Week 2: Part 1

Recursion, Recurrences & Running time

Chapter 4

- Divide and conquer VS iterative algorithms
- Recursion
- Solving Recurrences
- Binary Search
- Merge Sort
- Towers of Hanoi
- Tiling

Recall from Week 1

- Asymptotic Analysis: O, Ω, Θ
- Used to compare functions that represent the running times of different algorithms that can be used to solve a problem.
- How did we get the functions

Iterative Algorithm Analysis

```
for (i=1; i<=n*n; i++)     Executed n*n times
  for (j=0; j<i; j++)       Executed <= n*n times
    sum++;                   O(1)
```

Exact # of times `sum++` is executed:

$$\sum_{i=1}^{n^2} i = \frac{n^2(n^2+1)}{2}$$

$$= \frac{n^4+n^2}{2}$$

$$\in \Theta(n^4)$$

The Divide and Conquer Approach

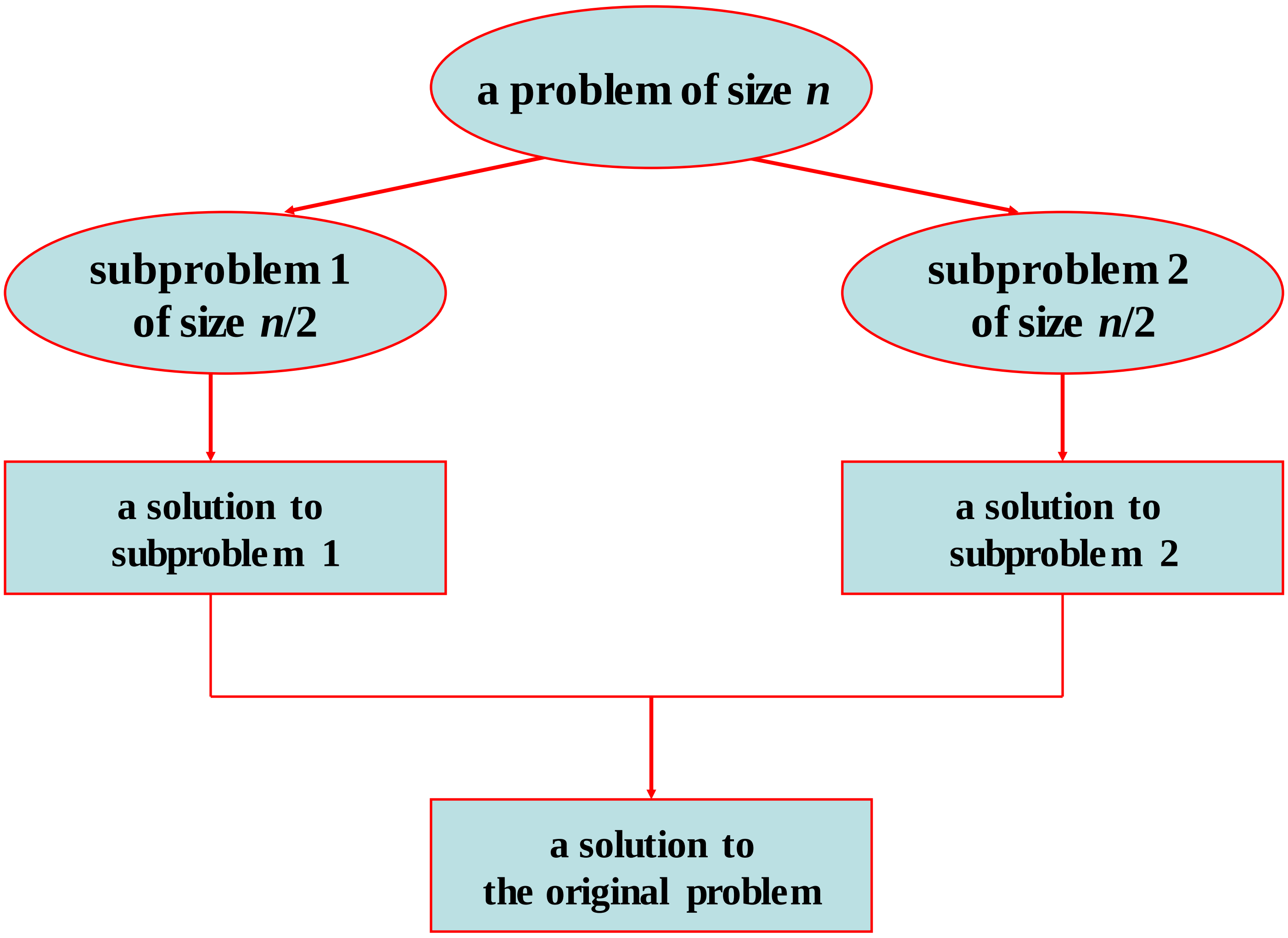
The most well known algorithm design strategy:

1. **Divide** the problem into two or more smaller subproblems.

