
Introduction to Linear Programming

LP Model Formulation

A Maximization Example

Resource 40 hrs of labor per day

Availability: 120 lbs of clay

Decision x_1 = number of bowls to produce per day

Variables: x_2 = number of mugs to produce per day

Objective Maximize $Z = \$40x_1 + \$50x_2$

Function: Where Z = profit per day

Resource $1x_1 + 2x_2 \leq 40$ hours of labor

Constraints: $4x_1 + 3x_2 \leq 120$ pounds of clay

Non-Negativity $x_1 \geq 0; x_2 \geq 0$

Constraints:

LP Model Formulation

A Maximization Example

Complete Linear Programming Model:

$$\text{Maximize } Z = \$40x_1 + \$50x_2$$

$$\begin{aligned} \text{subject to: } & 1x_1 + 2x_2 \leq 40 \\ & 4x_1 + 3x_2 \leq 120 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Feasible Solutions

A **feasible solution** does not violate **any** of the constraints:

Example: $x_1 = 5$ bowls

$x_2 = 10$ mugs

$$Z = \$40x_1 + \$50x_2 = \$700$$

Labor constraint check: $1(5) + 2(10) = 25 < 40$ hours

Clay constraint check: $4(5) + 3(10) = 50 < 120$ pounds

Infeasible Solutions

An **infeasible solution** violates **at least one** of the constraints:

Example: $x_1 = 10$ bowls

$x_2 = 20$ mugs

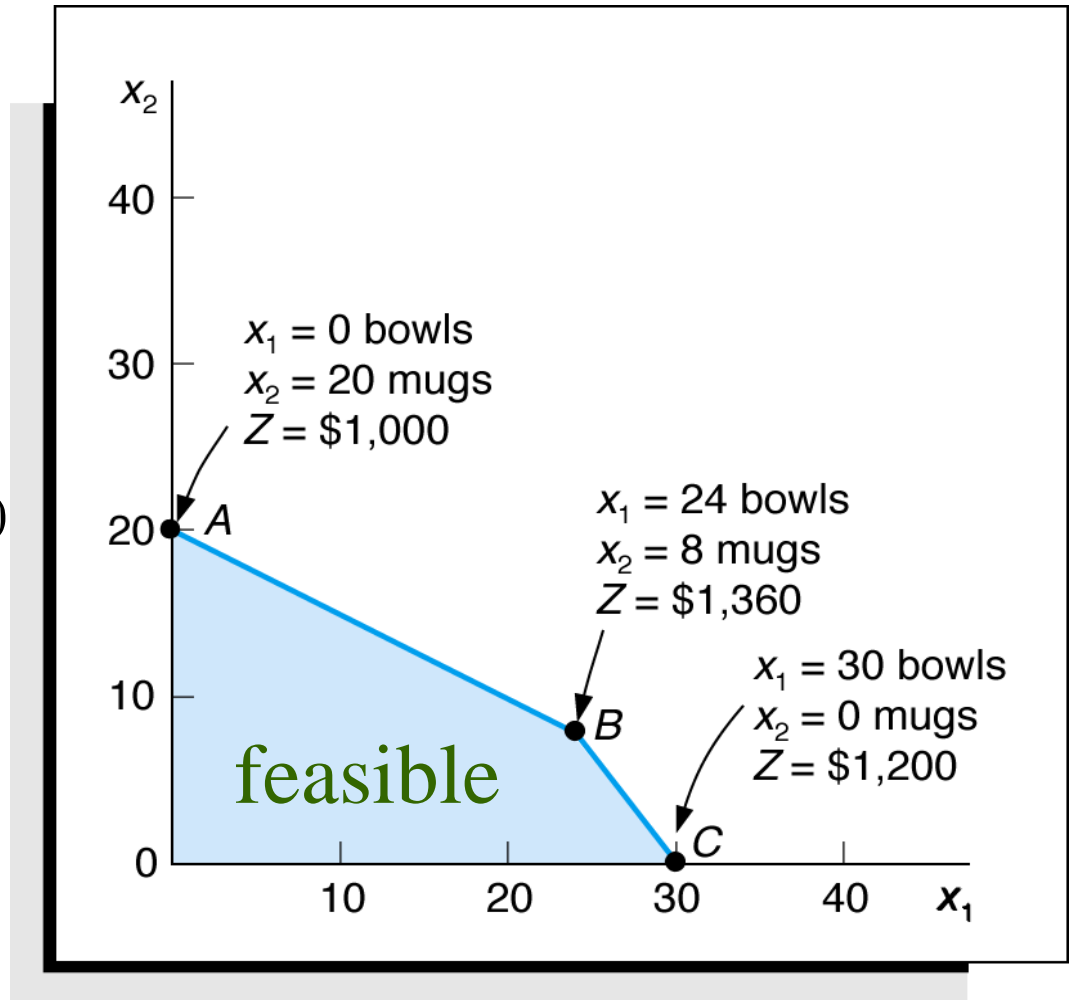
$$Z = \$40x_1 + \$50x_2 = \$1400$$

Labor constraint check: $1(10) + 2(20) = 50 > 40$ hours

Extreme (Corner) Point Solutions

Graphical Solution of Maximization Model

Maximize $Z = \$40x_1 + \$50x_2$
subject to:
 $1x_1 + 2x_2 \leq 40$
 $4x_1 + 3x_2 \leq 120$
 $x_1, x_2 \geq 0$

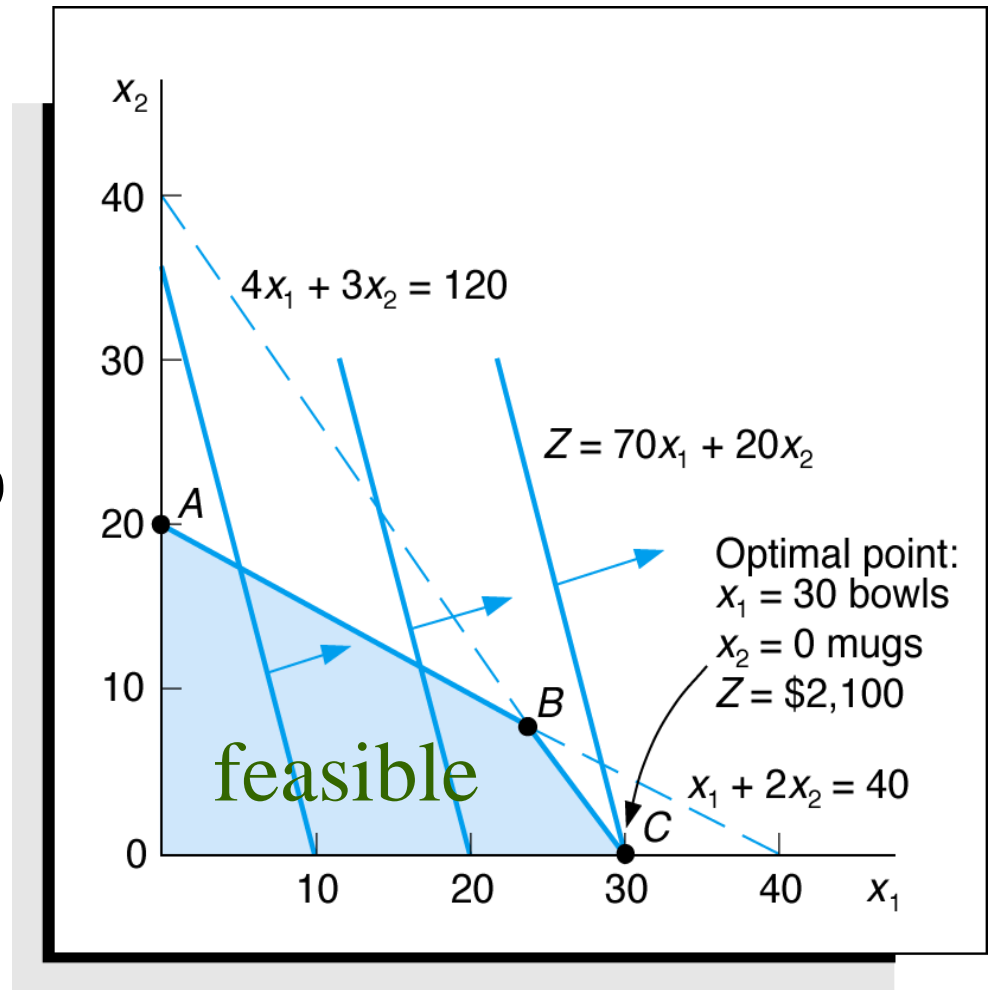


Solutions at All Corner Points

Optimal Solution for New Objective Function

Graphical Solution of Maximization Model

Maximize $Z = \$70x_1 + \$20x_2$
subject to:
 $1x_1 + 2x_2 \leq 40$
 $4x_1 + 3x_2 \leq 120$
 $x_1, x_2 \geq 0$



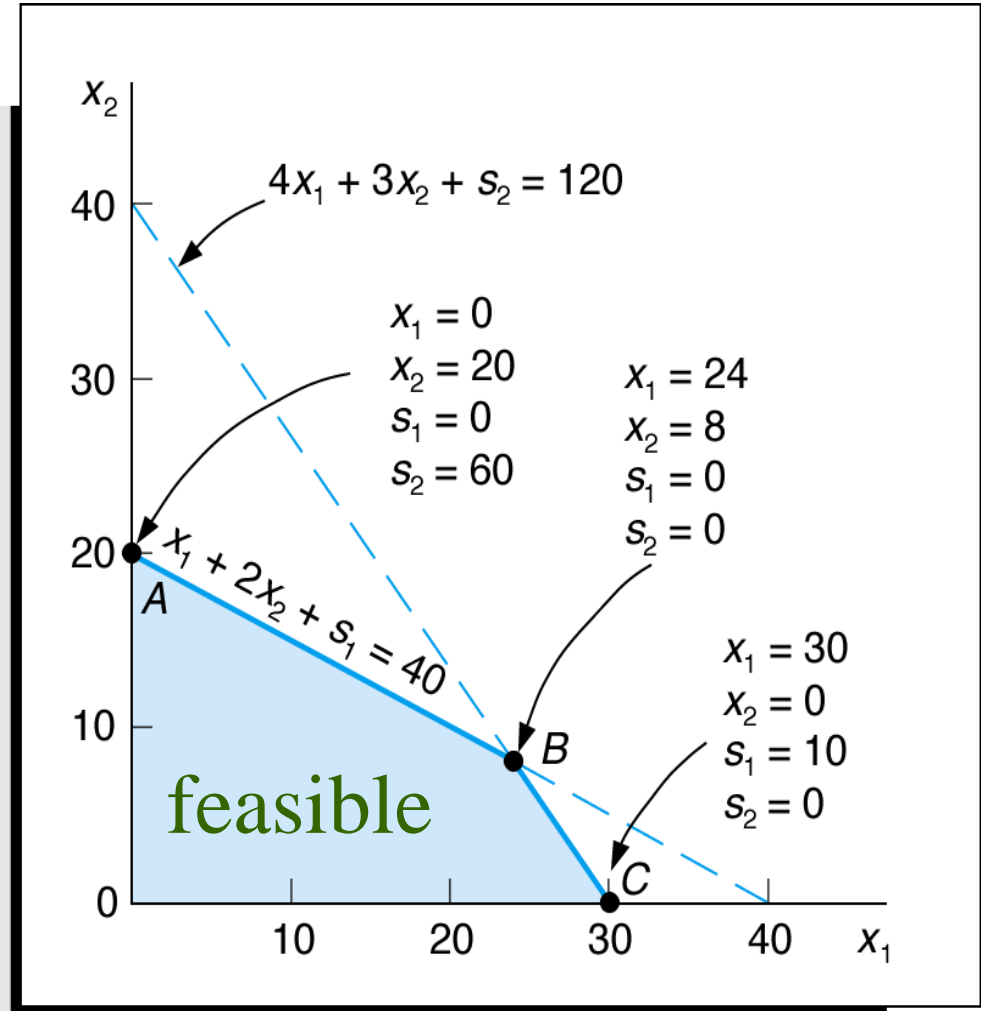
Optimal Solution with $Z = 70x_1 + 20x_2$

Linear Programming Model: Standard Form

$$\begin{aligned}\text{Max } Z &= 40x_1 + 50x_2 + 0s_1 + 0s_2 \\ \text{subject to: } &1x_1 + 2x_2 + s_1 = 40 \\ &4x_1 + 3x_2 + s_2 = 120 \\ &x_1, x_2, s_1, s_2 \geq 0\end{aligned}$$

Where:

x_1 = number of bowls
 x_2 = number of mugs
 s_1, s_2 are slack variables



Solution Points A, B, and C with Slack

What if we add a plate and 5 hours of labor?

