x+1)^2}$$

$$= \frac{x+1-x+2}{(x+1)^2} = \frac{3}{(x+1)^2}$$

$$\because x \neq -1 \qquad \therefore x+1 \neq 0$$

$$\therefore (x+1)^2 > 0 \qquad \therefore \frac{3}{(x+1)^2} > 0$$

∴ $f'(x) > 0$, for all $x \in R$, $x \neq -1$
Hence, the function f is increasing for all $x \in R$, where $x \neq -1$.

Question 4.
Show that the function $f(x) = \frac{3}{x} + 10$, $x \neq 0$ is decreasing.
Solution:
$f(x) = \frac{3}{x} + 10$

$$\therefore f'(x) = \frac{d}{dx}\left(\frac{3}{x} + 10\right) = 3\left(-\frac{1}{x^2}\right) + 0$$

$$= -\frac{3}{x^2}$$

$$\because x \neq 0 \qquad \therefore x^2 > 0 \qquad \therefore \frac{3}{x^2} > 0$$

$$\therefore -\frac{3}{x^2} < 0$$

∴ $f'(x) < 0$ for all $x \in R$, $x \neq 0$
Hence, the function f is decreasing for all $x \in R$, where $x \neq 0$.

Question 5.
If $x + y = 3$, show that the maximum value of x^2y is 4.
Solution:
$x + y = 3$
∴ $y = 3 - x$
∴ $x^2y = x^2(3 - x) = 3x^2 - x^3$
Let $f(x) = 3x^2 - x^3$
Then $f'(x) = \frac{d}{dx}(3x^2 - x^3)$
$= 3 \times 2x - 3x^2$
$= 6x - 3x^2$
and $f''(x) = \frac{d}{dx}(6x - 3x^2)$
$= 6 \times 1 - 3 \times 2x$
$= 6 - 6x$
Now, $f'(x) = 0$ gives $6x - 3x^2 = 0$
∴ $3x(2 - x) = 0$
∴ $x = 0$ or $x = 2$
$f''(0) = 6 - 0 = 6 > 0$
∴ f has minimum value at $x = 0$
Also, $f''(2) = 6 - 12 = -6 < 0$
∴ f has maximum value at $x = 2$
When $x = 2$, $y = 3 - 2 = 1$
∴ maximum value of $x^2y = (2)^2(1) = 4$.

Question 6.
Examine the function f for maxima and minima, where $f(x) = x^3 - 9x^2 + 24x$.
Solution:
$f(x) = x^3 - 9x^2 + 24x$
$\therefore f'(x) = \frac{d}{dx}(x^3 - 9x^2 + 24x)$
$= 3x^2 - 9 \times 2x + 24 \times 1$
$= 3x^2 - 18x + 24$
and $f''(x) = \frac{d}{dx}(3x^2 - 18x + 24)$
$= 3 \times 2x - 18 \times 1 + 0$
$= 6x - 18$
$f'(x) = 0$ gives $3x^2 - 18x + 24 = 0$
$\therefore x^2 - 6x + 8 = 0$
$\therefore (x - 2)(x - 4) = 0$
$\therefore$ the roots of $f'(x) = 0$ are $x_1 = 2$ and $x_2 = 4$.
(a) $f''(2) = 6(2) - 18 = -6 < 0$
$\therefore$ by the second derivative test,
f has maximum at x = 2 and maximum value of f at x = 2
$f(2) = (2) - 9(2)^2 + 24(2)$
$= 8 - 36 + 48$
$= 20$
(b) $f''(4) = 6(4) - 18 = 6 > 0$
$\therefore$ by the second derivative test, f has minimum at x = 4
and minimum value of f at x = 4
$f(4) = (4)^3 - 9(4)^2 + 24(4)$
$= 64 - 144 + 96$
$= 16.$