2. **Conquer** the subproblems by solving them recursively.

3. **Combine** the solutions to the subproblems into the solutions for the original problem.

A Typical Divide and Conquer Case



Recurrences and Running Time

- An equation or inequality that describes a function in terms of its value on smaller inputs.

$$T(n) = T(\frac{n}{4}) + 1$$

- Recurrences arise when an algorithm contains recursive calls to itself
- What is the actual running time of the algorithm?
- Need to solve the recurrence
  - Find an explicit formula of the expression
  - Bound the recurrence by an expression that involves n

Merge-Sort Example

- Merge-sort on an input sequence $\boldsymbol{S}$ with $\boldsymbol{n}$ elements consists of three steps:
  - Divide: partition $\boldsymbol{S}$ into two sequences $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ of about $\boldsymbol{n}/2$ elements each
  - conquer: recursively sort $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$
  - combine: merge $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ into a unique sorted sequence

Algorithm ***mergeSort***(***S, c***)

    Input sequence $\boldsymbol{S}$ with $\boldsymbol{n}$ elements, comparator $\boldsymbol{c}$

    Output sequence $\boldsymbol{S}$ sorted according to $\boldsymbol{c}$

    if ***S.size***() > 1

        $(\boldsymbol{S}_1, \boldsymbol{S}_2) \leftarrow \boldsymbol{partition}(\boldsymbol{S}, \boldsymbol{n}/2)$

        $\boldsymbol{mergeSort}(\boldsymbol{S}_1, \boldsymbol{c})$

        $\boldsymbol{mergeSort}(\boldsymbol{S}_2, \boldsymbol{c})$

        $\boldsymbol{S} \leftarrow \boldsymbol{merge}(\boldsymbol{S}_1, \boldsymbol{S}_2)$

Recurrence Equation

- The conquer step of merge-sort consists of merging two sorted sequences, each with $n/2$ elements takes at most cn steps, for some constant c.
- Likewise, the basis case ($n < 2$) will take at most b steps.
- Therefore, if we let $T(n)$ denote the running time of merge-sort:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + cn & \text{if } n \geq 2 \end{cases}$$

- We can analyze the running time of merge-sort by finding a **closed form solution** to the above equation. *That is, a solution that has $T(n)$ only on the left-hand side.*

Merge-Sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.

$$T(n) = 2\,T(n/2) + \Theta(n)$$

 # subproblems

subproblem size

work dividing and combining

Closed form: $T(n) = \Theta(n lg n)$

Binary Search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

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Example: Find 9

3	5	7	8	9	12	15

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3. ***Combine:*** Trivial.

Example: Find 9

3	5	7	8	$\boxed{9}$	12	15

Binary Search

$$T(n) = 1\,T(n/2) + \Theta(1)$$

 # subproblems

subproblem size

work dividing and combining

Closed form: $T(n) = \Theta(lgn)$

Power of a Number

Problem: Compute a^n, where $n \in \mathbb{N}$.

Naive algorithm: $\Theta(n)$.

Counting the number of operations which are multiplications

Example: $a^n = a*a*\ldots*a$

Example: $15^9 = 15*15*15*15*15*15*15*15*15$

Power of a Number

Problem: Compute a^n, where $n \in \mathbb{N}$.

Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

Base cases $a^0 = 1$ and $a^1 = a$

$$T(n) = T(n/2) + \Theta(1)$$

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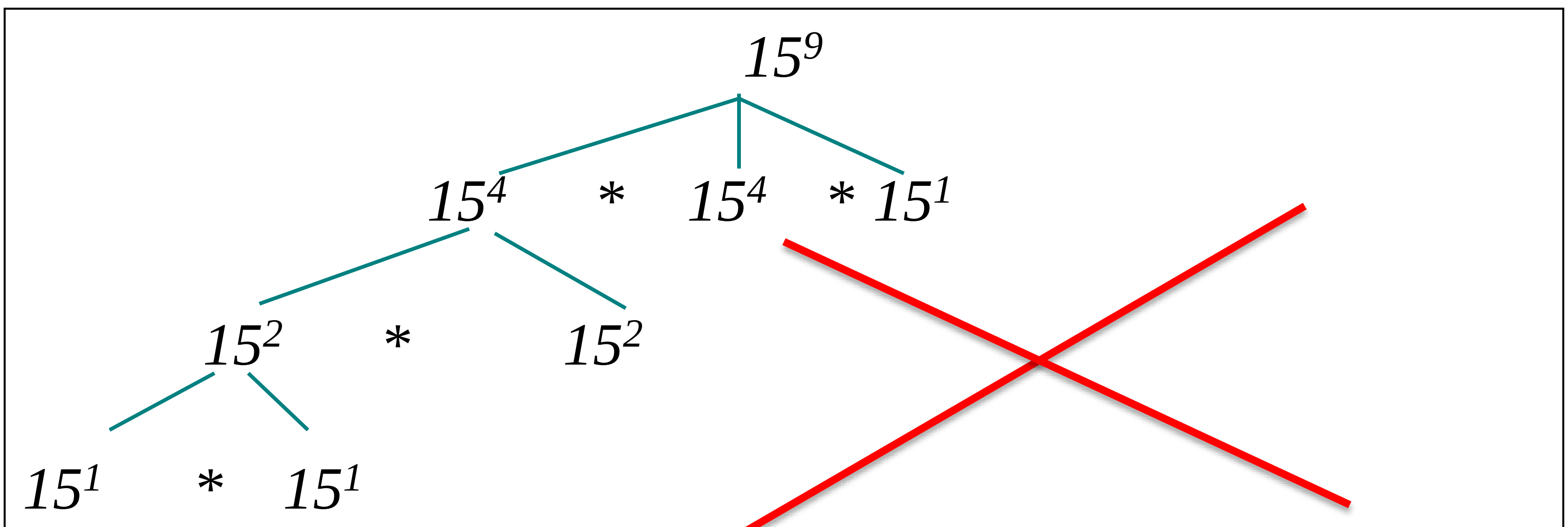


Recurrence Relations from Code

```
long power (long x, long n) {
    if(n == 0)
        return 1;
    else if(n == 1)
        return x;
    else if ((n % 2) == 0){
        temp = power(x, n/2);
        return temp*temp;
    }
    else {
        temp = power(x,(n-1)/2)
        return x * temp* temp;
    }
}
```

The recurrence relation is:

$$T(n) = 1 \qquad \text{if } n = 0 \text{ or } n = 1$$

$$T(n) = T(n/2) + c \qquad \text{if } n > 2$$

Running time $\Theta(\lg n)$

Extra Recursion

```
long power (long x, long n) {
    if(n == 0)
      return 1;
    else if(n == 1)
        return x;
    else if ((n % 2) == 0)
     return power (x, n/2) * power (x, n/2);
    else
     return x * power (x,(n-1)/2) * power (x,(n-1)/2);
}
```

The recurrence relation is:

$$T(n) = 1 \qquad \text{if } n = 0 \text{ or } n = 1$$

$$T(n) = 2T(n/2) + c \qquad \text{if } n > 2$$

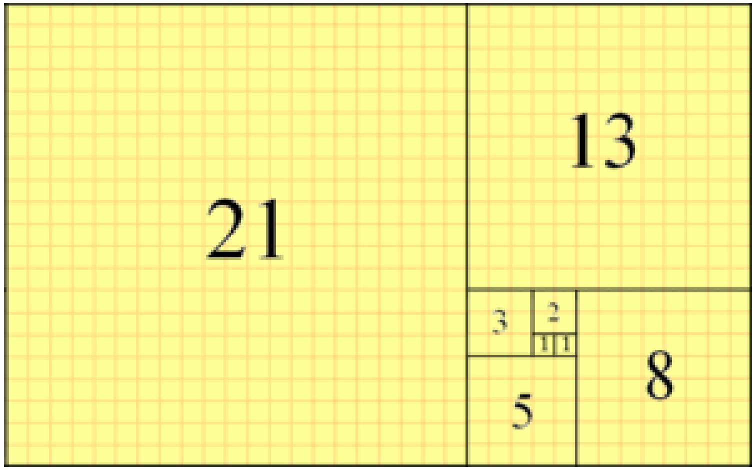
Running time $\Theta(n)$

Fibonacci Numbers

Recursive definition:

$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

$$0 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 8 \quad 13 \quad 21 \quad 34 \quad \cdots$$



Fibonacci

```
long fibonacci (int n) {
// Recursively calculates Fibonacci number
  if( n == 0)
      return 0;
  else if( n == 1)
      return 1;
  else
      return fibonacci(n – 1) + fibonacci(n – 2);
}
```