Profit \$30

Labor = 0.5 hours, clay = 2 lbs

Maximize $Z = \$40x_1 + \$50x_2 + \$30x_3$

subject to: $1x_1 + 2x_2 + 0.5x_3 \leq 45$
 $4x_1 + 3x_2 + 2x_3 \leq 120$
 $x_1, x_2 \geq 0$

$$\max 40x_1 + 50x_2 + 30x_3$$

ST

$$x_1 + 2x_2 + 0.5x_3 \leq 45$$

$$4x_1 + 3x_2 + 2x_3 \leq 120$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_3 \geq 0$$

OBJECTIVE FUNCTION VALUE

1) 1860.000

| VARIABLE | VALUE | REDUCED COST |
|----------|-----------|------------------|
| X1 | 0.000000 | 20.000000 |
| X2 | 12.000000 | 0.000000 |
| X3 | 42.000000 | 0.000000 |

| ROW | SLACK OR SURPLUS | DUAL PRICES |
|-----|------------------|-------------|
| 2) | 0.000000 | 4.000000 |
| 3) | 0.000000 | 14.000000 |
| 4) | 0.000000 | 0.000000 |
| 5) | 12.000000 | 0.000000 |
| 6) | 42.000000 | 0.000000 |

Reduced Cost

is the amount by which an objective function coefficient would have to improve (so increase for maximization problem, decrease for minimization problem) before it would be possible for a corresponding variable to assume a positive value in the optimal solution.

Only positive for decision variables with a value of 0.

What if mugs had a profit of \$60?

OBJECTIVE FUNCTION VALUE **max** $40x_1 + 50x_2 + 30x_3$

1) 1860.000

| VARIABLE | VALUE | REDUCED COST |
|----------|-----------------|------------------|
| X1 | 0.000000 | 20.000000 |
| X2 | 12.000000 | 0.000000 |
| X3 | 42.000000 | 0.000000 |

| ROW | SLACK OR SURPLUS | DUAL PRICES |
|-----|------------------|-------------|
| 2) | 0.000000 | 4.000000 |
| 3) | 0.000000 | 14.000000 |
| 4) | 0.000000 | 0.000000 |
| 5) | 12.000000 | 0.000000 |
| 6) | 42.000000 | 0.000000 |

OBJECTIVE FUNCTION VALUE **max** $60x_1 + 50x_2 + 30x_3$

1) 1860.000

| VARIABLE | VALUE | REDUCED COST |
|----------|-----------|--------------|
| X1 | 21.000000 | 0.000000 |
| X2 | 12.000000 | 0.000000 |
| X3 | 0.000000 | 0.000000 |

| ROW | SLACK OR SURPLUS | DUAL PRICES |
|-----|------------------|-------------|
| 2) | 0.000000 | 4.000000 |
| 3) | 0.000000 | 14.000000 |
| 4) | 21.000000 | 0.000000 |
| 5) | 12.000000 | 0.000000 |
| 6) | 0.000000 | 0.000000 |

Dual Price / Shadow Price

is the instantaneous change, per unit of the constraint, in the objective value of the optimal solution obtained by relaxing the constraint.

In other words, it is the marginal utility of relaxing the constraint, or, equivalently, the marginal cost of strengthening the constraint.

What if we add 5 hours of labor? The objective increases by $5 \times 4 = \$20$.

OBJECTIVE FUNCTION VALUE **max $40x_1 + 50x_2 + 30x_3$**

1) 1860.000

| VARIABLE | VALUE | REDUCED COST |
|----------|-----------|--------------|
| X1 | 0.000000 | 20.000000 |
| X2 | 12.000000 | 0.000000 |
| X3 | 42.000000 | 0.000000 |

| ROW | SLACK OR SURPLUS | DUAL PRICES |
|-----|------------------|-----------------|
| 2) | 0.000000 | 4.000000 |
| 3) | 0.000000 | 14.000000 |
| 4) | 0.000000 | 0.000000 |
| 5) | 12.000000 | 0.000000 |
| 6) | 42.000000 | 0.000000 |

OBJECTIVE FUNCTION VALUE **max $40x_1 + 50x_2 + 30x_3$**

1) 1880.000

| VARIABLE | VALUE | REDUCED COST |
|----------|-----------|--------------|
| X1 | 0.000000 | 20.000000 |
| X2 | 16.000000 | 0.000000 |
| X3 | 36.000000 | 0.000000 |

| ROW | SLACK OR SURPLUS | DUAL PRICES |
|-----|------------------|-------------|
| 2) | 0.000000 | 4.000000 |
| 3) | 0.000000 | 14.000000 |
| 4) | 0.000000 | 0.000000 |
| 5) | 16.000000 | 0.000000 |
| 6) | 36.000000 | 0.000000 |

LP Model Formulation – Minimization

Decision Variables:

x_1 = bags of Super-gro

x_2 = bags of Crop-quick

The Objective Function:

Minimize $Z = \$6x_1 + 3x_2$

Where: $\$6x_1$ = cost of bags of Super-Gro

$\$3x_2$ = cost of bags of Crop-Quick

Model Constraints:

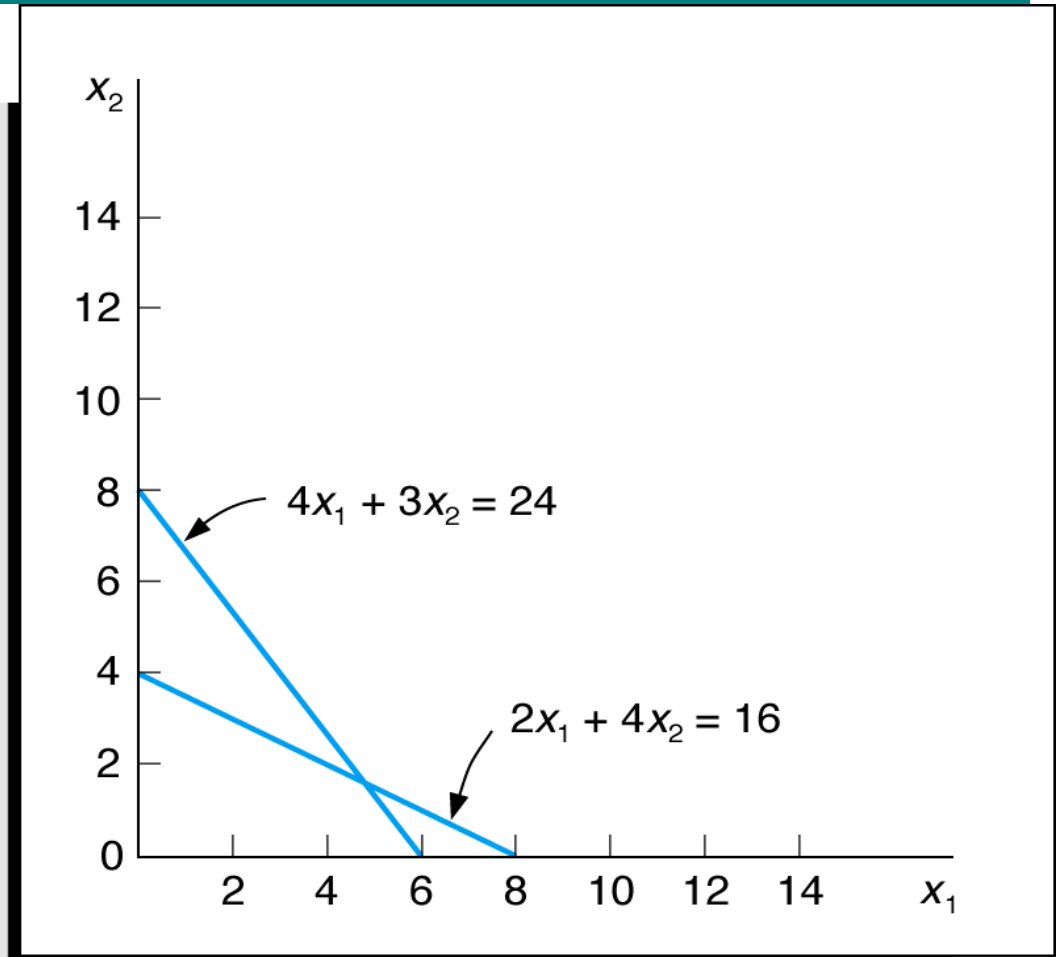
$2x_1 + 4x_2 \geq 16$ lb (nitrogen constraint)

$4x_1 + 3x_2 \geq 24$ lb (phosphate constraint)

$x_1, x_2 \geq 0$ (non-negativity constraint)

Constraint Graph – Minimization

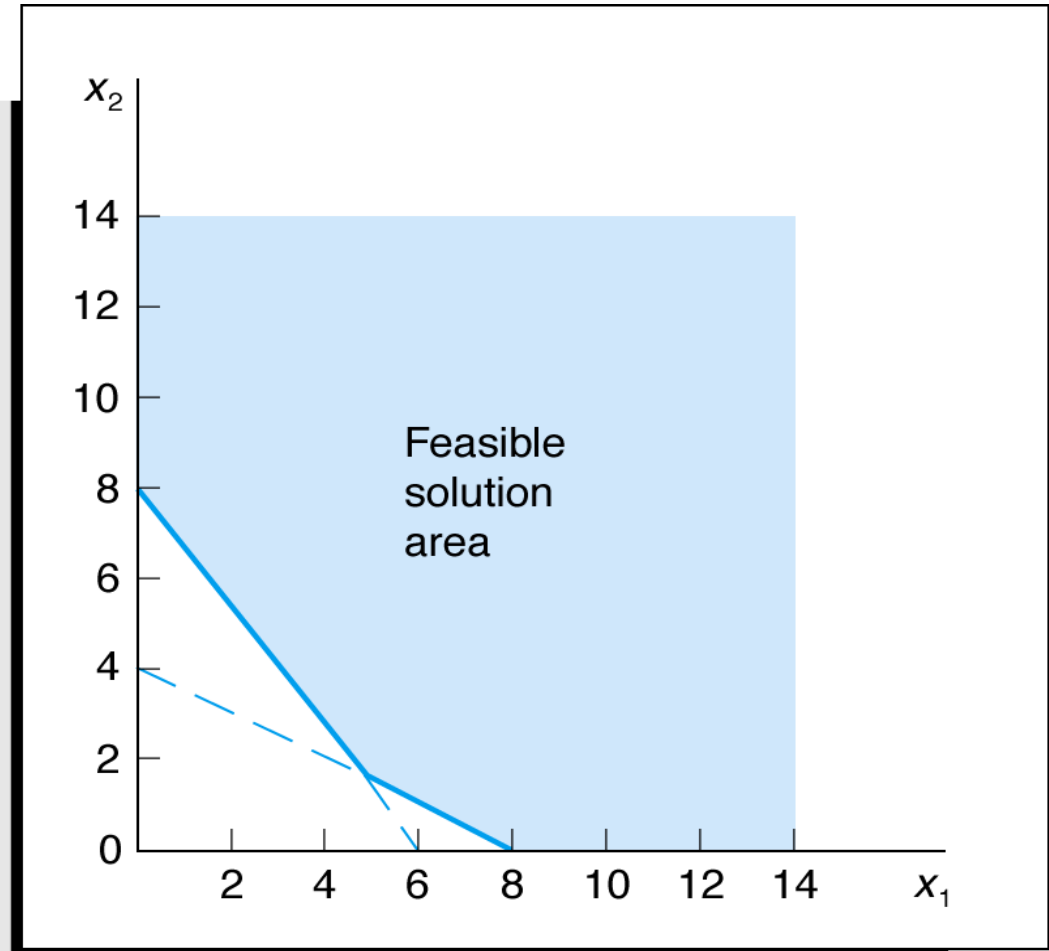
Minimize $Z = \$6x_1 + \$3x_2$
subject to:
 $2x_1 + 4x_2 \geq 16$
 $4x_1 + 3x_2 \geq 24$
 $x_1, x_2 \geq 0$



