UNIT 6: Differential equations

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SPECIFICATION REFERENCES

9.1 Find and use an integrating factor to solve differential equations of form $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ and recognise when it is appropriate to do so.

9.2 Find both general and particular solutions to differential equations.

9.3 Use differential equations in modelling in kinematics and in other contexts.

9.4 Solve differential equations of form $y'' + ay' + by = 0$, where a and b are constants, by using the auxiliary equation.

9.5 Solve differential equations of form $y'' + ay' + by = \mathrm{f}(x)$, where a and b are constants, by solving the homogeneous case and adding a particular integral to the complementary function (in cases where $\mathrm{f}(x)$ is a polynomial, exponential or trigonometric function).

9.6 Understand and use the relationship between the cases when the discriminant of the auxiliary equation is positive, zero and negative and the form of solution of the differential equation.

9.7 Solve the equation for simple harmonic motion $\ddot{x} = -\omega^2 x$ and relate the solution to the motion.

9.8 Model damped oscillations using second order differential equations and interpret their solutions.

9.9 Analyse and interpret models of situations with one independent variable and two dependent variables as a pair of coupled first order simultaneous equations and be able to solve them, for example predator-prey models.

PRIOR KNOWLEDGE

A level Mathematics – Pure content

7.1 – 7.6 Differentiation (See SoW Year 2 Unit 2)

8.2 – 8.8 Integration (See SoW Year 2 Unit 3)

KEYWORDS

Integrating, factor, complementary, function, differential, equation, order, auxiliary, discriminant, general, particular.

6a. Integrating factors to solve first order differential equations (9.1)(9.2)	Teaching time 6 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to identify the form of first order differential equations that can be solved by an integrating factor and carry out the solution;
- be able to find general and particular solutions of differential equations of this form.

TEACHING POINTS

It is a good point to start to revisit what a 1st Ordinary Differential Equations is. They should have already met one type in the A Level but it is just one of a family of differential equations. A good way to move on from here is to give examples that are the reverse to the product rule. This can be developed into the form of the Integrating factor. Good to show the proof of the integrating factor form. This will help with the development of the solution as will a firm reminder that the constant of integration should not be left out. It may also help to look at how the constant of integration can be chosen to help with the solution, especially where an answer is given.

Knowing the general form $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ will help identify when this technique is appropriate. It is important that $\mathrm{e}^{\int P(x)\mathrm{d}x}$ is known and used, but that's where the use of formulae should stop and each problem worked through once the IF is identified.

A solution process is a useful aide memoir:

1. Rearrange the differential equation into the form $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$
2. Find IF = $\mathrm{e}^{\int P(x)\mathrm{d}x}$
3. Multiply everything by the IF and verify the left hand side becomes the product rule.
4. Integrate both sides taking care when choosing the constant of integration.
5. Rearrange to an acceptable form/given answer.

The most interesting aspects to this approach is considering the graphs of the solutions, especially when the problems are more involved e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{2}y = 2\sin 5x$.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Considering different ranges for the constant of integration determines whether or not the solution is divergent or remains finite as $x \to \infty$. Online tools can be used to help graph the solution for various values of the constant of integration.

The convergence (or otherwise) of the solution can be further developed by using initial conditions to find the constant of integration and which initial conditions will determine that the solution remains finite.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Interpreting what is required in examinations is crucial here. Most mistakes occur when students do not realise the form of the equation and do not deploy the correct strategy for the initial method mark. Since the remaining marks typically depended on this mark, students will lose any marks available. It is wise to ask students to learn the form so they can recognise it, especially when some initial rearrangement is required.

Also you must emphasise that the constant of integration is important and should not be forgotten!

6b. Second order differential equations of the form $y'' + ay' + by = \mathrm{f}(x)$ (9.4)(9.5)(9.6)	**Teaching time** 7 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to solve second order differential equations of the form $y'' + ay' + by = \mathrm{f}(x)$ where $\mathrm{f}(x)$ is a polynomial, exponential or trigonometric function;
- be able to find general and particular solutions of second order differential equations of this form.

TEACHING POINTS

A methodical approach is essential to the successful completion of this sub-unit.

Begin with $\mathrm{f}(x) = 0$ and move quickly to the Principle of Superposition.

A good starting point is to give a solution and have the students simplify and substitute in the *y*", *y*' and *y* forms and develop the auxiliary equation, but then show how it can be written down without the initial working required. Consider each type based on the roots/discriminant of the auxiliary equation. It is useful if the form of the general solution is known for each type. This is a good opportunity to revise De Moivre's as the students can then be asked to develop the complex/imaginary solutions.

It is a good idea once the general solution is found to look at initial conditions/boundary value problems to determine the constants in the particular solution.

Move on to $\mathrm{f}(x) \neq 0$ and consider the particular integral required based on the form of $\mathrm{f}(x)$. It will help problem solving if each of these particular integrals are known. Take every opportunity to consider particular solutions using boundary value problems.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Develop a 'strategy sheet' that identifies what to do based on the problem being posed perhaps in the form of a flowchart.

Develop the need for the additional *x* in a particular integral and work out why needed.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Getting their variables mixed up in the complementary function is common as is the wrong sign on the power of e. The correct standard form of the particular integral and completing the necessary calculations to find the coefficients is usually well done. However, a common error in some examples is to then to ascribe them to the wrong trigonometric term.

6c. Modelling (9.3)(9.7)(9.8)(9.9)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use differential equations in modelling in kinematics and in other contexts;
- be able to solve the equation for simple harmonic motion $\ddot{x} = -\omega^2 x$ and relate the solution to the motion;
- be able to model damped oscillations using second order differential equations and interpret their solutions.

TEACHING POINTS

Take every opportunity to look at the physical situation that is to be modelled in theory e.g. weights bouncing on springs, pendulums etc. This will help with problem solving, especially if students are required to comment on the solution in context.

Contexts might include:

1 Vibrating springs (horizontal or vertical) attached to a fixed post with a mass on the end. Hooke's law and Newton's second law from physics will result in some fruitful SHM examples.

2 A shock absorber in a car is an example of damped vibrations that gives rise to $y'' + ay' + by = 0$ after applying Newton's second law. The damping force will need to be quoted. Overdamping, critical and underdamping can be considered and the corresponding solutions to the differential equation found.

3 Modelling forced vibrations can lead to $y'' + ay' + by = \mathrm{f}(x)$ with ma = damping force + external force + restoring force

4 Electrical circuits can also be modelled. Using Kirchhoff's voltage law for simple circuits can result in either first order or second order differential equations to be solved in context. Some simple physical relationships between variables will be required i.e. the equations typically covered in the topic of electricity in Physics.

5. Predator-prey equations in a closed eco-system. Consider historical development of Lotka-Volterra equations from the study of fish in the Mediterranean.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Problem solving in context bring differential equations to life. You may want to encourage those students studying physics to lead some practical group work using ideas from their physics course.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

To avoid misinterpretations of the physical context, the importance of a diagram cannot be underestimated. The more contexts that are encountered, the more it is likely that the interpretation of the context and solution will be successful.