UNIT 8: Differentiation

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SPECIFICATION REFERENCES

7.1c Differentiation from first principles for $\sin x$ and $\cos x$

7.1b Understand and use the second derivative as the rate of change of gradient; connection to convex and concave sections of curves and points of inflection

7.2 Differentiate e^{kx}, a^{kx}, $\sin kx$, $\cos kx$, $\tan kx$ and related sums, differences and constant multiples. Understand and use the derivative of $\ln x$

7.4 Differentiate using the product rule, the quotient rule and the chain rule, including problems involving connected rates of change and inverse functions

7.5 Differentiate simple functions and relations defined implicitly or parametrically, for first derivative only

7.6 Construct simple differential equations in pure mathematics and in context, (contexts may include kinematics, population growth and modelling the relationship between price and demand)

PRIOR KNOWLEDGE

Covered so far

- Functional notation including $f'(x)$

GCSE (9-1) in Mathematics at Higher Tier

G11 Coordinate geometry

A2, A6 Changing the subject of the formula, and substitution

A12 Graphs of linear, quadratic and trigonometric functions

AS Mathematics – Pure Mathematics content

2.7, 3.1, 3.2 Coordinate geometry (See Unit 2 of SoW)

5.5, 5.7 Trigonometric identities (See Unit 4b of SoW)

7 Differentiation (See Unit 6 of SoW)

KEYWORDS

Derivative, tangent, normal, turning point, stationary point, maximum, minimum, inflexion, parametric, implicit, differential equation, rate of change, product, quotient, first derivative, second derivative, increasing function, decreasing function.

NOTES

This topic builds on the differentiation covered in AS Mathematics – Pure Mathematics, see SoW Unit 6 and leads into integration.

8a. Differentiating $\sin x$ and $\cos x$ from first principles (7.1c)

Teaching time
2 hours

OBJECTIVES

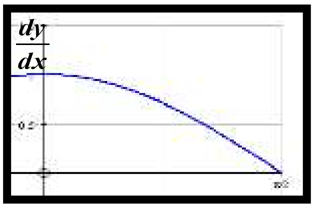
By the end of the sub-unit, students should:

- be able to find the derivative of $\sin x$ and $\cos x$ from first principles.

TEACHING POINTS

Review how to differentiate polynomials from first principles.

Sketch $y = \sin x$ and consider the gradient at key points by looking at slopes of tangents. If we plot the gradients then we get a shape which looks like the start of a cos graph:



This suggests that if $y = \sin x$, then $\frac{dy}{dx} = \cos x$, but this is not a proof or derivation!

Approach the differentiation from first principles in the same way as in AS Mathematics – Pure Mathematics, see SoW Unit 6.

Let's take a chord for $y = \sin x$ at $(x, \sin x)$ and $(x + \delta x, \sin(x + \delta x))$, the gradient of the chord is $\frac{\sin(x+\delta x)-\sin x}{\delta x}$

Using compound angle identity for $\sin(A + B)$ we find that $\frac{\sin x\cos\delta x+\cos x\sin\delta x-\sin x}{\delta x}$

By manipulation we obtain $\frac{\sin x(\cos\delta x-1)}{\delta x} + \cos x\frac{\sin\delta x}{\delta x}$

Since $\delta x \to 0$, $\frac{\sin\delta x}{\delta x} \to 1$ and $\frac{\cos\delta x-1}{\delta x} \to 0$ we conclude that $\lim_{\delta x\to 0}\frac{\sin(x+\delta x)-\sin x}{\delta x} = \cos x$

Therefore the gradient of the chord $\to$ gradient of the curve and we conclude that $\frac{dy}{dx} = \cos x$.

A similar argument with $y = \cos x$ as a starting point leads to:

$$\frac{\cos(x+\delta x)-\cos x}{\delta x} = \frac{\cos x\cos\delta x-\sin x\sin\delta x-\cos x}{\delta x}$$

and therefore finding the derivative to be $-\sin x$.

The alternative notations $h \to 0$ rather than $\delta x \to 0$ are acceptable.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Ask the students to experiment with a graph-drawing package to verify that the gradient functions of $\sin x$ and $\cos x$ match the result found using first principles. Students must understand that the differentiation of $\sin x$ and $\cos x$ can only be used when x is in radians and that they must use radians whether stated in the question or not.