Graph of Both Model Constraints

Feasible Region– Minimization

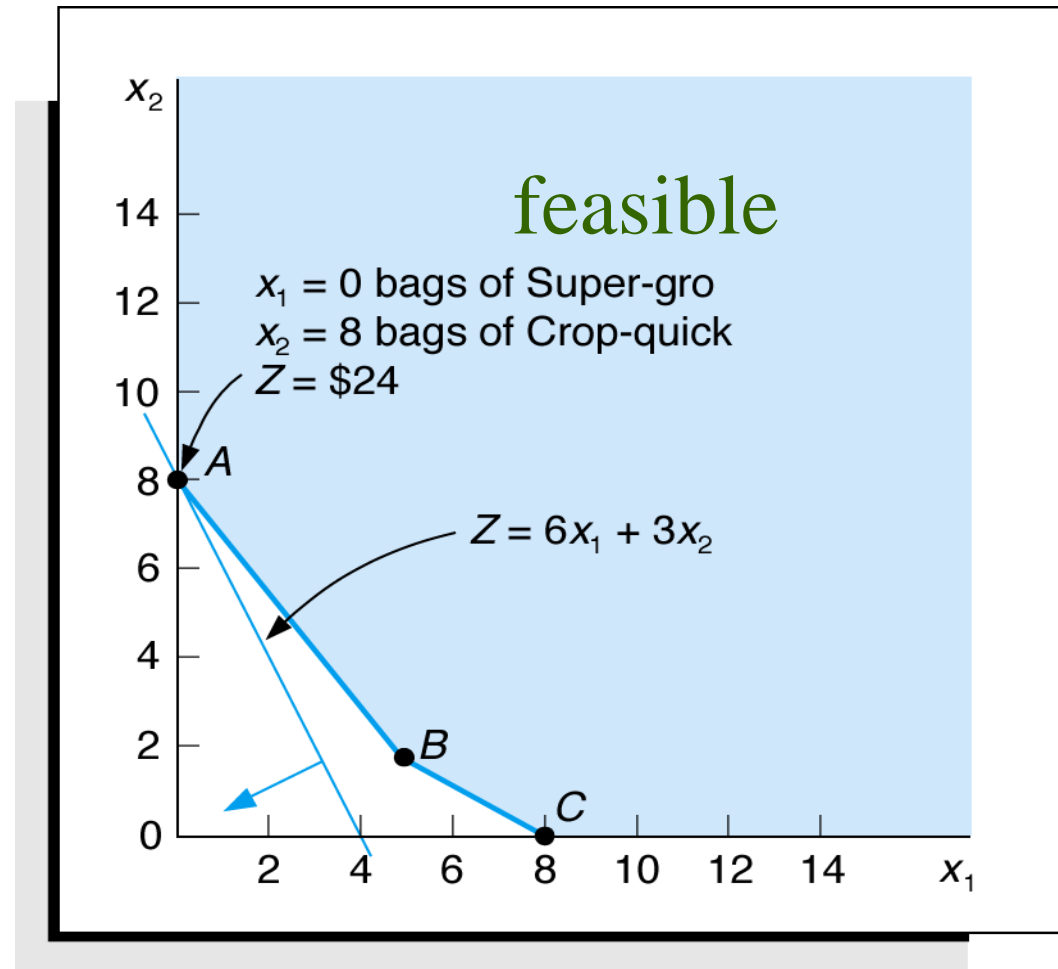
Minimize $Z = \$6x_1 + \$3x_2$
subject to: $2x_1 + 4x_2 \geq 16$
 $4x_1 + 3x_2 \geq 24$
 $x_1, x_2 \geq 0$



Feasible Solution Area

Optimal Solution Point – Minimization

Minimize $Z = \$6x_1 + \$3x_2$
subject to:
 $2x_1 + 4x_2 \geq 16$
 $4x_1 + 3x_2 \geq 24$
 $x_1, x_2 \geq 0$



Optimum Solution Point

Surplus Variables – Minimization

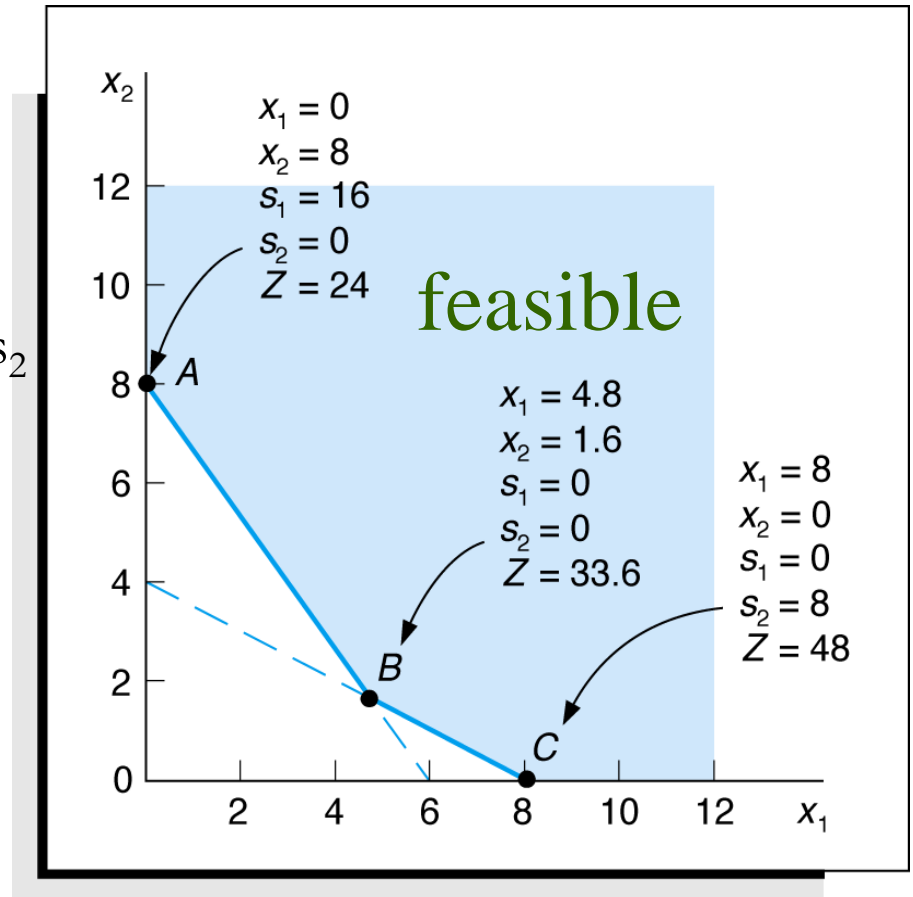
- A surplus variable is *subtracted from a \geq constraint* to convert it to an equation (=).
- A surplus variable *represents an excess* above a constraint requirement level.
- A surplus variable *contributes nothing* to the calculated value of the objective function.
- Subtracting surplus variables in the farmer problem constraints:

$$2x_1 + 4x_2 - s_1 = 16 \text{ (nitrogen)}$$

$$4x_1 + 3x_2 - s_2 = 24 \text{ (phosphate)}$$

Graphical Solutions – Minimization

Minimize $Z = \$6x_1 + \$3x_2 + 0s_1 + 0s_2$
subject to:
 $2x_1 + 4x_2 - s_1 = 16$
 $4x_1 + 3x_2 - s_2 = 24$
 $x_1, x_2, s_1, s_2 \geq 0$



Graph of Fertilizer Example

Irregular Types of Linear Programming Problems

For some linear programming models, the general rules do not apply.

Special types of problems include those with:

- Multiple optimal solutions
- Infeasible solutions
- Unbounded solutions

Multiple Optimal Solutions Beaver Creek Pottery

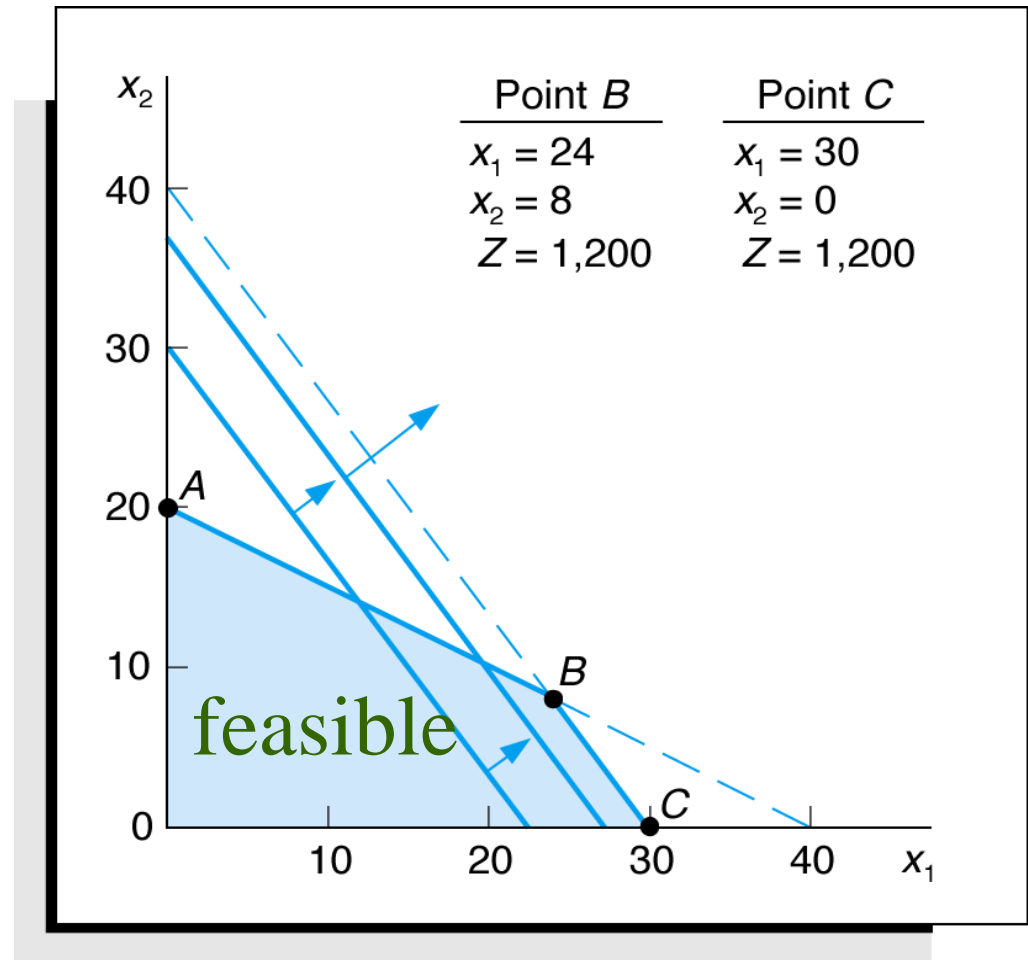
The objective function is **parallel** to a constraint line.

