UNIT 1: Algebra and Functions

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SPECIFICATION REFERENCES

2.1 Understand and use the laws of indices for all rational exponents

2.2 Use and manipulate surds, including rationalising the denominator

2.3 Work with quadratic functions and their graphs

The discriminant of a quadratic function, including the conditions for real and repeated roots Completing the square

Solution of quadratic equations, including solving quadratic equations in a function of the unknown

2.4 Solve simultaneous equations in two variables by elimination and by substitution, including one linear and one quadratic equation

2.5 Solve linear and quadratic inequalities in a single variable and interpret such inequalities graphically, including inequalities with brackets and fractions

Express solutions through correct use of 'and' and 'or', or through set notation Represent linear and quadratic inequalities such as $y > x + 1$ and $y > ax^2 + bx + c$ graphically

2.6 Manipulate polynomials algebraically, including expanding brackets, collecting like terms and factorisation and simple algebraic division; use of the factor theorem

2.7 Understand and use graphs of functions; sketch curves defined by simple equations including polynomials, $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$ (including their vertical and horizontal asymptotes)

Interpret algebraic solution of equations graphically; use intersection points of graphs to solve equations

2.9 Understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$ including sketching associated graphs:

$y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$, $y = \mathrm{f}(ax)$

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

A4 Collecting like terms and factorising

N8 Surds

A19 Solving linear simultaneous equations

A18 Solving quadratic equations (by factorising and completing the square)

A22 Working with inequalities

Solving quadratic inequalities

A12 Functional notation and shapes of standard graphs (e.g. parabola, cubic, reciprocal)

N7 Rules of indices

KEYWORDS

Expression, function, constant, variable, term, unknown, coefficient, index, linear, identity, simultaneous, elimination, substitution, factorise, completing the square, intersection, change the subject, cross-multiply, power, exponent, base, rational, irrational, reciprocal, root, standard form, surd, rationalise, exact, manipulate, sketch, plot, quadratic, maximum, minimum, turning point, transformation, translation, polynomial, discriminant, real roots, repeated roots, factor theorem, quotient, intercepts, inequality, asymptote .

1a. Algebraic expressions: basic algebraic manipulation, indices and surds (2.1) (2.2)	**Teaching time** 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to perform essential algebraic manipulations, such as expanding brackets, collecting like terms, factorising etc;
- understand and be able to use the laws of indices for all rational exponents;
- be able to use and manipulate surds, including rationalising the denominator.

TEACHING POINTS

Recap the skills taught at GCSE Higher Tier (9-1).

Emphasise that in many cases, only a fraction or surd can express the exact answer, so it is important to be able to calculate with surds.

Ensure students understand that $\sqrt{a} + \sqrt{b}$ is *not* equal to $\sqrt{a+b}$ and that they know that $a^{\frac{m}{n}}$ is equivalent to $\sqrt[n]{a^m}$ and that a^{-m} is equivalent to $\frac{1}{a^m}$.

Most students understand the skills needed to complete these calculations but make basic errors with arithmetic leading to incorrect solutions.

Questions involving squares, for example $(2\sqrt{3})^2$, will need practice.

Students should be exposed to lots of simplifying questions involving fractions as this is where most marks are lost in exams.

Recap the difference of two squares $(x+y)(x-y)$ and link this to $\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right) = x - y$, explaining the choice of term to rationalise the denominator.

Provide students with plenty of practice and ensure that they check their answers.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Include examples which involve calculating areas of shapes with side lengths expressed as surds. Exact solutions for Pythagoras questions is another place where surds occur naturally.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Common errors include: misinterpreting $\left(a\sqrt{b}\right)^2$ as $(a+\sqrt{b})^2$; evaluating $(\sqrt{2})^2$ as 4 instead of 2; slips when multiplying out brackets; basic arithmetic errors; and leaving surds in the denominator rather than fully simplifying fractions. Two examples of errors with indices are, writing $\frac{1}{3x}$ as $3x^{-1}$and writing $\frac{4}{\sqrt{x}}$ as $4x^{\frac{1}{2}}$; these have significant implications later in the course (e.g. differentiation).

