UNIT 11: Integration (part 2)

Return to overview

SPECIFICATION REFERENCES

8.3 Use a definite integral to find the area under a curve and the area between two curves

8.4 Understand and use integration as the limit of a sum

8.5 Carry out simple cases of integration by substitution and integration by parts; understand these methods as the inverse processes of the chain and product rules respectively

8.6 Integrate using partial fractions that are linear in the denominator

8.7 Evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions

8.8 Interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution; includes links to kinematics

9.3 Understand and use numerical integration of functions, including the use of the trapezium rule and estimating the approximate area under a curve and limits that it must lie between

PRIOR KNOWLEDGE

Covered so far

- Laws of logarithms
- Trigonometry
- Partial fractions
- Differentiation

GCSE (9-1) in Mathematics at Higher Tier

A15 Areas under curves

AS Mathematics – Pure Mathematics content

7, 8 Differentiation and integration (See Units 6 &7 of SoW)

6.4 Laws of logarithms (See Unit 8 of SoW)

5.1 Trigonometry (See Unit 4 of SoW)

AS Mathematics – Mechanics content

7.2 Kinematics (velocity–time graphs) (See Unit 7 of SoW)

KEYWORDS

Integral, definite integral, integrand, limit, indefinite integral, constant of integration, trapezium, substitution, by parts, area, differential equation, first order, separating variables, initial conditions, general solution.

NOTES

This section completes the calculus for this course. It is also the pre-requisite for the calculus in some of the Further Mathematics units.

11a. Integration by substitution (8.5)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate expressions using an appropriate substitution;
- be able to select the correct substitution and justify their choices.

TEACHING POINTS

Most students find integration by substitution challenging and will need to complete lots of different styles of questions. It is a good idea to start with an example which can be performed by inspection as the reverse of differentiation.

Students also like to have a step by step process.

1. Use the given substitution or decide on your own. The substitution is usually the contents of a bracket, square root or the ‘nasty’ bit! i.e. Let $u = \dots$
2. Differentiate the substitution i.e. $\frac{\mathrm{d}u}{\mathrm{d}x} = \dots$
3. Make dx the subject of the formula
4. Replace the dx and make the substitution into the integrand
5. Cancel out any remaining x*
6. Integrate the resulting (simpler) integral
7. Substitute back to get the answer in terms of x again

*If there are any remaining x, you can re-use the substitution making the x the subject

For expressions including trigonometric functions, the identities involving $\sin^2 x$, $\sec^2 x$ are often useful to simplify the integrand.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Try to encourage students to experiment with different substitutions, particularly types involving expressions such as $\sqrt{3x+4}$. Do we use $u^2 = 3x + 4$ or $u = 3x+4$? The former will require implicit differentiation.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Mistakes students make when attempting to integrate by substitution include not changing the dx correctly and simply writing it as du, and failing to substitute back to give an expression in x at the end.

NOTES

Return to this method when covering areas under curves as the limits need to be changed by substituting them into the required substitution.

11b. Integration by parts (8.5)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate an expression using integration by parts;
- be able to select the correct method for integration and justify their choices.

TEACHING POINTS

It is a good idea to show how the product rule for differentiation can be integrated on both sides to derive the 'by parts' formula (which is given in the formulae booklet).

Students are usually able to start questions using this method but struggle to get to full solutions and will require lots of practice with algebraic manipulation.

Time should be spent discussing the choice of u and dv. It is usually advisable to select the polynomial to be the u as it simplifies to a lower power after calculating du, thus making the second integral easier than the original question.

Students should recognise that $\ln x$ cannot be integrated simply and should therefore always be chosen as u.

$\ln x$ itself can be integrated using this method taking $u = \ln x$ and dv = 1 (as we cannot integrate $\ln x$, but can differentiate it to give $\frac{1}{x}$). The dv becomes more complicated, but then simplifies in the second integral with the $\frac{1}{x}$.

More able students should be able to access questions where it is necessary to use integration by parts twice (e.g. $u = x^2$).

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider the integral of $e^x \cos x$ and show that the application of 'by parts' loops back to the original question. Refer to the equation $x = 4 - x$ and contrast this with the structure of this example.

Let the original question be I (for integral) and this can lead to $2I = \ldots$.

[This is a pre-requisite for reduction formulae in Further Pure Mathematics.]

