UNIT 5: Elastic strings and springs and elastic energy

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SPECIFICATION REFERENCES

3.1 Elastic strings and springs. Hooke’s law.

3.2 Energy stored in an elastic string or spring.

PRIOR KNOWLEDGE

- Intuitive idea of elasticity in strings and springs

Covered so far

	KE, GPE and the work-energy principle (Unit 2)

AS Mathematics – Pure content

8.1 – 8.3	Basic integration (See SoW Year 1 Unit 4)

AS Mathematics – Mechanics content

8.1, 8.2, 8.4	Types of forces (in particular tension) and Newton’s laws (See SoW Year 1 Unit 9)

A level Mathematics – Mechanics content

8.2, 8.4, 8.5, 8.6	Resolving forces, friction, equilibrium and dynamics (See SoW Year 2 Units 4, 7 and 8)

KEYWORDS

String, spring, light, elasticity, modulus of elasticity (λ), extension, natural length, elastic potential energy (EPE), kinetic energy (KE), gravitational potential energy (GPE), joules (J), conservation of energy, equilibrium, work-energy principle, Newton’s 2nd law of motion, work done, joined or parallel strings/springs, friction, coefficient of friction, inclined plane.

NOTES

Specification states ‘Problems using the work-energy principle involving kinetic energy (KE), gravitational potential energy (GPE) and elastic potential energy (EPE) may be set.’

This forms a natural extension of the work and energy problems from AS Further Mechanics 1 Unit 2.

5a. Hooke's law and definition of modulus of elasticity (3.1). Deriving elastic potential energy formula (3.2)	Teaching time 7 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to investigate the ability of strings to stretch and springs to stretch and compress;
- be able to define the modulus of elasticity (λ), natural length (a) and extension (x);
- be able to use the above definitions to work out the tension in a stretched string or a stretched/compressed spring i.e. use Hooke's Law, $T = \frac{\lambda x}{a}$;
- be able to derive the elastic potential energy (EPE) from Hooke's Law by applying the work done in stretching a string/spring . i.e. EPE = $\frac{\lambda x^2}{2a}$.

TEACHING POINTS

This topic could be introduced by an experiment in which a string or spring has varying loads put on it. The load and amount of stretch (or extension) of the string can be plotted and should give a straight line graph (up to a limit). This shows that the extension is directly proportional to the applied load for a given string/spring.

What if we change to a different size or type of string/spring? The gradient should be different.

This investigation was performed by Robert Hooke (1635-1703) The unstretched length of a spring or string is defined as the natural length, a, the extension (or compression of a spring) is denoted x, and the force needed to *double* the unstretched length of a string (or halve the uncompressed length of a spring) is called the modulus of elasticity, λ.

[Take care that this is not confused with the spring constant k used in physics.]

If a string or spring is 'stronger', then λ is greater, as a greater force is needed to double the length.

Hooke's experiments found that the extension was directly proportional to the applied load (i.e. tension) and he proposed the following relationship connecting the modulus of elasticity, the extension and the natural length with the tension in a stretched string or spring:

$$T = \frac{\lambda x}{a}$$

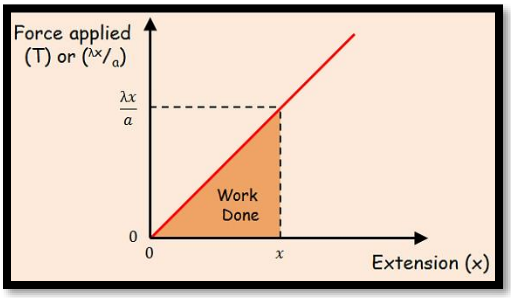
(In SI units both T and λ are measured in newtons)

This became known as Hooke's Law.

Note that if a string (or spring) is stretched too much it will deform and not return to its maximum length when released i.e. it has exceeded its 'elastic limit' and Hooke's Law will not apply. However, questions will refer to a 'perfectly elastic' string so that it can be assumed Hooke's Law is applicable.

Remember that Hooke's Law also applies to springs which are compressed with the thrust being proportional to the compression.