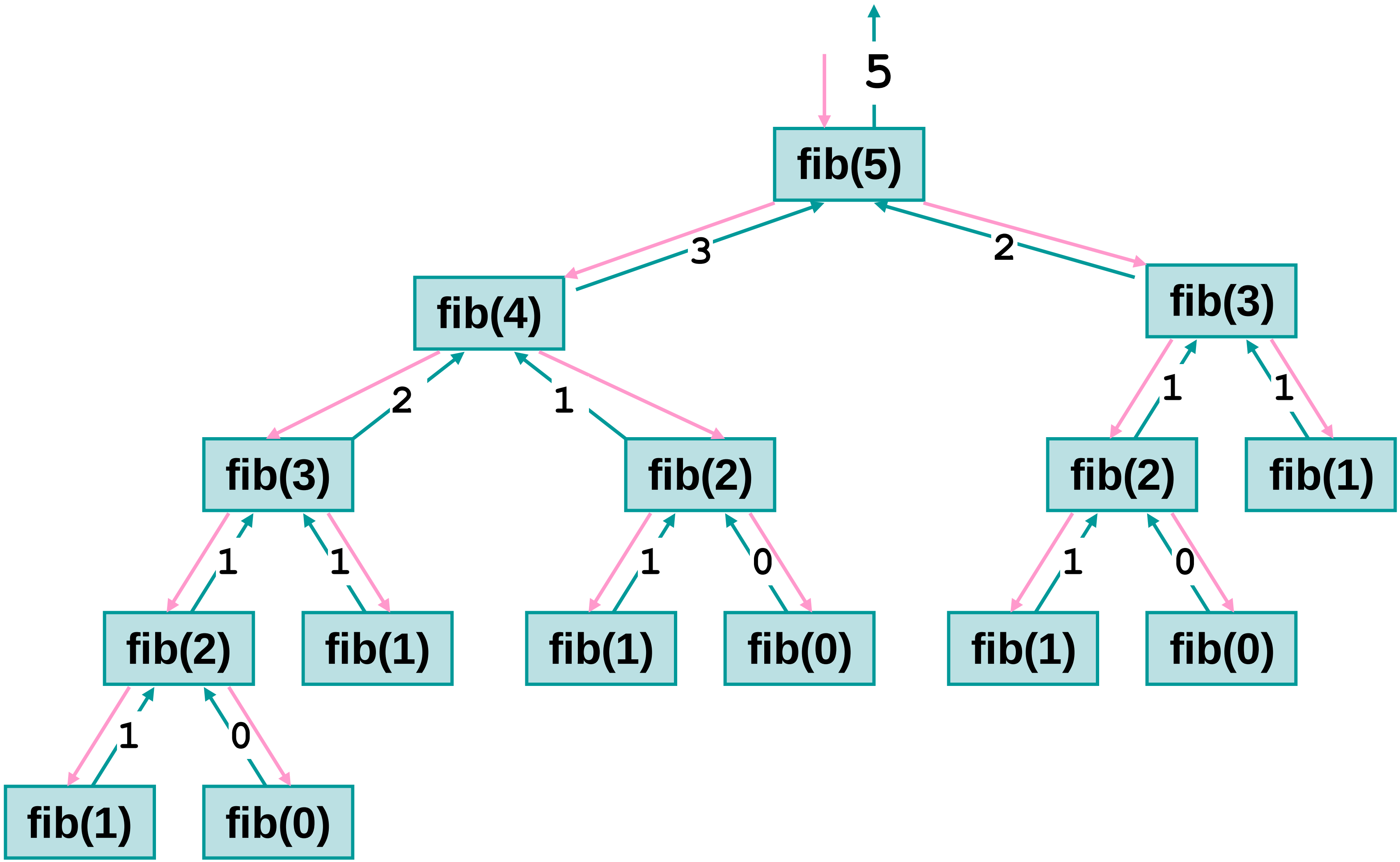
The recurrence relation is:

$$T(n) \;=\; 1 \qquad\qquad \text{if } n = 0 \text{ or } n = 1$$

$$T(n) \;=\; T(n-1) + T(n-2) + 1 \qquad\qquad \text{if } n \geq 2$$

$$\mathbf{T(n) = \Theta(\phi^n) \text{ where } \Phi = \frac{1+\sqrt{5}}{2} \text{ golden ratio} = 1.618..}$$

