UNIT 2: Hyperbolic functions

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SPECIFICATION REFERENCES

8.1 Understand the definitions of hyperbolic functions $\sinh x$, $\cosh x$ and $\tanh x$, including their domains and ranges, and be able to sketch their graphs.

8.2 Differentiate and integrate hyperbolic functions.

8.3 Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges.

8.4 Derive and use the logarithmic forms of the inverse hyperbolic functions.

8.5 Integrate functions of the form $(x^2 + a^2)^{-\frac{1}{2}}$ and $(x^2 - a^2)^{-\frac{1}{2}}$ and be able to choose substitutions to integrate associated functions.

PRIOR KNOWLEDGE

AS Mathematics – Pure content

6.1 – 6.7 Exponential functions (See SoW Year 1 Unit 8)

5.1 – 5.4 Trigonometry (See SoW Year 1 Unit 4)

A level Mathematics – Pure content

5.1 – 5.9 Trigonometry (See SoW Year 2 Unit 6)

KEYWORDS

Hyperbolic, sinh, cosh, tanh, domain, range, exponential, function, radical.

2a. sinh x, cosh x, tanh x and their inverses (8.1)(8.3)

Teaching time
5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know the definitions of sinh x, cosh x and tanh x including their domains and ranges;
- be able to sketch graphs of the hyperbolic functions;
- be able to differentiate and integrate the hyperbolic functions and know the standard results;
- understand and be able to use the inverse hyperbolic functions including domains and ranges.

TEACHING POINTS

Start with developing that the functions are exponential in definition but can behave like trigonometric functions, this can be left open to see why this is the case. The sketching of the exponentials using online function plotters and then add the functions it will see them develop and this will emphasise that knowing the definitions hyperbolic functions and their domains and ranges

Sketch the functions and interpret the domains and ranges as one would have done with the trig functions.

Look at the nature of odd and even function and that they are the same as trig.

Consider features such as sinh $(-x) = -$ sinh x using graph sketches.

A reminder that $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{-x}) = -\mathrm{e}^{-x}$ means the formulae for the derivatives and hence the integrals can be found as an exercise. This should also be looked at as comparison to the trigonometric functions.

Considering the graphs of each function in turn and reflecting each in the line $y = x$ can help develop the shape and domain and range of the inverse functions.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Look at hyperbolic identities and how they compare with familiar trigonometric identities; for example, $\cosh^2 x - \sinh x = 1$

Develop and use Osborne's rule to compare.

Good chance to look at Catenary curves and the identities, development and prevalence of them in the real world. This can be a good problem to approach as it will lead towards the calculus in the next section. Starting off with a differential equation.

How do the hyperbolic functions work in the complex plane? Can explore the nature of x replaced by a complex number and how does the domain and ranges change/adapt. This can be a good discussion point as to why are they needed.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Typically students can correctly differentiate hyperbolic functions, but the final stages of more complex solutions cause problems and a number of errors are seen often in processing terms down to a printed result.

2b. Logarithmic forms of the inverse hyperbolic functions and integrate functions of the form $(x^2 + a^2)^{-\frac{1}{2}}$ and $(x^2 - a^2)^{-\frac{1}{2}}$ (8.2)(8.4)(8.5)

Teaching time
5 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to derive, use and know the logarithmic forms of the inverse hyperbolic functions.

TEACHING POINTS

Start with $y = \sinh x \Rightarrow y = \frac{e^x - e^{-x}}{2}$ and rearrange to give a quadratic in e^x. This can be solved to give $e^x = \ln(y + \sqrt{y^2 + 1})$ which leads quickly to the formula for $\sinh^{-1} x$. Take care to consider the domains; for example, $\cosh x$ should be restricted to $x \geq 0$

When considering how to integrate functions such as $(x^2 + a^2)^{-\frac{1}{2}}$, draw parallels with earlier work on trigonometric substitutions. For example, substitute $x = a \tan\theta$ into a problem involving $\sqrt{x^2 + a^2}$ and compare with $x = a \sinh\theta$

It is a useful prompt if the appropriate substitution for each radical is known.

Give the functions to integrate and the substitutions and ask the students to match them. This will be a good way to make sure that can remember and apply the process correctly.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider the whole function without restricting the domain.

Noting that $x - \sqrt{x^2 - 1} = \frac{(x - \sqrt{x^2 - 1})(x + \sqrt{x^2 - 1})}{x + \sqrt{x^2 - 1}}$ and expanding the numerator can lead to a better understanding of $\cosh^{-1} x$ and its form.

More demanding problems can be set when radicals are involved that require hyperbolic trigonometric identities to produce a given answer e.g. $\int \sqrt{x^2 + 1}\,dx = \frac{1}{2}\ln(x + \sqrt{x^2 + 1}) + \frac{x}{2}\sqrt{x^2 + 1} + c$

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Students should understand that there are often a number of different approaches to solving problems and the choice of strategy can be crucial. Using the logarithmic form of inverse hyperbolic functions to obtain the final answer is often used, along with writing hyperbolic functions in terms of exponentials and proceeding to solve the resulting quadratics in e^x . It should be noted that sometimes this can end up with extra solutions that should be rejected. It should be noted that attempts to solve a given equation by expressing it in terms of exponentials can result in a dead end when, for example, a quartic in e^x is reached.