NP-Completeness

Part 1

NP-Completeness

- So far we’ve seen a lot of good news!
  - Many of the problems we have seen could be solved quickly (i.e., in close to linear time, or at least a time that is some small polynomial function of the input size)
- NP-completeness is a form of bad news!
  - Evidence that many important problems can not be solved quickly.
- NP-complete problems really come up all the time!
  - 0-1 Knapsack, Travelling Salesman, Bin Packing, Scheduling

NP-Completeness

- Some problems are *intractable*:
  as they grow large, we are unable to solve them in reasonable time
- What constitutes reasonable time? Standard working definition: *polynomial time*
  - On an input of size n the worst-case running time is $O(n^k)$ for some constant k
  - Polynomial time: $O(n^2)$, $O(n^3)$, $O(1)$, $O(n \lg n)$
  - Not in polynomial time: $O(2^n)$, $O(n^n)$, $O(n!)$

Why should we care?

- Knowing that they are hard lets you use other methods to solve them…
  - **Use a heuristic:** come up with a method for solving a reasonable fraction of the common cases.
  - **Solve approximately:** come up with a solution that you can prove that is close to right.
  - **Use an exponential time solution:** if you really have to solve the problem exactly and stop worrying about finding a better solution.

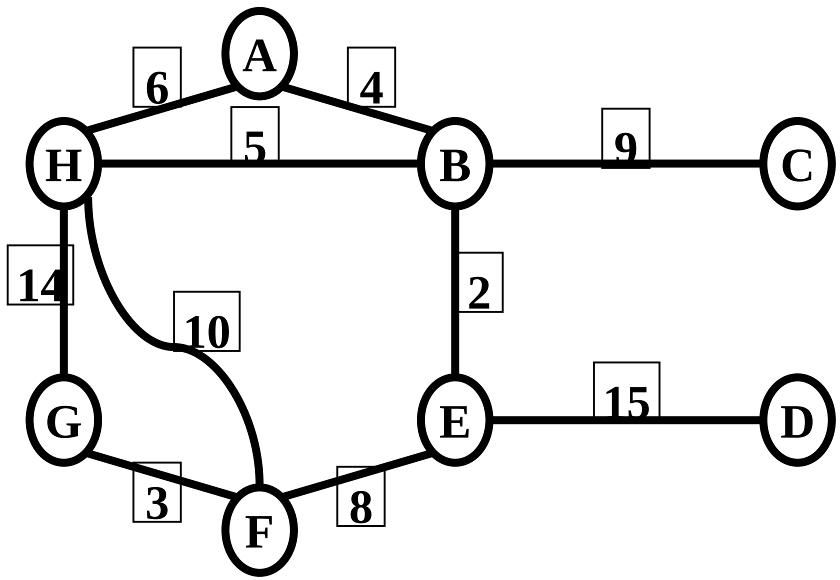
Optimization vs Decision Problems

- **Decision problems**
  - Given an input and a question regarding a problem, determine if the answer is yes or no
- **Optimization problems**
  - Find a solution with the “best” value
  - Optimization problems can be casted as decision problems that are easier to study

Optimization vs Decision Problems

Shortest Path given G =(V,E) and w(u,v) edge weights

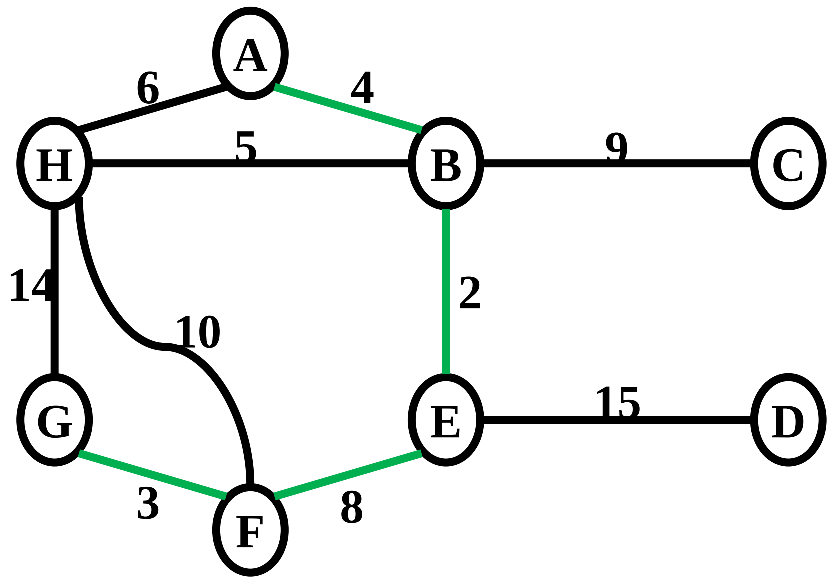
- Optimization: What is the minimum total weight of all paths between A and G?
- Decision: Does there exists a path between A and E with total weight at most 20?



Optimization vs Decision Problems

Shortest Path given G =(V,E) and w(u,v) edge weights

- Optimization: What is the minimum total weight of all paths between A and G? 17
- Decision: Does there exists a path between A and E with total weight at most 20? YES



Optimization vs Decision Problems

Problem: Knapsack (items, weights, benefits, W)

- Optimization: What is the maximum total benefit of all sets of items that can fit in a knapsack with capacity W.
- Decision: Does there exists a set of items having a total benefit of at least k that can fit in the knapsack with capacity W

Algorithmic vs Problem Complexity

- The ***algorithmic complexity*** of a computation is some measure of how *difficult* it is to perform the computation (i.e., specific to an algorithm)
- The **complexity of a computational *problem*** or *task* is the complexity of the algorithm with the **lowest** order of growth of complexity for solving that problem or performing that task.
  - *e.g.* the problem of searching an ordered list has *at most lgn* time complexity.
- **Computational Complexity**: deals with classifying problems by how hard they are.

Class of "P" Problems

- **Class P** consists of (decision) problems that are solvable in polynomial time
- Polynomial-time algorithms
  - Worst-case running time is $O(n^k)$, for some constant k
- Examples of polynomial time:
  - $O(n^2)$, $O(n^3)$, $O(1)$, $O(n \lg n)$
  - Searching and Sorting

Tractable/Intractable Problems

- Problems in P are also called **tractable**
- Problems **not** in P can be **or**
  - **intractable -** solved in reasonable time only for small inputs
  - **unsolvable -** can not be solved at all
- Are non-polynomial algorithms always worst than polynomial algorithms?

  - $n^{1,000,000}$ is *technically* tractable, but really impossible
  - $n^{\log \log \log n}$ is *technically* intractable, but easy

An Unsolvable Problem

- Turing discovered in the 1930's that there are problems **unsolvable** by *any* algorithm.
- The most famous of them is the ***halting problem***
  - Given an arbitrary algorithm and its input, will that algorithm eventually halt, or will it continue forever in an "*infinite loop*?"
- This is an interesting topic but we will not be covering unsolvable problems in this class. To learn more you can take CS321

Intractable Problems

- Can be classified in various categories based on their degree of difficulty, e.g.,
  - NP
  - NP-complete
  - NP-hard
- Let's define NP algorithms and NP problems …

Examples of Intractable Decision Problems

- Hamiltonian Cycle (HAM-CYCLE). Given a directed graph G = (V,E), does there exist a simple cycle C that visits every vertex?
- CIRCUIT-SAT. Given a combinational circuit built out of AND, OR, and NOT gates, is there a way to set the circuit inputs so that the output is 1?
- Travelling Salesman (TSP). Given a weighted graph G=(V,E)
  - Optimization Problem: Find a minimum weight Hamiltonian Cycle.
  - Decision Problem: Given a graph and an integer k, is there a Hamiltonian Cycle with a total weight at most k

Nondeterminism and NP Algorithms

Nondeterministic algorithm = two stage procedure:

1) Nondeterministic ("guessing") stage:

generate randomly an arbitrary string that can be thought of as a candidate solution ("certificate")

2) Deterministic ("verification") stage:

take the certificate and the instance to the problem and returns YES if the certificate represents a solution

NP algorithms (Nondeterministic polynomial)

verification stage is polynomial

Class of “NP” Problems

- **Class NP** consists of problems that could be solved by Nondetermistic Polynomial algorithms

  Or verifiable in polynomial time

- If we were given a “certificate” of a solution, we could verify/certify that the certificate is correct in time polynomial to the size of the input
- Warning: NP does **not** mean “non-polynomial”

Decision 0-1 Knapsack is in NP

Given a knapsack with capacity W = 20 is there a subset of items with total benefit at least $25?

Easy to verify in poly-time that

S = { 1, 3, 4, 5 } is a certificate solution.

