UNIT 5: Algebra and functions

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SPECIFICATION REFERENCES

4.1 **Understand and use the relationship between roots and coefficients of polynomial equations up to the quartic equations**

4.2 **Form a polynomial equation whose roots are a linear transformation of the roots of a given polynomial equation (of at least cubic degree)**

PRIOR KNOWLEDGE

GCSE (9-1) in Mathematics at Higher Tier

- Expanding and factorising quadratic

AS Mathematics - Pure content

2.6 Factor theorem (See SoW Year 1 Unit 2b)

KEYWORDS

Quadratic, cubic, quartic, polynomial, coefficient, degree, root, complex conjugate, degree, Vieta's formulas

5a. Roots of polynomial equations (4.1)	**Teaching Time** 4 Hours

OBJECTIVES

By the end of the sub-unit, students should:

- understand and use the relationship between roots and coefficients of polynomial equations up to quartic equations.

TEACHING POINTS

Although a polynomial may have complicated roots, we can often determine a lot about them using Vieta's formulas, which don't necessarily tell us what the roots are, but enable us to evaluate other expressions.

Go through Vieta's formulas, which tell us that:

- A quadratic equation with roots α and β can be written as $ax^2 + bx + c = a(x-\alpha)(x-\beta)$. Expanding and equating coefficients of x will yield $\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$. If $a = 1$, this simplifies to $\alpha + \beta = -b$ and $\alpha\beta = c$.
- A cubic equation with roots α, β and γ can be written as $ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$. This will yield $\alpha + \beta + \gamma = \frac{-b}{a}$, $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$ and $\alpha\beta\gamma = \frac{-d}{a}$. If $a = 1$, this simplifies to $\alpha + \beta + \gamma = -b$, $\alpha\beta + \beta\gamma + \alpha\gamma = c$ and $\alpha\beta\gamma = -d$.
- A quartic equation with roots α, β, γ and δ can be written as $ax^4 + bx^3 + cx^2 + dx + e = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$. This will yield $\alpha + \beta + \gamma + \delta = \frac{-b}{a}$, $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \beta\gamma = \frac{c}{a}$ and $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a}$.and $\alpha\beta\gamma\delta = \frac{e}{a}$. If $a = 1$, this simplifies to $\alpha + \beta + \gamma + \delta = -b$, $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \beta\gamma = c$ and $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -d$.and $\alpha\beta\gamma\delta = e$.

These formulas need to be learned. They follow an obvious pattern.

Show that these formulas can be used to evaluate expressions such as $\alpha^2 + \beta^2 + \gamma^2$, $(3+\alpha)(3+\beta)(3+\gamma)$ and $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$, without knowing the actual roots, which are most likely complex in many cases. Explain that the expressions should be manipulated to get them in terms of Vieta's formulas. For example, $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$. This can be rearranged to make $\alpha^2 + \beta^2 + \gamma^2$ the subject (see the next point), and the values from Vieta's formulas can be substituted in. Similarly $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ and $(3+\alpha)(3+\beta)(3+\gamma) = 27 + 9(\alpha + \beta + \gamma) + 3(\alpha\beta + \beta\gamma + \alpha\gamma) + \alpha\beta\gamma$.

Knowledge of the following identities will be helpful:

- $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$
- $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
- $\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3\alpha\beta(\alpha + \beta) - 3\alpha\gamma(\alpha + \gamma) - 3\beta\gamma(\beta + \gamma) - 6\alpha\beta\gamma$

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

The possible use of these to solve complex number problems. Explain that complex number polynomials are like normal ones and hence roots can be found may be but complex.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

It is usually more difficult to actually try to find the roots of the polynomial. 'A surprising number of candidates found the roots of the original equation and proceeded to answer the whole question using them. This often required lengthy manipulations and calculations where mistakes were common'.

Sometimes the question explicitly states 'without solving the equation', and students who answer the question ignoring this instruction will gain no marks.

'A number of candidates thought that $\alpha^3 + \beta^3$ was $(\alpha + \beta)^3$ and so lost two marks. A number of candidates worked with individual roots involving surds and were presumably unaware of the sum and product properties.'

Students must be careful with signs, particularly in the bracketed terms.

5b. Formation of polynomial equations (4.2)	**Teaching Time** 3 Hours

OBJECTIVES

By the end of the sub-unit, students should:

- be able to form a polynomial equation whose roots are a linear transformation of the roots of a given polynomial equation (of at least cubic degree).

TEACHING POINTS

The method used could be substitution but is most likely to be using sums and products of roots.

Start with quadratic equations, for example, if α and β are roots of the equation $3x^2 + 5x - 1 = 0$ find a quadratic equation that has roots of $\alpha + 3$, $\beta + 3$. Using sums and products of roots, this problem can be solved as follows: Using $\alpha + \beta = \frac{-5}{3}$ and $\alpha\beta = \frac{-1}{3}$, then the sum of new roots is $\alpha + 3 + \beta + 3 = \frac{-5}{3} + 6 = \frac{13}{3}$ and the product of new roots is $(\alpha + 3)(\beta + 3) = \alpha\beta + 3(\alpha + \beta) + 9 = \frac{-1}{3} - 5 + 9 = \frac{11}{3}$. So the quadratic equation with roots $\alpha + 3$, $\beta + 3$ becomes $z^2 - \frac{13}{3}z + \frac{11}{3} = 0$, ie. $3z^2 - 13z + 11 = 0$.

Now extend this to cubics and then quartics.

OPPORTUNITIES FOR REASONING/PROBLEM SOLVING

Synthetic division.

COMMON MISCONCEPTIONS/ EXAMINER REPORT QUOTES

Errors in forming the new quadratic equation include: the omission of "= 0", not giving integer coefficients and the omission of "x" in the middle term i.e. giving an answer as $8x^2 - 33 + 8 = 0$

'The most common error was a failure to perform basic algebraic manipulation on the sum of the roots'.