Work done in stretching a string



We know from the previous unit that, for a constant force: work done = force × distance

But here force varies as we stretch the string. In other words, the tension increases as the extension increases, so tension T is not constant.

For a variable force, work done = $\int$ F ds

So in this case we write:

Work done (in stretching the string from its natural length through an extension x)

$$= \int F\,\mathrm{d}s = \int T\,\mathrm{d}x = \int_0^x \frac{\lambda x}{a}\mathrm{d}x = \frac{\lambda x^2}{2a}$$

Note that this implies the work done is equivalent to area under the graph (as we would expect).

Therefore the work done in stretching the string from zero extension (natural length) to an extension of x is $\frac{\lambda x^2}{2a}$ and it is called the elastic potential energy (EPE). If SI units are used for force and length then the units of EPE are joules.

We can think of the EPE as energy stored in a string or spring, with the potential for it to be transformed into other forms of energy during the subsequent motion.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

It is important to point out that the EPE is the work done to reach an extension/compression of x but that if it is necessary to know the work done to reach an extension/compression of x_2 from one of x_1 then:

Work done $= \frac{\lambda({x_2}^2 - {x_1}^2)}{2l}$

More general problem solving will be considered in the next sub-unit.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

These questions are often well answered with the most common errors due to arithmetic slips and the misreading (or misunderstanding) of the question; for example, using 40g after being told the weight was 40 N.

NOTES

The final section of this topic will look at how we can use Hooke's law to solve problems involving equilibrium and the use of either the work-energy principle or the conservation of mechanical energy to solve problems involving the motion of a particle attached to an elastic string or a spring.

5b. Problem solving: equilibrium (3.1) and using the work-energy principle (3.2)

Teaching time
11 hours

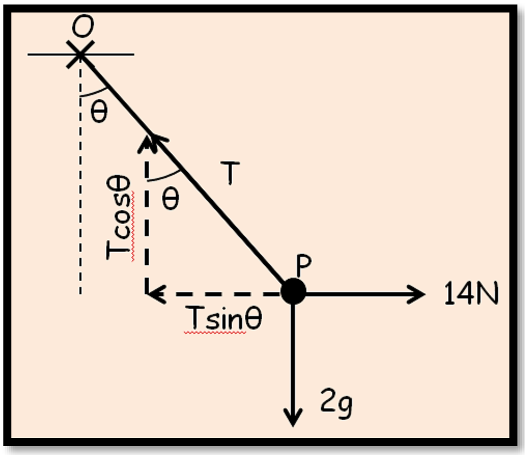
OBJECTIVES

By the end of the sub-unit, students should:

- be able to calculate the tension in a string or spring when a system is held in equilibrium;
- be able to include EPE when using the work-energy principle;
- know the conditions for conservation of mechanical energy;
- be able to solve string/spring problems involving work and energy (i.e. KE, GPE and EPE).

TEACHING POINTS

For systems in equilibrium (like the example below), you will need to resolve forces (making the resultant force = 0 N) and apply Hooke's law.



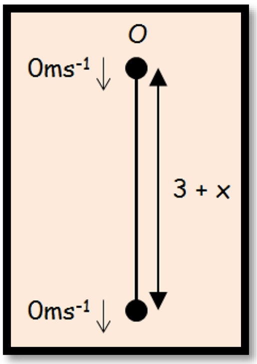
A particle could be attached to the ends of two different strings and held in equilibrium as in the example below:

> Fixed points A and B are on a horizontal ceiling, where $AB = 4a$. A light elastic string has natural length $3a$ and modulus of elasticity λ. One end of the string is attached to A and the other end is attached to B. A particle P of mass m is attached to the midpoint of the string. The particle hangs freely in equilibrium at the point C, where C is at a distance $\frac{3}{2}a$ vertically below the ceiling.
>
> Show that $\lambda = \frac{5mg}{4}$

Note that if a particle is **attached** to a point on a string then effectively it becomes a two string problem. The modulus of elasticity is unchanged for each part, and the sum of the natural lengths will be the natural length of the whole original string. In the above example the natural lengths of the 'half strings' will be 1.5a each. Vertical resolution and Hooke's Law lead to the required result.

