UNIT 3: Functions and modelling

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SPECIFICATION REFERENCES

2.7 The modulus of a linear function

2.8 Understand and use composite functions; inverse functions and their graphs

2.9 Understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$ including sketching associated graphs:

$y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$, $y = \mathrm{f}(ax)$

and combinations of these transformations

2.11 Use of functions in modelling, including consideration of limitations and refinements of the models

PRIOR KNOWLEDGE

<u>GCSE (9-1) in Mathematics at Higher Tier</u>

A7 Vocabulary and f(x) notation for functions

A13 Composite, inverse and transformations of polynomial functions

A12 Knowledge of polynomial, trigonometric, exponential and logarithmic functions, including their graphs

<u>AS Mathematics – Pure Mathematics content</u>

2.9 Transforming graphs (See Unit 1f of the SoW)

KEYWORDS

Function, mapping, domain, range, modulus, transformation, composite, inverse, one to one, many to one, mappings, $\mathrm{f}(x)$, $\mathrm{fg}(x)$, $\mathrm{f}^{-1}x$, reflect, translate, stretch.

NOTES

This topic very much builds on the functions section of the GCSE (9-1) Mathematics specification. The exponential function is an important function for modelling real-world problems such as growth and decay etc.

3a. Modulus function (2.7)

Teaching time
2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand what is meant by a modulus of a linear function;
- be able to sketch graphs of functions involving modulus functions;
- be able to solve equations and inequalities involving modulus functions.

TEACHING POINTS

Define the modulus of a set of numbers as being the positive values only. e.g. $|-2| = 2$ and $|5| = 5$.

Begin by using an ICT graph-drawing package (either using the whiteboard or students' individual devices) to sketch some linear graphs using both $y =$ and $f(x) =$ notation, e.g. $y = 2x - 1$ or $f(x) = 2x - 1$.

Display the graph of $y = |2x - 1|$ and discuss this with students, drawing comparisons with the 'non-modulus' graph and making sure everyone recognises that $y = |2x - 1|$ does not have any negative values of y (the graph 'bounces up' with the x-axis acting like a mirror).

Define the term modulus function and use the general notation $y = |f(x)|$.

Ask students to predict what the graph of $y = 2|x| - 1$ will look like and then plot it. This time the values of x that are substituted into the function cannot be negative. In other words the graph on the left of the y-axis is a reflection of the graph on the right (where the x-values are positive) with the y-axis being the line of symmetry.

The general notation for this type of function is $y = f|x|$.

Students should be able to sketch the graphs of $y =|ax + b|$ and use their graphs to solve modulus equations and inequalities.

Use the graph-drawing package to sketch the graph of $y = |2x - 1|$ and $y = x$ and use these to solve $|2x - 1| = x$ by considering the points of intersection. Ask students to think about how they might solve this equation algebraically without using a graph. Solving $2x - 1 = x$ gives one solution, but how would the 'modulus' part be represented algebraically? What is the equation of the straight-line graph that represents the 'bounced' part which is now above the y-axis?

Extend this idea to looking at inequalities, for example how to solve $|2x - 1| > x$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

What happens if we square a modulus? $|-2|^2 = 4$, so a modulus squared is always positive.

Apply this to the modulus equation above $|2x - 1|^2 = x^2$; leading to $3x^2 - 4x + 1 = 0$.

This quadratic gives the two solutions to the equation $y = |2x - 1|$ above. Does this always work? Does this work for inequalities?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students may find it difficult to sketch graphs involving modulus functions particularly if they are combined with other functions, for example logarithms.

In exam situations, often only the highest scoring students are able to solve modulus equations with x on both sides, or inequalities which involve the modulus function.

NOTES

The modulus function will also be used when expressing the validity of a binomial expansion in Unit 5.

3b. Composite and carte (2.8)	**Teaching time** 3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to work out the domain and range of functions;
- know the definition of a one-one and a many-one mappings;
- be able to work out the composition of two functions;
- be able to work out the inverse of a function and sketch its graph;
- understand the condition for an inverse function to exist.

TEACHING POINTS

The notation f: $x \mapsto \ldots$ and f(x) will be used as in GCSE (9-1) Mathematics.

Students will need to understand exactly what functions are and the notation associated with them.

Domain and range from $\mathbb{R}$ (or a subset of $\mathbb{R}$) to $\mathbb{R}$ are important terms for students to understand and should be used regularly. Link this to function machines and graphs (where the domain is the set of x-values and the range is the set of corresponding y-values).

Students should be aware of one-one and many-one mappings and know that a function cannot be one-many.

Definitions and examples of odd and even functions will need to be given

Students need to know how to find the inverse of a function and it is worth stressing the notation here as lots of students still differentiate when they see this in an exam.

Students should know that if f^{-1} exists, then $ff^{-1}(x) = f^{-1}f(x) = x$. It follows from this that the inverse of a many-one function can only exist if its domain is restricted to make it a one-one function.

Composite functions are also introduced here and it is worth spending some time going over why the order is very important. Students must know that fg means 'do g first and then f'. It may be helpful to use an an additional set of brackets in the notation for composite functions, e.g. f[g(x)].

Draw lots of examples of the above using graphing packages and relate the mappings to the graphs. Give an example of a quadratic in which the range is determined by the minimum or maximum point.

Students must also know that the graph of $f^{-1}(x)$ is the image of the graph of $y = f(x)$ after reflection in the line $y = x$. You could relate this to the reverse function machine and the algebraic approach for finding an inverse function (when you change the subject of the formula and rewrite it in terms of x as the final step).

Ask questions such as:

When does the function machine fail to find an inverse?

Do any functions have a self-inverse?

Is an inverse function always possible?

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The following activities are good for building familiarity and fluency with functions and notation.