NOTES

The rest of this unit covers differentiation of more complicated functions in which the derivatives of $\sin x$ and $\cos x$ are building blocks.

8b. Differentiating exponentials and logarithms (7.2)	**Teaching time** 3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to differentiate functions involving e^x, $\ln x$ and related functions such as $6e^{4x}$ and $5 \ln 3x$ and sketch the graphs of these functions;
- be able to differentiate to find equations of tangents and normals to the curve.

TEACHING POINTS

It is vital that students understand the functions e^x and $\ln x$ and do not just learn how to differentiate them. Use a graphing tool to show that the gradient of a special curve $y = a^x$ has a gradient which is exactly a^x. In other words its rate of growth is exactly the same as its value at that point. This models biological growth in nature (and decay if we consider a^{-x}) The curve sits between 2^x and 3^x and has a value of 2.718… We call this exponential e.

Therefore if $y = e^x$, $\frac{dy}{dx} = e^x$.

Explain that if $y = 2e^x$ then $\frac{dy}{dx} = 2e^x$.

The students could verify this on the graphs below as Fig. 1 is effectively a stretch parallel to the y-axis.

Fig. 2 shows that the graph of $y = e^{2x}$ is twice as steep as e^x, hence if $y = e^{2x}$ then $\frac{dy}{dx} = 2e^{2x}$.

These results will be deduced more formally in Unit 8c.

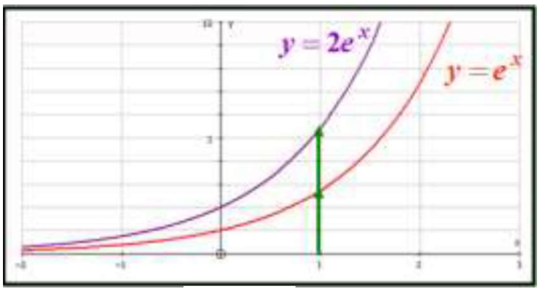


Fig. 1

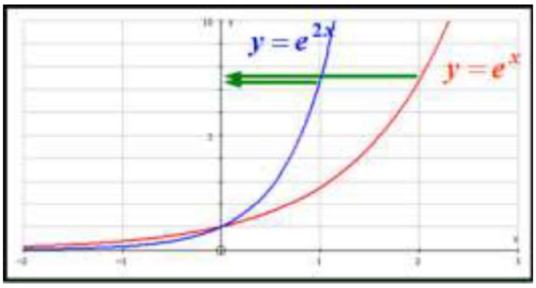


Fig. 2

For natural logarithms, recap the basic definition and graphs (from Pure Paper 1)

By looking at the graph we can see that the gradient of $y = \ln x$ at any particular point is the reciprocal of the x-coordinate of that point where the tangent is drawn. Therefore for $y = \ln x$, $\frac{dy}{dx} = \frac{1}{x}$.

This can be derived in the following way:

If $y = \ln x$, then, from our definition of logs, $x = e^y$. [Write $2 = \log_{10}100$ and $100 = 10^2$ to illustrate this.]

We can differentiate $x = e^y$ by finding $\frac{dx}{dy}$ instead of the usual $\frac{dy}{dx}$.

$\frac{dx}{dy} = e^y$, and taking the reciprocal of both sides gives $\frac{dy}{dx} = \frac{1}{e^y}$.

We know that $e^y = x$ from above, so this gives $\frac{dy}{dx} = \frac{1}{x}$ as the derivative of $y = \ln x$.

The graphical approach could then be used to investigate why, for example, $y = \ln(3x)$ also has a derivative of $\frac{1}{x}$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Find gradients and normals for exponential and log functions, using graphs to check and enhance the solutions.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students often miss out minus signs or add an extra x into the answer when differentiating expressions like $e^{-\frac{1}{4}x}$.

Some students mix up $\frac{dx}{dy}$ and $\frac{dy}{dx}$ and others struggle to differentiate functions involving ln. For example given when differentiating $y = \ln 6x$ they write $\frac{1}{6x}$ rather than $\frac{1}{x}$.

NOTES

Increasingly, exam questions focus on the ability to rearrange and solve equations involving e^x and $\ln x$.