Many of these errors can be avoided if students carefully check their work and have plenty of practice.

NOTES

Make use of matching activities (e.g. Tarsia puzzles)

1b. Quadratic functions: factorising, solving, graphs and discriminants (2.3)	Teaching time 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve a quadratic equation by factorising;
- be able to work with quadratic functions and their graphs;
- know and be able to use the discriminant of a quadratic function, including the conditions for real and repeated roots;
- be able to complete the square. e.g. $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$;
- be able to solve quadratic equations, including in a function of the unknown.

TEACHING POINTS

Lots of practice is needed as these algebraic skills are fundamental to all subsequent work. Students must become fluent, and continue to develop thinking skills such as choosing an appropriate method, and interpreting the language in a question. Emphasise correct setting out and notation.

Students will need lots of practice with negative coefficients for x squared and be reminded to always use brackets if using a calculator. e.g. $(–2)^2$.

Include manipulation of surds when using the formula for solving quadratic equations. [Link with previous sub-unit.]

Where examples are in a real-life contexts, students should check that solutions are appropriate and be aware that a negative solution may not be appropriate in some situations.

Students must be made aware that this sub-unit is about finding the links between completing the square and factorised forms of a quadratic and the effect this has on the graph. Use graph drawing packages to see the effect of changing the value of the '+ c' and link this with the roots and hence the discriminant.

Start by drawing $y = x^2$ and add different x terms followed by different constants in a systematic way. Then move on to expressions where the coefficient of x^2 is not 1.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Links can be made with Unit 3a – Proof:

Proof by deduction: e.g. complete the square to prove that $n^2 – 6n + 10$ is positive for all values of n.

Disproof by counter-example: show that the statement "$n^2 – n + 1$ is a prime number for all values of n" is untrue.

The path of an object thrown can be modelled using quadratic graphs. Various questions can be posed about the path:

- When is the object at a certain height?
- What is the maximum height?
- Will it clear a wall of a certain height, a certain distance away?

Areas of shapes where the side lengths are given as algebraic expressions.

Proof of the quadratic formula.

Working backwards, e.g. find a quadratic equation whose roots are $\frac{-5\pm\sqrt{17}}{4}$

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When completing the square, odd coefficients of x can cause difficulties. Students do not always relate finding the minimum point and line of symmetry to completing the square. Students should be provided with plenty of practice on completing the square with a wide range of quadratic forms.

Notation and layout can also be a problem; students must remember to show all the necessary working out at every stage of a calculation, particularly on 'show that' questions.

Examiners often refer to poor use of the quadratic formula. In some cases the formula is used without quoting it first and there are errors in substitution. In particular, the use of $-b \pm \frac{\sqrt{b^2-4ac}}{2a}$ so that the division does not extend under the "$-b$", is relatively common. Another common mistake is to think that the denominator is always 2. Also, students sometimes include x's in their expressions for the discriminant. Such methods are likely to lose a significant number of marks.

NOTES

Encourage the use of graphing packages or graphing Apps (e.g. Desmos or Autograph), so students can graph as they go along and 'picture' their solutions. You can link the discriminant with complex numbers if appropriate for students also studying Further Maths.

1c. Equations: quadratic/linear simultaneous (2.4)	**Teaching time** 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve linear simultaneous equations using elimination and substitution;
- be able to use substitution to solve simultaneous equations where one equation is linear and the other quadratic.

TEACHING POINTS

Simultaneous equations are important both in future pure topics but also for applied maths. Students will need to be confident solving simultaneous equations including those with non-integer coefficients of either or both variables.

The quadratic may involve powers of 2 in one unknown or in both unknowns, e.g. Solve $y = 2x + 3$, $y = x^2 - 4x + 8$ or $2x - 3y = 6$, $x^2 - y^2 + 3x = 50$.

