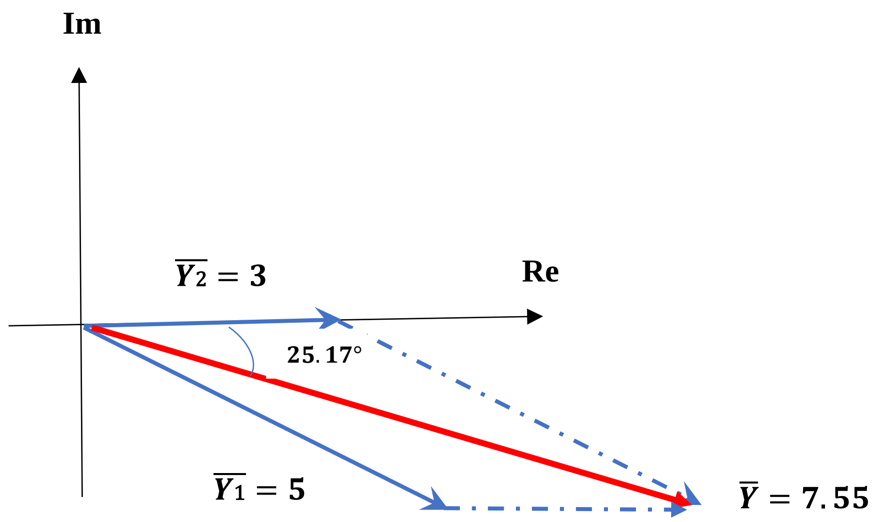
SAMPLE SOLUTION

QUESTION 1 [25 marks]

a. Graphical method

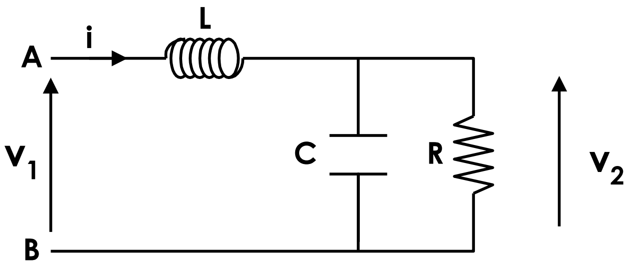
- Choose a scale (e.g. 1cm ≡ 1 volt (or 1 Ampere) depending on the nature of $\bar{Y}_1$ and $\bar{Y}_2$).
- Derive the phasor representation of the signals $Y_1(t)$ and $Y_2(t)$.
$Y_1(t) = 5sin(5t + 50^o) \rightarrow Y_1(t) = 5cos(5t + -40^o) \rightarrow \bar{Y}_1 = 5e^{-j40^o}$
$$Y_2(t) = 3cos(5t) \rightarrow \bar{Y}_2 = 3e^{j0^o} = 3$$
- Graph $\bar{Y}_1$ and $\bar{Y}_2$using the adopted scale and derive $\bar{Y}$, the resultant of $\bar{Y}_1$ and $\bar{Y}_2$.



- Measure both the magnitude and the phase of the resultant and write down its phasor representation.
$$\bar{Y} = 7.55\ e^{-j25.17°}$$
- Derive Y(t) from the phasor representation (remember that ω = 5 rad/sec).
$\bar{Y} = 7.55\ e^{-j25.17°} \rightarrow Y(t) = 7.55cos(5t - 25.17°)$

[8 marks]

b. Transfer Function



Using the voltage divider rule:

$$v_2 = \frac{Z_{RC}}{Z_{RC} + Z_L} x\ v_1$$

Where Z_{RC} is the equivalent resistance of the circuit elements Z_R *and* Z_C. Note that Z_R *and* Z_C are in parallel.

$$\frac{1}{Z_{RC}} = \frac{1}{Z_R} + \frac{1}{Z_C} \rightarrow \frac{1}{Z_{RC}} = \frac{1}{R} + Cs = \frac{1 + RCs}{R} \rightarrow Z_{RC} = \frac{R}{1 + RCs}$$

In the Laplace domain

$$V_2(s) = \frac{Z_{RC}}{Z_{RC} + Z_L}\, x\, V_1(s) = \frac{\frac{R}{1 + RCs}}{\frac{R}{1 + RCs} + Ls}\, x\, V_1(s) = \frac{\frac{R}{1 + RCs}}{\frac{R + Ls(1 + RCs)}{1 + RCs}}\, x\, V_1(s)$$

$$= \frac{\frac{R}{1 + RCs}}{\frac{RLCs^2 + Ls + R}{1 + RCs}}\, x\, V_1(s)$$

Simplifying the terms (1+RCs) and then dividing both the numerator and denominator by RLC leads to:

$$V_2(s) = \frac{\frac{R}{1 + RCs}}{\frac{RLCs^2 + Ls + R}{1 + RCs}}\, x\, V_1(s) = \frac{R}{RLCs^2 + Ls + R}\, x\, V_1(s) = \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}\, x\, V_1(s)$$

