UNIT 5: Further calculus

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SPECIFICATION REFERENCES

5.1 Derive formula for and calculate volumes of revolution

5.2 Evaluate improper integrals where either the integrand is undefined at a value in the range of integration or the range of integration extends to infinity.

5.3 Understand and evaluate the mean value of a function.

5.4 Integrate using partial fractions.

5.5 Differentiate inverse trigonometric functions.

5.6 Integrate functions of the form $(a^2 - x^2)^{-\frac{1}{2}}$ and $(a^2 + x^2)^{-1}$ and be able to choose trigonometric substitutions to integrate associated functions.

PRIOR KNOWLEDGE

<u>A level Mathematics – Pure content</u>

8.2 – 8.8 Integration (See SoW Year 2 Unit 3)

2.10 Partial fractions (See SoW Year 2 Unit 1b)

2.8 Inverse functions (See SoW Year 2 Unit 5b)

KEYWORDS

Improper, undefined, continuous, mean, integrate, partial, fraction, radical, inverse.

5a. Improper integrals (5.2)

Teaching time
6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know how to deal with infinity as a limit of a definite integral;
- be able to integrate functions across limits which include values when the function is undefined i.e. deal with discontinuous integrands.

TEACHING POINTS

A good start here is to look at the nature of functions that converge and diverge. This is a key point to how we find definite solutions to indefinite problems.

A good way to develop this topic is too look at the two types of problems that will develop the learning points.

Type A Not finite examples of which $\int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x$, $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$, $\int_{-\infty}^\infty \frac{1}{1+x^2}\mathrm{d}x$.

Type B Not continuous. Examples. - $\int_0^1 \frac{1}{x}\mathrm{d}x$, $\int_{-2}^2 \frac{1}{x^2-1}\mathrm{d}x$, $\int_0^\pi \tan x\,\mathrm{d}x$, where these functions are not continuous across the limits

It should be highlighted that the limits don't have to be using the infinity to be indefinite.

Sketching graphs of these functions will aid understanding. Include divergent integrals. Compare the process for doing each type and this will help with understanding and development of the problems when faced with unfamiliar functions.

Compare the approach for type 2 with the fundamental theorem of calculus e.g. $\int_0^\pi \sec^2 x\mathrm{d}x$ and show that it can't be done normally. This should remind students that they can't default to normal integration in this situation.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider integrals with two infinite limits.

Look at problems that involve both of these cases e.g. $\int_0^\infty \frac{1}{x^2}\mathrm{d}x$

Investigation into why $\frac{1}{x}$ is divergent and $\frac{1}{x^2}$ is convergent

Identify different functions where an indefinite integral can be found e.g. $\int \sin x^2\mathrm{d}x$, $\int \cos x^2\mathrm{d}x$, $\int \frac{\sin x}{x}\mathrm{d}x$

Linking to Stats and the use of the indefinite integral (the Gaussian function) to find the areas in Normal distribution.

How can it be shown that $\int_{-\infty}^\infty e^{-x^2}\mathrm{d}x = \sqrt{\pi}$

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Any problems posed should be considered carefully before a problem solving strategy is determined. Simply ignoring discontinuous integrands, for example, is unlikely to be awarded many marks.

5b. Mean value of a function (5.3)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to evaluate the mean value of a function.

TEACHING POINTS

Start with looking at Rolle's theorem and the basis of what the 'mean value' is. Rolle's is a special case f '(c) = 0

Need to make sure that a function is continuous and differentiable across the range.

Develop and prove the formula from Rolle's and emphasise the need to make sure that the working to solve problems is clear and well structured.

Sketching a graph of the function concerned and looking at what the answer represents can help students understand why this is the mean value. This is particularly important if the answer might be unexpected e.g. 0.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Good example to check in the use of the MVT in the real world. An average speed problem and how average speed cameras might work and proof that you can get a speeding fine. Testing out the accuracy of a speedometer.

Use the MVT to show how it can be applied in many situations and how important it is to Calculus e.g. proving inequalities $n\ln\left(1+\frac{1}{n}\right) \leq 1$ for a positive integers of n.

More advanced proofs with the MVT. Proving L'Hopital, Taylor's (these can be touched on but will not be covered until Further Pure Mathematics 1).

Statisticians may also want to consider how this average relates to measures of location.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

The formula should always be quoted in full before an attempt is made to substitute values. Without it, method marks may not be awarded if an error is made.

5c. Integrate using partial fractions (5.4)	**Teaching time** 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate functions which can be split into partial fractions up to denominators with quadratic factors.

TEACHING POINTS

A quick reminder of the work done in A level Mathematics covering the methodical techniques involved in decomposition into partial fractions may be useful to start, but start on the types of fractions that can be integrated easily. This can be linked and revised from the method of differences series problems.

