UNIT 2: Matrices

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SPECIFICATION REFERENCES

3.1 Add, subtract and multiply conformable matrices.
Multiply a matrix by a scalar.
3.2 Understand and use zero and identity matrices.
3.3 Use matrices to represent linear transformations in 2D.
Successive transformations.
Single transformations in 3D.
3.4 Find invariant points and lines for a linear transformation.
3.5 Calculate determinants of 2×2 and 3×3 matrices and interpret as scale factors, including the effect on orientation.
3.6 Understand and use singular and non-singular matrices.
Properties of inverse matrices.
Calculate and use the inverse of non-singular 2×2 matrices and 3×3 matrices.
3.7 Solve three linear simultaneous equations in three variables by use of the inverse matrix.
3.8 Interpret geometrically the solution and failure of solution of three simultaneous linear equations.

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

- Transformations (rotations, translations, reflections, enlargement)
- Trigonometric ratios
- Simultaneous equations

AS Mathematics - Pure content

10.1 Vectors in 2D (See SoW Year 1 Units 5a and 5b)

A level Mathematics - Pure content

10.1 Vectors in 3D (See SoW Year 2 Unit 11)

KEYWORDS

Array, dimension, rows, columns, elements, scalar, square matrices, commutative, associative, transformation, rotation, translation, reflection, enlargement, linear transformation, scale factor, vector, position vector, object, image, identity, determinant, inverse, transpose, symmetric, zero matrix, minor, cofactor, singular, non-singular, three-dimensional space, line, plane, parameter, vector equation, Cartesian equation, simultaneous equations, invariant point, invariant line, sheaf, prism,

2a. Matrix addition, subtraction and multiplication (3.1)

Teaching Time
2 Hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to find the dimension of a matrix;
- be able to add and subtract matrices of the same dimension;
- be able to multiply a matrix by a scalar;
- be able to multiply conformable matrices.

TEACHING POINTS

Stress the importance of thinking it terms of rows and columns. (You-tube had some great 'songs' to embed this!). An $n \times m$ matrix has n rows and m columns.

It is not always possible to find both **AB** and **BA** when multiplying matrices as the dimensions do not allow it. In the case of square matrices though, it will always be possible to find both **AB** and **BA**. However, do not assume **AB** = **BA**. The technical term for this is to say that matrix multiplication is not **commutative**. It is very important to place matrices in the correct order when finding the product of two matrices.

Do not assume $\mathbf{A}^2 - \mathbf{B}^2 = (\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B})$.

$\mathbf{A}^2$ means **A** x **A**.

Matrix multiplication is associative. This means that when evaluating the product of three or more matrices, provided the order is kept the same it does not matter which product pair is evaluated first. For example, when evaluating **ABC**, it does not matter is we evaluate **(AB)C** or **A(BC)**, although sometimes forward planning can make the resulting calculations easier.

The 2×2 identity matrix is given by $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and the 3×3 identity matrix is given by $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Multiplying by the identity matrix is equivalent to multiplying by 1 in arithmetic. So **AI** = **IA** = **A**.

Care must be taken with signs and arithmetic when multiplying two matrices.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Permutation matrices. $\mathbf{A}^n$ matrices. Prove that for all 2×2 matrices **A**, **B** and C (**AB**)**C** = **A**(**BC**).

Show that if **A** (1×2), **B** (2×3) and **C** (3×4) then (**AB**)**C** = **A**(**BC**)

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

'Sums, differences and/or products of matrices often had inaccurate elements which lost at least two marks'.

'A number of students did not take the importance of the order of matrix multiplication into account'.

'A few students failed to get the correct dimensional order of the answer'.

'Explanations for why **AB** and **BA** are not equal were many and varied and often correct. There were lots of references to transformations and matrix multiplication not being commutative, but these attempts were not usually sufficient for this mark. The most common approach was a discussion about the dimensions of the resulting matrices'.