Finally

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

[8 marks]

$$H(s) = \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}} \equiv \frac{K\omega_n^2}{s^2 + 2\zeta\, \omega_n s + \omega_n^2}$$

Knowing that R = 2.5 Ω, L = 1.25 H and C = 0.1 F. Mapping the two expressions leads to:

$$K = 1$$

$$\omega_n^2 = \frac{1}{LC} \rightarrow \omega_n = \sqrt{\frac{1}{LC}} = \sqrt{8} = 2.83\, rads/sec$$

$$2\zeta\, \omega_n = \frac{1}{RC} \rightarrow \zeta = \frac{1}{2\omega_n} x \frac{1}{RC} = 0.707$$

[3 marks]

c. $v_1(t) = 3\sin(2t + 35^o)$ ➔ ω = 2 rads/sec

To find the gain and the phase shift of the circuit, we will have to find the frequency response of the circuit. To do that, we will use the Laplace transfer function and replace s by jω.

Knowing that R = 2.5 Ω, L = 1.25 H and C = 0.1 F. The transfer function is then:

$$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{8}{s^2 + 4s + 8}$$

The frequency response is then

$$H(j\omega\ ) = \frac{V_2(j\omega\ )}{V_1(j\omega\ )} = \frac{8}{8 - \omega^2 + j4\omega}$$

The gain

$$|H(\omega\ )| = \frac{8}{\sqrt{(8-\omega^2)^2 + (4\omega)^2}}$$

[2 marks]

The phase

$$\angle H(\omega\ ) = -tan^{-1}\left(\frac{4\omega}{8-\omega^2}\right)$$

[2 marks]

Since ω = 2 rads/sec, then the gain and the phase are:

$$|H(\omega\ )| = \frac{8}{\sqrt{(8-\omega^2)^2 + (4\omega)^2}} = \frac{8}{\sqrt{(4)^2 + (8)^2}} = \frac{8}{\sqrt{80}} = 0.894$$

$$\angle H(\omega\ ) = -tan^{-1}\left(\frac{4\omega}{8-\omega^2}\right) = -tan^{-1}\left(\frac{4x2}{8-4}\right) = -tan^{-1}(2) = -63.43$$

Finally

$v_2(t)$ = 3x0.894 sin(2t + 35-63.43^o)= 2.682 sin(2t + -28.43^o)

[2 marks]

QUESTION 2 [25 marks]

a. Normalised form

$H(s) = \frac{30(s+10)}{(s+1)(s+20)} = \frac{30x10(\frac{s}{10}+1)}{1x20(\frac{s}{1}+1)(\frac{s}{20}+1)} = \frac{15(\frac{s}{10}+1)}{(\frac{s}{1}+1)(\frac{s}{20}+1)}$ **[2 marks]**

b. Gain and Phase

The frequency response is defined by (replace s by jω in the transfer function):

$$H(j\omega\ ) = \frac{15(\frac{j\omega}{10}+1)}{(\frac{j\omega}{1}+1)(\frac{j\omega}{20}+1)}$$

The gain

$$|H(j\omega\ )| = \frac{|15|x\left|\frac{j\omega}{10}+1\right|}{\left|\frac{j\omega}{1}+1\right|x\left|\frac{j\omega}{20}+1\right|} = \frac{15x\sqrt{\left(\frac{\omega}{10}\right)^2+1^2}}{\sqrt{\left(\frac{\omega}{1}\right)^2+1^2}x\sqrt{\left(\frac{\omega}{20}\right)^2+1^2}}$$

The gain in dB.

$$|H(j\omega\ )| = 20log(15) + 20log\sqrt{\left(\frac{\omega}{10}\right)^2+1} - 20log\sqrt{\left(\frac{\omega}{1}\right)^2+1} - 20log\sqrt{\left(\frac{\omega}{20}\right)^2+1}$$

