UNIT 2: Algebraic and partial fractions

Return to overview

SPECIFICATION REFERENCES

2.6 Simplify rational expressions including by factorising and cancelling, and algebraic division (by linear expressions only)

2.10 Decompose rational functions into partial fractions (denominators not more complicated than squared linear terms and with no more than 3 terms, numerators constant or linear)

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

A4 Algebraic fractions

AS Mathematics – Pure Mathematics content

2.6 Algebraic division, factor theorem (See Unit 3a of the SoW)

KEYWORDS

Polynomial, numerator, denominator, factor, difference of two squares, quadratic, power, index, coefficient, degree, squared, coefficients, improper, identity, algebraic fraction, partial fraction, rational.

NOTES

For algebraic fractions, denominators of rational expressions will be linear or quadratic,
e.g. $\frac{1}{ax+b}$, $\frac{ax+b}{px^2+qx+r}$, $\frac{x^3+a^3}{x^2-a^2}$.
Partial fractions to include denominators such as:
$(ax + b)(cx + d)(ex + f)$ and $(ax + b)(cx + d)^2$.
This work has applications in A level Pure Mathematics topics such as series expansions (Unit 5), differentiation (Unit 8) and integration (Unit11).

2a. Simplifying algebraic fractions (2.6)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to add, subtract, multiply and divide algebraic fractions;
- know how to use the factor theorem to shown a linear expression of the form $(a + bx)$ is a factor of a polynomial;
- know how to use the factor theorem for divisors of the form $(a + bx)$;
- be able to simplify algebraic fractions by fully factorising polynomials up to cubic.

TEACHING POINTS

Revise the basic rules of numerical fractions and start with simplifying some GCSE (9-1) Mathematics algebraic fractions.

Exam questions tend to focus on factorising polynomials and then cancelling common factors to simplify algebraic fractions. For example:

Simplify $\frac{x^2-5x-6}{x^2-10x+24} \div \frac{x^2-x-2}{x^2-4x}$

You can use function notation when referring to fractions. (This has been covered in GCSE (9–1) Mathematics and also links with Unit 3.) For example:

The function f is defined by

$$\mathrm{f}: x \rightarrow \frac{3(x+1)}{2x^2+7x-4} - \frac{1}{x+4}, \quad x \in \mathbb{R}, \ x > \frac{1}{2}$$

Show that $\mathrm{f}(x) = \frac{1}{2x-1}$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

End this section by showing the reverse process where a simplified rational function is split into two (or more) partial factions. This links to the next set of lessons.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students need to practise factorising quadratics as this is often done incorrectly.

The most common errors include failing to include all necessary brackets, casual miswriting of signs part way through calculations and not dealing correctly with factors. Particular care with signs needs to be taken when a fraction follows a minus sign.

NOTES

Students must be able to divide polynomials for use in the partial fractions next.

2b. Partial fractions (2.10)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to split a proper fraction into partial fractions;
- be able to split an improper fraction into partial fractions, dividing the numerator by the denominator (by polynomial long division or by inspection).

TEACHING POINTS

Stress the fact that when we break-up a fraction into two or more partial fractions, we use an identity (≡) sign, and not an equal sign, as the expressions are equivalent for all values of x.

Start with a pair of algebraic fractions that need to be added together. Stress that the single fraction answer may be simplified, but that it can often be difficult to work with. For example in order to integrate the fraction it may be necessary to split it back up into two (or more) partial fractions. In other words, the reverse process from the previous section above needs to be carried out.

The number of partial fractions and the format of the individual terms, is dependent on two factors.

1. The maximum power (or degree) of the polynomials of the numerator and denominator. The degree of the denominator must be *greater* than that of the numerator. If the degree is equal or the degree of the numerator is greater (i.e. the fraction is improper), then algebraic division must be carried out first, and then the partial fractions formed.
2. The type and power of denominator.

   If the denominator is, e.g. $(x + 2)^2$, then we call this a *repeated* factor. In order to cover all possibilities of factors this has to be set up as two partial fractions with denominators $(x + 2)$ and $(x + 2)^2$.

   Show a numerical example with a denominator of 25, and hence the denominators of the partial fractions are 5 and 25.)

Examples of each of the following types need to be covered.

Linear: $\frac{5x-5}{(x+3)(x-2)}$ $\quad \frac{2}{x^2-1}$ $\quad \frac{7x+3}{x(x+1)}$

Repeated: $\frac{4x^2-3x+5}{(x-1)^2(x+2)} \equiv \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{(x+2)}$

Improper: $\frac{2x^2+5x-6}{(2x-1)(1+x)} \equiv A + \frac{B}{2x-1} + \frac{C}{1+x}$

As students work through examples, encourage them to experiment with the choice of values they substitute. If necessary remind them that $x = 0$ is an option. Also show that equating coefficients can sometimes be a more efficient alternative, sometimes avoiding the necessity for simultaneous equations.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Are there any values which make the denominators zero? Make links with the graphs of the functions and talk about how these values will correspond to exceptions and special cases in future topics where partial factions need to be found as a simplifying step.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Some students will set up and solve simultaneous equations rather than using values of x to work out missing constants.

Ensure students are aware of the most efficient methods for solving different types of problem so they do not waste time in exam situations.

NOTES

The specification notes state, 'Denominators not more complicated than squared linear terms and with no more than 3 terms, numerators will be constant or linear'.

This unit has applications in Unit 5 – Series expansions, Unit 8 – Differentiation and Unit 11 – Integration.