UNIT 6: Elastic collisions in two dimensions

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SPECIFICATION REFERENCES

5.1 Oblique impact of smooth elastic spheres and a smooth sphere with a fixed surface. Loss of kinetic energy due to impact.

5.2 Successive oblique impacts of a sphere with smooth plane surfaces.

PRIOR KNOWLEDGE

Covered so far

- Momentum and impulse (Unit 1)
- Direct collisions, Newton's (experimental) law of restitution (Unit 3)
- Change in kinetic energy due to impact (Unit 3)
- Momentum and Impulse-momentum principle in vector form (Unit 4)
- Change in kinetic energy due to impact (Unit 5)

AS Mathematics – Pure content

10.1 – 10.5 Work with **i**, **j** vectors, magnitude & direction (See SoW Year 1 Unit 5)
5.3, 5.4 Trigonometric identities (See SoW Year 1 Unit 7)

A level Mathematics – Pure content

5.5 Trigonometric identities (See SoW Year 2 Unit 8)

AS Further Mathematics – Core Pure content

6.3 Scalar product of vectors in 2-D (See SoW Unit 7b)*

* Note that, although an understanding of scalar product is useful in problem solving, only vectors in two dimensions are being considered and so other techniques are possible.

KEYWORDS

Impact, momentum, impulse, magnitude, direction, sphere, equal radii, collision, oblique, smooth, coefficient of restitution, Newton's (experimental) law of restitution, rebound, conservation, perfectly elastic, inelastic, vector, component, parallel, perpendicular, normal, line of centres, deflection, scalar product, kinetic energy.

NOTES

This topic extends the ideas from AS Further Mechanics 1, Units 1 and 3, by considering oblique impacts in two dimensions. It was included in the legacy M4 Unit of A level Further Mathematics.

6a. Oblique impact of a smooth sphere with a fixed surface (5.1); Successive oblique impacts of a sphere with smooth plane surfaces (5.2)

Teaching time
9 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand that during an impact the impulse acts perpendicularly to the surface through the centre of the sphere;
- be able to apply Newton's (experimental) law of restitution in the direction of the impulse;
- appreciate that perpendicular to the impulse, the velocity component does not change;
- understand and be able to calculate an angle of deflection;
- be able to calculate the kinetic energy 'lost' in an impact;
- be able to work in speeds and angles or in velocity vectors (**i**, **j**).

TEACHING POINTS

First review Newton's experimental law of restitution in relation to a particle striking a fixed surface at right angles. Give a reminder of the equation in e (coefficient of restitution) given by $e = \frac{v}{u}$ in this case.

Now consider an oblique impact such as a snooker ball hitting the cushion at the edge of the table. By 'oblique' we mean that the velocity of approach is not at right angles to the cushion. Clearly the value of e must still be relevant but how exactly?

At the point of impact the reactions on the ball and the cushion are equal in magnitude and opposite in direction (Newton's third law). Moreover, they are perpendicular to the cushion through the centre of the ball (as long as we model the cushion and the ball as being smooth).

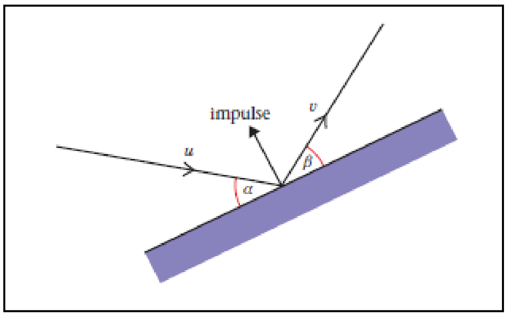
This has the important consequence that:

The impulse acts in a direction at right angles to the surface.

So what does this mean in the context of the snooker ball? By the impulse–momentum principle, only the component of velocity perpendicular (or normal) to the cushion will change. So we apply Newton's law to the velocity components in this direction only.

The velocity component parallel to the surface is not changed by the impact.

These two principles are fundamental; they allow us to formulate two equations which will be the basis for problem solving.



