UNIT 9: Numerical methods

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SPECIFICATION REFERENCES

9.1 Locate roots of $f(x) = 0$ by considering changes of sign of $f(x)$ in an interval of x on which $f(x)$ is sufficiently well-behaved
Understand how change of sign methods can fail

9.2 Solve equations approximately using simple iterative methods; be able to draw associated cobweb and staircase diagrams
Solve equations using the Newton-Raphson method and other recurrence relations of the form $x_{n+1} = g(x_n)$
Understand how such methods can fail

9.4 Use numerical methods to solve problems in context

PRIOR KNOWLEDGE

Covered so far

- Series, sequences and recurrence relations (Unit 4)
- Graphs, roots and functions
- Differentiation

GCSE (9-1) in Mathematics at Higher Tier

A15, A20 Iterations and approximate areas under curves
A15 Kinematics (velocity–time graphs)

AS Mathematics – Pure Mathematics content

2.7, 2.8 Graphs, roots and functions
7, 8 Differentiation and integration (See Units 6 &7 of SoW)

AS Mathematics – Mechanics content

7.2 Kinematics (velocity–time graphs) (See Unit 7 of SoW)

KEYWORDS

Roots, continuous, function, positive, negative, converge, diverge, interval, derivative, tangent, chord, iteration, Newton-Raphson, staircase, cobweb, trapezium rule.

NOTES

This topic extends the work done on iterations at GCSE (9-1) Mathematics and also links with graphs and functions.

9a. Location of roots (9.1)

Teaching time
1 hour

OBJECTIVES

By the end of the sub-unit, students should:

- be able to locate roots of $f(x) = 0$ by considering changes of sign of $f(x)$;
- be able to use numerical methods to find solutions of equations.

TEACHING POINTS

Students should be able to recognise that a root exists when there is a change of sign of $f(x)$. Students should recognise this and remember it. There is often an easy mark missed on the exam for this because it is phrased slightly differently.

Students should know that sign change is appropriate for continuous functions in a small interval.

When the interval is too large the sign may not change as there may be an even number of roots.

If the function is not continuous, the sign may change but there may be an asymptote (not a root) so the method will fail.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Look at continuous functions and then contrast this with say $y = \frac{1}{x}$ and $y = \tan x$, which will not have any roots in some intervals despite a change of sign. Use graph drawing packages to investigate similar behaviour in other functions.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students must define $f(x)$ before substituting x-values to find a root.

Most students can successfully identify the root of equations. However there are still many students who then write "change of sign therefore a root" without clarification of where the root lies and hence loose a mark.

Marks are sometimes lost unnecessarily if students do not give their answers to the specified number of significant figures or decimal places.

NOTES

Iterations may be suggested for solving equations which cannot be solved by analytic means (see the next section).

9b. Solving by iterative methods (knowledge of 'staircase and cobweb' diagrams) (9.2)	Teaching time 3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the principle of iteration;
- appreciate the need for convergence in iteration;
- be able to use iteration to find terms in a sequence;
- be able to sketch cobweb and staircase diagrams;
- be able to use cobweb and staircase diagrams to demonstrate convergence or divergence for equations of the form $x = g(x)$.

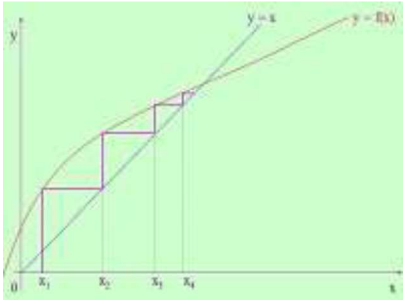
TEACHING POINTS

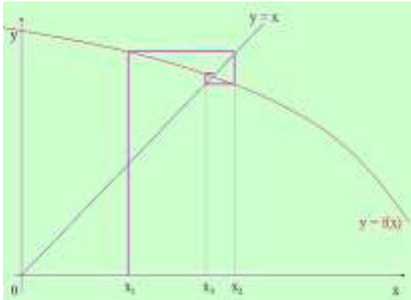
Students will have met iterations at GCSE (9-1) Mathematics, but will need to be introduced to some of the conditions for convergence and understand how the process works (and sometimes does not work).