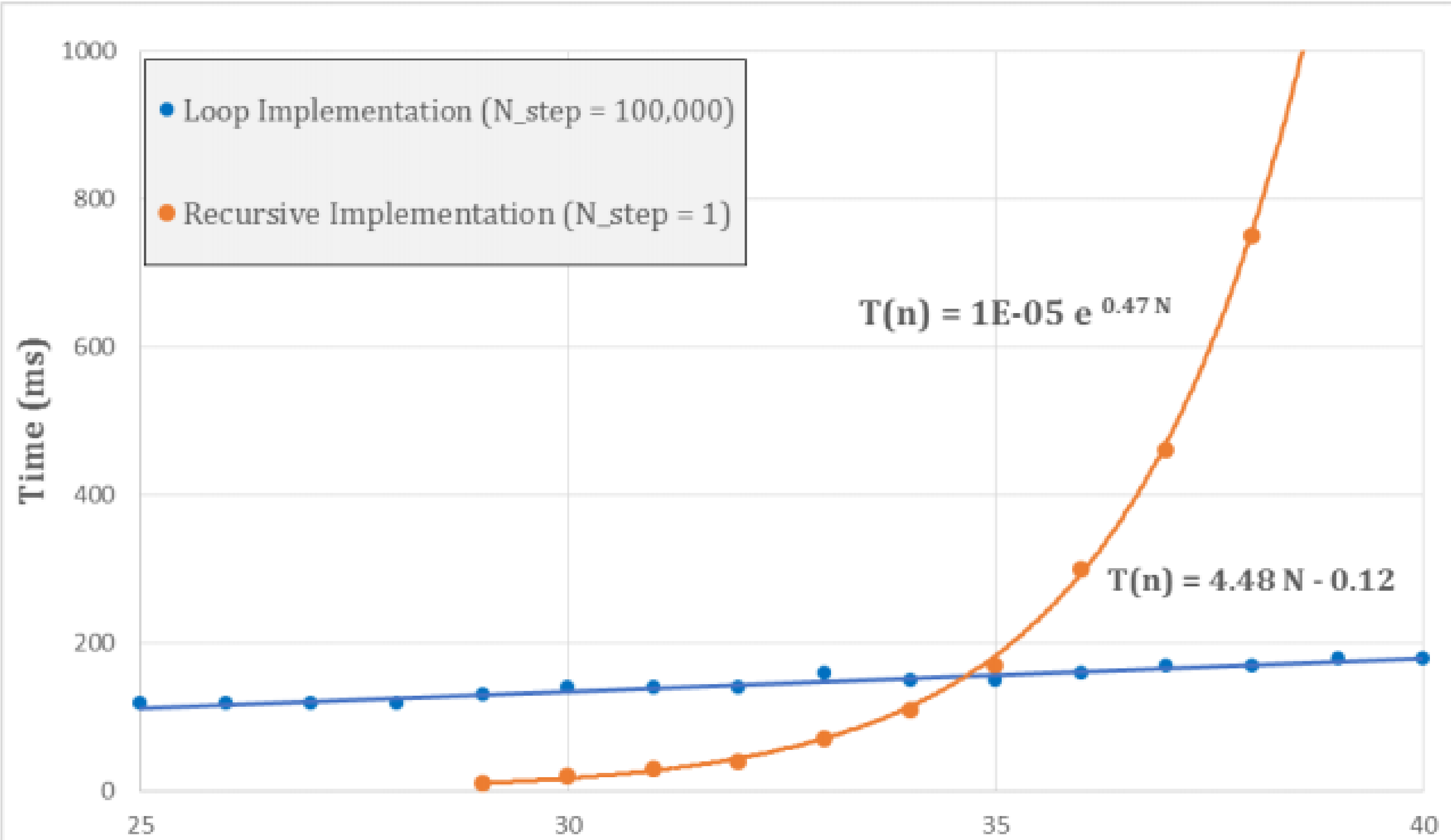
Function Analysis for call `fib(5)`



HW 1 Solution

Fibonnacci Performance Comparison:

Note: $e^{0.47n} \sim 1.60^n$



Tower of Hanoi

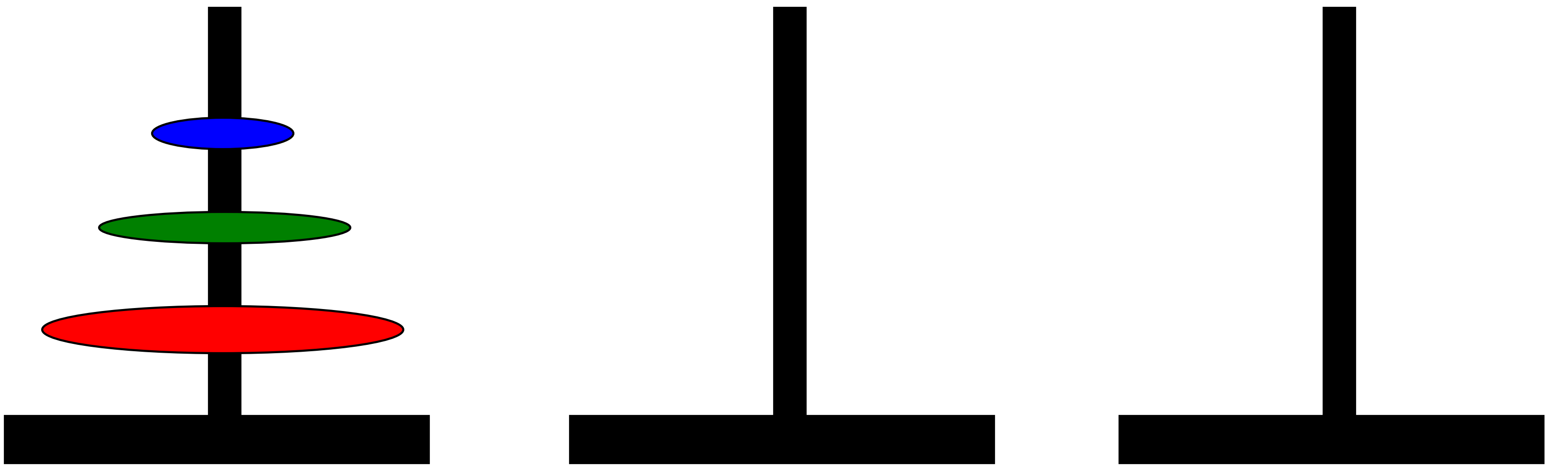
- There are three towers
- N gold disks, with decreasing sizes, placed on the first tower
- You need to move all of the disks from the first tower to the last tower
- Larger disks can not be placed on top of smaller disks
- The third tower can be used to temporarily hold disks

Tower of Hanoi

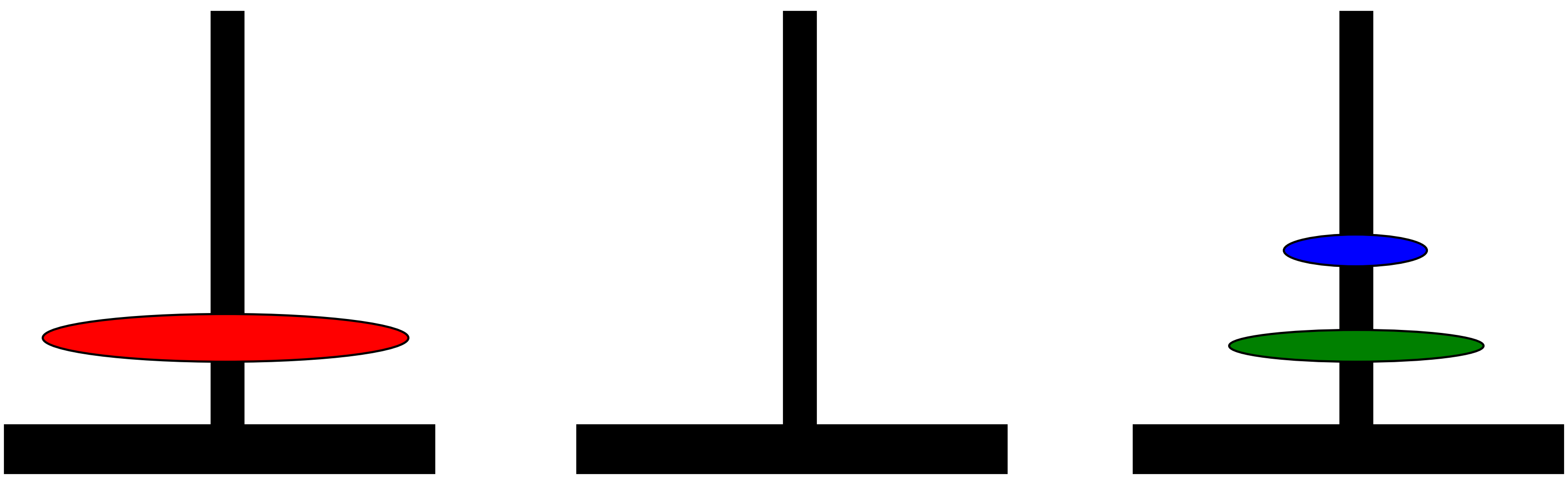
- The disks must be moved within one week. Assume one disk can be moved in 1 second. Is this possible?

- To create an algorithm to solve this problem, it is convenient to generalize the problem to the “N-disk” problem, where in our case N = 64.