The diagram represents the impact of a smooth sphere with a fixed plane.

Discuss why you would expect the angle β to be less than angle α.

Now introduce velocity components:

$u \cos \alpha$ and $v \cos \beta$ (parallel to plane)

$u \sin \alpha$ and $v \sin \beta$ (perpendicular to plane)

We can write down equations based on the two principles above, namely:

- $u \cos \alpha = v \cos \beta$ (no change parallel to surface)
- $eu \sin \alpha = v \sin \beta$ (Newton's law perpendicular to surface)

The impulse–momentum principle states: $\boldsymbol{I} = m(\boldsymbol{v} - \boldsymbol{u})$

So in this case, since we know the direction is perpendicular to surface, the magnitude of the impulse is given by:

$I = m(v \sin \beta + u \sin \alpha)$

Note the '+' sign since the direction of the velocity is reversed by the impact.

By dividing the first two equations, we find that:

$e \tan \alpha = \tan \beta$

A point of interest arising from this is that an experiment could be set up to measure the angles and hence deduce a value for the experimental constant, e. It can also be seen that when e=1, $\alpha = \beta$ and when e< 1, $\beta < \alpha$ (as expected)

Alternatively, we could eliminate the angle β from the above equations by squaring and adding, and then using the identity $\sin^2\beta + \cos^2\beta = 1$ to give:

$v^2 = u^2(e^2 \sin^2\alpha + \cos^2\alpha)$

It is possible to arrive at the required results without using trigonometry but to have components x and y (say) parallel and perpendicular to the surface where $x^2 + y^2 = v^2$

The **angle of deflection** is the angle between the continuation of the initial direction of motion and the final direction of motion, which here is $(\alpha + \beta)$. A common error is to just state the value of β. If an exact answer is required, the formula for $\tan(\alpha + \beta)$ is useful.

As considered in AS Further Mechanics 1, Unit 3, kinetic energy is 'lost' in any collision which is not perfectly elastic. (This energy is converted into other forms.) The important difference here is that we are working in two dimensions and so velocities have two components.

Remember that $\text{KE} = \frac{1}{2}mv^2$

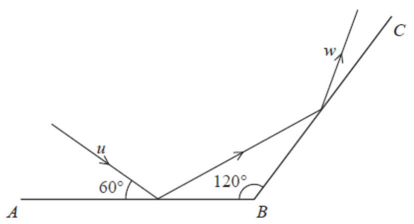
where v = speed (magnitude of velocity – obtained by squaring and adding components).

Note that since the component of velocity parallel to the surface is unaffected by the impact, we could find the change in energy by considering the perpendicular components only.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

It is possible to consider successive collisions as shown in the example below:

The plan view of part of a smooth horizontal floor, where *AB* and *BC* are smooth vertical walls is shown in the diagram. The angle between *AB* and *BC* is 120°. A ball is projected along the floor towards *AB* with speed u m s^{-1} on a path at an angle of 60° to *AB*. The ball hits *AB* and then hits *BC*.



The ball is modelled as a particle. The coefficient of restitution between the ball and each wall is $\frac{1}{2}$.

(a) Show that the speed of the ball immediately after it has hit *AB* is $\frac{\sqrt{7}}{4}u$.

The speed of the ball immediately after it has hit *BC* is w m s^{-1}.

(b) Find w in terms of u.

(***Answer***: *0.634u*)

Note that in (b), although there are various possible methods, it is easiest to calculate the angle of approach and then apply the same method as used in part (a) to find the final speed as a multiple of u. It is possible to use trig ratios and find an exact answer – this could be an interesting challenge!

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Questions on this topic are often answered well, though there is sometimes some confusion about the angle of deflection and occasionally students will apply the impact law parallel to the plane rather than in the perpendicular direction.

When forming the equation for the impulse, students must remember to take account of the change in direction of the motion perpendicular to the plane due to the impact.

NOTES

Problems may be set in terms of speeds and angles or in terms of vectors (**i**, **j**). The same principles are involved in both approaches and more practice at working in vectors will be available in the next sub-unit.