Total weight = 2 +4+5+9 = 20

Total benefit =

3 + 5 + 8 + 10 = 26 > 25

Item #	Weight w_i	Benefit b_i
1	2	3
2	3	4
3	4	5
4	5	8
5	9	10

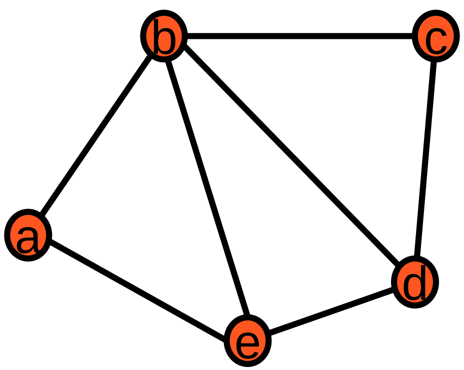
Hamiltonian Cycle is in NP

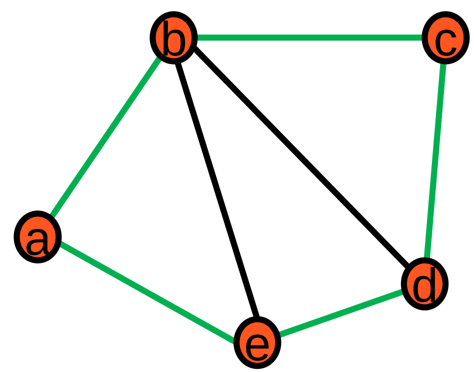
Given: a directed graph G = (V, E), determine a simple cycle that contains each vertex in V

- Each vertex can only be visited once

Certificate:

- Sequence: $\langle$a, e, d, c, b$\rangle$





certificate t

instance s

3-SAT is in NP

Given a CNF formula Φ with three literals per clause, is there a satisfying assignment?

Certificate. An assignment of truth values to the n boolean variables.

Certifier. Check that each clause in Φ has at least one true literal.

instance s

$$\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_1 \vee x_2 \vee x_4\right) \wedge \left(\overline{x_1} \vee \overline{x_3} \vee \overline{x_4}\right)$$

certificate t

$$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$$

3-SAT is in NP

Given a CNF formula Φ with three literals per clause, is there a satisfying assignment?

Certificate. An assignment of truth values to the n boolean variables.

Certifier. Check that each clause in Φ has at least one true literal.

instance s

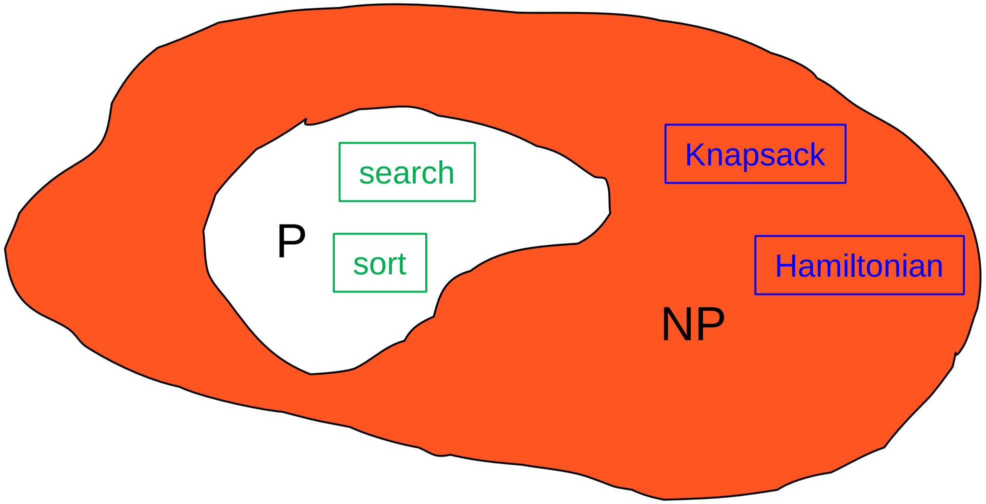
$$\left(\overline{1} \vee 1 \vee 0\right) \wedge \left(1 \vee \overline{1} \vee 0\right) \wedge \left(1 \vee 1 \vee 1\right) \wedge \left(\overline{1} \vee \overline{0} \vee \overline{1}\right)$$

$$(1) \wedge (1) \wedge (1) \wedge (1)$$

$$1$$

P vs NP???

All problems that can be solved in polynomial time can be verified in polynomial time.

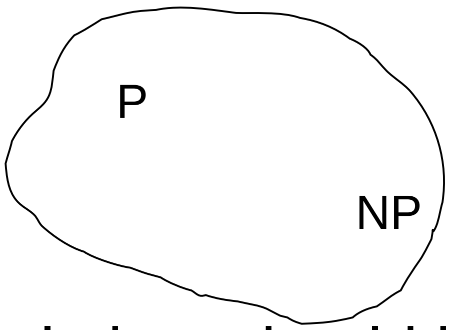


Any problem in P is also in NP: $P \subseteq NP$

Does P = NP?

The big (and **open question**)

is whether NP $\subseteq$ P or P = NP



– i.e., if it is always easy to check a solution, should it also be easy to find a solution?

- If yes: Efficient algorithms for KNAPSACK, TSP, FACTOR, SAT, …
- If no: No efficient algorithms possible for KNAPSACK, TSP, SAT, …
- Consensus opinion on P = NP? Probably no.

Most computer scientists believe that this is false but we do not have a proof …

NP-Complete Problems

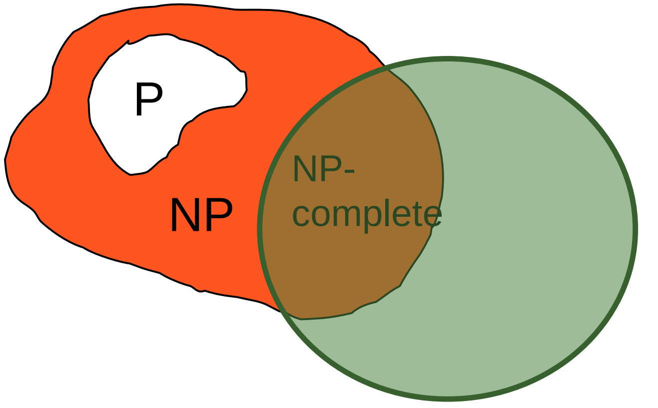
We will see that NP-Complete problems are the "hardest" problems in NP:

- If any *one* NP-Complete problem can be solved in polynomial time…
- …then *every* NP-Complete problem can be solved in polynomial time…
- …and in fact *every* problem in **NP** can be solved in polynomial time (which would show **P = NP**)
- Thus: solve Hamiltonian-cycle in O(n^{100}) time, you've proved that **P = NP**. Retire rich & famous.

NP-Completeness

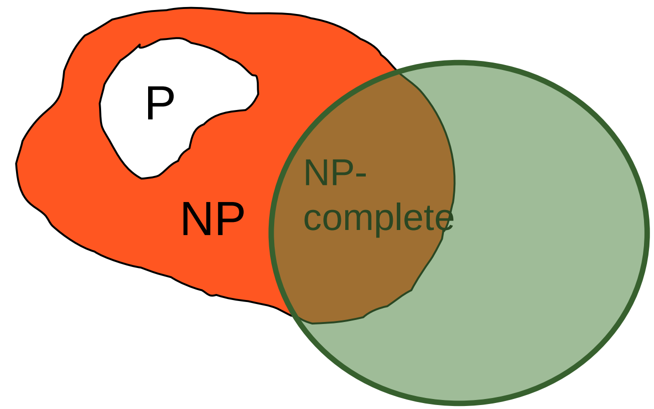
Part 2

NP-Completeness (informally)



- **NP-complete** problems are defined as the hardest problems in NP
- Most practical problems turn out to be either P or NP-complete.

NP-Completeness (formally)



- A problem B is **NP-complete** if:

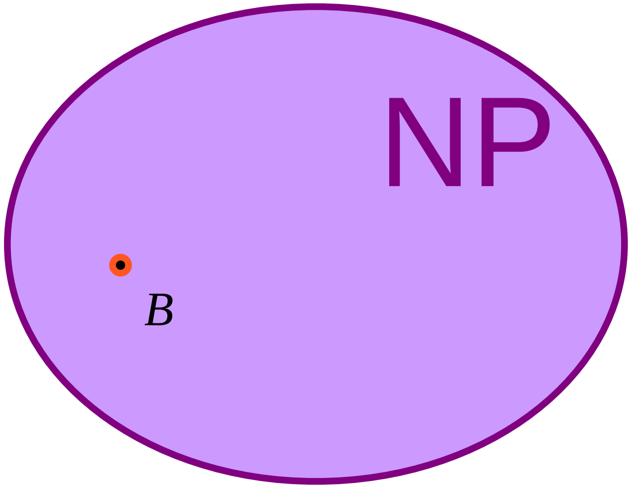
  (1) B $\in$ **NP**

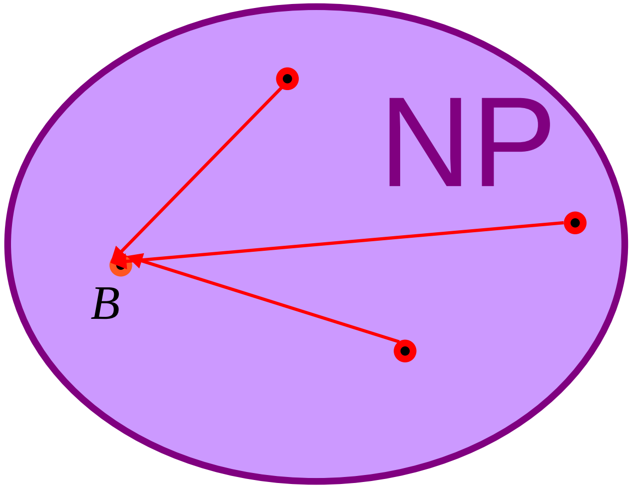
  (2) X $\leq_p$ B for all X $\in$ **NP**

- If B satisfies only property (2) we say that B is **NP-hard**
- No polynomial time algorithm has been discovered for an **NP-Complete** problem
- No one has ever proven that no polynomial time algorithm can exist for any **NP-Complete** problem