Students should integrate functions such as $\int x(x+3)^6 \mathrm{d}x$ using both 'by parts' and 'substitution' to show that they give the same answer. This is a good activity for discussion as initially they appear to be different, but after some algebraic manipulation give the same answer.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Common errors when integrating by parts include: choosing u and dv incorrectly (in particular $\ln x$ must always be chosen as u); algebraic errors – especially if they do not remove any common factors to outside the integral sign; incorrect coefficients when integrating dv; and sign errors where sin and cos are involved.

NOTES

The method of integration by parts may be specified in the question.

Revisit this method when finding areas under curves (introducing limits) and/or the trapezium rule (for approximate areas).

11c. Use of partial fractions (8.6)	Teaching time 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to integrate rational expressions by using partial fractions that are linear in the denominator;
- be able to simplify the expression using laws of logarithms.

TEACHING POINTS

Revise the simplification of rational expressions into partial fractions. We have already seen that this technique is useful in binomial expansions.

Often the first part of an integration question of this sort will ask students to split the fraction into two (or more) partial fractions.

The next part will then ask for the integration to be carried out. For example:

Integrate $\int \frac{5}{(x-1)(3x+2)} \mathrm{d}x$.

This will lead to $\int \frac{5}{(x-1)(3x+2)} \mathrm{d}x = \int \left(\frac{1}{x-1} - \frac{3}{3x+2}\right) \mathrm{d}x = \ln(x-1) - \ln(3x+2)\ (+\,c)$

It is sometimes sufficient to leave the answer in this form, but 'Show that' questions will influence the further simplification using laws of logs.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Although the specification states 'linear in the denominator', you may want to cover repeated factors, which will lead to, for example, $(x-2)^2$ in the denominator, which will not be a log integral.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Partial fractions questions are generally done well though some students attempt to integrate the numerator and denominator separately without using partial fractions.

NOTES

These integrals will sometimes be tested via a differential equation later in the course and laws of logs will form a vital role in finding the general solution. Definite integrals may also need to be calculated and simplified numerically. e.g. $\ln 6 - \ln 2 = \ln \frac{6}{2} = \ln 3$.

11d. Areas under graphs or between two curves, including understanding the area is the limit of a sum (using sigma notation) (8.3) (8.4)

Teaching time
4 hours

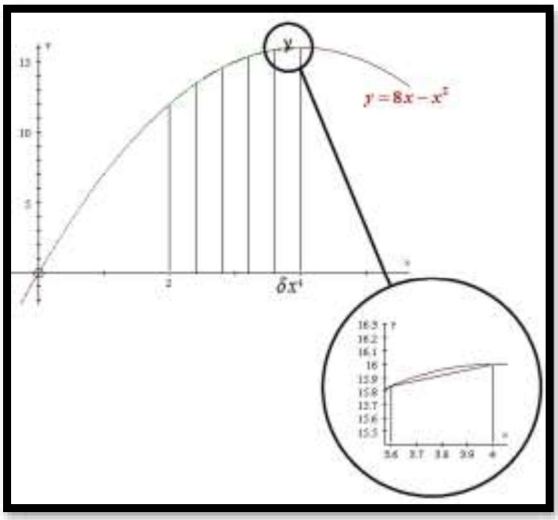
OBJECTIVES

By the end of the sub-unit, students should:

- understand and be able to use integration as the limit of a sum;
- understand the difference between an indefinite and definite integral and why we do not need + c;
- be able to integrate polynomials and other functions to find definite integrals, and use these to find the areas of regions bounded by curves and/or lines;
- be able to use a definite integral to find the area under a curve and the area between two curves.

TEACHING POINTS

Begin by showing a sketch of the curve and spit the area below it into thin strips, as shown below.



Now each strip is of elemental width δx, so the approximate area of each strip is $y\delta x$, where y is the height of each strip measured on the y-axis. If we sum all the strips, this would give us the total area below the curve. If the first strip starts at the point (2, 0) and the last strip ends at (4, 0), these become the limits on the definite integral. We can think of '4' as the area up to 4 and '2' as the area up to 2 (both measured across from the y-axis or $x = 0$).

We have seen from work on series, that we can use the sigma notation for sums so we can represent the area as $\sum y\delta x$. As δx gets thinner and thinner, the area becomes more accurate as the strips become more like rectangles. (This links nicely with the trapezium rule in the next sub-unit.)