When a system is not in equilibrium and the particle attached to an elastic string/spring is moving, the '$F = ma$' equation is complicated by the fact that the acceleration is not constant. In this specification the approach used to solve these problems is to consider the work-energy principle or conservation of mechanical energy. This is illustrated in the question below.

A particle of mass 0.6 kg is attached to one end of a light elastic string, of natural length 3 m and modulus of elasticity 9.8 N. The other end of the elastic string is attached to the fixed point O.
The particle is released from the point O.
Find the greatest distance reached by the particle below O.



The greatest distance the particle reaches below O occurs when the particle is instantaneously at rest.
In this case, there are no external forces (other than gravity) and the KE at the start and end is zero (greatest distance below O), so energy is conserved and we can equate the sum of GPE and EPE at the start with that at the end. In other words the loss in GPE = gain in EPE
It is of interest to note that the maximum speed of the particle occurs as it passes through its equilibrium position.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Further problem solving may include motion on a rough plane and consequently the work done against friction in addition to changes in energy. The first example below involves a horizontal plane and the second example involves an inclined plane. Both examples require the derivation of an inequality; it is important that students realise that $F = \mu R$ can only be assumed if the particle is moving or in limiting equilibrium; otherwise the inequality $F \leq \mu R$ must be applied.

A particle P of mass m is attached to one end of a light elastic string of natural length l and modulus of elasticity $3mg$. The other end of the string is attached to a fixed point O on a rough horizontal table. The particle lies at rest at the point A on the table, where $OA = \frac{7}{6}l$.
The coefficient of friction between P and the table is μ.

(a) Show that $\mu \geq \frac{1}{2}$

The particle is now moved along the table to the point B, where $OB = \frac{3}{2}l$, and released from rest.

(b) Given that $\mu = \frac{1}{2}$, find the speed of P at the instant when the string becomes slack.

* See Examiners Report below

One end A of a light elastic string AB, of modulus of elasticity mg and natural length a, is fixed to a point on a rough plane inclined at an angle θ to the horizontal. The other end B of the string is attached to a particle of mass m which is held at rest on the plane. The string AB lies along a line of greatest slope of the plane, with B lower than A and $AB = a$. The coefficient of friction between the particle and the plane is μ, where $\mu < \tan\theta$. The particle is released from rest. (*a*) Show that when the particle comes to rest it has moved a distance $2a(\sin\theta - \mu\cos\theta)$ down the plane. (*b*) Given that there is no further motion, show that $\mu \geq \frac{1}{3}\tan\theta$ * See Examiners Report below

In part (b) of the above example it is important to realise that the particle has already slid down the plane and come to rest so any further motion will be up the plane. It is therefore necessary to show that the tension in the string at the point where the particle stopped is less than or equal to the sum of the component of the weight down the plane and the maximum possible friction force.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The following quote relates to the first example above:

The first part highlighted the difficulty many candidates have in setting out a plausible proof, especially when an inequality is involved. All too often the tension in the string was correctly equated to the frictional force, which was then incorrectly stated to be equal to (rather than less than or equal to) μR. This showed a poor understanding of friction forces.

Once $\mu = \frac{1}{2}$ had been obtained it was common to see the required inequality appear. Sometimes an attempt to explain the inequality was given, but only in rare cases would this be satisfactory.

Most attempted the second part using the change in energy, successfully equating elastic potential energy, kinetic energy and work done against friction although there were some sign errors in the equations. However, many omitted the work done against friction from their calculation, in spite of the attention drawn to friction by completing the first part of the question.

The following quote relates to the second example above:

Many students did not recognise part (a) as a work-energy problem and tried to find an acceleration which they could use, ignoring the fact that the acceleration was in fact variable; others mixed forces and energy terms in their equation. This is a "show that" question and so students must show every step of their working.

NOTES

Many of the legacy (M3) exam papers, in which this topic was tested, include simple harmonic motion which is not in this course.