Maximize $Z = \$40x_1 + 30x_2$
subject to: $1x_1 + 2x_2 \leq 40$
 $4x_1 + 3x_2 \leq 120$
 $x_1, x_2 \geq 0$

Where:

x_1 = number of bowls

x_2 = number of mugs

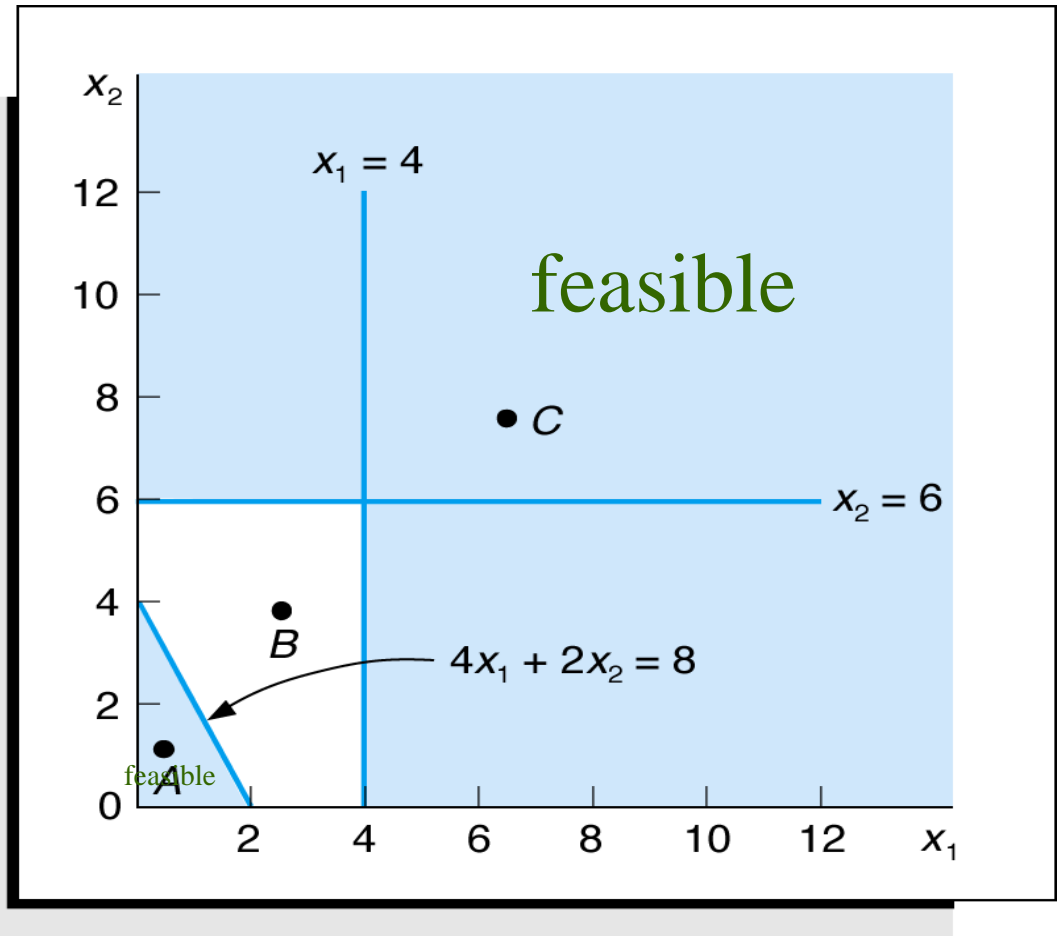


Example with Multiple Optimal Solutions

An Infeasible Problem

Every possible solution
violates at least one constraint:

Maximize $Z = 5x_1 + 3x_2$
subject to: $4x_1 + 2x_2 \leq 8$
 $x_1 \geq 4$
 $x_2 \geq 6$
 $x_1, x_2 \geq 0$

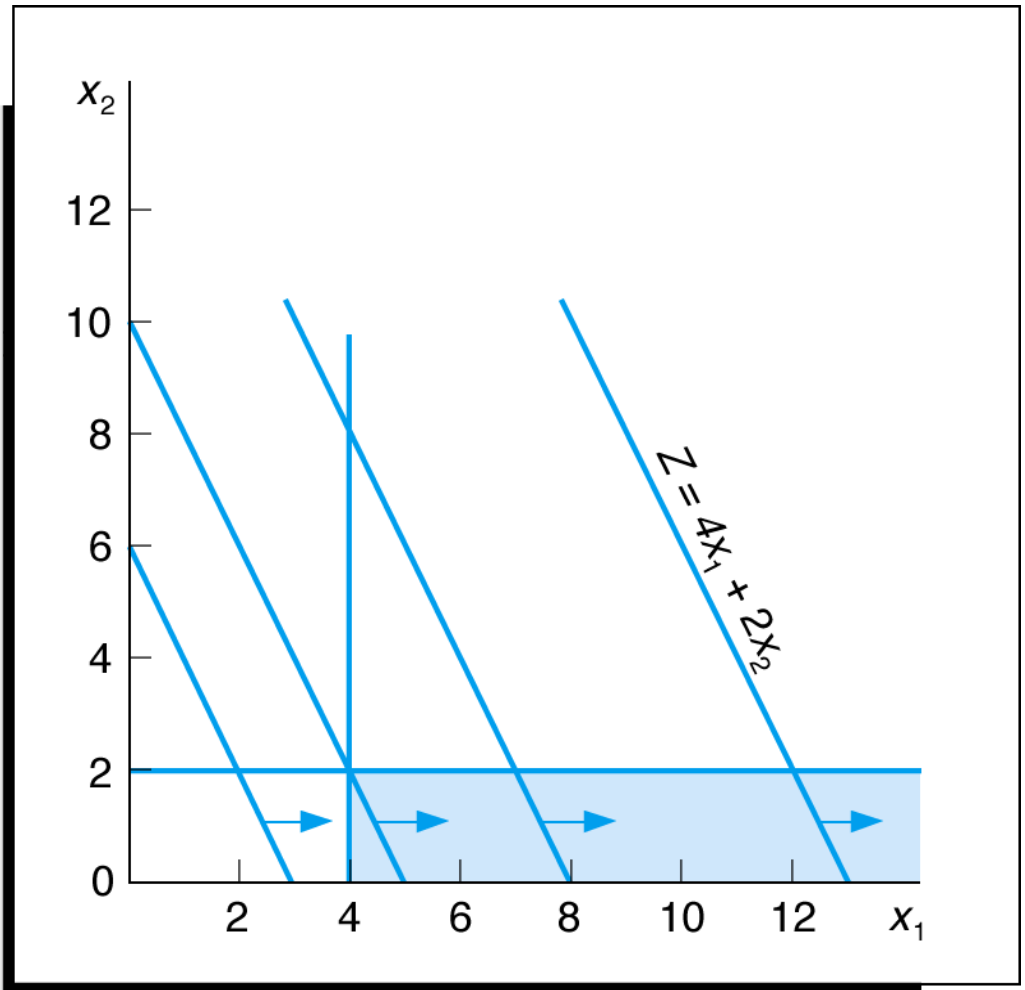


Graph of an Infeasible Problem

An Unbounded Problem

Value of the objective function increases indefinitely

$$\begin{aligned} \text{Maximize } Z &= 4x_1 + 2x_2 \\ \text{subject to: } x_1 &\geq 4 \\ x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Graph of an Unbounded Problem

Simplex Method

A large variety of Simplex-based algorithms exist to solve LP problems.

Other (polynomial time) algorithms have been developed for solving LP problems:

- Khachian algorithm (1979)

- Kamarkar algorithm (AT&T Bell Labs, mid 80s)

- See Section 4.10

BUT,

none of these algorithms have been able to beat Simplex in actual practical applications.

HENCE,

Simplex (in its various forms) is and will most likely remain the most dominant LP algorithm for at least the near future

Extreme point theorem:

If the maximum or minimum value of a linear function defined over a polygonal convex region exists, then it is to be found at the boundary of the region.

Convex set:

A set (or region) is convex if, for any two points (say, x_1 and x_2) in that set, the line segment joining these points lies entirely within the set.

A point is by definition convex.

What does the extreme point theorem imply?

- A finite number of extreme points implies a finite number of solutions!
- Hence, search is reduced to a finite set of points
- However, a finite set can still be too large for practical purposes
- Simplex method provides an efficient systematic search guaranteed to converge in a finite number of steps.

Formulations

There are many ways to formulate linear programs:

- **objective (or cost) function**

maximize $c^T x$, or

minimize $c^T x$, or

find any feasible solution

- **(in)equalities**

$Ax \leq b$, or

$Ax \geq b$, or

$Ax = b$, or any combination

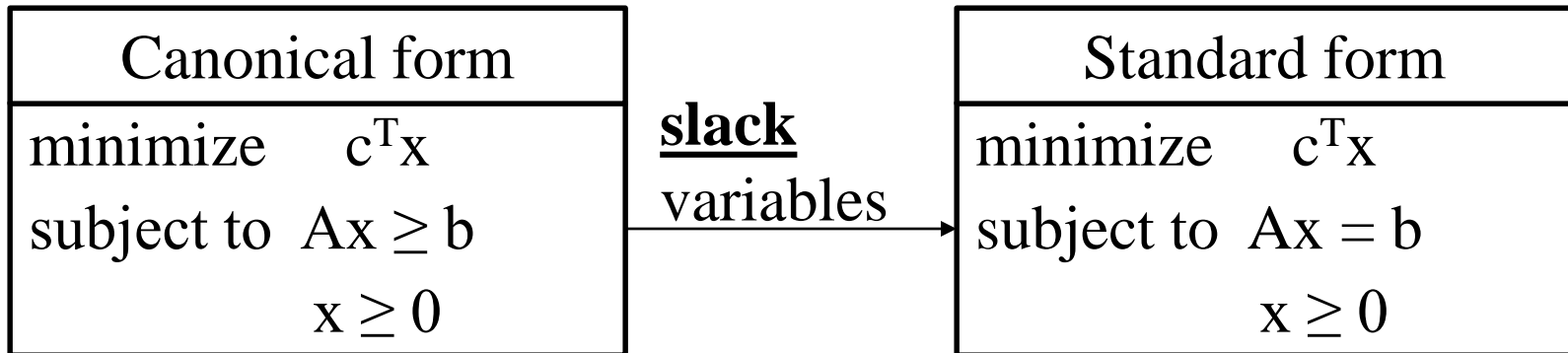
- **nonnegative variables**

$x \geq 0$, or not

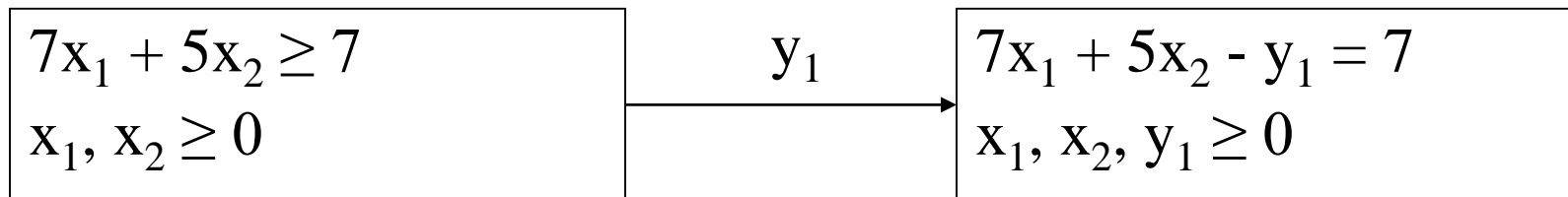
Fortunately it is pretty easy to convert among forms

Formulations

The two **most common** formulations:



e.g.