2b. Inverse of 2×2 and 3×3 matrices (3.5) (3.6)

Teaching Time
6 Hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to calculate determinants of 2×2 and 3×3 matrices;
- understand and use singular and non-singular matrices;
- be able to know the properties of inverse matrices;
- be able to calculate the inverse of non-singular 2×2 and 3×3 matrices.

TEACHING POINTS

Whilst understanding the process of finding the inverse of a matrix is required, students should be able to use a calculator to calculate the inverse of a matrix.

Students should show that they understand the process of inverting a matrix and the change in complexity of inverting a 2×2 to a 3×3. Adding dimensions to a matrix add significant complexity to find the inverse.

Generally should be using a calculator to find the inverse of a 3×3.

Opportunity to show matrix algebra.

If det(**A**) =0, then $\mathbf{A}^{-1}$ cannot be found, and **A** is a **singular** matrix.

Note that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

If **A** and **B** are non-singular matrices, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Show determinant properties for a $n\times n$ matrix.

- If A has a row or column which is all 0's then $|A| = 0$
- If A has 2 identical rows of columns then $|A| = 0$
- if B is obtained from A due to one row or column multiplying by a scalar $k \in \mathbb{R}$ then $|B| = k|A|$
- for an combination of 2 matrices. $|AB| = |A||B|$
- If B is obtained from A by interchanging 2 rows or columns then $|B| = -|A|$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The use of matrices for Encryption.

Cramer's rule using determinants to find the solution to a system of simultaneous equations. Could be used later on.

Investigating $(A^{-1})^{-1} = A$, $(kA)^{-1} = 1/k\, A^{-1}$, $(A^n)^{-1} = (A^{-1})^n$ where n is a positive integer.

Investigate the determinant properties.

Matrix Algebra. How to solve matrix equations. AB = C using the inverse.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUO1TES

‘A significant number of candidates thought that the determinant of **A** is $\frac{1}{\det\mathbf{A}}$.’

‘A large group could not find the inverse of a 2×2 matrix accurately’.

‘There were some errors in the calculation of the determinant and also some errors in the positions and signs of the elements within the inverse matrix’.

‘A significant number of students took the time to either factorise the determinant or to multiply the determinant into their matrix, neither of which were necessary to get all the available marks’.

‘There were some candidates who attempted to write down the inverse matrix in one step. With errors in their inverse in these cases it was difficult to identify a convincing method’.

NOTES

Students should become proficient at using their calculators to find the inverse of a matrix.

2c. Simultaneous equations (3.7, 3.8)	**Teaching Time** 8 Hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use matrices and their inverses to solve linear simultaneous equations, including three linear simultaneous equations in three variables;
- be able to interpret geometrically the solution and failure of solution of three simultaneous linear equations.

TEACHING POINTS

Write the simultaneous equations in matrix form and then use inverse matrices to solve. For example, to solve the equations $-3x + 4y - 2z = -17$, $2x - 3y + 5z = 23$ and $-5x + 6y + z = -14$, write the equations as $\begin{pmatrix} -3 & 4 & -2 \\ 2 & -3 & 5 \\ -5 & 6 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -17 \\ 23 \\ -14 \end{pmatrix}$, ie. $\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -17 \\ 23 \\ -14 \end{pmatrix}$, then the solution will be given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} -17 \\ 23 \\ -14 \end{pmatrix}$.

Show the development of the solution using matrices and matrix algebra as well as using Gaussian Elimination to show what solution profile you can have. 1 unique solution, no solution and an infinite number of solutions.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Linear programming.

Looking towards vector planes and the solution to the intersection of 3 planes.

Develop a system of equations that technology (GDC) can solve and then link this with the algebra of the solution.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Candidates should be encouraged to check their answers.

