Unit 9: Linear Programming (part 2)

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SPECIFICATION REFERENCES

5.3 The Simplex algorithm and tableau for maximising and minimising problems with ≤ constraints.

5.4 The two-stage Simplex and big-M methods for maximising and minimising problems which may include both ≤ and ≥ constraints.

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

A22 Solve linear inequalities; represent the solution set using set notation and on a graph

KEYWORDS

Decision variables, constraints, objective function, slack variables, Simplex method, surplus variables, artificial variables, basic and non-basic variables, Simplex tableau, feasible solution, feasible region, optimal solution, pivotal column, pivotal row, pivot, optimality condition, big-M

9a. Formulation of problems (5.1)

Teaching time
1 hour

OBJECTIVES

By the end of the sub-unit, students should:

- understand and use slack, surplus and artificial variables.

TEACHING POINTS

Introduce slack variables as the method of changing inequalities to equalities in order to use the Simplex method. For example, consider the inequality

$2x + 4y \leq 4$

We can also think of this as an equation, but we need to make the left hand side *equal* to 4. We do this by adding a non-negative amount to the left hand side to increase it enough to make it equal to 4. The inequality then becomes

$2x + 4y + s_1 = 4$

This process is repeated for all subsequent constraint inequalities.

The variable s_1 is called a slack variable because it measures the amount of slack (idle) resources still remaining in stock at any point in time during the production process. Since it is not possible to have negative slack, non- negativity constraints also apply to slack variables, so $s_1 \geq 0$.

To convert "$\geq$" constraints to an equality constraint, a surplus variable is subtracted on the left-hand side of the constraint. For example, consider the inequality constraint

$x_1 + y_1 \geq 2$

Since it is possible to exceed the minimum required production quantity, the only way to obtain an equation when converting to an equality constraint is to subtract the excess production (the quantity produced in excess of the minimum requirement):

$x_1 + y_1 - s_2 = 2$

where s_2 is the number of items produced in excess of showroom requirements. Non-negativity constraints are also required for surplus variables. Thus, $s_2 \geq 0$.

Artificial variables are used in order to use the Simplex method with mixed constraints. They are used to obtain a basic feasible solution. Artificial variables are introduced into each equation that has a surplus variable. They also have to satisfy non-negativity constraints to ensure that we only consider basic feasible solutions. In the example above, we introduce an artificial variable t_1 into the equation involving the surplus variable s_2:

$x_1 + y_1 - s_2 + t_1 = 2$

A large 'penalty' is introduced into the objective function to prevent an artificial variable from becoming part of an optimal solution. This penalty, M is chosen so large that the artificial variable is forced to be zero in any final optimal solution. We add the term $-Mt_1$ to the objective function.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Use questions where students have to change units before writing their constraints. Many students struggle to write and simplify the constraint inequalities. Use as varied a selection of problems as possible, for example include constraints given as percentages.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Students can be easily put off when things look different, for example when inequalities are written using percentages.

Weak algebra skills can cause difficulties when dealing with inequalities.

Common mistakes include omitting the instruction to maximise the objective, omitting non-negativity constraints, trying to combine several conditions into one inequality, wasting time by starting to solve the LP problem.

NOTES

For example,

$$3x + 2y \leq 20 \Rightarrow 3x + 2y + s_1 = 20$$

$$2x + 5y \leq 35 \Rightarrow 2x + 5y + s_2 = 35$$

$$x + y \geq 5 \Rightarrow x + y - s_3 + t_1 = 5$$

where s_1, s_2 are slack variables,

s_3 is a surplus variable and t_1 is an artificial variable.

9b. Simplex algorithm (5.3)	Teaching time 7 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use slack variables to write inequality constraints as equations;
- know how to rewrite LP problems so that each equation contains all the variables *x, y, s,* and *t*;
- be able to put the information in an initial tableau;
- be able to find the pivot and use it to form a new tableau;
- be able to identify if a tableau satisfies the optimality condition.