Geometric View of Canonical Form

A **polytope** in n -dimensional space

Each inequality corresponds to a half-space.

The “feasible set” is the intersection of the half-spaces

This corresponds to a polytope

Polytopes are **convex**: if x, y is in the polytope, so is the line segment joining them.

The optimal solution is at a vertex (i.e., a corner).

Geometric View of Canonical Form

minimize:

$$z = -2x_1 - 3x_2$$

subject to:

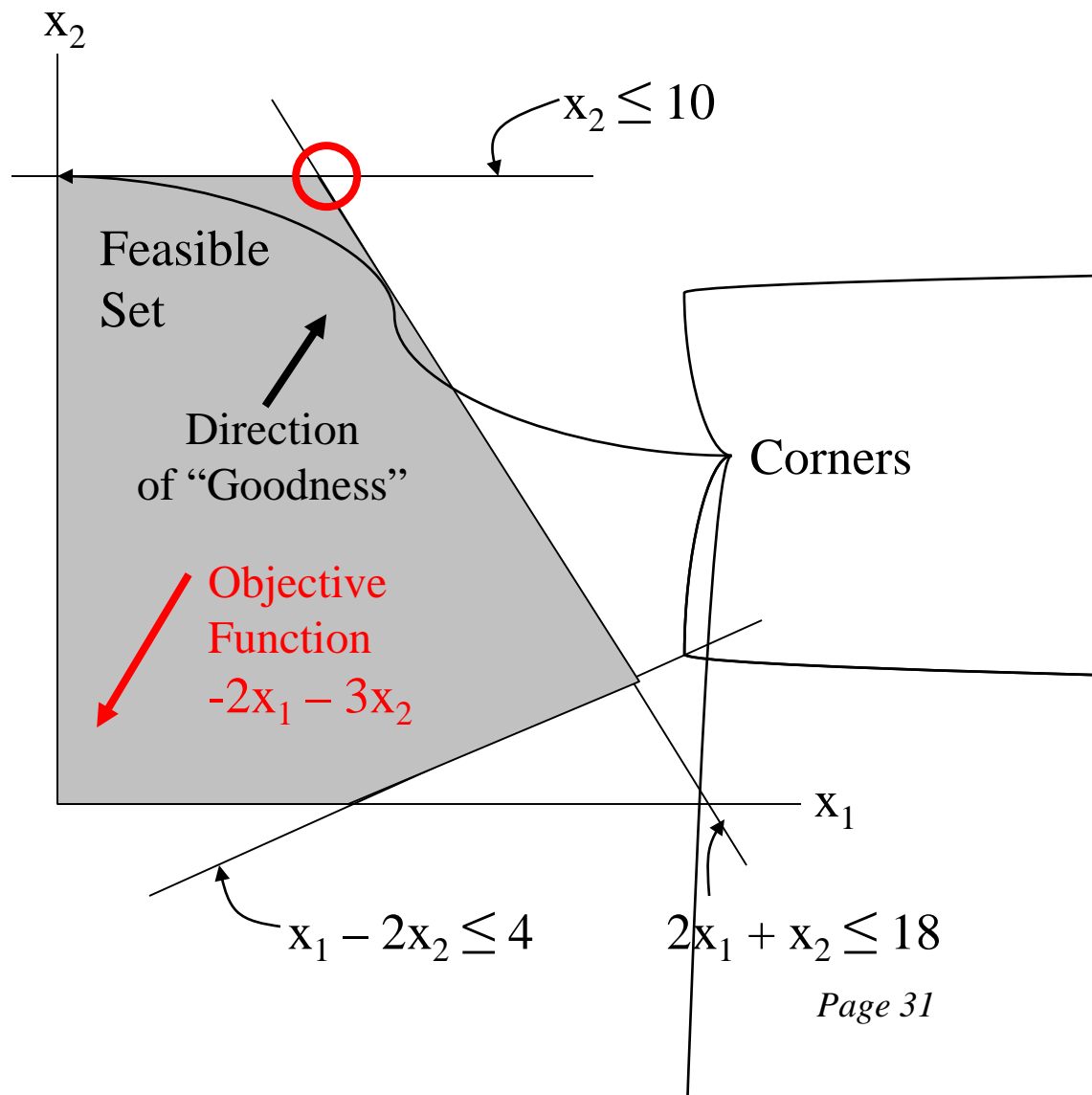
$$x_1 - 2x_2 \leq 4$$

$$2x_1 + x_2 \leq 18$$

$$x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

An intersection of 5
halfspaces



Geometric View of Canonical Form

A **polytope** in n -dimensional space

Each inequality corresponds to a half-space.

The “feasible set” is the intersection of the half-spaces.

This corresponds to a polytope

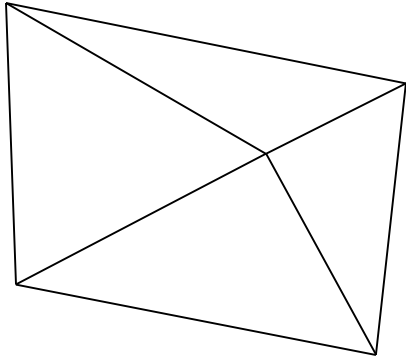
The optimal solution is at a corner.

Simplex moves around on the surface of the polytope

Interior-Point methods move within the polytope

The Simple Essence of Simplex

Polytope P



Input: $\min f(x) = cx$
s.t. $x \text{ in } P = \{x: Ax \leq b, x \geq 0\}$

Consider Polytope P from canonical form as a graph $G = (V, E)$ with
 $V =$ polytope vertices,
 $E =$ polytope edges.

- 1) Find *any* vertex v of P .
- 2) While there exists a neighbor u of v in G with $f(u) < f(v)$, update v to u .
- 3) Output v .

Choice of neighbor if several u have $f(u) < f(v)$?

Basic Steps of Simplex

1. Begin the search at an extreme point (i.e., a basic feasible solution).
2. Determine if the movement to an adjacent extreme can improve on the optimization of the objective function. If not, the current solution is optimal. If, however, improvement is possible, then proceed to the next step.
3. Move to the adjacent extreme point which offers (or, perhaps, *appears* to offer) the most improvement in the objective function.
4. Continue steps 2 and 3 until the optimal solution is found or it can be shown that the problem is either unbounded or infeasible.

Simplex Running Time

For dense matrices, takes $O(n(n+m))$ time per iteration

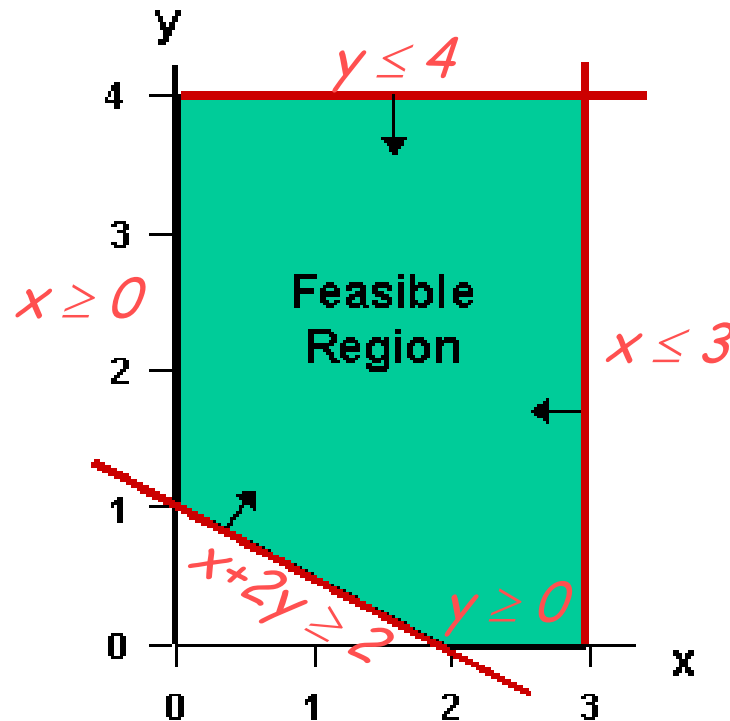
Can take an **exponential** number of iterations.

In practice, sparse methods are used for the iterations.

Linear Programming in 2 dimensions

| Name | Vars | Constraints | Objective |
|-------------------------|------|---------------------|-----------------|
| linear programming (LP) | real | linear inequalities | linear function |

*2 variables:
feasible region is a
convex polygon*

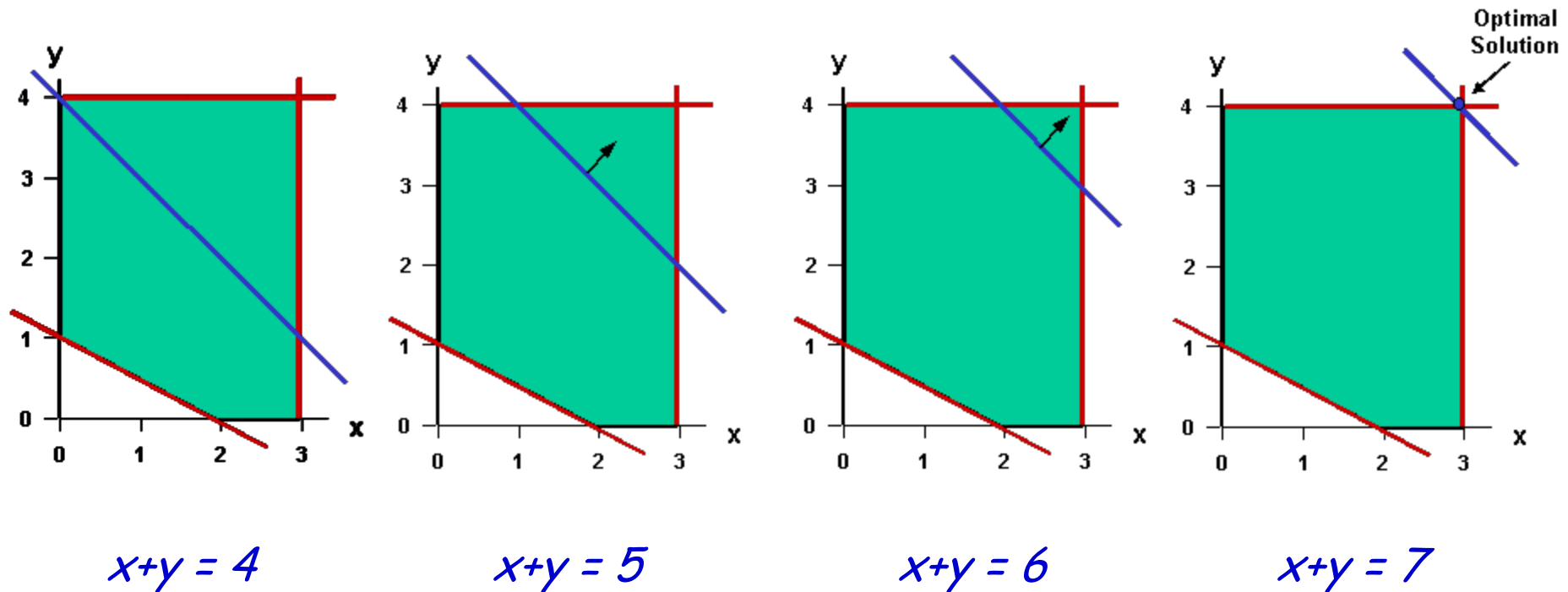


*boundary of
feasible region
comes from
the constraints*

Linear Programming in 2 dimensions

| Name | Vars | Constraints | Objective |
|-------------------------|------|---------------------|-----------------|
| linear programming (LP) | real | linear inequalities | linear function |

