UNIT 6: Trigonometry

Return to overview

SPECIFICATION REFERENCES

5.1 Work with radian measure, including use for arc length and area of sector

5.2 Understand and use the standard small angle approximations of sine, cosine and tangent i.e. $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$, $\tan\theta \approx \theta$ where θ is in radians

5.3 Know and use exact values of sin and cos for 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π and multiples thereof, and exact values of tan for 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π and multiples thereof

5.4 Understand and use the definitions of secant, cosecant and cotangent and of arcsin, arccos and arctan; their relationships to sine, cosine and tangent; understanding of their graphs; their ranges and domains

5.5 Understand and use $\sec^2\theta = 1 + \tan^2\theta$ and $\text{cosec}^2\theta = 1 + \cot^2\theta$

5.6a Understand and use double angle formulae; use of formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$; understand geometrical proofs of these formulae

5.6b Understand and use expressions for $a\cos\theta + b\sin\theta$ in the equivalent forms of $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$

5.8 Construct proofs involving trigonometric functions and identities

5.9 Use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

G22 Sine and cosine function

G18 Length of arc and area of sector

AS Mathematics – Pure Mathematics content

2.6 Algebraic division, factor theorem (See Unit 3a of the SoW)

5.7 Solving trigonometric equations (See Unit 4 of the SoW)

5.5 $\sin^2 x + \cos^2 x = 1$ and $\frac{\sin x}{\cos x} = \tan x$ (See Unit 4 of the SoW)

5.3 Properties of graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ (See Unit 4 of the SoW)

KEYWORDS

Pythagoras, Pythagorean triple, right-angled triangle, opposite, adjacent, hypotenuse, trigonometry, sine, cosine, tangent, secant, cosecant, cotangent, SOHCAHTOA, exact, symmetry, periodicity, identity, equation, interval, quadrant, degree, radian, circular measure, infinity, asymptote, small angles, approximation, identity, proof.

NOTES

This unit is fundamental to future study of trigonometry in Further Maths and also links to mechanics. For example, the path of a projectile requires the identity $1 + \tan^2 x = \sec^2 x$.

6a. Radians (exact values), arcs and sectors (5.1) (5.3)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the definition of a radian and be able to convert between radians and degrees;
- know and be able to use exact values of sin, cos and tan;
- be able to derive and use the formulae for arc length and area of sector.

TEACHING POINTS

Ensure all students know how to change between radian and degree mode on their own calculators and emphasise the need to check which mode it is in.

Radian measure will be new to students and it is important that they understand what 1 radian actually is.

Make sure students know that 'exact value' implies an answer must be given in surd form or as a multiple of π. They need to know the exact values of sin and cos for 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π (and their multiples) and exact values of tan for 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π (and their multiples).

Emphasise the need to always put a scale on both axes when drawing trigonometric graphs; students must be able to do this in radians.

Make links between writing the trig ratio of any angle (obtuse/reflex/negative) to the trig ratio of an acute angle and to the trig graphs. (Do not rely on the CAST method as this tends to show a lack of understanding.)

Derive the formulae for arc length and area of a sector by replacing the $\frac{\theta}{360°}$ in the GCSE formulae with $\frac{\theta}{2\pi}$.

The πs cancel giving length of arc = $r\theta$ and area of sector = $\frac{1}{2}r^2\theta$.

Cover examples which will involve finding the area of a segment by subtracting a triangle from a sector.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

One radian can be defined as 'the angle at the centre of a circle which measures out exactly one radius around the circumference.' Therefore, using $C = 2\pi r$, we can conclude that the full circumference, C is made up of 2π radians. This means 360 is equivalent to 2π radians.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

A common exam mistake is for students to have their calculators set in the wrong mode resulting in the loss of accuracy marks.

6b. Small angles (5.2)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use the standard small angle approximations for sine, cosine and tangent.

TEACHING POINTS

The Specification states:- $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$, $\tan\theta \approx \theta$ (where θ is in radians)

Experiment with trigonometric graphs and a graph-drawing package by reading off values near the origin and zooming into small angles so the students get a feeling for this new concept.
The formal proof is based on considering the area of a sector in which the angle is so small, the shape becomes a right-angled triangle (since the curved part is straightened).
By considering the area of the triangle within the sector, the area of the sector and the area of the right angled triangle we can see that

$$\frac{1}{2}r^2\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2\tan\theta$$

Cancelling $\frac{1}{2}r^2$ gives $\sin\theta < \theta < \tan\theta$

Dividing by $\sin\theta$ gives $1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

As θ tends to 0, $\frac{1}{\cos\theta}$ tends to 1, and so $\frac{\theta}{\sin\theta}$ must tend to 1 as it is fixed between two values which tend to 1.