8c. Differentiating products, quotients, implicit and parametric functions. (7.2) (7.4) (7.5)	**Teaching time** 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to differentiate composite functions using the chain rule;
- be able to differentiate using the product rule;
- be able to differentiate using the quotient rule;
- be able to differentiate parametric equations;
- be able to find the gradient at a given point from parametric equations;
- be able to find the equation of a tangent or normal (parametric);
- be able to use implicit differentiation to differentiate an equation involving two variables;
- be able to find the gradient of a curve using implicit differentiation;
- be able to verify a given point is stationary (implicit).

TEACHING POINTS

Most students will be able to differentiate simple instances of e^{3x}, sin $3x$ and ln $3x$ without needing formal methods such as $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$.

Many will also be able to differentiate expressions such as $(3x + 7)^5$ without using the formal method $\frac{d}{dx}(f(x))^n = n(f(x)^{n-1})f'(x)$.

When using the chain rule and the formula $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, initially u can be given to students, but they must be able to choose their own u and should move onto this quickly. Encourage students to lay work out carefully, using correct notation and $\frac{dy}{du}$ and $\frac{du}{dx}$, not always $\frac{dy}{dx}$.

Teaching should focus on *how* students know a function needs to be differentiated using the chain rule (or function of a function) and *why* a particular u is selected.

As an introduction for the product rule, ask the students to differentiate x^4. If you rewrite this as the product $(x^2)(x^2)$ and differentiate each part separately, it does not match $4x^3$. Using the product rule will give that match.

In a similar way, writing x^4 as $\frac{x^3}{x}$ can lead into the quotient rule.

Work involving the product and quotient rule often breaks down because of weak algebraic skills and this needs plenty of practice. Students should practice fully simplifying their answers as they may be asked to give a solution in a particular form. Encourage students to lay work out carefully. Good notation is vital to achieve success.

Show that the product rule and the quotient rule give the same answers on functions that can be written in two ways, for example, $y = \frac{x+1}{x+2}$ and $y = (x + 1)(x + 2)^{-1}$.

Also show that the chain rule and the product rule give the same derivative for $\cos^2 x$ and $\sin^2 x$.

Use the product and quotient rules to derive the differentials of some key trigonometric expressions. For example $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right)$ using the quotient rule giving $\sec^2 x$.

For parametric differentiation, make links with the chain rule to give $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$

Stress that we often substitute in the value of the parameter t at the point which we need to find the gradient

Many questions will involve trigonometric functions, so students must be fluent at differentiating these.

For implicit differentiation, consider the equation of a circle, $x^2 + y^2 = 16$. To differentiate this function we would have to make y the subject of the formula. Sometimes this can be difficult or even impossible.

Make sure students can confidently differentiate terms like x^2y using implicit differentiation. Finally, stress that we need to substitute in *both* x and y coordinates to find the gradient at a certain point.

Students may have to apply the product or quotient rules in implicit differentiation questions and should be given examples of this. In exam questions students are almost always required to find the gradient through implicit differentiation.

Take a point on a circle or another type of curve and find the gradient using two both parametric and implicit differentiation. Then find the equation of tangent and/or normal and see that both methods give the same answer.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Although repeated chain rule questions rarely appear in the exam they provide good extension material and provide an excellent test of good method and correct mathematical notation. Extend the students further by asking them to look up a proof or derivation of the product and quotient rules.

Use the methods above to work out the derivative of a general exponential function, i.e. $\frac{\mathrm{d}}{\mathrm{d}x}(a^{kx}) = ka^{kx}\ln a$

Give the students lots of mixed questions which will enable them to select the correct method. Discussion should focus on why they have selected a particular method and quick ways of identifying the correct method.

Students must be able to use all methods as a particular method is sometime specified in the exam.

Some questions require a trigonometric identity in order to simplify the solution.

The specification states that 'differentiation of $\arcsin x$, $\arccos x$, and $\arctan x$ are required'.

Cover questions involving finding tangents, turning points and normals (this links with Unit 6d).

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Common errors involve: not using the method specified; algebraic errors when manipulating expressions; and being unable to identify the need of the product rule and instead simply differentiating the separate parts and multiplying.

NOTES

Check which differentials are in the formula book and which must be learnt.

This work links to Unit 11f (Differential equations), after more integration skills have been developed.