NP-Complete

A problem B is in NP-complete if:



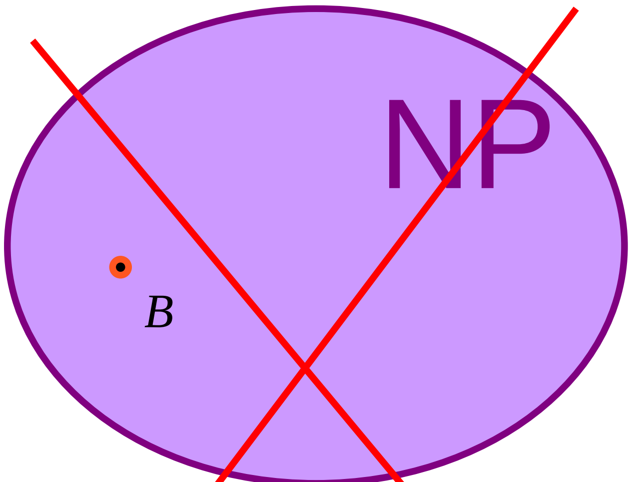


1. $B \in \mathbf{NP}$

2. There is a polynomial-time reduction from every problem $A \in \mathbf{NP}$ to B.

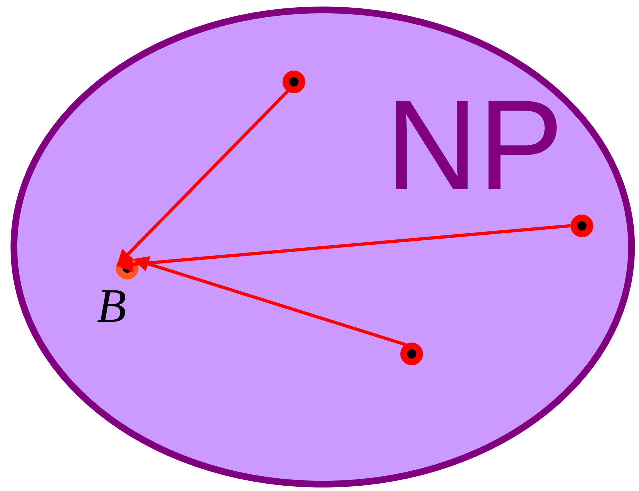
NP-~~Complete~~ Hard

A problem B is in NP-Hard if:



1. $\cancel{B \in \mathbf{NP}}$

Not *necessary* for NP-Hard



There is a polynomial-time reduction from every problem $A \in \mathbf{NP}$ to B.

P & NP-Complete Problems

- **Shortest simple path**
  - Given a graph G = (V, E) find a **shortest** path from a source to all other vertices
  - Polynomial solution: O(VE)
- **Longest simple path**
  - Given a graph G = (V, E) find a **longest** path from a source to all other vertices
  - NP-complete

P & NP-Complete Problems

- **Euler tour**
  - G = (V, E) a connected, directed graph find a cycle that traverses <u>each edge</u> of G exactly once (may visit a vertex multiple times)
  - <u>Polynomial solution O(E)</u>
- **Hamiltonian cycle**
  - G = (V, E) a connected, directed graph find a cycle that visits <u>each vertex</u> of G exactly once
  - <u>NP-complete</u>

Why Prove NP-Completeness?

- Though nobody has proven that **P ≠ NP**, if you prove a problem NP-Complete, most people accept that it is probably intractable
- Therefore it can be important to prove that a problem is NP-Complete
  - Don't need to come up with an efficient algorithm
  - Can instead work on *approximation algorithms*

Using Reductions

- If A is *polynomial-time reducible* to B, we denote this $A \leq_p B$
- Definition of NP-Complete:
  - If A is NP-Complete, $A \in$ **NP** and all problems X are reducible to A
  - Formally: $X \leq_p A \ \forall X \in$ **NP**
- If $A \leq_p B$ and A is NP-Complete, then B is NP-Hard
  - If $B \in$ **NP** too then B is NP-Complete

Reduction

The crux of NP-Completeness is *reducibility* $\leq_p$

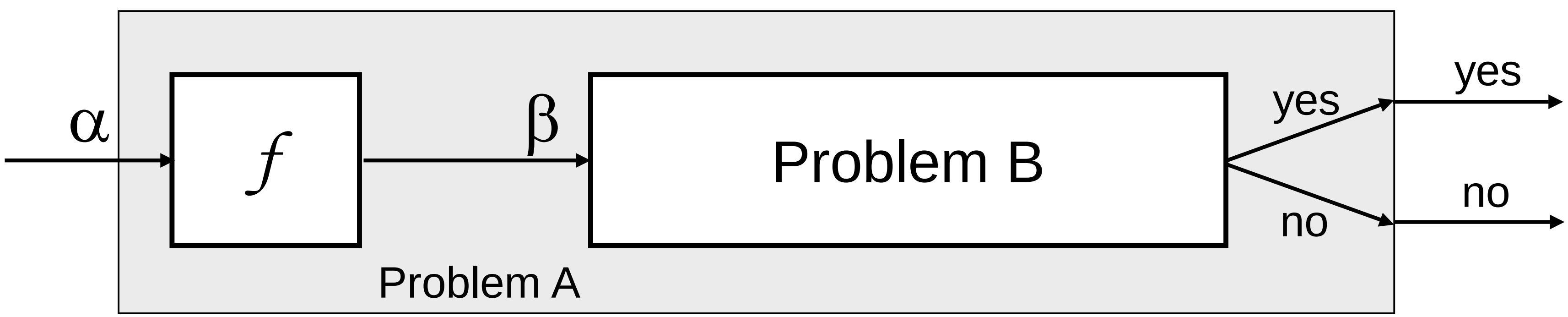
- Informally, a problem A can be reduced to another problem B if *any* instance of A can be "easily rephrased" as an instance of B, the solution to which provides a solution to the instance of A
  - *What do you suppose "easily" means?*
  - This rephrasing is called *transformation*
- Intuitively: If A reduces to B,
  - A $\leq_p$ B
  - A is "no harder to solve" than B

Reductions

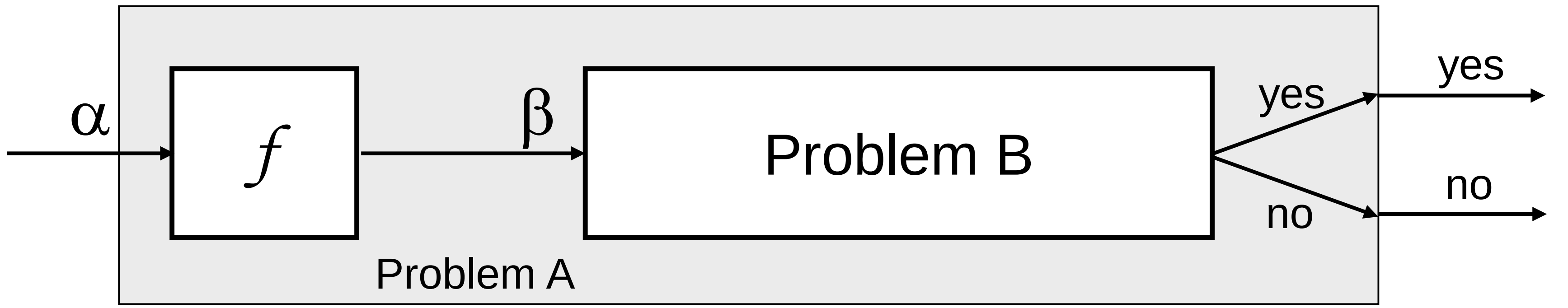
- Reduction is a way of saying that one problem is **"easier"** than another.
- We say that problem A is no harder than problem B,

  (i.e., we write **"A $\le_p$ B"**)

  if we can solve A using the algorithm that solves B.
- **Idea:** transform the inputs of A to inputs of B



Implications of Reduction



- If $A \leq_p B$ and $B \in P$, then $A \in P$

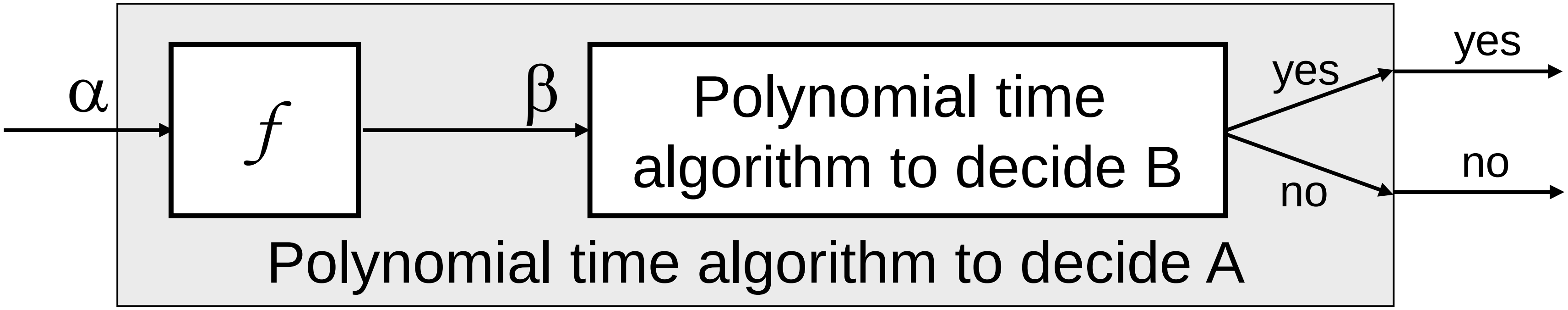
- if $A \leq_p B$ and $A \notin P$, then $B \notin P$