So $\frac{\theta}{\sin\theta}$ is approximately equal to 1 for small values of θ (the small angle was the assumption at the start).
Rearranging gives $\sin\theta \approx \theta$.

Following a similar process, but dividing by $\tan\theta$ at the start gives $\tan\theta \approx \theta$.

Using the identity $\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$ (which is covered in a later sub-unit), and substituting $\sin\frac{1}{2}\theta \approx \frac{1}{2}\theta$, gives the third approximation $\cos\theta \approx 1 - \frac{\theta^2}{2}$.

The small angle approximations can be used to give estimated values of trigonometric expressions. For example, $\frac{\cos 3x - 1}{x\sin 4x}$ approximates to $-\frac{9}{8}$ (when x is small)

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

These approximations only work when the small angles are measured in radians. Why don't the approximations work in degrees?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students may try to use these approximations when angles are measured in degrees rather than radians.

NOTES

Small angles are also used in further mechanics work when dealing with simple pendulums.

6c. Secant, cosecant and cotangent (definitions, identities and graphs) & inverse trigonometrical functions & inverse trigonometrical functions (5.4) (5.5)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the secant, cosecant and cotangent functions, and their relationships to sine, cosine and tangent;
- be able to sketch the graphs of secant, cosecant and cotangent;
- be able to simplify expressions and solve involving sec, cosec and cot;
- be able to solve identities involving sec, cosec and cot;
- know and be able to use the identities $1 + \tan^2 x = \sec^2 x$ and $1 + \cot^2 x = \text{cosec}^2 x$ to prove other identities and solve equations in degrees and/or radians
- be able to work with the inverse trig functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$;
- be able to sketch the graphs of $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

TEACHING POINTS

Introduce students to the reciprocal trigonometric functions secant θ, cosecant θ and cotangent θ.

A good way to introduce these as reciprocal trig functions is to start by asking whether there is another way of writing x^{-1}. This should lead to the answer $\frac{1}{x}$. If we try this with $\sin^{-1}\theta$ it is not the same meaning as $\frac{1}{\sin\theta}$, so we need to name a different function cosec θ. (Contrast this with inverse trig functions looked at later in this section)

To help students remember which reciprocal function goes with sin, cos and tan, point out that the **third** letter of these new functions, gives the name of the trig function in the denominator, i.e.

$$\text{se}\mathbf{c}\ \theta = \frac{1}{\cos\theta} \qquad \text{co}\mathbf{s}\text{ec}\ \theta = \frac{1}{\sin\theta} \qquad \text{co}\mathbf{t}\ \theta = \frac{1}{\tan\theta}$$

You should also point out that cot θ can be written as the reciprocal of tan θ to give $\frac{\cos\theta}{\sin\theta}$.

Students will be expected to know what the graphs of each of the reciprocal and inverse functions look like and their key features, including domains and ranges. The relationships between the graphs and their originals can be explored on graphical calculators or graphing Apps.

Show students how to work out new trigonometric identities by dividing $\sin^2\theta + \cos^2\theta = 1$ (from AS Mathematics – Pure Mathematics) by $\cos^2\theta$ or by $\sin^2\theta$ to give the two new identities: $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \text{cosec}^2\theta$.

This is a good alternative to simply remembering the identities and lessens the chance of mixing them up.

It is a good idea to use the new identities to solve trigonometric equations (which are often quadratic look-a-likes) before proving identities. Sub-unit 6f covers proving identities when all the available formulae have been covered.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

To contrast **reciprocal** trig functions students will also need to be familiar with the **inverse** functions of $\sin\theta$,$\cos\theta$ and $\tan\theta$. They will again need an understanding of the graphs of $\arcsin\theta$, $\arccos\theta$ and $\arctan\theta$. Refer back to the work on functions and emphasise that for arcsin, arccos and arctan to be true functions there must be a one-one relationship between domain and range and so the domains must be restricted to $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The most common errors in these questions involve using wrong notation, for example $\sin x^2$ instead of $\sin^2 x$, or making algebraic mistakes. Students sometimes struggle to deal with more complicated functions such as cosec $(3x + 1)$ and do not always recognise where trigonometric identities can be used.