TEACHING POINTS

The graphical method has its limitations as it requires a feasible region to be drawn, and this is only possible if there are two decision variables. The Simplex method is an algebraic method of finding the optimal solution, which will occur, if it exists, at the extreme points of the feasible region.

Return to the method of introducing slack variables to rewrite the inequalities as equations and define basic solutions.

The Simplex method has two steps:

1. finding out if a given solution corresponding to an extreme point, is an optimal solution
2. finding an adjacent extreme point with a larger value for the objective function.

Define the standard form for an LP problem, and demonstrate how to summarise this information in an initial tableau.

Use the optimality condition to determine if an optimal solution has been reached and, if not, use pivots to form a new tableau.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Discuss if the values in the tableau could ever be negative.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Mistakes which examiners have commented on include:

"…selecting a negative pivot, with a consequent negative 'theta' value, for their second pivot choice, which is not acceptable. Students must remember that negatives in the value column (apart from the P row) are a certain sign that something has gone wrong. Basic variables can never be negative at any point in the algorithm."

"…reversed inequalities and slack variables are inappropriately introduced and then often mishandled. Most students were able to identify the first pivot and then deal with the pivot row correctly, although some forgot to change the basic variable. The row operations were then usually correctly carried out, but arithmetic errors often started to creep in at this point."

It is also important that students remember to use the context of the question when asked to define variables in a tableau.

NOTES

Students must be able to write down the equations represented by a tableau.

Problems will be restricted to those with a maximum of four variables (excluding slack variables), and four constraints in addition to non-negativity constraints.

9c. Big-M method and two-stage; Simplex algorithm (5.4)

Teaching time
7 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know how to use slack and surplus variables;
- understand and be able to use artificial variables;
- be able to use the two-stage simplex algorithm;
- be able to use the big-M method;
- be able to relate the solution to the original problem.

TEACHING POINTS

Both methods rely on introducing an alternative objective function to maximise or minimise, and then relating this solution back to the original problem.

Two-Stage Simplex:

Explain the case for the two-stage Simplex. The Simplex algorithm relies on (0, 0) being a feasible solution. If it's not, we need to add artificial variables t_1, t_2, etc. to all $\geq$ constraints to move from (0, 0) into the feasible region.

Subtract surplus variables from the constraints, and introduce a new objective function $Q = t_1 + t_2 + \ldots$ which you must minimise before completing the simplex algorithm in the usual way.

If the minimum value of the new objective is always greater than zero, then there is no feasible solution to the original problem.

Big-M:

If problem constraints have negative constants on the right hand side multiply through by –1, taking care to reverse the inequality sign (if present).

For each $\leq$ constraint, add a slack variable. Introduce a surplus variable and an artificial variable in each $\geq$ constraint, as well as an artificial variable in each = constraint.

For each artificial variable t, add $-Mt$ to the objective function, using the same constant M for all artificial variables.

Solve the modified problem using tableau.

When relating the solution to the modified problem to the original;

1. if the modified problem has no optimal solution, the original problem has no optimal solution
2. if all artificial variables are zero in the optimal solution to the modified problem, delete them and find an optimal solution to the original problem
3. if any artificial variables are non-zero in the optimal solution, the original problem has no optimal solution.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Explore the three types of solutions outlined above – what does the solution of the modified problem tell you?

Students must be able to interpret their results in the context of the problem; at the beginning by accurately defining their variables, and then by being able to explain the variables used in the tableau in the context of the problem.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Students may choose the wrong first pivot. This always leads to a negative element appearing in the last column. If the ratio test is correctly applied, this problem is avoided.

It is important to remember both steps when interpreting a final tableau: to identify the values of the variables, and to say what this means in the given context.

NOTES

Problems will be restricted to those with a maximum of four variables (excluding slack, surplus and artificial variables) and four constraints, in addition to any non-negativity conditions.