Emphasise that simultaneous equations lead to a pair or pairs of solutions, and that both variables need to be found.

Make sure students practise examples of worded problems where the equations need to be set up.

Students should be encouraged to check their answers using substitution.

Sketches can be used to check the number of solutions and whether they will be positive or negative. This will be reviewed and expanded upon as part of the curve sketching topic.

Use graphing packages or graphing Apps (e.g. Desmos or Autograph), so students can visualise their solutions e.g. straight lines crossing an ellipse or a circle.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Simultaneous equations in contexts, such as costs of items given total cost, can be used. Students must be aware of the context and ensure that the solutions they give are appropriate to that context.

Simultaneous equations will be drawn on heavily in curve sketching and coordinate geometry.

Investigate when simultaneous equations cannot be solved or only give rise to one solution rather than two.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Mistakes are often due to signs errors or algebraic slips which result in incorrect coordinates. Students should be encouraged to check their working and final answers, and if the answer seems unlikely to go back and look for errors in their working. Examiners often notice that it is the more successful candidates who check their solutions.

Students should remember to find the values of both variables as stopping after finding one is a common cause of lost marks in exam situations. Students who do remember to find the values of the second variable must take care that they substitute into a correct equation or a correctly rearranged equations.

1d. Inequalities: linear and quadratic (including graphical solutions) (2.5)	Teaching time 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve linear and quadratic inequalities;
- know how to express solutions through correct use of 'and' and 'or' or through set notation;
- be able to interpret linear and quadratic inequalities graphically;
- be able to represent linear and quadratic inequalities graphically.

TEACHING POINTS

Provide students with plenty of practice at expressing solutions in different forms using the correct notation. Students must be able to express solutions using 'and' and 'or' appropriately, or by using set notation. So, for example:

$x < a$ **or** $x > b$ is equivalent to $\{x: x < a\} \cup \{x: x > b\}$

and $\{x: c < x\} \cap \{x: x < d\}$ is equivalent to $x > c$ **and** $x < d$.

Inequalities may contain brackets and fractions, but these will be reducible to linear or quadratic inequalities. For example, $\frac{a}{x} < b$ becomes $ax < bx^2$.

Students' attention should be drawn to the effect of multiplying or dividing by a negative value, this must also be taken into consideration when multiplying or dividing by an unknown constant.

Sketches are the most commonly used method for identifying the correct regions for quadratic inequalities, though other methods may be used. Whatever their method, students should be encouraged to make clear how they obtained their answer.

Students will need to be confident interpreting and sketching both linear and quadratic graphs in order to use them in the context of inequalities.

Make sure that students are also able to interpret combined inequalities. For example, solving

$ax + b > cx + d$

$px^2 + qx + r \geq 0$

$px^2 + qx + r < ax + b$

and interpreting the third inequality as the range of x for which the curve $y = px^2 + qx + r$ is below the line with equation $y = ax + b$.

When representing inequalities graphically, shading and correctly using the conventions of dotted and solid lines is required. Students using graphical calculators or computer graphing software will need to ensure they understand any differences between the conventions required and those used by their graphical calculator.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Financial or material constraints within business contexts can provide situations for using inequalities in modelling. For those doing further maths this will link to linear programming.

Inequalities can be linked to length, area and volume where side lengths are given as algebraic expressions and a maximum or minimum is given.

Following on from using a quadratic graph to model the path of an object being thrown, inequalities could be used to find the time for which the object is above a certain height.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students may make mistakes when multiplying or dividing inequalities by negative numbers.

In exam questions, some students stop when they have worked out the critical values rather than going on to identify the appropriate regions. Sketches are often helpful at this stage for working out the required region.

It is quite common, when asked to solve an inequality such as $2x^2 - 17x + 36 < 0$ to see an incorrect solution such as $2x^2 - 17x + 36 < 0 \Rightarrow (2x - 9)(x - 4) < 0 \Rightarrow x < \frac{9}{2}, x < 4$.