8d. Second derivatives (rates of change of gradient, inflections) (7.1b)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find and identify the nature of stationary points and understand rates of change of gradient.

TEACHING POINTS

The specification states 'Understand and use the second derivative as the rate of change of gradient; connection to convex and concave sections of curves and points of inflection' and 'know that at an inflection point $f''(x)$ changes sign.'

The basic principle is usually

	$\frac{dy}{dx}$ or $f'(x)$	$\frac{d^2y}{dx^2}$ or $f''(x)$
maximum	$= 0$	< 0
minimum	$= 0$	> 0

However show examples of curves in which $\frac{d^2y}{dx^2}$ or $f''(x) = 0$, where there could beva point of inflexion (or not). i.e. The rate of change of gradient is zero.

We would need to work out $f'(x)$ and scrutinise gradient either side of the point x. There may be positive or negative inflexion or neither (depending on the nature of the curve, which could be convex or concave).

Use graph drawing packages to investigate the shapes and turning points of various curves of the type $y = ax^n$ $(n > 2)$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Look at $y = \frac{a}{x}$, e^x, $\ln x$ etc.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students should be encouraged to state "$\frac{dx}{dy} = \ldots$ when $x = \ldots$", especially when finding a given answer.

An easy mistake students may make is to mix up maxima and minima.

NOTES

The types of functions and complexity of expressions are consistent with the functions covered earlier in this unit.

8e. Rates of change problems* (including growth and decay) (7.6) *see also Integration (part 2) – Differential equations	Teaching time 3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use a model to find the value after a given time;
- be able to set up and use logarithms to solve an equation for an exponential growth or decay problem;
- be able to use logarithms to find the base of an exponential;
- know how to model the growth or decay of 2D and 3D objects using connected rates of change;
- be able to set up a differential equation using given information which may include direct proportion.

TEACHING POINTS

This content links to kinematics, where velocity is considered as $\frac{\mathrm{d}s}{\mathrm{d}t}$ and acceleration as $\frac{\mathrm{d}v}{\mathrm{d}t}$.

The example below is from the original SAMs:

A team of conservationists is studying the population of meerkats on a nature reserve.

The population is modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{22}P(11-2P), \quad t \geq 0$$

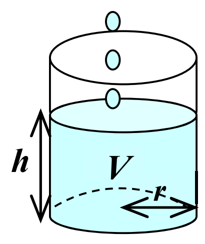
where P, in thousands, is the population of meerkats and t is the time measured in years since the study began.

Given that there are 1000 meerkats on the nature reserve when the study began,

(a) determine the time taken, in years, for this population of meerkats to double,

(b) show that the population cannot exceed 5500.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING



Consider water entering this cylinder. To work out the rate at which the height is increasing we need to calculate $\frac{\mathrm{d}h}{\mathrm{d}t}$.

In exam questions, the rate that the volume of water increases at is often given as $\frac{\mathrm{d}V}{\mathrm{d}t}$.

Therefore, we need to use the chain rule to create $\frac{\mathrm{d}h}{\mathrm{d}t}$ from $\frac{\mathrm{d}V}{\mathrm{d}t}$.

$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \times \frac{\mathrm{d}h}{\mathrm{d}V}$ so we need a formula connecting h and V.

$V = \pi r^2 h$ and from this we can work out $\frac{\mathrm{d}V}{\mathrm{d}h}$ and then $\frac{\mathrm{d}h}{\mathrm{d}V}$ etc.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Most students are able to substitute correctly into a formula for exponential growth and decay.

When required to set up an inequality most students showed that they understood the information given and wrote down a correct opening expression, although there was uncertainty over which way the inequality should go. Some then simplified and solved using logarithms efficiently to get the correct answer. Some resorted to trial and improvement which was accepted for full marks if done correctly, but was worth no marks otherwise.

When solving equations involving exponentials, knowledge of using logarithms varied widely. Many were unable to deal properly with the coefficient and the exponential term and wrote down equations in which t actually should have cancelled out.

Some care needs to be taken when interpreting the answers to exponential growth and decay questions to ensure they are given in the correct form e.g. to the nearest year, second etc.

NOTES

For first order differential equations (which require separating variables) see Unit 11 – Integration (part 2)