Polynomial Reductions

Given two problems A, B, we say that A is polynomially **reducible** to B ($A \leq_p B$) if:

1. There exists a function f that converts the input of A to inputs of B in polynomial time
2. $A(i) = YES \Leftrightarrow B(f(i)) = YES$

Proving Polynomial Time

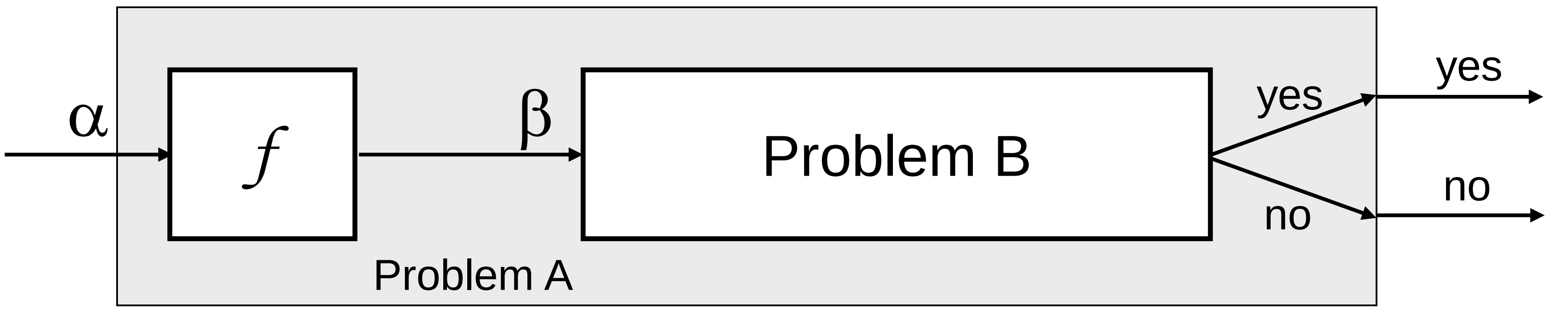


1. Use a **polynomial time** reduction algorithm to transform A into B
2. Run a known **polynomial time** algorithm for B
3. Use the answer for B as the answer for A

Reducibility

- An example:
  - A: Given a set of Booleans, $(x_1, x_2, \ldots, x_n)$, is at least one TRUE?
  - B: Given a set of integers, $(y_1, y_2, \ldots, y_n)$, is their sum positive?
  - Transformation: $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ where $y_i = 1$ if x_i = TRUE, $y_i = 0$ if x_i = FALSE

Reductions



Proving NP-Completeness

What steps do we have to take to prove a problem B is NP-Complete?

1. Pick a known NP-Complete problem A. Reduce A to B
   - Describe a polynomial time transformation/reduction that maps instances of A to instances of B, s.t. “yes” for B = “yes” for A
   - Prove the transformation works
   - Prove it runs in polynomial time

   By proving step 1 you have proved that problem B is NP-Hard
2. Prove B $\in$ **NP**
   - Show that a solution to B can be verified in polynomial time

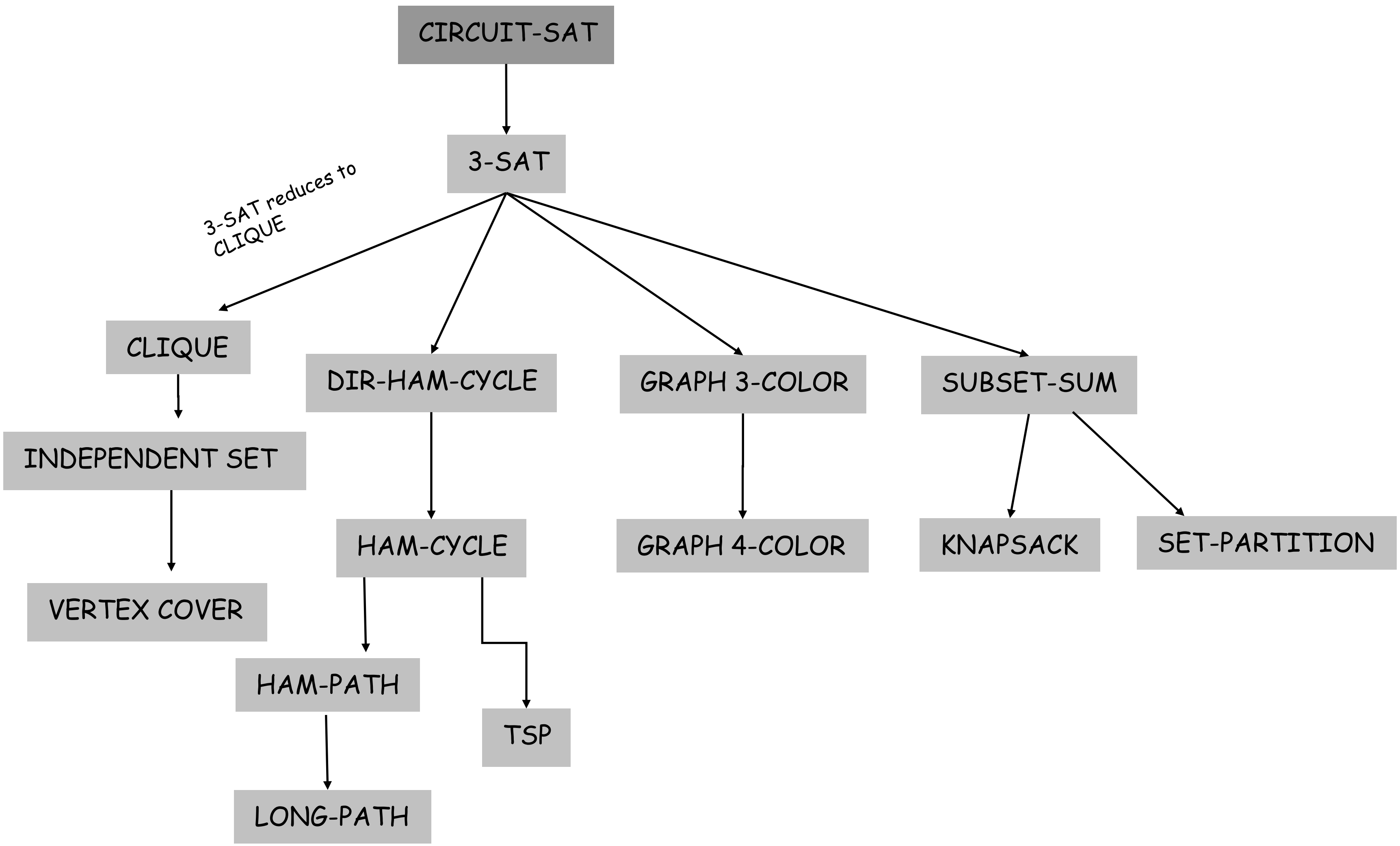
If you can prove steps 1 and 2 you have proven that B is NP-complete.