NOTES

These trigonometric functions will be useful tools for the calculus units that follow later in the course.

6d. Compound* and double (and half) angle formulae (5.6a) *geometric proofs expected	Teaching time 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to prove geometrically the following compound angle formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$;
- be able to use compound angle identities to rearrange expressions or prove other identities;
- be able to use compound angle identities to rearrange equations into a different form and then solve;
- be able to recall or work out double angle identities;
- be able to use double angle identities to rearrange expressions or prove other identities;
- be able to use double angle identities to rearrange equations into a different form and then solve.

TEACHING POINTS

A good introduction is to ask the class to work out $\sin(30 + 60)°$.It is equal to $\sin(90)° = 1$. Go on to ask whether $\sin 30° + \sin 60°$ gives the same value (either using a calculator or using surds). They should discover that the values are different. Explain that the reason for this is that you can't simply multiply out functions in this way.

This leads in to explaining why compound angle formulae are needed to calculate $\sin(A + B)$.

Unit 1 above gives an example of a geometric proof by deduction for $\sin(A + B)$.

Care needs to be taken when using the result to extend to $\sin(A – B)$ for negative values. Students will need to remember that $\cos(–B) = \cos B$ and that $\sin(–B) = –\sin(B)$.

Extend these formulae by substituting $A = B$ to derive the double angle formulae

Show that there is only one version of $\sin 2x = 2 \sin x \cos x$, but the basic version of $\cos 2x = \cos^2 x – \sin^2 x$, can be re-written by substituting $\cos^2 x + \sin^2 x = 1$ (from AS Mathematics – Pure Mathematics) into two different versions (exclusively in $\sin x$ or $\cos x$).

A critical part of future questions and proofs involves choosing the correct version of the compound and/or double angle formulae.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Derive and cover examples using half angle formulae by adapting the double angle versions.

The next sub- unit will look at how to solve equations of the type $a \cos\theta + b \sin\theta = C$, using compound angles to rewrite and simplify the expression on the left hand side.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The most common errors are sign errors when using the compound and double angle formulae.

NOTES

$t\,(\tan\frac{1}{2}\theta)$ formulae will *not* be required.

You should cover reading off obtuse and reflex values by considering a right-angled triangle and assigning a negative or positive sign depending on which quadrant the angle lies in.

Double angle formulae will be a vital substitution when presented in calculus later in the course

6e. R cos $(x \pm \alpha)$ or R sin $(x \pm \alpha)$ (5.6b)	**Teaching time** 3 hours

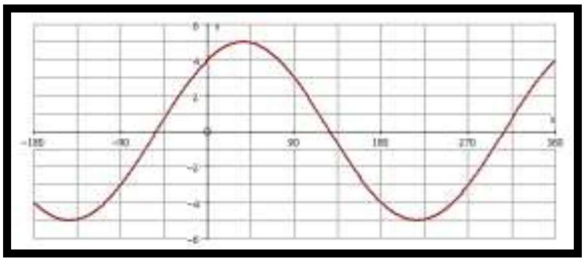
OBJECTIVES

By the end of the sub-unit, students should:

- be able to express $a \cos \theta + b \sin \theta$ as a single sine or cosine function;
- be able to solve equations of the form $a \cos \theta + b \sin \theta = c$ in a given interval.

TEACHING POINTS

Start by drawing a graph of, say, 4 cos x + 3 sin x to show that is has the basic sin–cos shape. Where are the coordinates of the maximum or minimum points? It approximately fits 5 cos (x – 40°).



Equating 4 cos x + 3 sin x to an expanded form of R cos (x – α) gives:

$$4 \cos x + 3 \sin x \equiv R \cos x \cos \alpha + R \sin x \sin \alpha$$

Equating coefficients leads to:

$$R \sin \alpha = 3 \text{ and } R \cos \alpha = 4.$$

By squaring and adding we obtain $R = 5$, and by dividing we obtain $\alpha = 36.9°$. (This confirms the approximate fit above.)

Move on to solving equations of the type $a \cos \theta + b \sin \theta = c$ using $R \cos (x \pm \alpha)$ or $R \sin (x \pm \alpha)$ as the first step. Effectively, the question reduces to a trigonometry equation like those done in Pure Paper 1, but at this level the angles could be in radians.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Ask students whether they can relate the R and α to the basic properties of the curve.

