UNIT 1: Complex numbers

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SPECIFICATION REFERENCES

2.8 Understand de Moivre’s theorem and use it to find multiple angle formulae and sums of series.

2.9 Know and use the definition $e^{i\theta} = \cos\theta + i\sin\theta$ and the form $z = re^{i\theta}$.

2.10 Find the n distinct nth roots of $re^{i\theta}$ for $r \neq 0$ and know that they form the vertices of a regular n-gon in the Argand diagram.

2.11 Use complex roots of unity to solve geometric problems.

PRIOR KNOWLEDGE

AS Further Mathematics – Core Pure content

2.1 – 2.7 Complex numbers (See SoW Unit 1)

The GCSE (9-1) prior knowledge detailed in this scheme of work and the AS and A level Mathematics – Pure content references also apply here.

KEYWORDS

Conjugate, real part, imaginary part, complex conjugate, nth root, distinct root, discriminant, Argand diagram, Cartesian coordinates, vector, magnitude, modulus, argument, principal argument, radians, modulus-argument form, polynomial, coefficient, quadratic, quartic, cubic, de Moivre, unity, exponential, multiple angle.

1a. Know and use $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ (2.9)	**Teaching time** 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to multiply and divide complex numbers in modulus-argument and exponential form;
- know and use cosine and sine in terms of the exponential form.

TEACHING POINTS

Link exponential form and modulus-argument form and $z = a + ib$ introduced in AS Further Mathematics – Core Pure content with an Argand diagram. Link modulus-argument form and exponential form using series expansions of $\sin x$, $\cos x$ and e^x. Emphasise the importance of this relationship and move on to consider cosine and sine in terms of the exponential function.

Ensure the relationship between modulus and argument when multiplying or dividing complex numbers is understood with specific examples and the proof of the general form. Relate back to Cartesian form to ensure the links to FP1 content are well understood.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Prove $e^{i\pi} = -1$ and consider the history and importance of this result.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Encourage students to use an Argand diagrams whenever possible to illustrate problems and facilitate approaches to solutions.

Final answers where the modulus and argument are known should be evaluated and not left in forms such as $4096(\cos 2\pi + i\sin 2\pi)$.

1b. De Moivre's theorem (2.8)	**Teaching time** 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand, remember and be able to use de Moivre's theorem: $z^n = r^n e^{in\theta} = r^n(\sin n\theta + \mathrm{i}\cos n\theta)$;
- be able to derive multiple angle formulae/expressions e.g. $\cos 3\theta$ in terms of powers of $\cos\theta$, and $\sin^3\theta$ in terms of multiple angles of $\sin\theta$;
- be able to apply de Moivre's theorem to sum a geometric series.

TEACHING POINTS

Derive the general statement of de Moivre's theorem by developing expressions for z^2, z^3 and z^4, and spotting the pattern to support the more obvious approach using $(e^{i\theta})^n = e^{in\theta}$. Make a clear distinction between the approach to derive e.g. $\cos 3\theta$ in terms of powers of $\cos\theta$ using de Moivre and the approach to derive e.g. $\sin^3\theta$ in terms of multiple angles of $\sin\theta$ using $\left(z^n + \frac{1}{z^n}\right) = 2\cos n\theta$ etc. This distinction is important, so students do not confuse the two methods.

It is useful to remind students of the formulae for summing a geometric series before dealing with the final topic.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider the Argand diagram in more detail whenever possible. Look at the relationship between rectangular and polar forms of a complex number: $x = r\cos\theta$, $y = r\sin\theta$ etc. and illustrate solutions whenever appropriate. This will support the ideas that are introduced later.

Link with proof by induction to prove De Moivre's

Student can look at using the Binomial to show the $\left(z^n + \frac{1}{z^n}\right) = 2\cos n\theta$. This will let them see the distinct difference between the two methods.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Reading the question carefully and reflecting on the approach required is a common theme in examiners' reports. If the question asks for the use of de Moivre, then attempting alternative approaches by expanding brackets etc. may not gain any credit.

Encourage students to draw Argand diagrams to reflect on the demands of the question whenever the opportunity arises.

1c. The *n*th roots of $z = re^{i\theta}$ (2.10) and complex roots of unity (2.11)	**Teaching time** 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know how to solve completely equations of the form $z^n - a - ib = 0$, giving special attention to cases where $a = 1$, $b = 0$

TEACHING POINTS

Approach each problem by finding the modulus and argument of $a + ib$ i.e. (r, θ) and then writing z^n as $r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$. The development of nature of the repeat solutions and why we use $2k\pi$ it is important to show how you can find the additional solutions. Emphasis should also be placed on the principle argument. As the solutions here will only show solutions from 0.

Applying de Moivre's theorem after taking the *n*th root will help students to understand the process rather than jumping to a formula. It also helps relate the solutions to the Argand diagram i.e. *n*th roots lie on a circle radius $\sqrt[n]{r}$, evenly spaced at $\frac{2\pi}{n}$ intervals.

Look at the nature of the problem and that you have 3 situations that can be considered.

z^n = real, z^n = imaginary and z^n = complex. These will all have unique starting points and then the following solutions can be looked at in more detail with the argand diagram.

Consider roots of unity as a special case and take care to develop this idea as it can be instrumental in problem solving (see below).

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Look at an alternative method to solve $z^n - a = 0$ by using the roots of unity i.e. express a in the form $re^{i\theta}$ then find one solution $\sqrt[n]{r}e^{\frac{i\theta}{n}}$. Find other roots by multiplying the first root by the n complex roots of unity in turn. For example, roots of $z^4 = -16$ so one root is $2e^{\frac{i\theta}{4}} = \sqrt{2} + i\sqrt{2}$ and so four roots are $\sqrt{2} + i\sqrt{2}$, $i(\sqrt{2} + i\sqrt{2})$, $-1(\sqrt{2} + i\sqrt{2})$, $-i(\sqrt{2} + i\sqrt{2})$. The latter three roots should then be simplified.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Care should be taken when finding the argument of the complex number. A typical error is to choose the value of θ in the wrong quadrant. The order of the operations to be carried out is important too i.e. $2k\pi$ should be added to their argument and then divided by n to generate more values rather than the other way around. The latter error usually results in a significant loss of accuracy marks.

One examiner commented: From the drawings produced, it seemed that students did not appreciate the geometrical relationship between the roots in the Argand diagram. It was rare to see the basic circle drawn. The angular placement of the points showed that many students had little concept of the size of (the angle in radians) or an appreciation that the arguments differed and how to show that on a diagram. Students need to appreciate that a sketch needs to be reasonably accurate in terms of distances and angles to get credit.