NP-Completeness

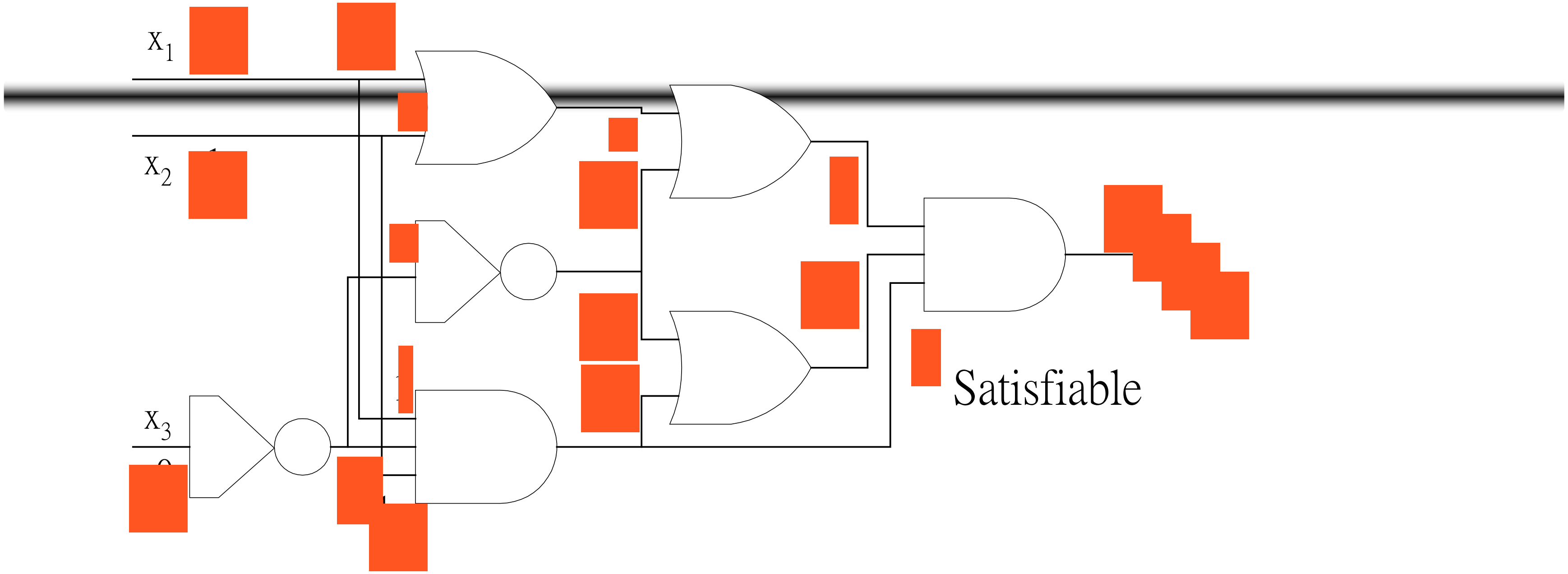
Part 3

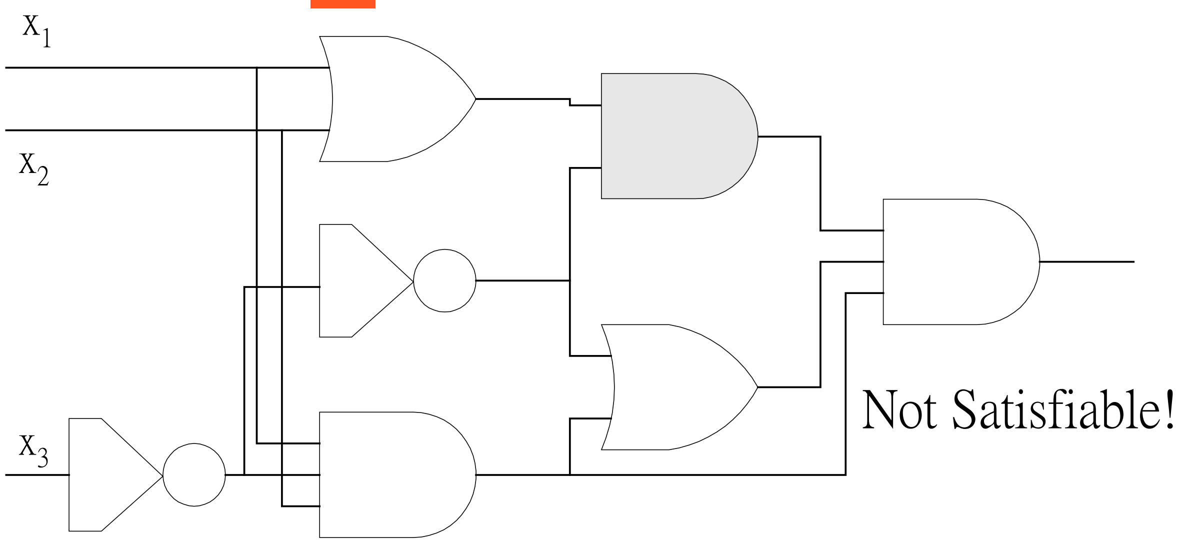
NP-Completeness

All problems below are NP-complete and polynomial reduce to one another!

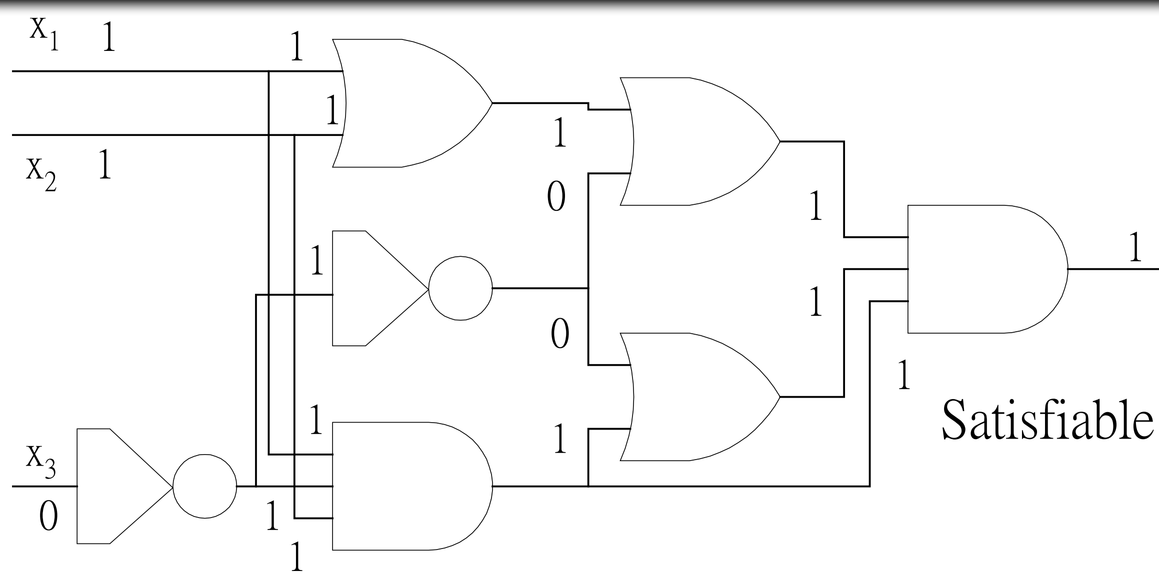


Circuit satisfiability is in NP

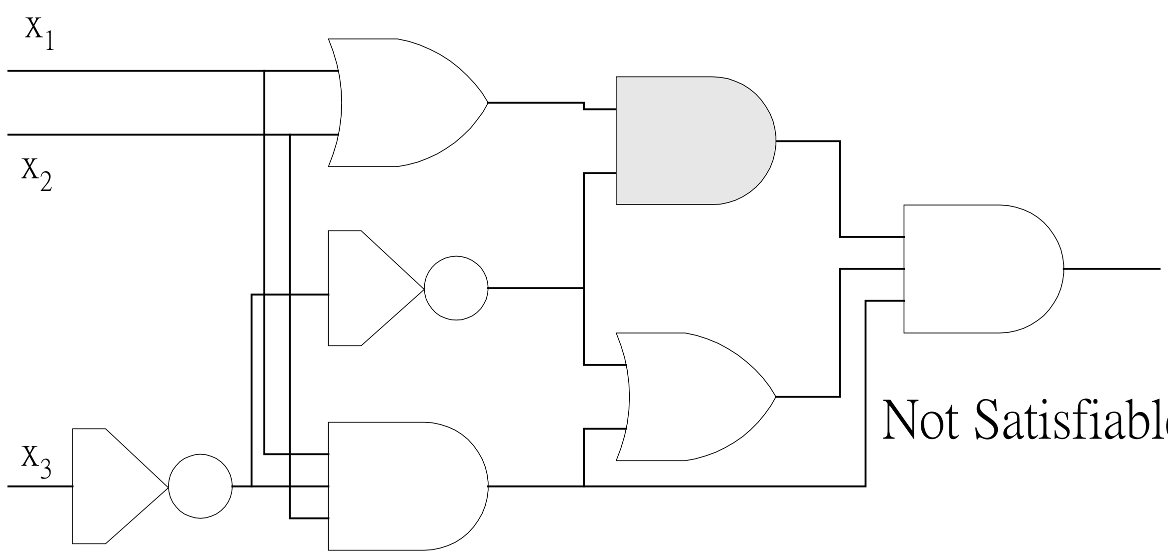




Circuit satisfiability



Satisfiable



Not Satisfiable!

Satisfiability Problem (SAT)

- **Satisfiability problem:** given a logical expression Φ, find an assignment of values (F, T) to variables x_i that causes Φ to evaluate to T

$$\Phi = x_1 \vee \neg x_2 \wedge x_3 \vee \neg x_4$$

- SAT was the first problem shown to be NP-complete!

CNF Satisfiability

- CNF is a special case of SAT
- Φ is in "Conjuctive Normal Form" (CNF)
  - "AND" of expressions (i.e., clauses)
  - Each clause contains only "OR"s of the variables and their complements

$$\Phi = (x_1 \vee x_2) \wedge (x_1 \vee \neg x_2) \wedge (\neg x_1 \vee \neg x_2)$$

clauses

3-CNF Satisfiability

A subcase of CNF problem:

– Contains three clauses

$$\Phi = (x_1 \vee \neg x_1 \vee \neg x_2) \wedge (x_3 \vee x_2 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$$

- **3-CNF (3-SAT)** is NP-Complete
- Interestingly enough, **2-CNF** is in P!

Clique

Clique Problem:

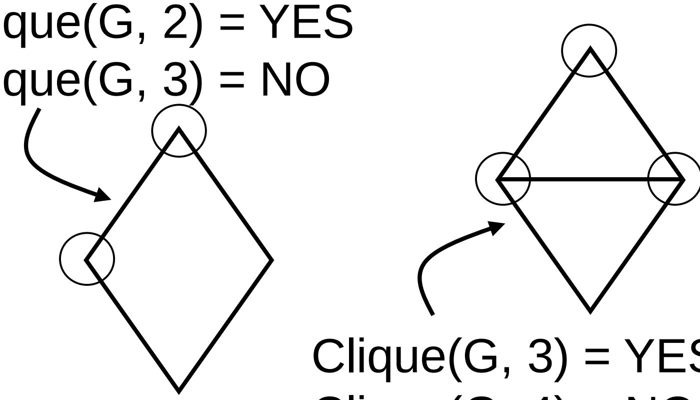
- Undirected graph G = (V, E)
- **Clique:** a subset of vertices in V all connected to each other by edges in E (i.e., forming a complete graph)
- **Size of a clique:** number of vertices it contains

Optimization problem:

- Find a clique of maximum size

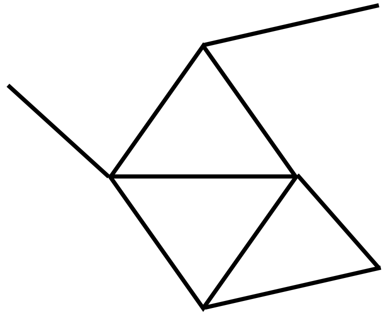
Decision problem:

- Does G have a clique of size k?