We say that 'in the limit, as δx approaches zero' the sum becomes continuous rather than discrete and we can replace y with f(x) and $y\delta x$ becomes f(x)δx.

It happens that the rule for integration (which so far has only been used as the reverse of differentiation) gives the exact area under the curve. We can substitute in a and b, where the area's strips began and ended, as the limits of integration. The $y\delta x$ becomes f(x)δx and for the integral becomes f(x)dx. In other words the δx is the dx we have always understood as 'with respect to x'.

This leads to, $\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x$

Do lots of work on finding areas that require more than just a simple integral to be evaluated, for example when some of the area is below the x-axis or when finding the area between a line and a curve.

For example:

Find the finite area bounded by the curve $y = 6x - x^2$ and the line $y = 2x$.

Find the finite area bounded by the curve $y = x^2 - 5x + 6$ and the curve $y = 4 - x^2$.

Encourage students to always do a sketch or use a graph drawer to help with such questions.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Consider questions which have part of the graph below the x-axis, in which the area is negative. This time the roots are vital as we have to create two separate regions to calculate the total area. Show that just integrating between the start and end points will give a wrong result as the areas will subtract form each other.

Sometimes, you can create a new equation by subtracting the two areas *before* you integrate (when you have two curves and have to find the area between them):

$$\int_a^b y_1\,dx - \int_a^b y_2\,dx = \int_a^b (y_1 - y_2)\,dx$$

Care is needed with this method, and you should emphasise to students that they need to sketch it first making sure y_1 is higher than y_2.

Include questions where the area is found between a curve and the y-axis using $\int x\,dy$, with y-coordinates as limits.

Finally, consider areas which are bounded by curves defined by other types of functions, e.g. $y = e^{2x}$ or $y = \ln x$.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

The method for answering these types of exam questions is often understood, but many students loose accuracy marks due to arithmetical errors or using incorrect limits.

NOTES

Link this section to the trapezium rule which follows next.

11e. The trapezium rule (9.3)	Teaching time 2 hours

OBJECTIVES

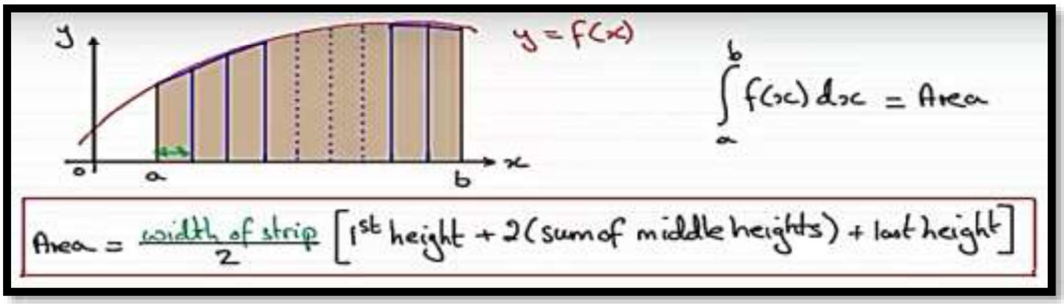
By the end of the sub-unit, students should:

- be able to use the trapezium rule to find an approximation to the area under a curve;
- appreciate the trapezium rule is an approximation and realise when it gives an overestimate or underestimate.

TEACHING POINTS

Make a direct link with the previous section and how to find an estimate for the area under a curve by dividing it into a finite number of strips. Sometimes an estimate is all that we need, and sometimes the integral is very complicated (or sometimes impossible) to integrate and so we have to estimate the area numerically.

The trapezium rule is given in the formula book (and may have also been covered in GCSE (9-1)). Students who struggle with algebra sometimes prefer to use the word version below:



Some students may be able to derive the rule by adding all the individual strips areas (i.e. $\frac{1}{2}h(y_0 + y_1) + \frac{1}{2}h(y_1 + y_2) + \ldots$) and then factorising to give the trapezium rule as in the formula book.

Ask students to calculate $\int_0^1 x\mathrm{e}^{2x}$ by using integration by parts and also by completing the table and using the trapezium rule (this is the quicker method). They should compare the answers they get using the different methods.

x	0	0.2	0.4	0.6	0.8	1
$y = xe^{2x}$	0	0.29836		1.99207		7.38906

Another example of the type of question that may be asked is:

Evaluate $\int_0^1 \sqrt{2x+1}\,\mathrm{d}x$ using the values of $\sqrt{2x+1}$ at $x = 0, 0.25, 0.5, 0.75$ and 1.