Revise the method to make one of the x's the subject of the formula, leading to $x = f(x)$. Use graph-drawing packages to look at the function and decide where would be appropriate for the first iteration value (i.e. x_0).

The method at A level is to consider the roots of the function $y = f(x)$ as the intersection of the two functions $y = x$ and $y = f(x)$ (hence $x = f(x)$).

Use an iteration of the form $x_{n+1} = f(x_n)$ to find a root of the equation $x = f(x)$ and show how the convergence can be understood in geometrical terms by drawing cobweb and staircase diagrams like those shown here.



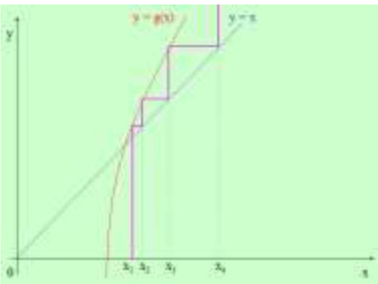


OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Which iterations converge or diverge?

Are there any values which cannot be substituted into certain iterations?

Why does this staircase diagram fail?



COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Marks will be lost due to using degrees (instead of radians) if functions involve trigonometric terms.

Choosing an unsuitable interval will also prevent progress in these questions.

NOTES

Students should understand that many mathematical problems cannot be solved analytically, but that numerical methods permit a solution to be found to a required level of accuracy.

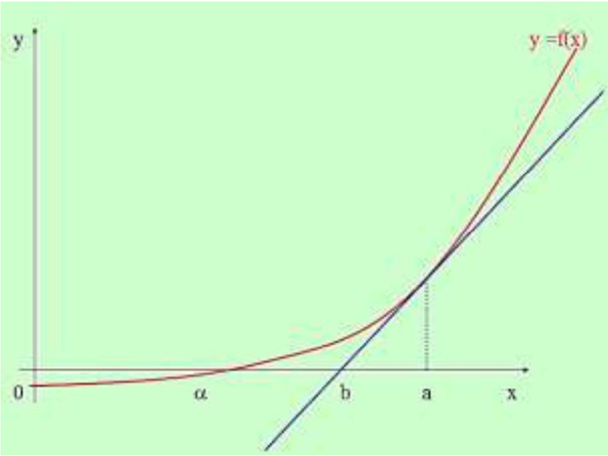
9c. Newton-Raphson method (9.2)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve equations approximately using the Newton-Raphson method;
- understand how the Newton-Raphson method works in geometrical terms.

TEACHING POINTS

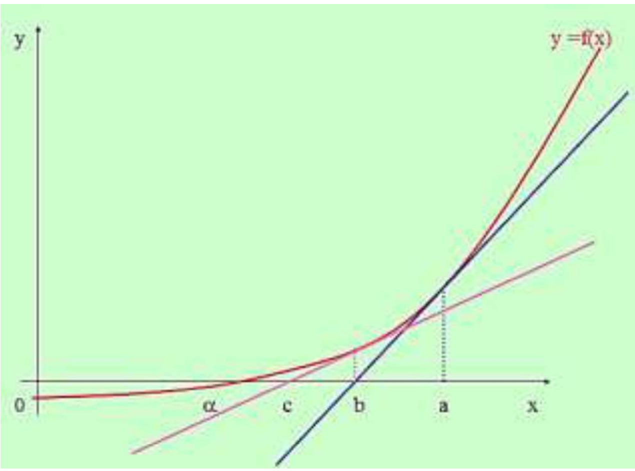


Consider the diagram above. The tangent crosses the x-axis at b (which is quite near the actual root α).