Clique Verifier

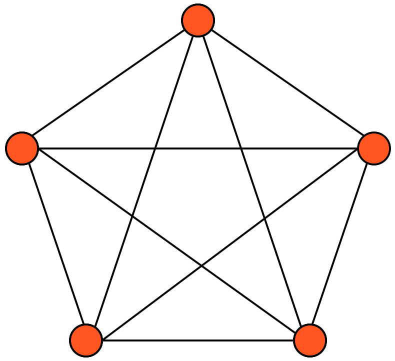
- **Given**: an undirected graph G = (V, E)
- **Problem**: Does G have a clique of size k?
- **Certificate**:
  - A set of k nodes



- **Verifier**:
  - Verify that for all pairs of vertices in this set there exists an edge in E

Example: Clique

- CLIQUE = { <G,k> | G is a graph with a clique of size k }
- A clique is a subset of vertices that are all connected
- Why is CLIQUE in NP?



3-CNF $\leq_p$ Clique

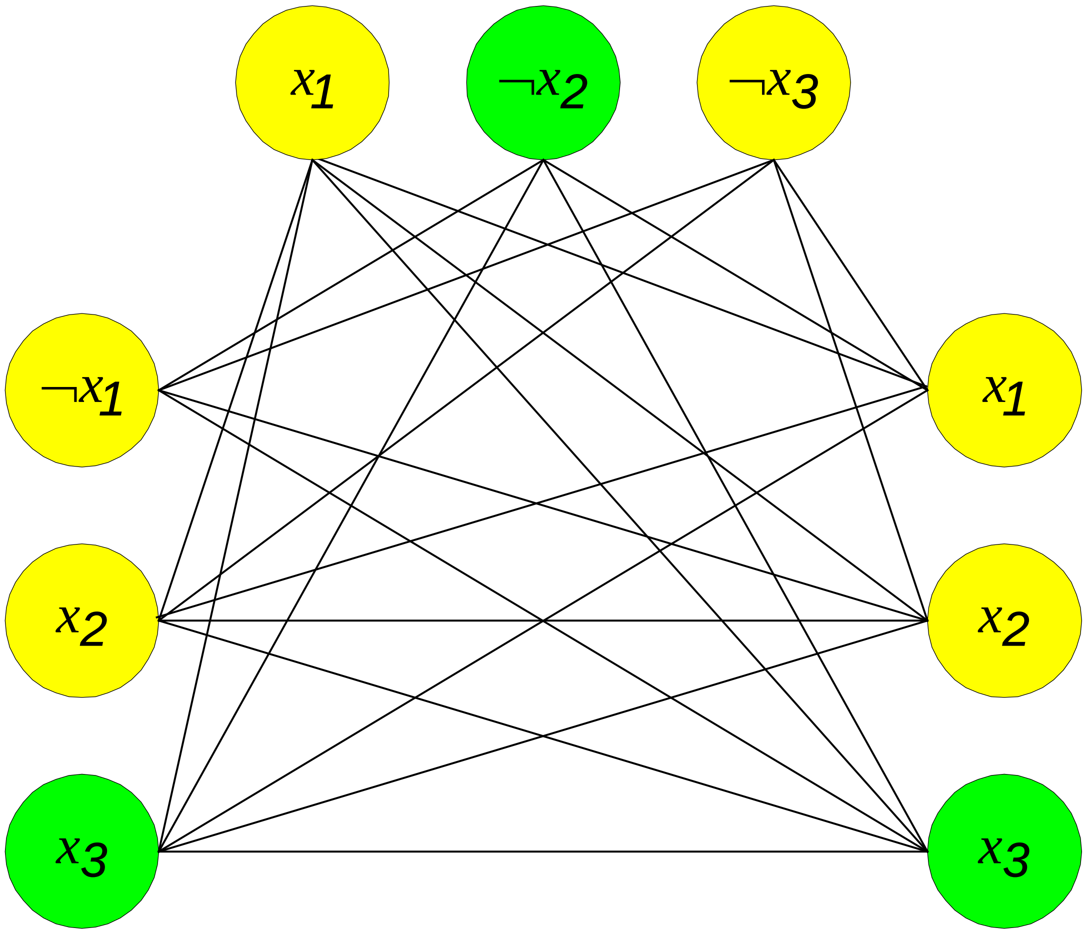
- **Idea:**
  - Construct a graph G such that Φ is satisfiable only if G has a clique of size k

Reduce 3-SAT to Clique

- Pick an instance of 3-SAT, Φ, with *k* clauses
- Make a vertex for each literal
- Connect each vertex to the literals in other clauses that are not the negation
- Any k-clique in this graph corresponds to a satisfying assignment

Example

$C_1 = x_1 \vee \neg x_2 \vee \neg x_3$

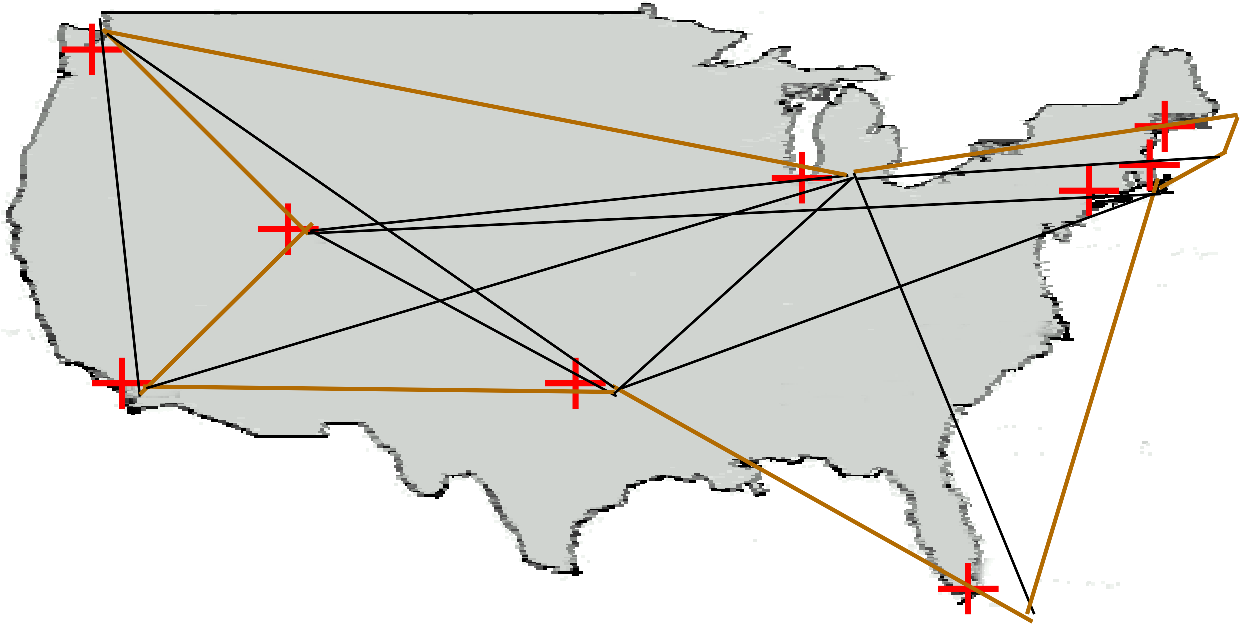


$C_2 = \neg x_1 \vee x_2 \vee x_3$

$C_3 = x_1 \vee x_2 \vee x_3$

Traveling Salesman Problem

- A traveling salesperson needs to visit *n* cities
- Is there a route of at most *d* length? (decision problem)
  - Optimization-version asks to find a shortest cycle visiting all vertices once in a weighted graph



NP-Completeness Proof Method

To show that B is NP-Complete:

1. Show that B is in NP.

   Give a polynomial time algorithm for verifying a solution.