6b. Oblique impact of two smooth spheres of equal radius (5.1)	Teaching time 11 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand that, during a collision between two smooth spheres, total momentum is conserved and the impulse acts in the direction of the line of centres;
- be able to apply Newton's (experimental) law of restitution in the direction of the line of centres;
- appreciate that perpendicular to the line of centres, velocity components do not change;
- understand and be able to calculate an angle of deflection;
- be able to calculate the kinetic energy 'lost' in a collision;
- be able to work in speeds and angles or in velocity vectors (**i**, **j**).

TEACHING POINTS

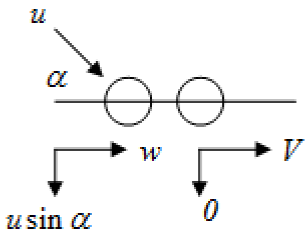
We consider an oblique collision between two spheres of equal radius. 'Oblique' in this context means that immediately before impact the spheres are not moving along the same straight line. The spheres are modelled as 'smooth' to ensure that the impulse is entirely in the direction of the line of centres (and there is no 'spin' created). Observing a few collisions of snooker balls should soon give a picture of the idea.

The general approach is similar to that adopted in the previous section; we work in velocity components parallel and perpendicular to the direction of the impulse (which this time is the line of centres). However, now the law of conservation of linear momentum is relevant. So we can identify three principles which lead to three equations:

- conservation of linear momentum parallel to line of centres (or for the whole system)
- Newton's (experimental) law of restitution parallel to line of centres
- no change in velocity components perpendicular to line of centres.

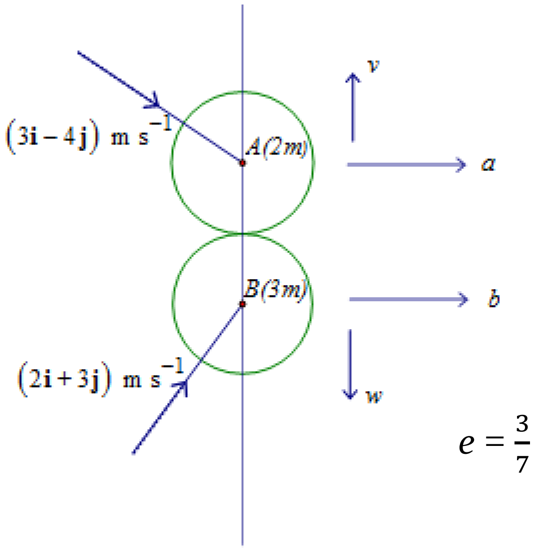
Clearly, linear momentum is also conserved perpendicular to the line of centres but this leads to a redundant equation since we already have the stronger condition of no change in velocity components.

A clear diagram showing components before and after collision is very important. The velocities could be specified as vectors or as speeds and directions. First we have an example of a diagram where a smooth sphere with velocity u at angle α to line of centres collides with an identical smooth sphere at rest.



Note that the components perpendicular to line of centres do not change and have been entered on the diagram directly. Note that, when one of the spheres is at rest before the impulse, this sphere moves off along the line of centres; it is important to know this in games such as billiards, snooker and pool.

Next is an example where the velocities before impact are given as vectors and the line of centres is parallel to the vector **j**. It is possible to calculate the velocities after impact using the three principles previously stated; the solution is outlined below:



Perpendicular to line of centres (no change): $a = 3$, $b = 2$

Parallel to line of centres – Conservation of linear momentum: $-4 \times 2 + 3 \times 3 = 2v - 3w$

Parallel to line of centres – Newton's law of restitution: $v + w = e \times 7$ (where $e = \frac{3}{7}$)

Solving simultaneously gives $v = 2$ and $w = 1$

In this example the line of centres was conveniently in the direction of the vector **j**. However it is worth considering a case where it is in neither **i** or **j** direction, and might not even be specified. If the velocity of one sphere is given before and after impact, a calculation of the impulse will give the direction of line of centres. For example, if the velocity of one sphere is (6**i** - 5**j**) before impact and (3**i** - 9**j**) after impact, then the impulse (and therefore the line of centres) will be in the direction (3i + 4j). A neat way of finding the components of velocities in this direction (and perpendicular) is to make use of the scalar product; for example the component of, say, 6**i** – 5**j** in the direction 3**i** + 4**j** is given by $(6\mathbf{i} - 5\mathbf{j})\,\frac{(3\mathbf{i} + 4\mathbf{j})}{5} = -\frac{2}{5}$ (since the magnitude of 3**i** + 4**j** is 5).

