UNIT 4: Series and sequences

Return to overview

SPECIFICATION REFERENCES

4.2 Work with sequences including those given by a formula for the *n*th term and those generated by a simple relation of the form $x_{n+1} = \text{f}(x_n)$; increasing sequences; decreasing sequences; periodic sequences

4.3 Understand and use sigma notation for sums of series

4.4 Understand and work with arithmetic sequences and series, including the formulae for *n*th term and the sum to *n* terms

4.5 Understand and work with geometric sequences and series including the formulae for the *n*th term and the sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $|r| < 1$; modulus notation

4.6 Use sequences and series in modelling

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

A23 Generate terms of a sequence from either a term-to-term or a position-to-term rule

A24 Use simple arithmetic and geometric progression and geometric sequence

A25 Finding expressions for the *n*th term of linear and quadratic sequences

KEYWORDS

Sequence, series, finite, infinite, summation notation, Σ(sigma), periodicity, convergent, divergent, natural numbers, arithmetic series, arithmetic progression (AP), common difference, geometric series, geometric progression (GP), common ratio, *n*th term, sum to *n* terms, sum to infinity (S_∞), limit.

NOTES

Specification states: 'The proof of the sum formula should be known' and 'Given the sum of a series students should be able to use logs to find the value of *n*'. So this unit links to Unit 1 above (Proof) and to AS Mathematics – Pure Mathematics, see SoW Unit 8 (Exponentials and logarithms).

4a. Arithmetic and geometric progressions (proofs of 'sum formulae') (4.4) (4.5) (4.6)

Teaching time
4 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know what a sequence of numbers is and the meaning of finite and infinite sequences;
- know what a series is;
- know the difference between convergent and divergent sequences;
- know what is meant by arithmetic series and sequences;
- be able to use the standard formulae associated with arithmetic series and sequences;
- know what is meant by geometric series and sequences;
- be able to use the standard formulae associated with geometric series and sequences;
- know the condition for a geometric series to be convergent and be able to find its sum to infinity;
- be able to solve problems involving arithmetic and geometric series and sequences;
- know the proofs and derivations of the sum formulae (for both AP and GP).

TEACHING POINTS

Start by recapping the work students did on sequences at GCSE (9-1) Mathematics before moving on to the new A level content, paving the way for the sigma notation in the following section.

Use practical situations, for example involving money, to illustrate APs and GPs and contrast the different ways they grow.

Find the nth term of a given arithmetic sequences and also use the rule to find the next two terms.

The Gauss problem (1 + 2 + … + 1000) is a good numerical way to lead into the full proof of the sum of an AP. Students will need to know the proof and derivation of the formula for the sum of an arithmetic sequence.

Illustrate how arithmetic sequences are different to geometric sequences, and explain that the common difference (a) becomes the common ratio (r). Students need to be aware that not all geometric sequences converge.

Cover problems where the n in the nth term formula (ar^{n-1}) is to be found using logarithms. (Show that it works if we use either base 10 or e.)

Illustrate when to use $\frac{a(1-r^n)}{(1-r)}$ and when to use $\frac{a(r^n-1)}{(r-1)}$ (depending on the value of r).

Show that $\frac{a}{(1-r)}$ can be derived if we illustrate on a calculator that r^n tends to zero when $-1 < r < 1$.

A way of illustrating the sum to infinity is to imagine hammering in a nail into a piece of wood, where each strike makes the nail sink in exactly half its remaining distance. There will be a limit to how many times it will need to be hit, as it surely will end up being 'flush' to the surface of the wood and have a distance of zero above the wood. (You can link this to Zeno's paradox.)

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

This topic can be linked to mechanics by investigating, for example, a ball which is dropped from 2 m and bounces to $\frac{3}{4}$ of its height after each bounce.

Challenge students to come up with a rule to determine which series will have a sum to infinity and which won't.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When working with formulae for sequences and series, it is important that students state the relevant formula before substituting so that method marks can be awarded even if there is a numerical slip.

NOTES

Move onto general notation of series by using the sigma and recurrence notations in the next sessions.

4b. Sigma notation (4.3)	**Teaching time** 2 hours

OBJECTIVES

By the end of the sub-unit, students should:

- be familiar with $\sum$ notation and how it can be used to generate a sequence and series;
- know how this notation will lead to an AP or GP and its sum;
- Know that $\sum_{1}^{n} 1 = n$.

TEACHING POINTS

The key to understanding the concept of $\sum$ is to look at the limit values and substitute them into the nth term formula to generate the terms of the sequence.

Emphasise to students that they must take care when finding the starting point and never assume it starts with $n = 1$.

Students may initially find the $\sum$ notation tricky, particularly if they are not asked to find the sum of first n terms, but instead asked to find, e.g. the 7th to the 20th.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Challenge students to try to work out whether a sequence is an AP, GP or neither from just looking at the structure of the sigma version of a series?

Ask students to write a series in sigma notation

Show that $\Sigma n = \frac{1}{2}n(n + 1)$ is the sum of n natural numbers and relate this to the sum formula derived in the previous section.

Think about what to do if the upper limit is infinity.

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

A fairly common error is to mix up the formulae for sums and terms, for example finding S_n rather than U_n and vice-versa.

NOTES

Students will need to be clear on the meanings and the usage of the various notations covered in this unit.

4c. Recurrence and iterations (4.2) (4.6)

Teaching time
3 hours

OBJECTIVES

By the end of the sub-unit, students should:

- know that a sequence can be generated using a formula for the *n*th term or a recurrence relation of the form $x_{n+1} = \mathrm{f}(x_n)$;
- know the difference between increasing, decreasing and periodic sequences;
- understand how a recurrence relation of the form $U_n = \mathrm{f}(U_{n-1})$ can generate a sequence;
- be able to describe increasing, decreasing and periodic sequences.

TEACHING POINTS

Work with sequences including those given by a formula for the *n*th term and those generated by a simple relation of the form $x_{n+1} = \mathrm{f}(x_n)$ and link this with the work done on iterations in GCSE (9-1) Mathematics.

Explore $x_{n+1} = \mathrm{f}(x_n)$ type series using graphics calculators or spreadsheets. (You can draw links between this work and Unit 9 – Numerical methods.)

Move on to general recurrence relations of the form $U_n = \mathrm{f}(U_{n-1})$ and investigate which sequences are increasing, decreasing and periodic. Spend some time looking at the different forms of notation for recurrence relations, making sure you cover examples of increasing, decreasing and periodic sequences. For example:

$u_n = \frac{1}{3n+1}$ describes a decreasing sequence as $u_{n+1} < u_n$ for all integers n

$u_n = 2^n$ is an increasing sequence as $u_{n+1} > u_n$ for all integers n

$u_{n+1} = \frac{1}{u_n}$ for $n > 1$ and $u_1 = 3$ describes a periodic sequence of order 2.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Cover questions in which sequences can be used to model a variety of different situations. For example, finance, growth models, decay, periodic (tide height for example) etc.

Can you tell from the structure of a recurrence relation how it will behave, and the type of sequence it will generate?

COMMON MISCONCEPTIONS/EXAMINER REPORT QUOTES

When asked to find the limit of u_n some candidates use the sum to infinity of a geometric series.

NOTES

Encourage the use of the ANS button on a calculator to obtain the terms for a recurrence relation.