2. Show that $A \leq_P B$ for some $A \in$ NP-Complete

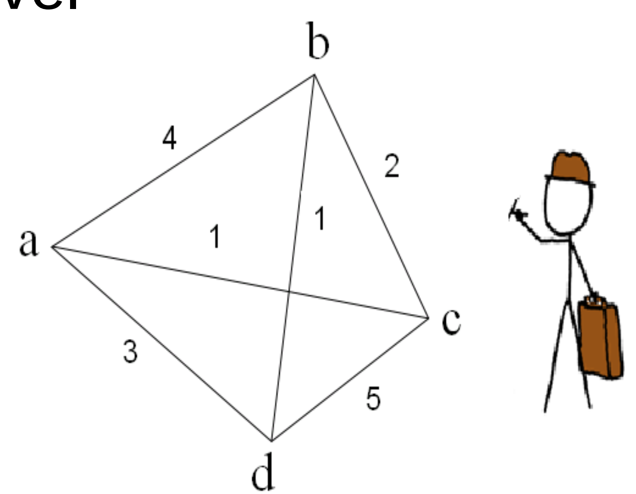
   Pick an instance, A, of your favorite NP-Complete problem

   Show a polynomial algorithm to transform A into an instance of B

   Step 2 alone shows that a problem is NP-Hard

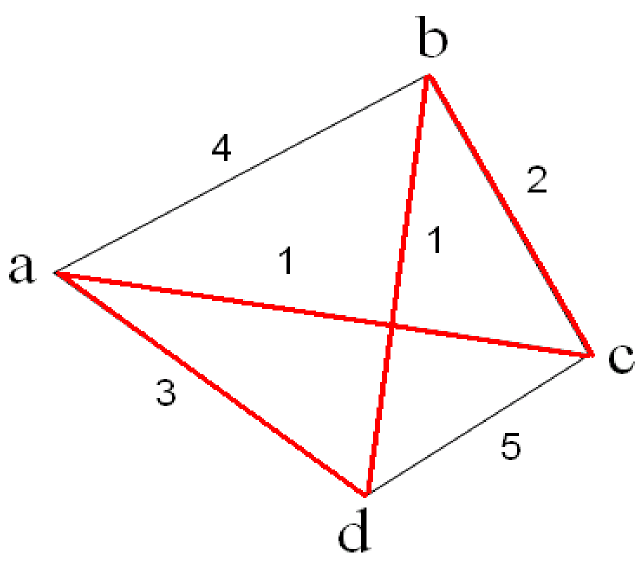
Traveling Salesman Problem

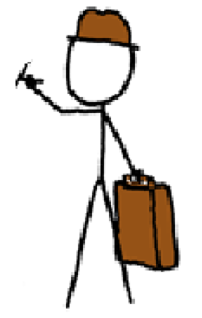
- In the traveling salesman problem, a salesman must visit ***n*** cities.
- Salesman wishes to make a tour visiting each city exactly once and finishing at the city he started.
- There is an integer cost ***c(i,j)*** to travel from city ***i*** to city ***j***.
- For example, the salesman must travel to ***a, b, c, d*** locations.
- Travel costs are given



Optimization TSP

- The salesman wishes to make the tour whose total cost is minimum.
- The total cost is sum of the individual costs along the edges of the tour
- In the example the minimum cost tour is a-c-b-d
- The cost of this tour is 1+2+1+3 = 7





Decision TSP

The formal language:

TSP = { <*G,c,k*>: *G=(V,E)* is a complete graph,

c is a function from $(V \times V) \rightarrow N$, $k \in$ N and

G has a traveling salesman tour with cost at most *k*}

G = ({a, b, c, d},{(a,b),..(c,d)}) c(a,b) = 4, c(a,c) = 1,…c(c,d) = 5

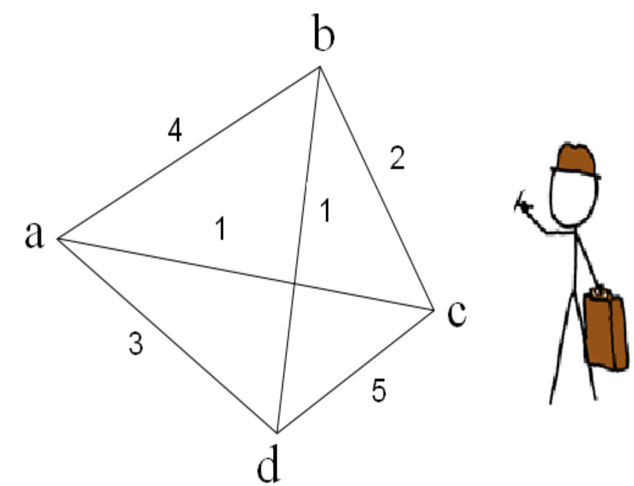
Suppose k = 15

Can we verify the solution certificate

(a,d,c,b) - yes

c(a,d) + c(d,c) + c(c,b) + c(b,a) =

3 + 5 + 2 +4 = 14 < 15



Prove TSP-Decision is NP-complete

1) Show that TSP belongs to NP.

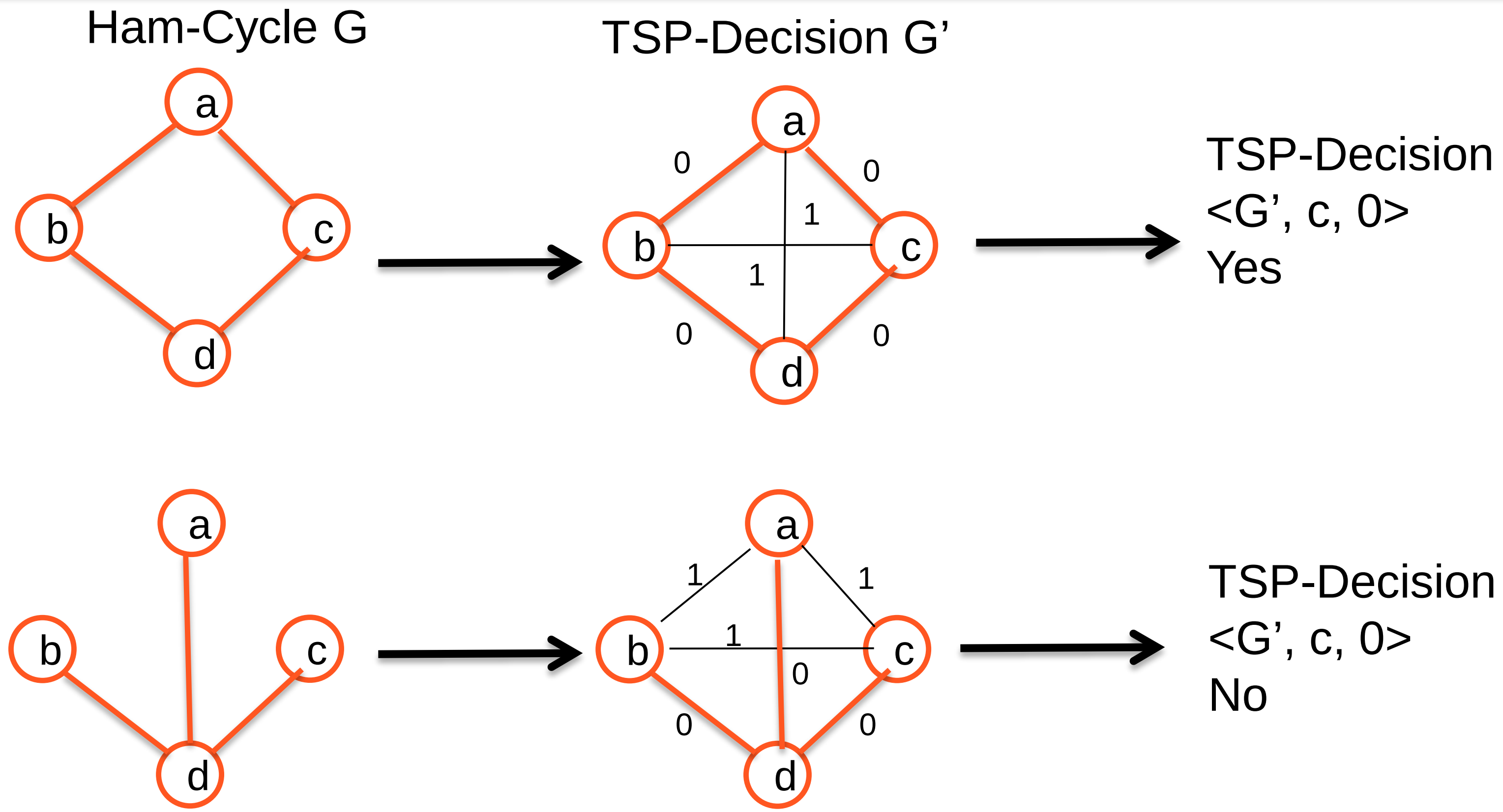
Given an instance of the problem the certificate is the sequence of *n* vertices (cities) in the tour.

The certifier (verification algorithm) checks that

- this sequence contains each vertex exactly once,
- sums up the edge costs and checks whether the sum is at most ***k***.

This process can be done in polynomial time.

Therefore TSP-Decision is in NP


Ham-Cycle G
TSP-Decision G’
a
b
c
d
0
0
1
1
0
0
TSP-Decision
<G’, c, 0>
Yes
a
b
c
d
1
1
1
0
0
0
TSP-Decision
<G’, c, 0>
No

Prove TSP-Decision is NP-complete

2) Prove that TSP is NP-hard. We can show that

$$\text{Ham-cycle} \leq_p \text{TSP}.$$

where Ham-cycle $\in$ NP-Complete

Let $G=(V,E)$ be an instance of Ham-cycle. We construct an instance of TSP as follows

- Form the complete graph $G' = (V,E')$ where $E' = \{ (i,j) : i, j \in V \text{ and } i \neq j\}$ and
- Define the cost function c by $c(i,j) = \{ 0 \text{ if } (i,j) \in E, 1 \text{ if } (i,j) \notin E \}$

The instance of TSP is then $<G',c,0>$ which is easily formed in polynomial time.

By proving 2) TSP-Decision is NP-Hard. Since 1) held too then we have shown that TSP-Decision is NP-Complete

We now show that graph ***G*** has a Hamiltonian cycle if and only if graph ***G′*** has a tour of cost at most 0**.**

Suppose the graph ***G*** has a Hamiltonian cycle ***h***.

Each edge in ***h*** belongs to ***E*** and thus has a cost 0 in ***G‘***

Thus ***h*** is a tour in ***G′*** with cost 0

Conversely suppose that graph ***G′*** has a tour ***h′*** of cost at most 0.

