UNIT 5: The binomial theorem

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SPECIFICATION REFERENCES

4.1 Understand and use the binomial expansion of $(a + bx)^n$ for rational n, including its use for approximation;

be aware that the expansion is valid for $\left|\frac{bx}{a}\right| < 1$ (proof not required)

PRIOR KNOWLEDGE

Covered so far

- Series and sequences (See Unit 4 of the SoW)

GCSE (9-1) in Mathematics at Higher Tier

A4 Algebraic fractions

AS Mathematics – Pure Mathematics content

2.6 Algebraic division, factor theorem (See Unit 3a of the SoW)

4.1 Binomial expansion of the form $(a + bx)^n$, where n is a positive integer (See Unit 3b of the SoW)

KEYWORDS

Binomial, expansion, theorem, integer, rational, power, index, coefficient, validity, modulus, factorial, ${}^nC_r$, combinations, Pascal's triangle, partial fractions, approximation, converges, diverges, root.

NOTES

The formula book includes formulae for the binomial expansion:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{1\times 2\times \ldots \times r}x^r + \cdots \qquad (|x| < 1,\ n \in \mathbb{R})$$

This unit links with the binomial distribution in the statistics section of the remaining A level Mathematics content of statistics.

5a. Expanding $(a + bx)^n$ for rational n; knowledge of range of validity (4.1)	**Teaching time** 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the binomial expansion of $(1-x)^{-1}$ for rational values of n and $|x| < 1$;
- be able to find the binomial expansion of $(1+x)^n$ for rational values of n and $|x| < 1$;
- be able to find the binomial expansion of $(1+bx)^n$ for rational values of n and $|x| < \frac{1}{|b|}$;
- be able to find the binomial expansion of $(a+x)^n$ for rational values of n and $|x| < a$;
- be able to find the binomial expansion of $(a+bx)^n$ for rational values of n and $\left|\frac{bx}{a}\right| < 1$;
- know how to use the binomial theorem to find approximations (including roots).

TEACHING POINTS

Begin by reviewing the expansion of $(a + b)^n$ when n is a positive integer.

Ask students to expand $(1 + x)^4$ and then try $(1 + x)^{-2}$. Why does it fail to work? Which coefficient calculation breaks down?

Explain how the binomial theorem allows us to expand *any* power. (Explain the reasoning behind the factorial notation using the explanation in the *Reasoning and problem solving* section below.)

Consider why the expansions are infinite when the power is *not* a positive integer. How far do we need to expand and to which term? (For example, up to and including coefficients of x^3.)

Take care to show the precision needed when dealing with negative calculations by demonstrating examples such as $(1-2x)^{-\frac{1}{2}}$.

If we expanded $(1+x)^{\frac{1}{2}}$ then substituted $x = -0.1$, we would be effectively finding the square root of 0.9. Ask students to use a calculator to find an accurate value for $\sqrt{9}$. How many terms of the expansion would we need to substitute into in order to get a 4 decimal place version of the accurate value?

What happens when we substitute $x = 3$? Does this find the square root of 4?

Explain that if we raise a number > 1 to a positive power, it 'grows' and diverges out of control. This means that the value of x must be such that $-1 < x < 1$ or $|x| < 1$ in order to use the expansion of $(1 + x)^n$. The validity of the expansion is dependent upon the value of x we substitute into the terms.

Cover examples that build-up the expansions listed in the objectives above, ending with $(a + bx)^n$ for rational values of n and valid for $\left|\frac{bx}{a}\right| < 1$.

Introduce the concept of expansions of expressions which start with a rather than 1. Begin by showing that if we have $(2 + x)$ and if we want to make this start with a 1 in the bracket, we must take out the factor of 2, giving $2(1 + \frac{x}{2})$.

Now show for example, that $2(1 + 4)$ gives the same result as $(2 + 8)$ if we multiplied this out, but that if the bracket were squared the result would not be the same i.e. $2(1 + 4)^2 \neq (4 + 8)^2$.

However, $2^2(1 + 4)^2 = (4 + 8)^2$, so we need to raise the factor to the same power of the bracket and $(a + bx)^n = a^n(1 + \frac{bx}{a})^n$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Show how ${}^nC_r$ will only work on a calculator for positive integer values of n (as was done in Unit 3 of Pure).

However, we can instead use the definition and formula for selections (shown here for 'choose 2 from n different objects').

$$
\begin{aligned}
{}^nC_2 &= \frac{n!}{2!\,(n-2)!} \\
&= \frac{n(n-1)\cancel{(n-2)}\cancel{(n-3)}\ldots}{2!\,\cancel{(n-2)}\cancel{(n-3)}\ldots} \\
&= \frac{n(n-1)}{2!}
\end{aligned}
$$

This formula works for *all* values of n and follows the pattern of the binomial theorem as stated in the formula book.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When expanding $(1+4x)^{\frac{1}{2}}$ most students got the first two terms of the expansion correct, but often there was a mistake in the x^2 term, with $4x$ becoming just x being the common error. Some students made arithmetic errors with 4^2, by failing to actually square the 4, and others failed to simplify the binomial coefficient correctly.

When expanding an expression of the form $(a+x)^n$ a common error is to write this as $a(1+\frac{x}{a})^n$ rather than $a^n(1+\frac{x}{a})^n$.

Other errors include algebraic errors when combining two expansions, doing more work than is necessary when, for example, only terms up to x^2 are required, including the equality in the expression for the range of valid values for x and lack of understanding when using the modulus symbol (writing expressions such as $|x| < -4$).

NOTES

Link this section to the next part of this unit: expanding functions by first using partial fractions.

5b. Expansion of functions by first using partial fractions (4.1)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use partial fractions to write a rational function as a series expansion.

TEACHING POINTS

This sub-unit links with sub-unit 2b above (Partial Fractions) and gives the students a purpose for learning how to break-up a rational function into two or more partial fractions.

If we consider the 'complicated' fraction below, it needs to be simplifies into two simpler fractions each of which only involve a *single* algebraic bracket.

$$\frac{2x^2+5x-10}{(x-1)(x+2)} \equiv A + \frac{B}{x-1} + \frac{C}{x+2}$$

We can now rewrite each term as a binomial series. (It is important to demonstrate that the $\frac{B}{x-1}$ term will become $B(x-1)^{-1}$.)

Particular care needs to be taken when working with brackets that don't start with 1, and also when multiplying out all the terms to arrive at the final simplified series (up to and including the power required).

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

You will need to assess all the separate validities for the individual binomial terms to declare the validity for the final series.

Include examples in which one of the terms is not a binomial and just multiplies without expansion. For example, $(2x-1)(1+3x)^{-2}$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

(These all relate directly to the example given above.)

Nearly all students were able to make the connection between the parts of the question, but there were many errors in expanding both $(x-1)^{-1}$ and $(2+x)^{-1}$.

Few were able to write $(x-1)^{-1}$ as $-(1-x)^{-1}$ and the resulting expansions were incorrect in the majority of cases, both $1+x-x^2$ and $1-x-x^2$ being common errors.

However, $(2+x)^{-1}$ was handled better, but the constant $\frac{1}{2}$ in $\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$ was frequently incorrect.

NOTES

Inform the students that partial fractions are also required to break down rational functions before they are differentiated (Unit 8) and integrated (Units 10 and 11).