Look at the well-known, basic integral formulae, especially $\int \frac{1}{x} \mathrm{d}x = \ln|x| + c$, and see how these apply to integrating each partial fraction.

Then look to develop the nature of the fractions and how the integral can be developed, e.g. $\frac{a}{(x-1)} + \frac{bx+c}{(x^2+1)}$ where both of the fractions might not be ln integrals.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider fractions with a higher power in the numerator than the denominator when the denominator is already factorised.

Develop fractions that cannot be easily integrated and can lead to some investigation how they can be and this could lead to the next section.

Look at problems that involve a substitution for a trigonometric function which leads into partial fractions e.g. $\int \frac{\sec^2 x}{\tan^3 x - \tan^2 x} \mathrm{d}x$

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Consider partial fraction denominators carefully and always divide out first if necessary.

5d. Differentiate inverse trigonometric functions (5.5) and integrate using trigonometric substitutions (5.6)	Teaching time 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to differentiate inverse trigonometric functions such as $\frac{1}{2}\arctan x^2$;
- know how to integrate functions of the form $(a^2 - x^2)^{-\frac{1}{2}}$ and $(a^2 + x^2)^{-1}$ and be able to choose trigonometric substitutions to integrate associated functions.

TEACHING POINTS

Develop the formulae by using implicit differentiation: e.g. $y = \sin^{-1} x \Rightarrow \sin y = x$ and then $(\cos y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1$. This leads quickly to the formula $\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ using $\cos^2 y + \sin^2 y = 1 \Rightarrow \cos y = \sqrt{1-x^2}$. The other formulae can be shown using a similar approach and then $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}\frac{x}{a}\right)$.

Consider integrating $(a^2 - x^2)^{-\frac{1}{2}}$ and associated functions as the reverse of the above process.

This is an opportunity to revise the trigonometric identities and the differentiation of sec θ, tan θ etc.

Choosing the appropriate trigonometric substitution is easier if the following prompts are learnt:

$\sqrt{a^2 - b^2x^2} \Rightarrow x = \frac{a}{b}\sin\theta$

$\sqrt{b^2x^2 - a^2} \Rightarrow x = \frac{a}{b}\sec\theta$

$\sqrt{a^2 + b^2x^2} \Rightarrow x = \frac{a}{b}\tan\theta$

These should be shown how they are developed so making the understanding easier to use for integration and identifying the correct substitution.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Developing the formulae for $\frac{\mathrm{d}}{\mathrm{d}x}(\sec^{-1}x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosec}^{-1}x)$ involving $|x|$ and are more challenging.

Linking with the Series section and Maclaurin's and the differentiation on arctan θ for finding the approximation of the proof of π.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Algebraic techniques involving radicals are very important and are a common source of errors.

5e. Further volumes of revolution (5.1)	**Teaching time** 3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to derive formulae for and calculate volumes of revolution about both the x and y-axes.
- be able to find volumes of revolution for functions given in parametric form.

TEACHING POINTS

Revise parametric equations from P2, and conversion between parametric and Cartesian forms.

Both volume for rotation about the x-axis $V = \pi \int y^2 \; \mathrm{d}x$ and the y-axis $V = \pi \int x^2 \; \mathrm{d}y$ are required.

Care should be taken when the equation is given in parametric form as it may be easier to integrate the resulting function in either Cartesian or parametric form.

Stress the importance of quoting the general formula for volume before substituting values.

Look at using more complex functions for y or x so when squaring will challenge and develop their integration skills.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Finding the volume of a cone.

Finding the volume of a sphere.

Finding the volume of a section rotated about the axes. Subtracting one volume from the other. Torus?

Finding the volume of shapes rotated around a line that is parallel to one of the axes.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

'It was expected that students should at least quote the general formula for volume and then substitute their derivative. It is good practice to quote formulae before substitution. When an error is made on substitution the examiner needs to be sure that the correct formula is being used before the method mark can be awarded'.

Students must know the formula for volume. 'The majority of candidates were able to apply volume formula $\pi \int y^2 \, \mathrm{d}x$. A number of candidates, however, used incorrect formulae such as $2\pi \int y^2 \, \mathrm{d}x$ or $\int y^2 \, \mathrm{d}x$ or even $\int y \, \mathrm{d}x$'.

Students must be sure of the correct formula for volume when the variables are given parametrically. 'Some rewrote the volume formula as $\pi \int y^2 \, \mathrm{d}\theta$, with dx being replaced by $d\theta$. Those who did not apply $\pi \int y^2 \frac{dx}{d\theta} \mathrm{d}\theta$ gained little access to this question, and it was also disappointing to see some students who attempted to apply an incorrect $\pi \int \left(y \frac{dx}{d\theta}\right)^2 \mathrm{d}\theta$'.