Recursive Solution



Recursive Solution



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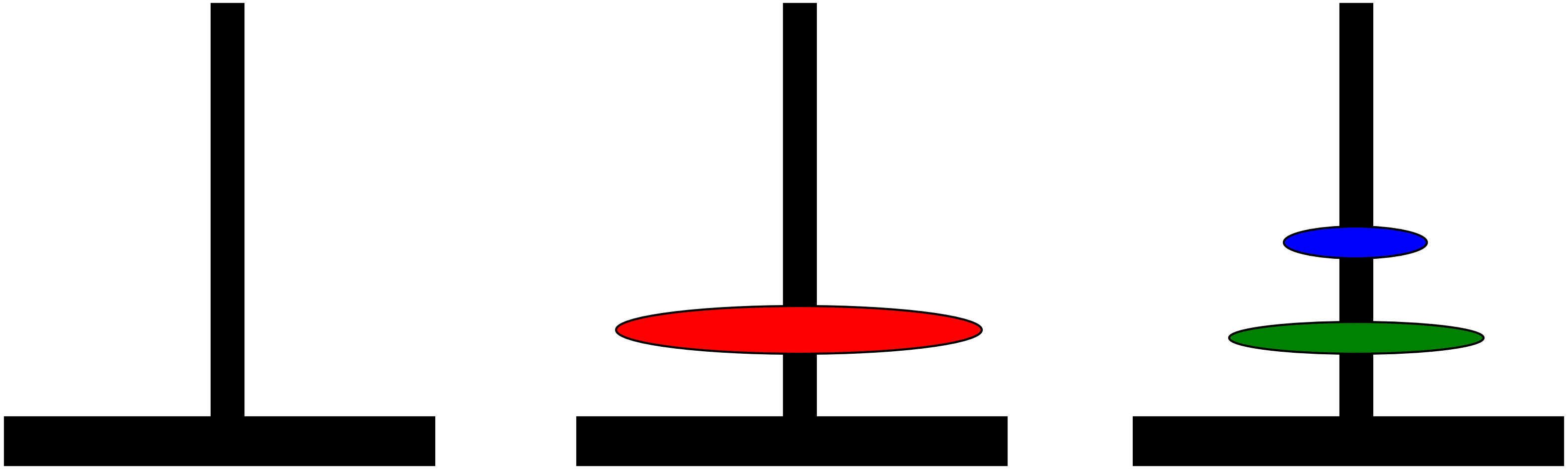

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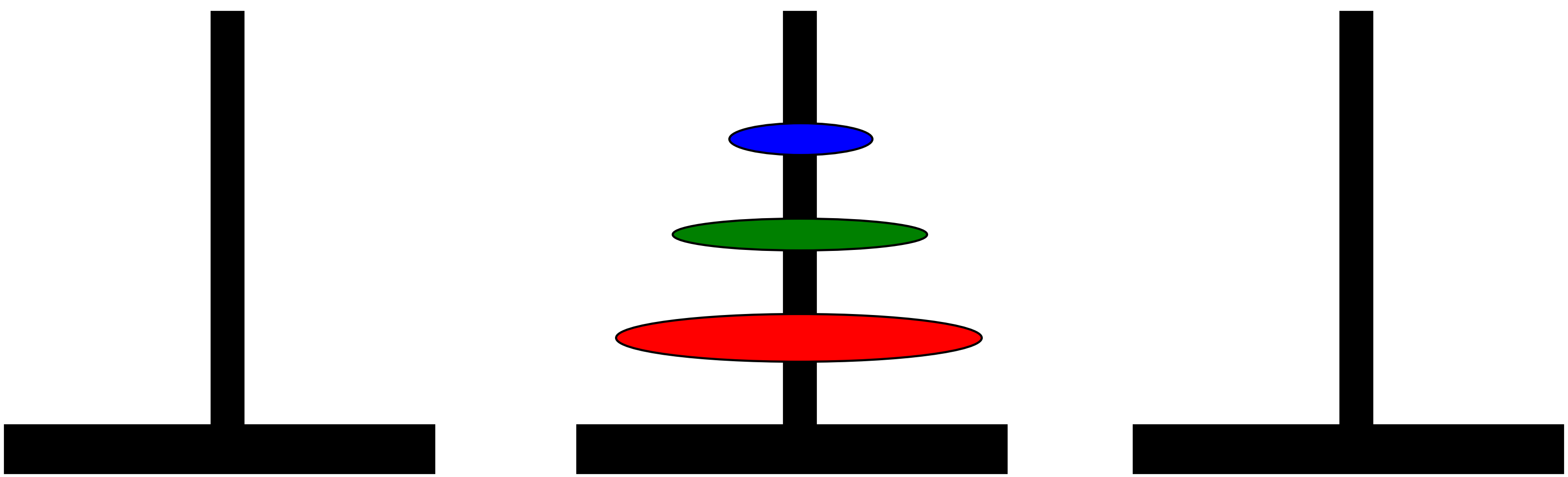
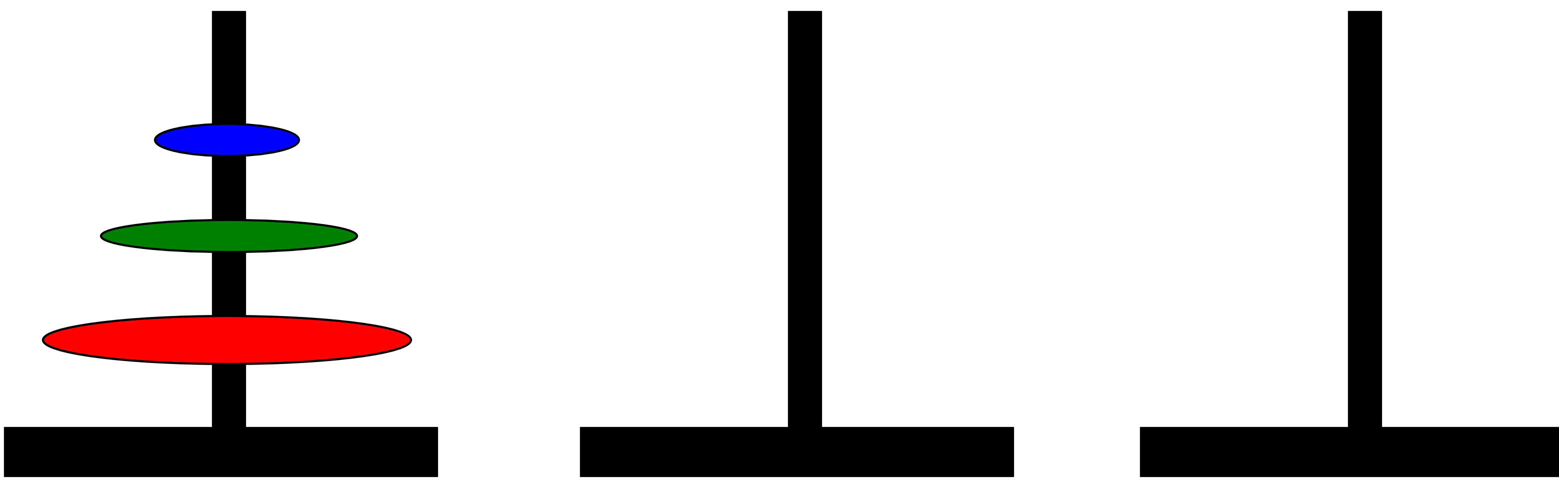