2d. Linear transformations (3.2, 3.3, 3.4)	**Teaching Time** 10 Hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to use matrices to represent 2D rotations, reflections, enlargements and translations;
- understand and use zero and identity matrices;
- be able to use matrix products to represent combinations of transformations;
- be able to use matrices to represent linear transformations in three dimensions;
- be able to use inverse matrices to reverse the effect of a linear transformation;
- be able to use the determinant of a matrix to determine the area scale factor of a transformation;
- be able to find invariant points and lines for a linear transformation.

TEACHING POINTS

Students will need to know the exact trigonometric ratios for sin, cos, tan of 0°, 30°, 45°, 60°, 90°.

To calculate the coordinates of a point after a transformation, multiply the transformation matrix by the coordinates.

To identify a 2D transformation from its matrix, consider how the points (1, 0) and (0, 1) in the form of the matrix $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ are transformed. For example, for the matrix **T**=$\begin{bmatrix}6 & 0\\0 & 3\end{bmatrix}$, the point (1, 0) is transformed to (6, 0), and the point (0, 1) is transformed to (0, 3) so the matrix T represents a stretch scale factor 6 parallel to the *x*-axis and a stretch scale factor 3 parallel to the *y*-axis.

The order matters: **A** followed by **B** followed by **C** is represented by **CBA**. For example, is the transformation *T* is represented by the matrix **T** and the transformation *U* is represented by the matrix **U**, then the matrix **UT** represents the combined transformation of the transformation *T* followed by the transformation *U*.

For 2D, identification and use of the matrix representation of single and combined transformations will be confined to:

- **reflection** in coordinate axes or the lines $y = \pm x$
- **rotation** about (0, 0) through any angle
- **stretches** parallel to the *x*-axis and *y*-axis
- **enlargement** about centre (0, 0), with scale factor k, ($k \neq 0, k \in \mathbb{R}$)

The identity matrix does not carry out any transformation.

When describing an enlargement, the centre will always be (0, 0), and this should be stated, along with the scale factor.

When describing a rotation, the centre will also always be (0, 0), and, again, this should always be stated, along with the angle and direction (anticlockwise is positive).

When describing a reflection, the mirror line must be stated.

Drawing diagrams to show the images of the original vectors is very useful.

The determinant of a matrix gives the area scale factor of a transformation. If the determinant is negative, the transformation involves a reflection.

To find the original points before a transformation was applied, transform each point separately using the inverse of the transformation matrix.

3D transformations will be confined to **reflection** in one of $x = 0$, $y = 0$ or $z = 0$, or **rotation** about one of the coordinate axes.

To identify the matrix representing a particular 3D transformation, consider the effect of the matrix or the transformation on three simple vectors; $\begin{pmatrix}1\\0\\0\end{pmatrix}$ (also denoted **i**), $\begin{pmatrix}0\\1\\0\end{pmatrix}$ (also denoted **j**) and $\begin{pmatrix}0\\0\\1\end{pmatrix}$ (also denoted **k**). Given any matrix $\mathbf{M} = \begin{pmatrix}a & b & c\\d & e & f\\g & h & i\end{pmatrix}$, the image of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ is the first column of **M**, the image of $\begin{pmatrix}0\\1\\0\end{pmatrix}$ is the second column of **M** and the image of $\begin{pmatrix}0\\0\\1\end{pmatrix}$ is the third column of **M**.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Use of successive transformations to prove trigonometric identities e.g. the addition formulae.

Taking a unit square using matrix transformations how many quadrilaterals can you generate, using rotations and stretches (shears). What would the matrix be for each.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

‘The most common error was failing to state the centre of rotation, the origin’.

The order of the transformations is important: ‘The majority of candidates used their answer from (b) correctly to multiply two matrices in the correct order, but a significant number multiplied them in the wrong order, gaining no marks’.

‘The majority knew the determinant property for area but some divided by det **A** instead of multiplying.’

‘Many students could not find the matrices securely to represent basic transformations’.

‘Some students left the matrix in terms of trigonometric ratios’.

NOTES

Transformation matrices are often used to create 3-D computer graphics.