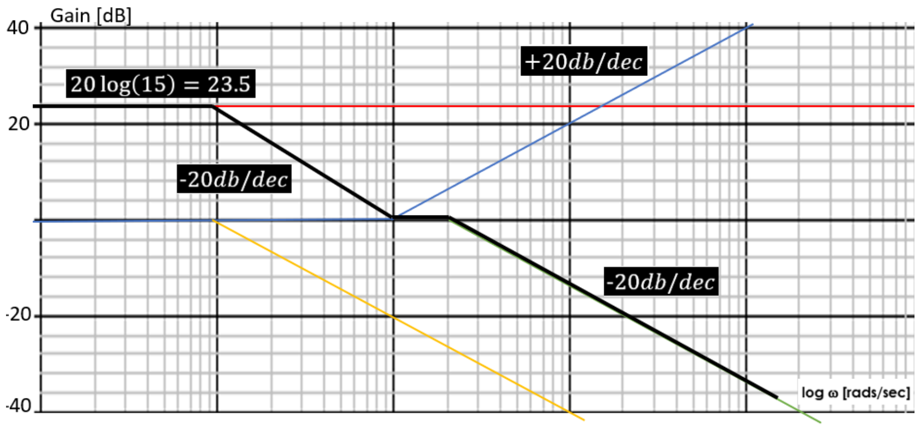
[3 marks]

The phase

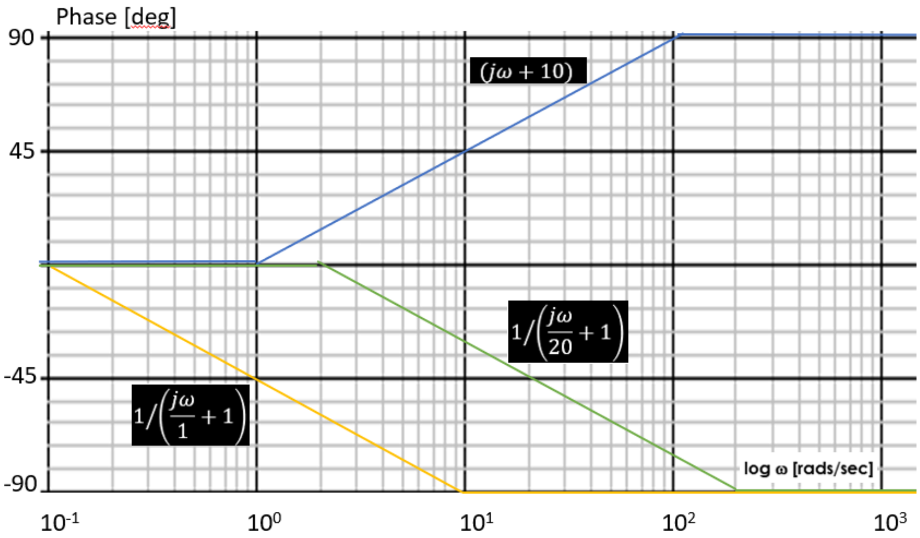
$$\angle H(j\omega\ ) = \angle 15 + \angle\left(\frac{j\omega}{10}+1\right) - \angle\left(\frac{j\omega}{1}+1\right) - \angle\left(\frac{j\omega}{20}+1\right)$$
$$= tan^{-1}\left(\frac{\omega}{10}\right) - tan^{-1}(\omega) - tan^{-1}\left(\frac{\omega}{20}\right)$$

[2 marks]

c. Bode Plots



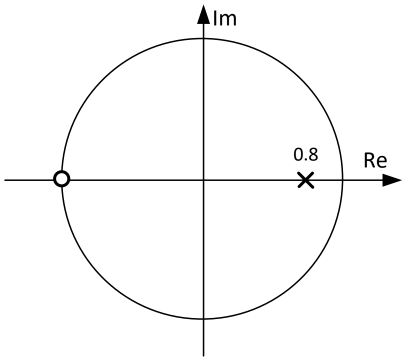
[4 marks]



[4 marks]

d.

i. Pole-zero diagram



[2 marks]

ii. The filter's difference equation

$$H(z) = \frac{Y(z)}{X(z)} = \left(\frac{z+1}{z-0.8}\right) x \frac{z^{-1}}{z^{-1}} = \frac{1+z^{-1}}{1-0.8z^{-1}}$$

[1 mark]

Cross multiplying

$$Y(z)(1-0.8z^{-1}) = X(z)(1+z^{-1})$$

$$Y(z) - 0.8Y(z)z^{-1} = X(z) + X(z)z^{-1}$$

[1 mark]