Make a sketch of the graph to determine whether the trapezium rule gives an over-estimate or an under-estimate of the exact value of the integral.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The following exam question shows a modelling example:

A river, running between parallel banks, is 20 m wide. The depth, y metres, of the river, measured at a point x metres from one bank, is given by the formula:

$y = \frac{1}{10}x\sqrt{20 - x}$, $0 \le x \le 20$

(*a*) Complete the table below, giving values of y to 3 decimal places.

x	0	4	8	12	16	20
y	0		2.771			0

(*b*) Use the trapezium rule with all the values in the table to estimate the cross-sectional area of the river.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When using the trapezium rule students sometimes mix up the number of strips and the number of x or y values.

The other main place marks are lost is not giving the final answer to three significant figures.

NOTES

Make sure that you use the same form for the trapezium rule as that given in the formula book.

11f. Differential equations (including knowledge of the family of solution curves) (8.7)(8.8)

Teaching time
4 hours

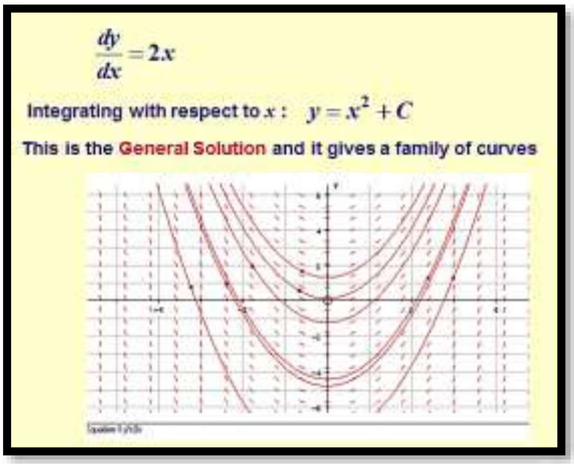
OBJECTIVES

By the end of the sub-unit, students should:

- be able to write a differential equation from a worded problem;
- be able to use a differential equation as a model to solve a problem;
- be able to solve a differential equation;
- be able to substitute the initial conditions or otherwise into the equation to find + c and the general solution.

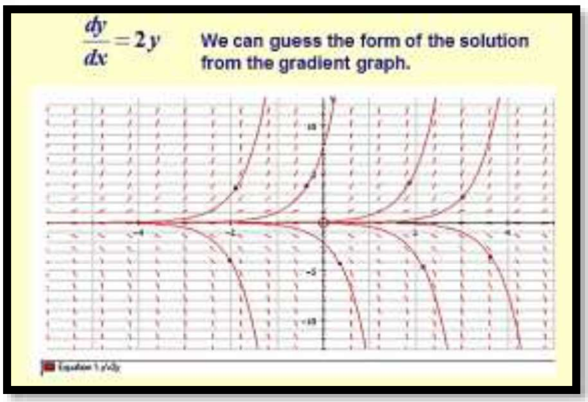
TEACHING POINTS

Begin by considering the simplest possible differential equation (defined as first order) as below.



Notice that the graph drawing tool can plot the differential equation to give a family of curves which mirror the solution (family of parabolas)

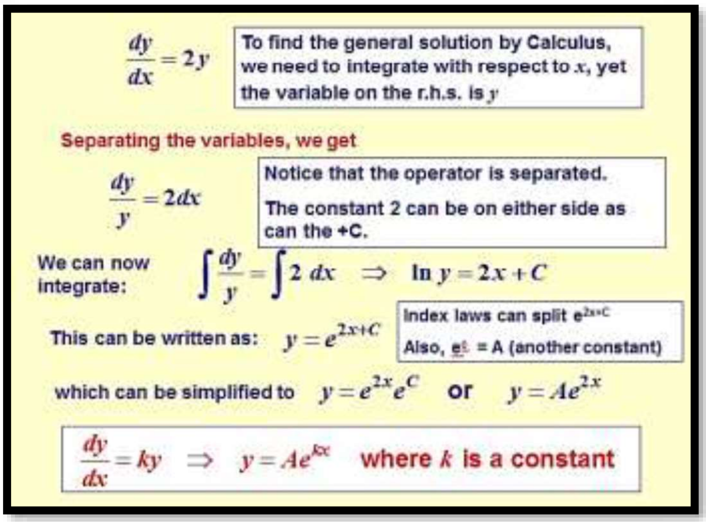
The next differential equation is more difficult as we cannot integrate directly because the variable is y rather than x. But looking at the family of curves may give us a clue about the eventual solution.