1. Play a game where the teacher gives clues about a function and the student have to work out what the function is.

   Begin with numerical examples, e.g. $f(3) = 10$, $f(5) = 26$, $f(-2) = 5$ … until the correct $f(x)$ is given.

   Then to make it more relevant the clues should become algebraic, e.g. $f(4) = 14$, $f(2a) = 4a^2 - 2$, $f(x^2) = x^4 - 2$ etc.

2. Put all these ideas together by giving students two functions to explore. For example $f(x) = \frac{1}{x}$ and $g(x) = x + 1$ or $f(x) = |x|$ and $g(x) = x - 2$ or $f(x) = e^x$ and $g(x) = 2x - 1$.

   Students should explore the following using graphs etc.

   a) compare and contrast the graphs of $fg(x)$ and $gf(x)$
   b) work out if there are any one-one functions here
   c) find the inverses of any one-one functions (relating the inverses to the originals by sketching)

3. Students should investigate whether the following properties of functions are sometimes true, never true or always true.

   $fg(x) = gf(x)$ $\quad$ $g(x) = g^{-1}(x)$ $\quad$ $(fg)^{-1}(x) = g^{-1}f^{-1}(x)$ $\quad$ $(fg)^{-1}(x) = f^{-1}g^{-1}(x)$

An extension activity could be to find as many functions as possible such that $fg(x) = gf(x)$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students can often successfully find the range in exam questions, but some give their answer in terms of x rather than $f(x)$.

When finding inverse functions, students need to remember to swap x and y. When describing why a function does not have an inverse, students should be advised to answer this question as “because it is not one to one” or “because it is many to one”.

NOTES

Relate and link this work on functions to exponentials and natural logarithms covered in AS Pure Mathematics (Unit 8).

3c. Transformations (2.9)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$ including sketching associated graphs and *combinations* of the transformations: $y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$, $y = \mathrm{f}(ax)$;
- be able to transform graphs to produce other graphs;
- understand the effect of composite transformations on equations of curves and be able to describe them geometrically.

TEACHING POINTS

Students should have some understanding of graph transformations from GCSE (9-1) Mathematics and AS Mathematics – Pure Mathematics, but this will not necessarily include combinations of transformations.

Students need to be able to sketch the transformations $y = a\mathrm{f}(x) + b$, $a\mathrm{f}(x + b)$ and $\mathrm{f}(ax) + b$, but will not be required to sketch $\mathrm{f}(ax + b)$

Use graph drawing packages to investigate the properties of familiar functions (such as trigonometric and exponential functions) when you apply the above transformations. Relate the geometry of the transformation to the algebra. For example, $\mathrm{f}(x) + a$ adds a to all the y-coordinates, hence the graph moves 'up' by a units (translation vector).

Pose the question, "Does the order in which transformations are applied matter?" Ask students to explore this and present their findings to the class.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Students can explore the difference between transforming x *before* it goes through the function and transforming it afterwards.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Students often score well on questions which involve describing geometrical transformations, but incorrect use of terminology will lose marks. Students must use the correct terms: stretch, scale factor and translation.

Students also need to be aware that the order of transformations is often important.

NOTES

Link with the work on transformations in AS Mathematics – Pure Mathematics, see SoW Unit 1f.

3d. Modelling with functions (2.11)	Teaching Time 3 Hours

OBJECTIVES

By the end of the sub-unit, students should:

- use functions in modelling, including consideration of limitations and refinements of the models.

TEACHING POINTS

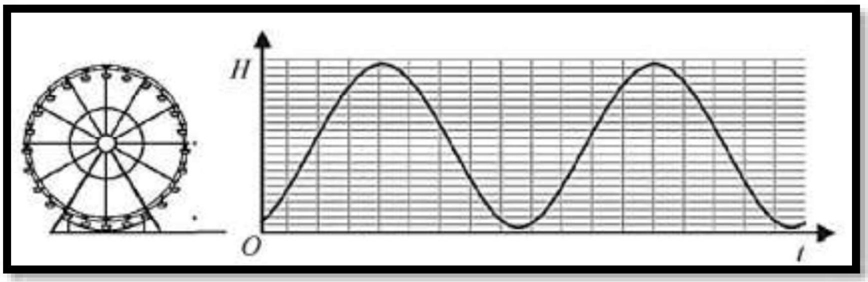
The specification gives some possible contexts in which functions can be used to model real-life situations. These are:

- Use of trigonometric functions for modelling tides, hours of sunlight, etc.
  (See the example in Unit 6g)
- Use of exponential functions for growth and decay (See AS Mathematics content - Pure Mathematics, Section 6.7).
- Use of reciprocal function for inverse proportion (e.g. Pressure and volume)

OPPORTUNITIES FOR REASONING/PROBLEM

Trigonometry example:

The height above the ground of a passenger on a Ferris wheel is modelled by the equation
$H = 11 - 10\cos(80t)° + 3\sin(80t)°$
where the height of the passenger above the ground is H metres, t minutes after the wheel starts turning.
Figure 3 below shows the graph of H against t for two complete cycles of the wheel.



Use the model to find the maximum height above the ground reached by the passenger

Exponential example:

The mass, m grams, of a radioactive substance t years after first being observed, is modelled by the equation

$$m = 25e^{-0.05t}$$

According to the model,

(a) state the value of m when the radioactive substance was first observed,

(b) show that $\frac{dm}{dt} = km$, where k is a constant that should be found.

(c) With reference to the model, interpret the significance of the sign of the value of k found in part (b).

NOTES

Rather like the 'Proof' (Unit 1), the applications of functions in context can appear throughout the specification. Link with 'Exponential Functions' (AS Mathematics - Pure Mathematics, see SoW Unit 8) & 'Trigonometry' in Unit 6.