Since the cost of edges in ***E′*** are 0 and 1, the cost of tour ***h′*** is exactly 0 and each edge on the tour must have cost 0.

Thus ***h′*** contains only edges in ***E***.

Hence we conclude that ***h′*** is a Hamiltonian cycle in graph ***G***.

By proving 2) TSP-Decision is NP-Hard. Since 1) held too then we have shown that TSP-Decision is NP-Complete

SUBSET-SUM

Instance: A set of numbers denoted S and a target number t.

Problem: To decide if there exists a subset $S' \subseteq S$, s.t $\Sigma_{y \in S'} y = t$.

SUBSET-SUM is in NP

On input S , t:

- Guess $S' \subseteq S$
- Accept iff $\Sigma_{y \in S'} y = t$.

The length of the certificate: O(n) (n=|S|)

Time complexity: O(n)

Examples of SUBSET-SUM

$\langle \{2,4,8\}, 10 \rangle \in \text{SUBSET-SUM}$… because 2+8=10

$\langle \{2,4,8\}, 11 \rangle \notin \text{SUBSET-SUM}$

… because 11 cannot be made out of {2,4,8}

$$\text{SUBSET-SUM} = \{\langle S, t \rangle \mid S = \{x_1, \ldots, x_k\} \text{ there is a subset } R \subseteq S \text{ such that } \sum_{y \in R} y = t\}$$

SUBSET-SUM is NP-Complete

Proof: We'll show $3SAT \leq_p$ SUBSET-SUM.

Reducing 3SAT to SubSet Sum

Proof idea:

- Choosing the subset numbers from the set S corresponds to choosing the assignments of the variables in the 3CNF formula.

- The different digits of the sum correspond to the different clauses of the formula.

- If the target t is reached, a valid and satisfying assignment is found.

Subset Sum

3CNF formula:

$$(x_1 \vee x_2 \vee x_3) \wedge$$
$$(\overline{x}_1 \vee x_2 \vee x_4) \wedge$$
$$(\overline{x}_2 \vee \overline{x}_2 \vee \overline{x}_3) \wedge$$
$$(x_1 \vee \overline{x}_3 \vee \overline{x}_4)$$

Make the number table, and the 'target sum' t

					clause: 1	2	3	4	
$+x_1$	1	0	0	0	1	0	0	1	
$-x_1$	1	0	0	0	0	1	0	0	
$+x_2$		1	0	0	1	1	0	0	
$-x_2$		1	0	0	0	0	1	0	
$+x_3$			1	0	1	0	0	0	
$-x_3$			1	0	0	0	1	1	
$+x_4$				1	0	1	0	0	
$-x_4$				1	0	0	0	1	
					1	0	0	0	dummies
					1	0	0	0	dummies
						1	0	0	dummies
						1	0	0	dummies
							1	0	dummies
							1	0	dummies
								1	dummies
								1	dummies
+	1	1	1	1	3	3	3	3	

Reducing 3SAT to SubSet Sum

- Let $\phi \in$3CNF with k clauses and ℓ variables $x_1,\ldots,x_\ell$.
- Create a Subset-Sum instance $\langle S_\phi, t\rangle$ by: $2\ell+2k$ elements of
  $S_\phi = \{y_1,z_1,\ldots,y_\ell,z_\ell,g_1,h_1,\ldots,g_k,h_k\}$
  - y_j indicates positive x_j literals in clauses
  - z_j indicates negated x_j literals in clauses
  - g_j and h_j are dummies
  - and

Subset Sum

$$(x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee x_4) \wedge$$
$$(\overline{x}_2 \vee \overline{x}_2 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_3 \vee \overline{x}_4)$$

Note 1: The "1111" in the target forces a proper assignment of the x_i variables.

Note 2: The target "3333" is only possible if each clause is satisfied. (The dummies can add maximally 2 extra.)

$+x_1$	1	0	0	0	1	0	0	1
$-x_1$	1	0	0	0	0	1	0	0
$+x_2$		1	0	0	1	1	0	0
$-x_2$		1	0	0	0	0	1	0
$+x_3$			1	0	1	0	0	0
$-x_3$			1	0	0	0	1	1
$+x_4$				1	0	1	0	0
$-x_4$				1	0	0	0	1
					1	0	0	0
					1	0	0	0
						1	0	0
						1	0	0
							1	0
							1	0
								1
	1	1	1	1	3	3	3	3

Subset Sum

$$(x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee x_4) \wedge$$
$$(\overline{x}_2 \vee \overline{x}_2 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_3 \vee \overline{x}_4)$$

1	0	0	0	1	0	0	1
	1	0	0	0	0	1	0
		1	0	0	0	1	1
			1	0	1	0	0
				1	0	0	0
				1	0	0	0
					1	0	0
					1	0	0
						1	0
							1
1	1	1	1	3	3	3	3

+

$+x_1$
$-x_1$
$+x_2$
$-x_2$
$+x_3$
$-x_3$
$+x_4$
$-x_4$

1	0	0	0	1	0	0	1
1	0	0	0	0	1	0	0
	1	0	0	1	1	0	0
	1	0	0	0	0	1	0
		1	0	1	0	0	0
		1	0	0	0	1	1
			1	0	1	0	0
			1	0	0	0	1
				1	0	0	0
				1	0	0	0
					1	0	0
					1	0	0
						1	0
						1	0
							1
1	1	1	1	3	3	3	3

$x_1, \overline{x}_2, \overline{x}_3, x_4$ is a satisfying assignment

Subset Sum

$$(x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee x_4) \wedge$$
$$(\overline{x}_2 \vee \overline{x}_2 \vee \overline{x}_3) \wedge (x_1 \vee \overline{x}_3 \vee \overline{x}_4)$$

1	0	0	0	0	1	0	0
	1	0	0	0	0	1	0
		1	0	0	0	1	1
			1	0	1	0	0
				1	0	0	0
				1	0	0	0
					?	?	?
1	1	1	1	2	?	?	?

+

$+x_1$	1	0	0	0	1	0	0	1
$-x_1$	1	0	0	0	0	1	0	0
$+x_2$		1	0	0	1	1	0	0
$-x_2$		1	0	0	0	0	1	0
$+x_3$			1	0	1	0	0	0
$-x_3$			1	0	0	0	1	1
$+x_4$				1	0	1	0	0
$-x_4$				1	0	0	0	1
					1	0	0	0
					1	0	0	0
						1	0	0
						1	0	0
							1	0
							1	0
								1
	1	1	1	1	3	3	3	3

is not a satisfying assignment

Proof 3SAT $\leq_P$ Subset Sum

- For every 3CNF ϕ, take target t=1…13…3 and the corresponding set S_ϕ.

- If $\phi \in$3SAT, then the satisfying assignment defines a subset that reaches the target.

- Also, the target can only be obtained via a set that gives a satisfying assignment for ϕ.

$$\boxed{\phi \in 3SAT \text{ if and only if } \left\langle \mathrm{S}_{\phi}, 1...13...3 \right\rangle \in \mathrm{SubsetSum}}$$

0-1 KNAPSACK

Prove the following knapsack problem to be NP complete

- Optimization Version:

n objects, each with a weight $w_i > 0$ and a benefit $b_i > 0$

capacity of knapsack : W

Maximize $\sum_{1 \leq i \leq n} b_i x_i$

Subject to $\sum_{1 \leq i \leq n} w_i x_i \leq W$

$x_i = 0$ or 1, $1 \leq i \leq n$

- Decision version :

Given K, $\exists \sum_{1 \leq i \leq n} b_i x_i \geq K$?

KNAPSACK is NP-Complete

1) Show NP – Verify a solution in polynomial time

2) Show NP-Hard. Reduce SUBSET-SUM to KNAPSACK

Show NP-Hard. Reduce SUBSET-SUM to KNAPSACK. How can you use Knapsack to solve Subset-Sum

Reduction from *SUBSET-SUM* to Decision-*KNAPSACK*:

SUBSET-SUM = { <*S*, *t*> | *S* = $\{x_1, \ldots, x_k\}$ *and for some* $\{y_1, \ldots, y_l\} \subseteq S$, $\Sigma y_i = t$ }

Set $b_i = w_i = x_i$

Set $W = k = t$

Then for any subset $T \subseteq S$

$$\sum_{i \in T} x_i = t \quad \text{if and only if} \quad \sum_{i \in T} b_i = \sum_{i \in T} x_i \geq t \quad \text{and} \quad \sum_{i \in T} w_i = \sum_{i \in T} x_i \leq t$$

Bin Packing Problem

- Given a set of items $S = \{x_1 \ldots x_n\}$ each with some weight w_i, pack maximum number of items into a collection of finite number of bins each with some capacity B_i using minimum number of bins.

- Knapsack problem is a particular case of Bin-packing when the number of bins is 1 and its capacity is K

SUBSET-SUM to PARTITION

- PARTITION = { $x_1, x_2, \ldots x_k$ | we can split the integers into two sets which sum to half }
- SUBSET-SUM = { $\langle x_1, x_2, \ldots x_k, t \rangle$ | there exists a subset which sums to t }

- 1) If I can solve SUBSET-SUM, how can I use that to solve an instance of PARTITION?
- 2) If I can solve PARTITION, how can I use that to solve an instance of SUBSET-SUM?

Polynomial Reductions

- 1) Partition REDUCES to Subset-Sum
  - Partition $\leq_p$ Subset-Sum
- 2) Subset-Sum REDUCES to Partition
  - Subset-Sum $\leq_p$ Partition
- Therefore they are equivalently hard