The curves look like exponentials.

The solution can be performed by using a method called ‘separating variables’, in which we rearrange and split up the $\frac{\mathrm{d}y}{\mathrm{d}x}$ as if it is a fraction. It is vital to keep all the y’s and dy’s *and* the x’s and dx’s together, but also the dx and dy must be in the numerator on each side.

The full solution is shown below.



As suspected, the family of curves were exponential curves.

$y = Ae^{2x}$ is a general solution, but how do we find the value of the constant A? We need to have some information about the data from which the differential equation originates. Something along the lines of ‘when $x = 0$, $y = 2$’.

Substituting this pair of values into the general solution and finding the value of A, will lead to a particular solution.

Sometimes we may have a choice of pairs to substitute or we may have two pairs of values in order to work out two constants.

Explain that questions may be set in a context and, in these cases, students need to interpret the solution of the differential equation in the context of the problem. This may including identifying limitations of the solution.

The following example is typical:-

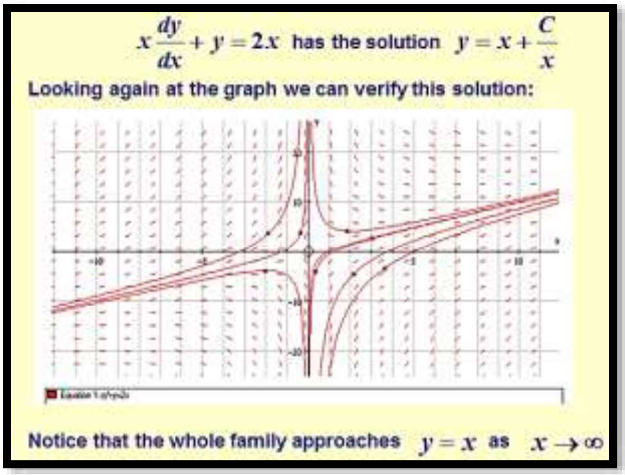
> The population of a town was 50 000 in 2010 and had increased to 55 000 by 2015. Assuming that the population is increasing at a rate proportional to its size at any time, estimate the population in 2020 giving your answer to the nearest hundred.

$\frac{db}{dt} = kn \;\; \Rightarrow \;\; n = Ae^{kt}$ as above, but now n is the number of people and t is the time in years.

The validity of the solution for large values should be considered, for example, if the question was modelling population growth; would it be realistic for the value to keep increasing forever?

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The example below has a family of curves which has elements of both $y = x$ and $y = \frac{1}{x}$, but it seems that the $y = x$ is trying to win!



Also, for separating variables and finding the particular solution, encourage the more able students to use the initial conditions as the limits of integration, thus avoiding the + c.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

Examiner comments indicate that this can prove a difficult topic for some students:

When forming a differential equation some students wrote down the correct differential equation apparently fully understanding all the information given and interpreting it correctly. However, all sorts of errors abounded in other attempts, some not even involving a derivative, and some with derivatives in x and y.

Many had a spurious t and/or h, either as a multiple or power, and the k appeared in a variety of places. Some students did not even form an equation, leaving a proportionality sign in their answer.

When solving a differential equation most students knew they were expected to separate the variables and did it correctly, although there were some notation errors in the positioning of dx, at the front rather than the rear of the integrand. Those who failed to separate the variables, just produced nonsense. Many students struggled with the fact that integration by parts or substitution was needed. All students, no matter what their attempt at the integral, could obtain a method mark if they included a constant and tried to find it using the given initial conditions.

NOTES

Link this topic to kinematics. For example solving differential equations of the form $\frac{\mathrm{d}v}{\mathrm{d}t} = 3t^2$ (when $t = 0$, $v = 4$). Separating variables leads to $v = t^3 + c$ etc.

$\frac{\mathrm{d}v}{\mathrm{d}t}$ is the acceleration and this shows that if we integrate the acceleration, we get the velocity.