UNIT 9: Kinematics 2 (variable acceleration)

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SPECIFICATION REFERENCES

7.4 Use calculus in kinematics for motion in a straight line.

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

A11 Identify and interpret roots, intercepts, turning points of quadratic functions graphically; deduce roots algebraically and turning points by completing the square

A14 Plot and interpret graphs (including reciprocal graphs and exponential graphs) and graphs of non-standard functions in real contexts to find approximate solutions to problems such as simple kinematic problems involving distance, speed and acceleration

A15 Calculate or estimate gradients of graphs and area under graphs (including quadratic and non-linear graphs), and interpret results in cases such as distance-time graphs, velocity-time graphs and graphs in financial contexts

AS Mathematics – Pure Mathematics content

7, 8 Differentiation and integration of polynomials (See Units 6 and 7 of the SoW)

KEYWORDS

Distance, displacement, velocity, speed, constant acceleration, variable acceleration, retardation, deceleration, gradient, area, differentiate, integrate, rate of change, straight-line motion, with respect to time, constant of integration, initial conditions.

NOTES

All the functions in this section are functions of time, so the differentials and integrals are always with respect to time.

9a. Variable force; Calculus to determine rates of change for kinematics (differentiation) (7.4)	Teaching time 3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use calculus (differentiation) in kinematics to model motion in a straight line for a particle moving with variable acceleration;
- understand that gradients of the relevant graphs link to rates of change;
- know how to find max and min velocities by considering zero gradients and understand how this links with the actual motion (i.e. acceleration = 0).

TEACHING POINTS

Start by stating that the *suvat* formulae from Unit 7 can only be used when acceleration is constant and the motion is in a straight line. This means the speed-time or velocity-time graphs are made up of straight lines.

Draw the graph of say, $v = 2t^2 + 2t + 1$ (for $t > 0$). This is part of a parabola where the gradient is increasing so as time passes the object is accelerating more quickly. As acceleration is not constant, the *suvat* formulae will not work for this model.

Make links (using AS Pure Mathematics calculus) to the rate of change of velocity explaining that $\frac{\mathrm{d}v}{\mathrm{d}t}$ = gradient = acceleration. This idea that the gradient of a velocity–time graph gives acceleration should be familiar from previous work in Unit 7 and also from GCSE (9-1) in Mathematics.

Summarise the situation by talking about, velocity as the rate of change of displacement and acceleration as the rate of change of velocity.

Express these statements in the notation of calculus: $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ and $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2s}{\mathrm{d}t^2}$.

Students will also need to relate the fact that the gradient = 0 at the max or min point to this mathematical model i.e. if $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$, then acceleration = 0, so the particle must be at max or min velocity, as it cannot accelerate (or get any faster or slower) any more at this point in time.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

You could extend the calculus approach to relate double differentiation and signs of $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}$ to indicate if it is a min or max displacement.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students who draw sketches of the situation are often more successful in reaching the correct solution, so you should continue to encourage this wherever possible.

Students often ignore or don't recognise the difference between displacement and distance and so may end up discarding negative values without considering how they should be interpreted.

NOTES

The level of calculus will be consistent with the contents of AS Pure Mathematics.

The specification states the following:– $v = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$, $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}$ using '$\boldsymbol{r}$' to represent displacement 's'.
r will become the vector notation of displacement when we later analyse 2D kinematics using the **i**, **j** system (A level Mathematics – Mechanics section, see SoW Unit 8).

9b. Use of integration for kinematics problems i.e. $\boldsymbol{r} = \int \boldsymbol{v}\,\mathrm{d}t$, $\boldsymbol{v} = \int \boldsymbol{a}\,\mathrm{d}t$ (7.4)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use calculus (integration) in kinematics to model motion in a straight line for a particle moving under the action of a variable force;
- understand that the area under a graph is the integral, which leads to a physical quantity;
- know how to use initial conditions to calculate the constant of integration and refer back to the problem.

TEACHING POINTS

Return to the graph of $v = 2t^2 + 2t + 1$ (for $t > 0$) introduced at the start of Unit 9a.

From earlier work in Unit 7 and from GCSE (9-1) in Mathematics, students should know that the area under a velocity–time graph equals the displacement.

Remind students that, from their work for Pure Mathematics, the area under a curve can be found using integration. This means that the integral of the velocity expression (with respect to time) gives the displacement.

By linking integration with the reverse of differentiation, displacement and velocity can be found by integrating expressions for velocity and acceleration respectively:

$$\boldsymbol{r} = \int v\,\mathrm{d}t \text{ and } v = \int a\,\mathrm{d}t$$

(Again 's' can be used in place of '$\boldsymbol{r}$' for straight line motion in this section)

Move on to explain that the constant of integration, c needs to be found by referring back to the problem and using some (usually initial) information about the body. For example knowing that the particle starts from O at rest means that when $t = 0$ (initially), $s = 0$ (at O) and $v = 0$ (at rest). These values can be substituted to calculate c.

OPPORTUNITIES FOR PROBLEM SOLVING/MODELLING

Students need to be able to know when to differentiate and/or integrate and how acceleration = 0 gives a maximum velocity so questions like the following are useful.

A particle moves so that it's motion is modelled by the following equation, $v = 6t(3 - t)$ m s^{-1}.

Find: **a** the times when it is at rest, **b** its maximum velocity, **c** an expression for its acceleration, **d** the total distance it travels between the times it is stationary.

Extension: Starting with constant a, students can derive the earlier equations of uniform motion. Stress constants of integration, which produce u in $v = u + at$ and the s_0 (s when $t = 0$) in $s = ut + \frac{1}{2}at^2 + s_0$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students can easily forget that if the velocity becomes negative, for example when a particle stops and changes direction, they need to split the integral to calculate distance rather than displacement.

NOTES

The following diagram may help students decide whether to differentiate or integrate to solve a problem.

'**d**' for the **d**own arrow means '**d**ifferentiate'. Hence, down from 's' gives 'v' or $\frac{\mathrm{d}s}{\mathrm{d}t} = v$.

Integration is the opposite of differentiation so up is integrate, so up from 'a' gives 'v' or integral of 'a' with respect to t gives 'v'.

$\downarrow s$

diff $\downarrow v \uparrow$ int

$a \uparrow$