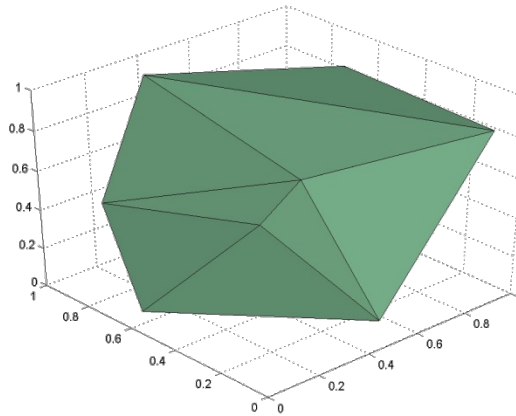
"level sets" of the objective $x+y$ (sets where it takes a certain value)



Linear Programming in n dimensions

| Name | Vars | Constraints | Objective |
|-------------------------|------|---------------------|-----------------|
| linear programming (LP) | real | linear inequalities | linear function |

*3 variables:
feasible region is a
convex polyhedron*

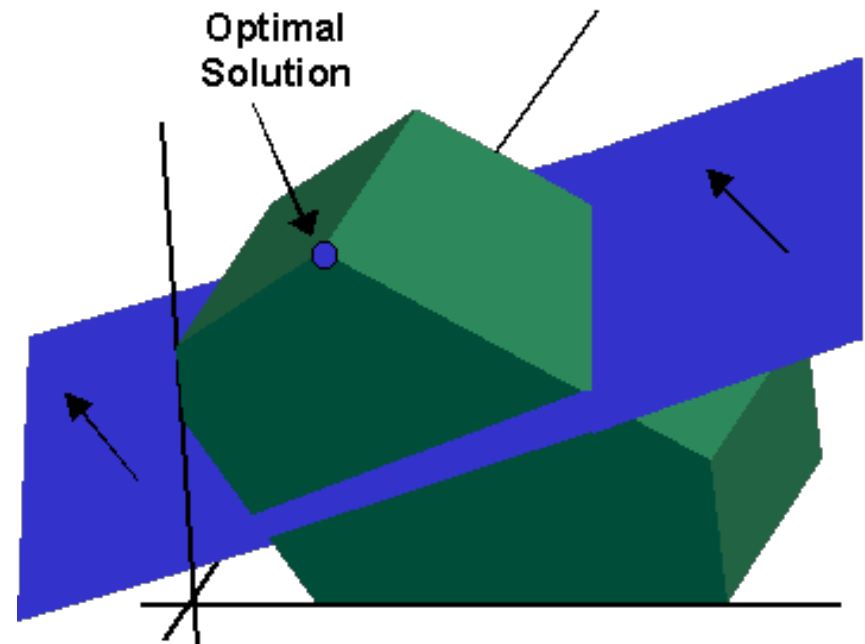


Linear Programming in n dimensions

| Name | Vars | Constraints | Objective |
|-------------------------|------|---------------------|-----------------|
| linear programming (LP) | real | linear inequalities | linear function |

*here level set is a plane
(in general, a hyperplane)*

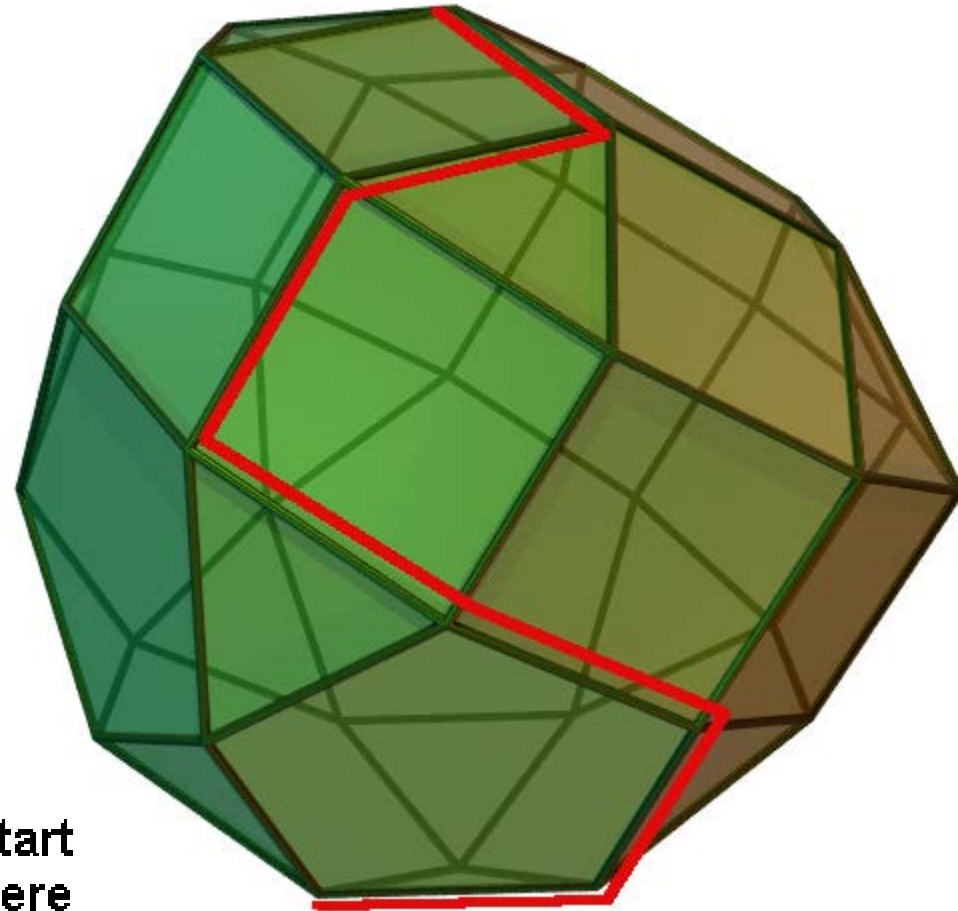
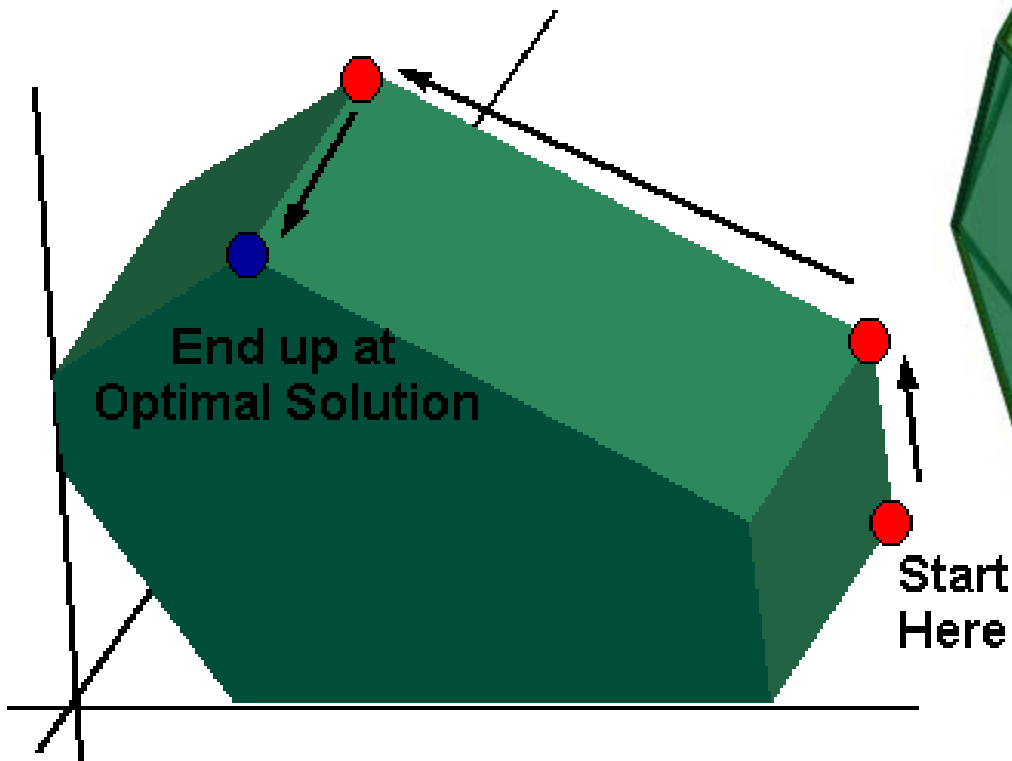
*If an LP optimum is finite,
it can always be achieved
at a corner ("vertex") of
the feasible region.*



(Can there be infinite solutions? Multiple solutions?)

Simplex Method for Solving an LP

At every step, move to an adjacent vertex that improves the objective.



Trail Mix Problem

- Acme Snacks produces three different trail mix blends by mixing the following raw ingredients: peanuts, raisins, soynuts, and pretzels.
- These ingredients contain the following nutrients: vitamins, protein, calcium, and fat in the following quantities:

| Ingredient, i | Nutrient, k | | | |
|---------------|-------------|---------|---------|-----------|
| | Vitamins | Protein | Calcium | Crude Fat |
| peanuts | 8 | 10 | 6 | 8 |
| raisins | 6 | 5 | 10 | 6 |
| soynuts | 10 | 12 | 6 | 6 |
| pretzels | 4 | 18 | 6 | 9 |

Let a_{ik} = quantity of nutrient k per kg of ingredient i

Constraints

- The company contracts for the following demands.*

| Demand (kg) | Mix 1 | Mix 2 | Mix 3 |
|-------------|--------|-------|-------|
| d_j | 10,000 | 6,000 | 8,000 |

- There are limited availabilities of the raw ingredients.

| Supply (kg) | peanuts | raisins | soynuts | pretzels |
|-------------|---------|---------|---------|----------|
| s_i | 6,000 | 10,000 | 4,000 | 5,000 |

- The different mixes have “quality” bounds per kilogram.

| | <i>Vitamins</i> | | <i>Protein</i> | | <i>Calcium</i> | | <i>Crude fat</i> | |
|--------------|-----------------|------------|----------------|------------|----------------|------------|------------------|------------|
| | <i>min</i> | <i>max</i> | <i>min</i> | <i>max</i> | <i>min</i> | <i>max</i> | <i>min</i> | <i>max</i> |
| <i>Mix 1</i> | 6 | -- | 6 | -- | 7 | -- | 4 | 8 |
| <i>Mix 2</i> | 6 | -- | 6 | -- | 6 | -- | 4 | 8 |
| <i>Mix 3</i> | 4 | 6 | 6 | -- | 6 | -- | 4 | 8 |

The above values represent bounds: l_{jk} and u_{jk}

Costs and Notation

Cost per kg of the raw ingredients is as follows:

| | <i>peanuts</i> | <i>raisins</i> | <i>soynuts</i> | <i>pretzels</i> |
|----------------------------------|----------------|----------------|----------------|-----------------|
| <i>cost/kg, c_i</i> | <i>20¢</i> | <i>12¢</i> | <i>24¢</i> | <i>12¢</i> |

Formulate problem as a linear program whose solution yields desired trail mix production levels at minimum cost.