The **angle of deflection** in this context is the same as defined in the previous sub-unit: the angle between the continuation of the initial direction of motion and the final direction of motion. This can be calculated for each sphere and a clear diagram which shows the velocities before and after impact is important for identifying the relevant angle. Also, as before, the scalar product can be used.

As previously stated, kinetic energy is lost as a result of a collision (unless the collision is perfectly elastic). In the case of the two spheres, this means a loss in the TOTAL KE (but not necessarily for each individual sphere). Again, remind students that energy is a scalar quantity and v^2 in '$\frac{1}{2}mv^2$' is the square of the magnitude of the velocity of one sphere. Although energy lost only depends on the components parallel to the line of centres, it might be good practice to calculate total KE for each sphere (before and after impact).

Sometimes students are required to express the loss as a fraction of the original KE, in which case the full values **must** be used.

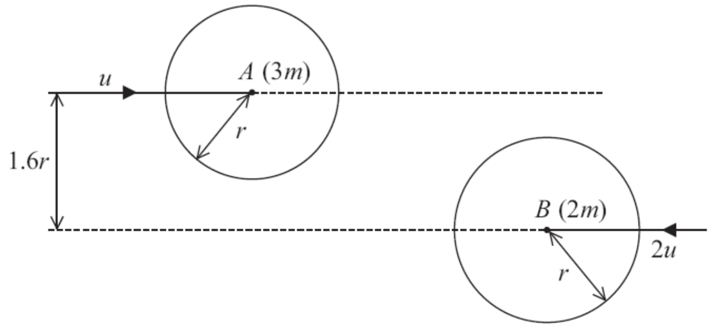
It is important not to confuse the above (where we subtract scalar quantities) with calculating the magnitude of the impulse (where we subtract vector quantities before finding the magnitude). Remember that impulse' is a vector quantity given by: $\boldsymbol{I} = m(\boldsymbol{v} - \boldsymbol{u})$

In a collision, the impulse received by each sphere is equal and opposite, so could be found by considering either sphere. When working in vectors, the components of the impulse must be found before calculating the magnitude. However, if using the fact that the impulse is in the direction of the line of centres, then the difference in the velocity components in this direction will lead to the magnitude but care must be taken with signs (which depend on directions).

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The problem below requires angles of approach to be calculated by noting that at the point of collision the centres are $2r$ apart. A diagram should clarify the situation but some clear thinking is required about directions.

Two smooth uniform spheres A and B, of equal radius r, have masses $3m$ and $2m$ respectively. The spheres are moving on a smooth horizontal plane when they collide. Immediately before the collision they are moving with speeds u and $2u$ respectively. The centres of the spheres are moving towards each other along parallel paths at a distance $1.6r$ apart. The coefficient of restitution between the two spheres is $\frac{1}{6}$.



Find, in terms of m and u, the magnitude of the impulse received by B in the collision.

(***Answer***: *2.52mu*)

Note the examiner report comments for this question:

Choosing simple names for the components parallel to the line of centres after collision rather than introducing two unknown angles led to simpler equations to work with, and usually less confused working. Most candidates knew the appropriate formula for impulse, but they did not always take account of the change in direction of motion and some candidates used a mixture of speeds and components of velocity.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Again, this is a topic where students will benefit from drawing clear diagrams showing the angles and components of velocity.

Close attention should be given to the signs used as this can be a cause of lost marks.

Finding the angle of deflection can prove challenging for some students.

NOTES

Only smooth surfaces and smooth (uniform) colliding spheres of the same radius will be considered.

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