1e. Graphs: cubic, quartic and reciprocal (2.7)	**Teaching time** 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and use graphs of functions;
- be able to sketch curves defined by simple equations including polynomials;
- be able to use intersection points of graphs to solve equations.

TEACHING POINTS

Students should be familiar with the general shape of cubic curves from GCSE (9-1) Mathematics, so a good starting point is asking students to identify key features and draw sketches of the general shape of a positive or negative cubic. Equations can then be given from which to sketch curves.

Quartic equations will be new to students and they may benefit from initially either plotting graphs by hand or using a graphical calculator or graphing software to look at the shape of the curve.

Cubic and quartic equations given at this point should either already be factorised or be easily simplified (e.g. $y = x^3 + 4x^2 + 3x$) as students will not yet have encountered algebraic division.

The coordinates of all intersections with the axes will need to be found. Where equations are already factorised, students will need to find where they intercept the axes. Repeated roots will need to be explicitly covered as this can cause confusion.

Students should also be able to find an equation when given a sketch on which all intersections with the axes are given. To do this they will need to be confident multiplying out multiple brackets.

Reciprocal graphs in the form $y = \frac{a}{x}$ are covered at GCSE but those in the form $y = \frac{a}{x^2}$ will be new. When sketching reciprocal graphs such as $y = \frac{a}{x}$ and $= \frac{a}{x^2}$, the asymptotes will be parallel to the axes.

Intersecting points of graphs can be used to solve equations, a curve and a line and two curves should be covered. When finding points of intersection students should be encouraged to check that their answers are sensible in relation to the sketch.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students should be able to justify the number of solutions to simultaneous equations using the intersections of two curves.

Students can be given sketches of curves or photographs of curved objects (e.g. roller coasters, bridges, etc.) and asked to suggest possible equations that could have been used to generate each sketch.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When sketching cubic graphs, most students are able to gain marks by knowing the basic shape and sketching it passing through the origin. Recognising whether the cubic is positive or negative sometimes causes more difficulty. Students sometimes fail to recognise the significance of a square factor in the factorised form of a polynomial.

When sketching graphs, marks can easily be lost by not labelling all the key points or labelling them incorrectly e.g. (0, 6) instead of (6, 0).

1f. Transformations: transforming graphs (2.9), f(x) notation (2.8)	**Teaching time** 5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$;
- be able to sketch the result of a simple transformation given the graph of any function $y = \mathrm{f}(x)$.

TEACHING POINTS

Transformations to be covered are: $y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$ and $y = \mathrm{f}(ax)$.

Students should be able to apply one of these transformations to any of the functions listed and sketch the resulting graph:

quadratics, cubics, quartics, reciprocals, $y = \frac{a}{x^2}$, $\sin x$, $\cos x$, $\tan x$, e^x and a^x.

Students will need to be able to transform points and asymptotes both when sketching a curve and to give either the new point or the equation of the line.

Given a curve or an equation that has been transformed students should be able to state the transformation that has been used.

Links can be made with sketching specific curves. Students should be able to sketch curves like $y = (x - 3)^2 + 2$ and $y = \frac{2}{x-3} + 2$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Examples can be used in which the graph is transformed by an unknown constant and students encouraged to think about the effects this will have.

The use of graphing packages or graphing Apps (e.g. Desmos or Autograph) can be invaluable here.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

One of the most common errors is translating the curve in the wrong direction for $\mathrm{f}(x + a)$ or $\mathrm{f}(x) + a$. Students sometimes also apply the wrong scale factor when sketching $\mathrm{f}(ax)$.

Other errors involve algebraic mistakes and incomplete sketches, or sketches without key values marked.

Students should be encouraged to check any answers they have calculated against their sketches to check they make sense

NOTES

Dynamic geometry packages can be used to help students investigate and visualise the effect of transformations.