Indices/sets

| | |
|-----------|--|
| $i \in I$ | <i>ingredients { peanuts, raisins, soynuts, pretzels }</i> |
| $j \in J$ | <i>products { mix 1, mix 2, mix 3 }</i> |
| $k \in K$ | <i>nutrients { vitamins, protein, calcium, crude fat }</i> |

Data

| | |
|----------|---|
| d_j | <i>demand for product j (kg)</i> |
| s_i | <i>supply of ingredient i (kg)</i> |
| l_{jk} | <i>lower bound on number of nutrients of type k per kg of product j</i> |
| u_{jk} | <i>upper bound on number of nutrients of type k per kg of product j</i> |
| c_i | <i>cost per kg of ingredient i</i> |
| a_{ik} | <i>number of nutrients k per kg of ingredient i</i> |

Decision Variables

| | |
|----------|--|
| x_{ij} | <i>amount (kg) of ingredient i used in producing product j</i> |
|----------|--|

LP Formulation of Mix Problem

$$\min \quad \sum_{i \in I} \sum_{j \in J} c_i x_{ij}$$

$$\text{s.t.} \quad \sum_{i \in I} x_{ij} = d_j \quad \forall j \in J$$

$$\sum_{j \in J} x_{ij} \leq s_i \quad \forall i \in I$$

$$\sum_{i \in I} a_{ik} x_{ij} \geq l_{jk} d_j \quad \forall j \in J, k \in K$$

$$\sum_{i \in I} a_{ik} x_{ij} \leq u_{jk} d_j \quad \forall j \in J, k \in K$$

$$x_{ij} \geq 0 \quad \forall i \in I, j \in J$$

LP:Labor Planning

Addresses staffing needs over a specific time period.

Hong Kong Bank of Commerce:

- 12 Full time workers available, but may fire some.
- Use part time workers who has to work for 4 consecutive hours in a day.
- Lunch time is one hour between 11a.m. and 1p.m. shared by full time workers.
- Total part time hours is less than 50% of the day's total requirement.
- Part-timers earn \$4/hr (=\$16/day) and full timers earn \$50/day.

LP:Labor Planning (Cont'd.)

| Time Period | Minimum labor required |
|---------------|------------------------|
| 9a.m.-10a.m. | 10 |
| 10a.m.-11a.m. | 12 |
| 11a.m.-noon | 14 |
| Noon-1p.m. | 16 |
| 1p.m.-2p.m. | 18 |
| 2p.m.-3p.m. | 17 |
| 3p.m.-4p.m. | 15 |
| 4p.m.-5p.m. | 10 |

LP:Labor Planning (cont'd.)

F : # Full time tellers per day

P_i : # Part time tellers who start work at time slot $i, i = 1, 2, \dots, 5$.

$$\text{Min Daily Personnel Cost} = \$50F + \$16 \sum P_i$$

$$F + P_1 \geq 10$$

$$F + P_1 + P_2 \geq 12$$

$$0.5F + P_1 + P_2 + P_3 \geq 14$$

$$0.5F + P_1 + P_2 + P_3 + P_4 \geq 16$$

$$F + P_2 + P_3 + P_4 + P_5 \geq 18$$

$$F + P_3 + P_4 + P_5 \geq 17$$

$$F + P_4 + P_5 \geq 15$$

$$F + P_5 \geq 10$$

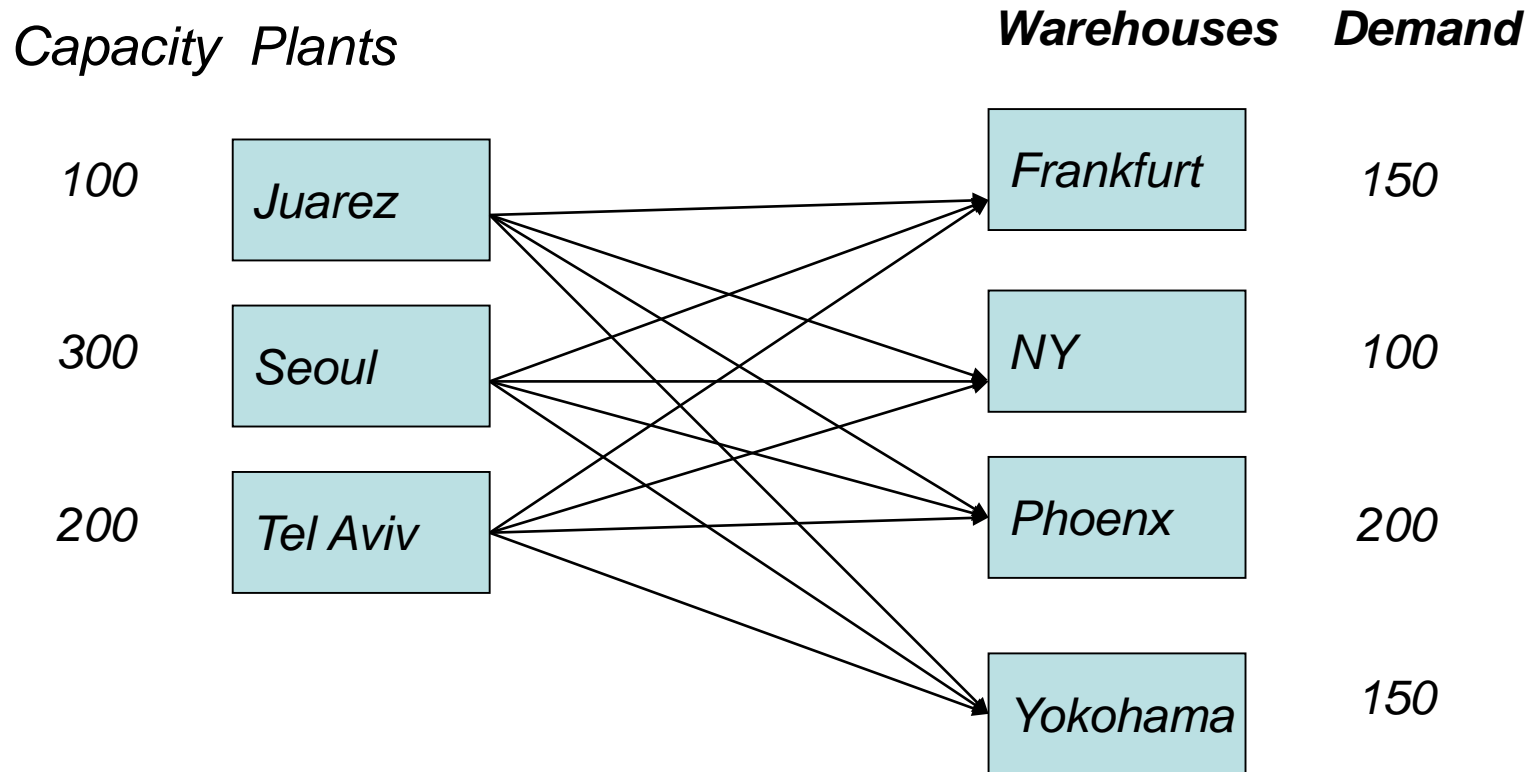
$$F \leq 12$$

$$4 \sum P_i \leq 0.5(10 + 12 + 14 + \dots + 10)$$

$$F, P_i \geq 0$$

Applications of LP: Transportation Models

Sporting goods company



LP:Transportation Models (cont'd.)

What are the optimal shipping quantities from the plants to the warehouses, if the demand has to be met by limited capacities while the shipping cost is minimized?

Shipping Costs per pair of skis

| From Plant | Destination | | | |
|------------|-------------|-----|---------|----------|
| | Frankfurt | NY | Phoenix | Yokohama |
| Juarez | \$19 | \$7 | \$3 | \$21 |
| Seoul | 15 | 21 | 18 | 6 |
| Tel Aviv | 11 | 14 | 15 | 22 |

LP:Transportation Models (cont'd.)

X_{ij} : Number of units shipped from plant i to warehouse j . $i=1,2,3$
and $j=1,2,3,4$.

Minimize shipping costs= $19X_{11}+7X_{12}+3X_{13}+21X_{14}$
 $+15X_{21}+21X_{22}+18X_{23}+6X_{24}$
 $+11X_{31}+14X_{32}+15X_{33}+22X_{34}$

| From Plant | Destination | | | | Capacity |
|------------|-------------|-----|---------|----------|----------|
| | Frankfurt | NY | Phoenix | Yokohama | |
| Juarez | X11 | X12 | X13 | X14 | 100 |
| Seoul | X21 | X22 | X23 | X24 | 300 |
| Tel Aviv | X31 | X32 | X33 | X34 | 200 |
| Demand | 150 | 100 | 200 | 150 | 600 |

LP:Transportation Models (cont'd.)

subject to

#shipped from a plant can not exceed the capacity:

$$X_{11}+X_{12}+X_{13}+X_{14}\leq 100 \text{ (Juarez Plant)}$$

$$X_{21}+X_{22}+X_{23}+X_{24}\leq 300 \text{ (Seoul Plant)}$$

$$X_{31}+X_{32}+X_{33}+X_{34}\leq 200 \text{ (Tel Aviv Plant)}$$

#shipped to a warehouse can not be less than the demand:

$$X_{11}+X_{21}+X_{31}\geq 150 \text{ (Frankfurt)}$$

$$X_{12}+X_{22}+X_{32}\geq 100 \text{ (NY)}$$

$$X_{13}+X_{23}+X_{33}\geq 200 \text{ (Phoenix)}$$

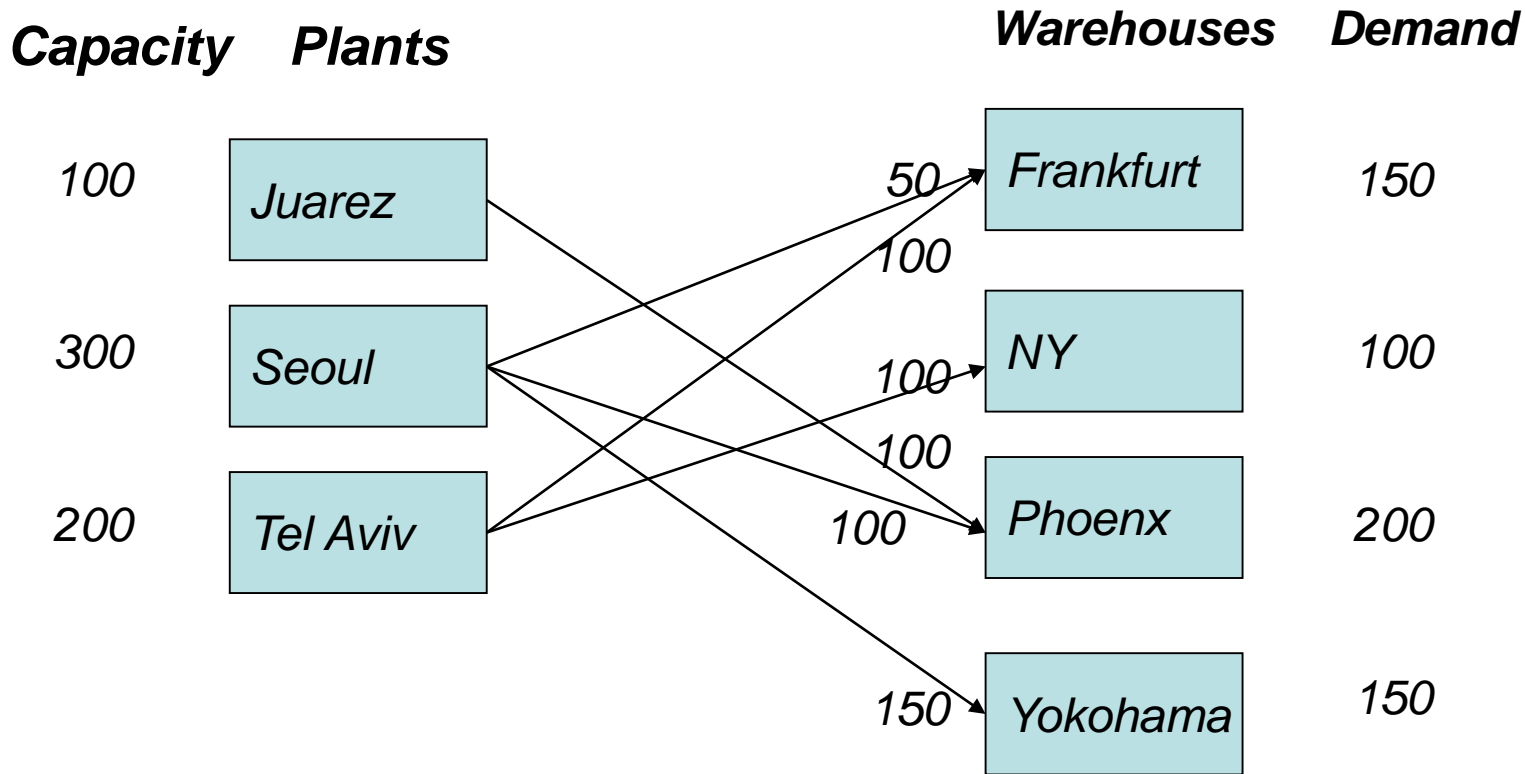
$$X_{14}+X_{24}+X_{34}\geq 150 \text{ (Yokohama)}$$

Nonnegativity

$$X_{ij}\geq 0 \text{ for all } i,j.$$

LP:Transportation Models (cont'd.)

Optimal Solution: Optimal cost=\$6,250



Decision Problem

$$\min(\max(x_1, x_2, x_3))$$

$$\text{ST } 3x_1 + 2x_2 - 5x_3 \leq 8$$

Decision Problem

$$\min(\max(x_1, x_2, x_3))$$

$$\text{ST } 3x_1 + 2x_2 - 5x_3 \leq 8$$

$$\min = t$$

$$\text{ST } 3x_1 + 2x_2 - 5x_3 \leq 8$$

$$x_1 \leq t \quad x_1 - t \leq 0$$

$$x_2 \leq t \quad x_2 - t \leq 0$$

$$x_3 \leq t \quad x_3 - t \leq 0$$

Decision Problem

$$\min = t$$

$$\text{ST } 3x_1 + 2x_2 - 5x_3 \leq 8$$

$$x_1 \leq t \quad x_1 - t \leq 0$$

$$x_2 \leq t \quad x_2 - t \leq 0$$

$$x_3 \leq t \quad x_3 - t \leq 0$$

Unbounded: x 's can keep getting smaller

min (max(x1, x2, x3))

| | | | max | 3x1+2x2- 5x3 <= 8 | |
|-----|-----|-----|-----|-------------------|------|
| x1 | x2 | x3 | 0 | constraint | met |
| 2 | 1 | 5 | 5 | -17 | TRUE |
| -2 | -3 | 8 | 8 | -52 | TRUE |
| 3 | -2 | 2 | 3 | -5 | TRUE |
| -10 | -20 | -7 | -7 | -35 | TRUE |
| -10 | -20 | -8 | -8 | -30 | TRUE |
| -10 | -20 | -9 | -9 | -25 | TRUE |
| -12 | -22 | -10 | -10 | -30 | TRUE |
| -14 | -24 | -11 | -11 | -35 | TRUE |
| -16 | -26 | -12 | -12 | -40 | TRUE |
| -18 | -28 | -13 | -13 | -45 | TRUE |
| -20 | -30 | -14 | -14 | -50 | TRUE |
| -22 | -32 | -15 | -15 | -55 | TRUE |
| -24 | -34 | -16 | -16 | -60 | TRUE |
| -26 | -36 | -17 | -17 | -65 | TRUE |
| -28 | -38 | -18 | -18 | -70 | TRUE |
| -30 | -40 | -19 | -19 | -75 | TRUE |
| -32 | -42 | -20 | -20 | -80 | TRUE |
| -34 | -44 | -21 | -21 | -85 | TRUE |
| -36 | -46 | -22 | -22 | -90 | TRUE |
| -38 | -48 | -23 | -23 | -95 | TRUE |
| -40 | -50 | -24 | -24 | -100 | TRUE |
| -42 | -52 | -25 | -25 | -105 | TRUE |
| min | | | -25 | | |

Shortest Paths

We can compute the length of the shortest path from s to t in a weighted directed graph by solving the following very simple linear programming problem.

$$\begin{array}{ll}\text{maximize} & d_t \\ \text{subject to} & d_s = 0 \\ & d_v - d_u \leq \ell_{u \rightarrow v} \quad \text{for every edge } u \rightarrow v\end{array}$$

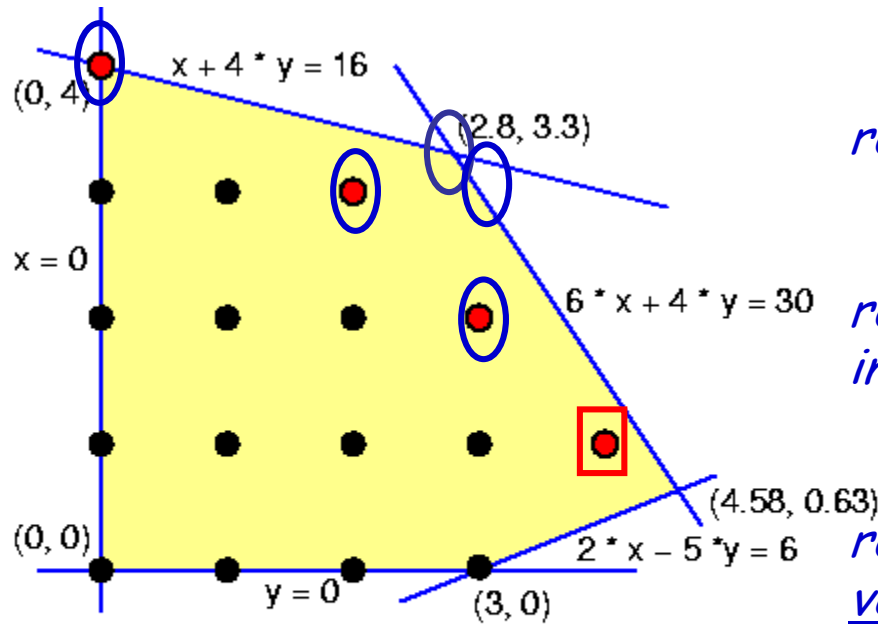
Here, $\ell_{u \rightarrow v}$ is the length of the edge $u \rightarrow v$. Each variable d_v represents a tentative shortest-path distance from s to v . The constraints mirror the requirement that every edge in the graph must be relaxed. These relaxation constraints imply that in any feasible solution, d_v is *at most* the shortest path distance from s to v . Thus, somewhat counterintuitively, we are correctly *maximizing* the objective function to compute the *shortest* path! In the optimal solution, the objective function d_t is the actual shortest-path distance from s to t , but for any vertex v that is not on the shortest path from s to t , d_v may be an underestimate of the true distance from s to v . However, we can obtain the true distances from s to every other vertex by modifying the objective function:

Variations

- Integer Programming
- Mixed Integer Programming

Integer Linear Programming (ILP)

| Name | Vars | Constraints | Objective |
|----------------------------|---------|---------------------|-----------------|
| integer linear prog. (ILP) | integer | linear inequalities | linear function |



*round to nearest int (3,3)?
No, infeasible.*

*round to nearest feasible
int (2,3) or (3,2)?
No, suboptimal.*

*round to nearest integer
vertex (0,4)?
No, suboptimal.*

Function to maximize: $f(x, y) = 6 * x + 5 * y$

Optimum LP solution $(x, y) = (2.8, 3.3)$

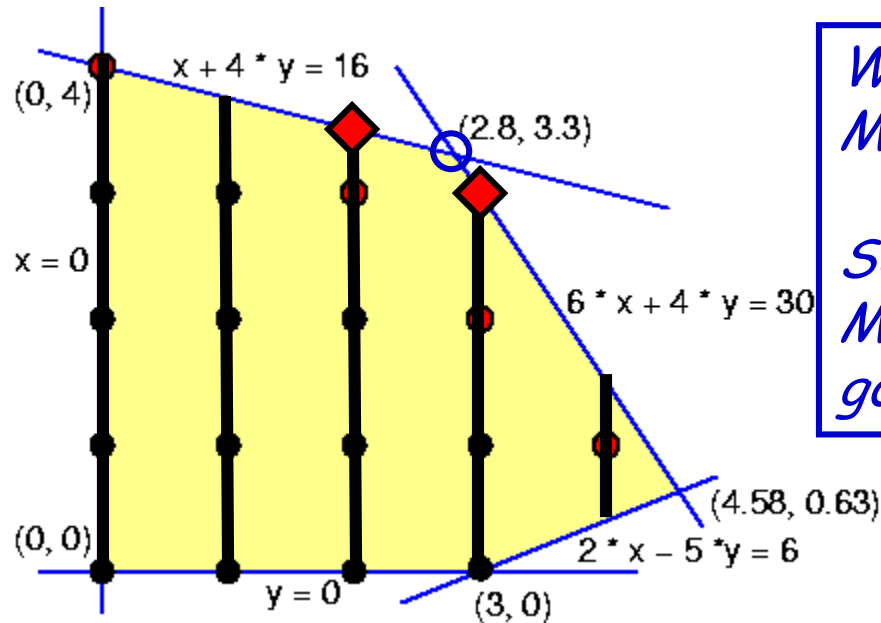
Pareto optima: (0, 4), (2, 3), (3, 2), (4, 1)

Optimum ILP solution $(x, y) = (4, 1)$

Mixed Integer Programming (MIP)

| Name | Vars | Constraints | Objective |
|----------------------------|----------|---------------------|-----------------|
| linear programming (LP) | real | linear inequalities | linear function |
| integer linear prog. (ILP) | integer | linear inequalities | linear function |
| mixed integer prog. (MIP) | int&real | linear inequalities | linear function |

*x still integer
but y is now real*



*We'll be studying
MIP solvers.*

*SCIP mainly does
MIP though it
goes a bit farther.*

Figure 7.2 The feasible polyhedron for a three-variable linear program.