Think about the maximum/minimum value and where it occurs.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Examiner comments suggest that the part of the calculation which causes most problems is working out the angle α:

When writing $a \cos \theta + b \sin \theta$ into the form $R \sin(\theta - \alpha)$ most students found the value of R correctly, the same was not true of the angle α. Some students seemingly failed to notice that α was given as an acute angle.

When solving an equation of the form $a \cos \theta + b \sin \theta = c$ many students seemingly could not cope with the result of –39.23° that their calculator gave them and could not get the first solution. In addition some students found the third quadrant solution only, whereas some found more than two solutions. However many students did give a fully correct solution, often by using a sketch graph to help them decide where the solutions lay.

NOTES

On the legacy specifications, the form of expression to use was given in the question. Encourage students to choose which form to use. It is better to choose the version which, when expanded, gives the same signs for the corresponding terms as the original expression.

6f. Proving trigonometric identities (5.8)	**Teaching time** 4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to construct proofs involving trigonometric functions and previously learnt identities.

TEACHING POINTS

Proving trigonometric identities is something that challenges many students and is considered by some to be the most challenging part of the course.

The basic principles are the same as in Unit 1 (Proof): manipulate the LHS and use logical steps to make it to match the RHS or vice-versa. (Sometimes both sides can be manipulated to reach the same expression.)

Make sure you explain why we use $\equiv$ rather than $=$.

In the example below, the most efficient method is to start with the LHS and use $\sec^2\theta = 1 + \tan^2\theta$ to replace the numerator. The vital step is to multiply top and bottom of the resulting fraction by $\cos^2\theta$, this leads to the two familiar identities involving $\sin^2\theta$ and $\cos^2\theta$.

(a) Prove that

$$\frac{\sec^2\theta}{1-\tan^2\theta} = \sec 2\theta, \qquad \theta \neq \frac{n\pi}{4}, n \in \mathbb{Z}$$

(b) Hence state a reason why the equation

$$\frac{\sec^2\theta}{1-\tan^2\theta} = \frac{1}{2}$$

does not have any solutions.

The final step has 'Hence', so students should be encouraged to use the result in part (a) and write $\sec 2\theta = \frac{1}{2}$, which leads to $\cos 2\theta = 2$.

Students now need to explain fully that $-1 \leq \cos 2\theta \leq 1$, and so $\cos 2\theta = 2$ has no solutions.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The specification says 'Students need to prove identities such as $\cos x \cos 2x + \sin x \sin 2x \equiv \cos x$.'

Sub-unit 6g gives some examples of where trigonometry is used for problem solving.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

These questions often prove to be the most demanding on the paper and serve to differentiate between students.

Students need to make sure they include all steps in the proof with full explanation.

NOTES

It is a essential that students know which formulae are provided in the formulae book and which have to be learnt.

6g. Solving problems in context (e.g. mechanics) (5.9)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces.

TEACHING POINTS

Links can be made with simple harmonic motion in further mechanics, where a sin and/or cos curve could model the height of the tide against a harbour wall. When is it safe for the ship to come into the port?

For kinematics the velocity equation could be expressed as $v = 3\sin(2t)$ m s^{-1}. The times at which the object is stationary or at maximum speed could be analysed (no calculus at this stage).

An oscillating share price could be modelled using trigonometric equations. Ask students: when is the best time to buy and sell?

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

$h = a\cos t + b\sin t$ will model the tide height, h, and makes a good link with the previous section. ($t \geq 0$) Is t in degrees or radians?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The following question and comments come from a paper set in June 2014.

> A student records the number of hours of daylight each Sunday throughout the year. She starts on the last Sunday in May with a recording of 18 hours, and continues until her final recording 52 weeks later. She models her results with the continuous function given by
>
> $$H = 12 + 6\cos\left(\frac{2\pi t}{52}\right) + 2.5\sin\left(\frac{2\pi t}{52}\right), \quad 0 \leq t \leq 52$$
>
> where, H is the number of hours of daylight and t is the number of weeks since her first recording.
>
> Use this function to find the maximum and minimum values of H predicted by the model.

This was probably the least successful question. Although a good number of students could write down the maximum and minimum easily, some of those who had a correct value for the maximum then gave either –18.5 or 12 as the minimum. There were quite a number of students who had no idea how to tackle this part, often using values of H when $t = 0$ and $t = 52$.

Many students showed very little working at this stage, so it was sometimes unclear how much of the work was accurate.

NOTES

The specification says: 'Problems could involve (for example) wave motion, the height of a point on a vertical circular wheel, or the hours of sunlight throughout the year. Angles may be measured in degrees or in radians.'