By considering the gradient of the tangent, we get $f'(a) = \frac{f(a)}{a-b}$ which can be rearranged to give $b = a - \frac{f(a)}{f'(a)}$.

We therefore have an expression for an approximation of the root (b), which uses the equation of the curve and its derivative at the point a.

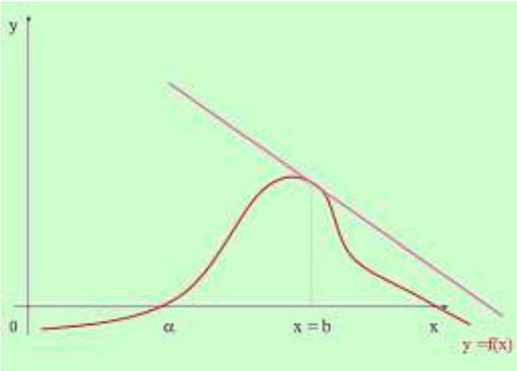
If we now go up from the point b, hit the curve and then construct another tangent (as in the diagram below) then, a similar argument, gives a better approximate root at c (nearer than b). Therefore we would get $c = b - \frac{f(a)}{f'(a)}$.



So if we continued this process we would get $d = c - \frac{f(a)}{f'(a)}$ and generally $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Sometimes the process fails for some curves or starting points.

What happens to the tangent if we try to apply the process here?



An example of the type question which may be seen:

> f $(x) = x^3 + 8x - 19$.
>
> Obtain an approximation to the real root of f(x) = 0 by performing two applications of the Newton-Raphson procedure to f(x), using $x = 2$ as the first approximation.
>
> Give your answer to 3 decimal places.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Try different methods to find the roots of the same function. Which is the most efficient method or leads to the more accurate approximation? Consider, for example, iteration vs Newton–Raphson.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Marks are often lost for sign errors and other numerical slips.

Students must show full working leading to the correct answer for full marks. Giving a correct answer either without working or following wrong working will result in zero marks.

NOTES

Graph drawing packages are an essential way to 'look' at the curve and the potential position of the roots depending on the first approximation of the root.

There will be a rich source of questions from the legacy FP1 papers as this topic was part of that specification.

Functions used will be consistent with the differentiation unit, e.g. e^{2x}, etc.

9d. Problem solving (9.4)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use numerical methods to solve problems in context.

TEACHING POINTS

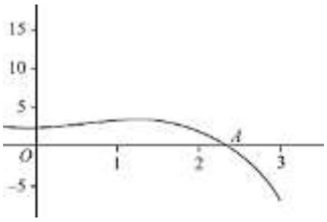
Recurrence relations, iterations and Newton-Raphson methods can be used obtain approximate solution(s) to an equation set in a context. The important point to make is that the original equation is too difficult to solve algebraically (e.g. the roots are decimal and/or the functions will not factorise or contain terms which are non-polynomials).

The choice of degree of accuracy is dependent upon the context of the problem, e.g. nearest minute or number of years.

An example of a possible question is as follows.

The equation $P = -t^3 + 2t^2 + 2$ $(t > 0)$ represents a share price p, at time t months after the money was invested.

The iteration $t_{n+1} = \frac{2}{(t_n)^2} + 2$ represents the solution to the above equation.



Taking $t_0 = 2.5$ months, show that the root gives an approximation to when the share price has zero value.
Use the iteration to find the (converged) time at which the shares lose their value before going negative.
When were the shares at their highest value?

Can Newton-Raphson be used to find the approximate solution of the above relationship?

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Which approximation method (when a choice is possible) gives the most efficient solution?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Questions in context (other than the trapezium rule) are not in the legacy specs so no examination data is available.

NOTES

The specification states: ‘iterations may be suggested for the solution of equations not soluble by analytic means’.

For approximate areas under curves to find the displacement (distance travelled) under a velocity (speed)-time graph, see Unit 11e – Trapezium rule.