UNIT 1: Proof Examples including proof by deduction and proof by contradiction (1.1)	Teaching time 3 hours

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SPECIFICATION REFERENCES

1.1 Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including proof by deduction. Proof by contradiction (including proof of the irrationality of $\sqrt{2}$ and the infinity of primes, and application to unfamiliar proofs)

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

G20	Pythagoras Theorem
	Trigonometry
A18	Algebraic manipulation including completing the square
N4, N8, N10	Surds, prime and irrational numbers

AS Mathematics – Pure Mathematics content

1.1 Proof (See Unit 3a of the SoW)

KEYWORDS

Proof, verify, deduction, contradict, rational, irrational, square, root, prime, infinity, square number, quadratic, expansion, trigonometry, Pythagoras.

NOTES

Proof may also be tested throughout the specification through other topics e.g. trigonometry, series, differentiation, etc.

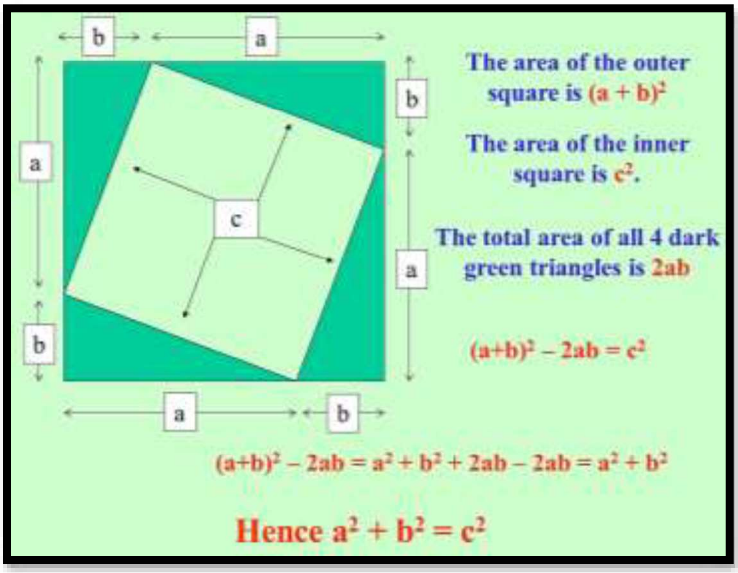
OBJECTIVES

By the end of the sub-unit, students should:

- understand that various types of proof can be used to give confirmation that previously learnt formulae are true, and have a sound mathematical basis;
- understand that there are different types of proof and disproof (e.g. deduction and contradiction), and know when it is appropriate to use which particular method;
- be able to use an appropriate proof within other areas of the specification later in the course.

TEACHING POINTS

Introduce using areas and the expansion of $(a + b)^2$ to prove Pythagoras' theorem as an example of using a logical sequence of steps in order to deduce a familiar result.



Explain how verification for a set number of values in *not* a proof of a general result (for all values of n).

Show how different methods can be used to prove a statement, including:

- Manipulating the LHS of a result and using logical steps (normally algebraic) to make it match the RHS or vice versa (or, sometimes, manipulating both sides to reach the same expression).
- Manipulating an expression to show it holds true for all values. For example, an inequality can always be ≥ 0 if we manipulate the LHS to be in the form of [something]^2 since anything squared will always be bigger or equal to zero. This argument can be used on a gradient function to prove a function is increasing.

Provide standard examples of proof by contradiction, e.g., $\sqrt{2}$ is irrational:

Assuming it can be written as a rational number $\frac{a}{b}$ which has been written in its lowest terms. It follows that $\frac{a^2}{b^2} = 2$ and $a^2 = 2b^2$. Therefore, a^2 is even because it is equal to $2b^2$. It follows that a must be even (as squares of odd integers are never even). Because a is even, there exists an integer k that fulfills: $a = 2k$. Substituting $2k$ for a above gives $2b^2 = (2k)^2 = 4k^2$, so $b^2 = 2k^2$. Because $2k^2 = b^2$, it follows that b^2 is even and b is also even.

Hence a and b are both even, which contradicts that $\frac{a}{b}$ is in its simplest form

Another example of proof by contradiction is the proof that there is an infinite number of primes:

Assume there is an integer p, such that p is the largest prime number.

Now $p! + 1 > p$ and is not divisible by p or any other number less than p*

*If 2 is a factor of n, then 2 is not a factor of $n + 1$. Similarly if 3 is a factor of n, 3 is not a factor of $n + 1$. Now 2, 3, … p are all factors of $p!$, so none are factors of $p! + 1$.

So, either $p! + 1$ is not divisible by an integer other than 1 or $p! + 1$ which means $p! + 1$ is prime, or $p! + 1$ is divisible by some number between p and $p! + 1$ which implies there is a prime number larger than p.

These both contradict our initial assumptipn, which proves there are an infitie number of primes.

Illustrate proof by exhaustion e.g. Prove that $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$ for the positive integers from 1 to 5 inclusive.

This can be proved if you substitute (exhaust) all the possible values of n from 1 to 5. Note that this type of proof can only be used for proving something for a set of given values.

You should also talk about disproof by counter-example.

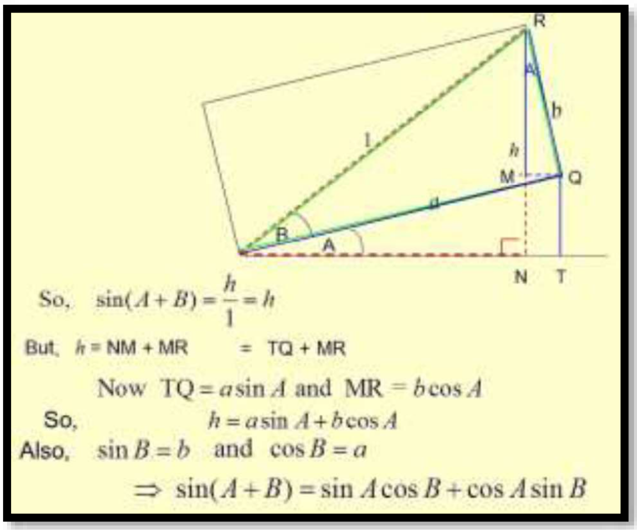
Explain that all we have to do is find *one* example where the statement does not hold and this is enough to show that it is not always true. This method can be used to disprove trigonometric identities as well as statements such as $a > b \Rightarrow a^2 > b^2$:

Choose any pair of negative numbers with $a > b$ e.g. $a = -2$ and $b = -3$.

Hence $a > b$, but if we square the numbers $a^2 < b^2$ (as $4 < 9$) and so this disproves the statement.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Link with Trigonometry (Unit 6d) and provide a deduction of the compound-angle formula for e.g. $\sin(A + B)$



COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Some students mistakenly think that substituting several values into an expression is sufficient to prove the statement for all values.

Similarly, for example, referring to a graph to prove that the gradient is always positive rather than completing the square will not gain marks for a proof.

NOTES

Proof may be tested throughout the specification in other topics such as trigonometry, series, differentiation, etc.