The difference equation

$$y[n] - 0.8y[n-1] = x[n] + x[n-1]$$

$$y[n] = 0.8y[n-1] + x[n] + x[n-1]$$

[1 mark]

iii. Magnitude frequency response over the range 0< Ω<π

Frequency response

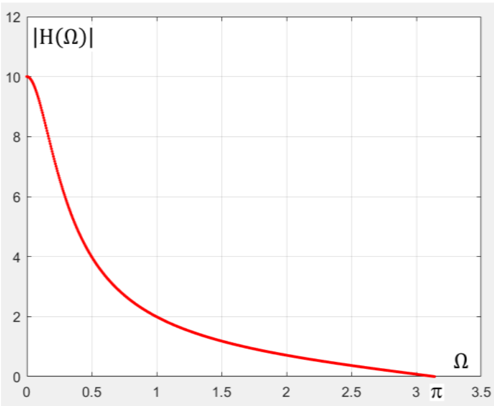
$$H(\Omega) = \frac{e^{j\Omega}+1}{e^{j\Omega}-0.8} = \frac{\cos(\Omega)+j\sin(\Omega)+1}{\cos(\Omega)+j\sin(\Omega)-0.8}$$

[1 mark]

Magnitude frequency response

$$H(\Omega) = \frac{\sqrt{(\cos(\Omega)+1)^2 + \left(\sin(\Omega)\right)^2}}{\sqrt{(\cos(\Omega)-0.8)^2 + \left(\sin(\Omega)\right)^2}} = \frac{\sqrt{2+2\cos(\Omega)}}{\sqrt{1.64-1.6\cos(\Omega)}}$$

[1 mark]



A rough sketch with 4/5 values of Ω should be OK.

[3 marks]

QUESTION 3 [25 marks]

a) Fourier series

The signal x(t) is odd, then $a_0 = a_n = 0$

[2 marks]

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,dt = \frac{2}{T}\int_{\frac{T}{4}}^{0} -A\sin(n\omega_0 t)\,dt + \frac{2}{T}\int_0^{\frac{T}{4}} A\sin(n\omega_0 t)\,dt$$

$$b_n = \frac{2A}{n\pi}\left\{1-\cos\left(\frac{n\pi}{2}\right)\right\}$$

[5 marks]

The 4 first non-zero harmonics

$$b_1 = \frac{2A}{\pi}\left\{1-\cos\left(\frac{\pi}{2}\right)\right\} = \frac{2A}{\pi}$$
$$b_2 = \frac{2A}{2\pi}\left\{1-\cos\left(\frac{2\pi}{2}\right)\right\} = \frac{2A}{\pi}$$
$$b_3 = \frac{2A}{3\pi}\left\{1-\cos\left(\frac{3\pi}{2}\right)\right\} = \frac{2A}{3\pi}$$

[3 marks]

b) $x(t) = a_0 + \sum_{n=1}^{n=3} a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t) = b_1\sin\left(\frac{\pi}{2}t\right) + b_2\sin\left(2\frac{\pi}{2}t\right) + b_3\sin\left(3\frac{\pi}{2}t\right)$ $\left(\omega_0 = \frac{\pi}{2}\right)$

[2 marks]

c) Consider $H(s) = \frac{Y(s)}{X(s)} = \frac{0.25s}{1+0.25s}$

i. Gain and phase

Gain: $|H(\omega)| = \frac{0.25\omega}{\sqrt{1+(0.25\omega)^2}}$ Phase: $\angle H(\omega) = 90 - \tan^{-1}(0.25\omega)$

[3 marks]

ii. The output y(t):

For $\omega = \omega_0 = \frac{\pi}{2}$

Gain: $|\mathrm{H}(\omega)| = \frac{0.25\omega}{\sqrt{1+(0.25\omega)^2}} = 0.365$

Phase: $\angle\mathrm{H}(\omega) = 90 - \tan^{-1}(0.25\omega) = 89.62$

[2 marks]

For $\omega = 2\omega_0 = 2\frac{\pi}{2}$

Gain: $|\mathrm{H}(\omega)| = \frac{0.25\omega}{\sqrt{1+(0.25\omega)^2}} = 0.617$

Phase: $\angle\mathrm{H}(\omega) = 90 - \tan^{-1}(0.25\omega) = 89.33$

[2 marks]

For $\omega = 23\omega_0 = 3\frac{\pi}{2}$

Gain: $|\mathrm{H}(\omega)| = \frac{0.25\omega}{\sqrt{1+(0.25\omega)^2}} = 0.76$

Phase: $\angle\mathrm{H}(\omega) = 90 - \tan^{-1}(0.25\omega) = 89.13$

[2 marks]

$$\mathrm{y(t)} = 0.36\mathrm{b}_1 \sin\left(\frac{\pi}{2}\mathrm{t} + 89.62\right) + 0.62\mathrm{b}_2 \sin\left(2\frac{\pi}{2}\mathrm{t} + 89.33\right) + 0.76\mathrm{b}_3 \sin\left(3\frac{\pi}{2}\mathrm{t} + 89.13\right)$$

[2 marks]

The transfer function is that of a high pass filter having a cut-off frequency of 4 rad/sec. Therefore, it is attenuating harmonics present in x(t).

[2 marks]