Tower of Hanoi



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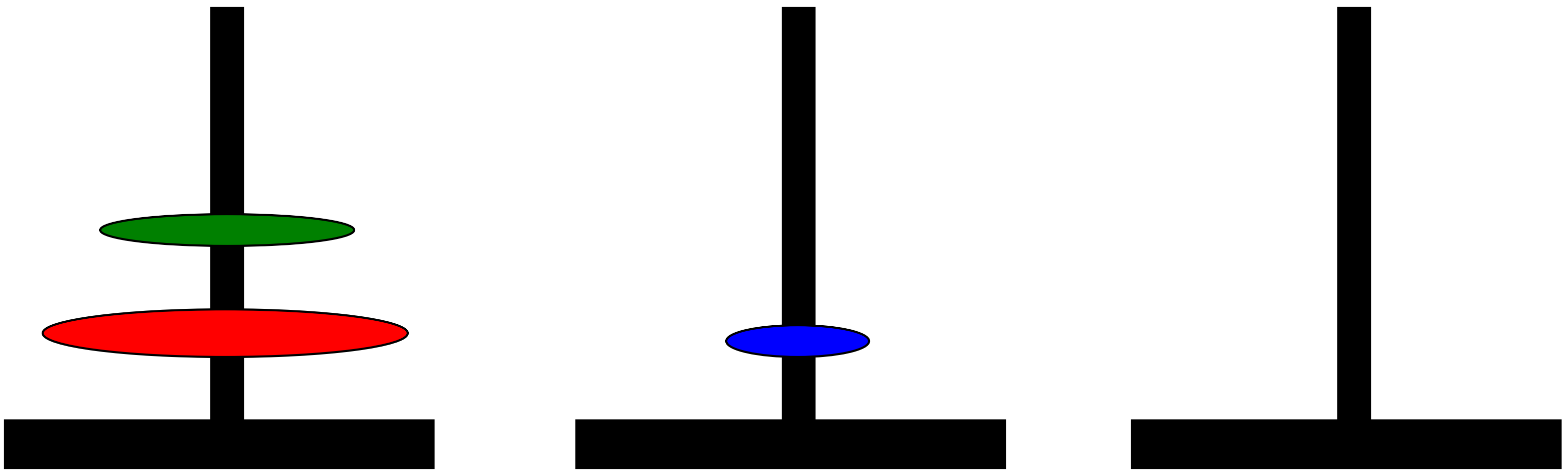
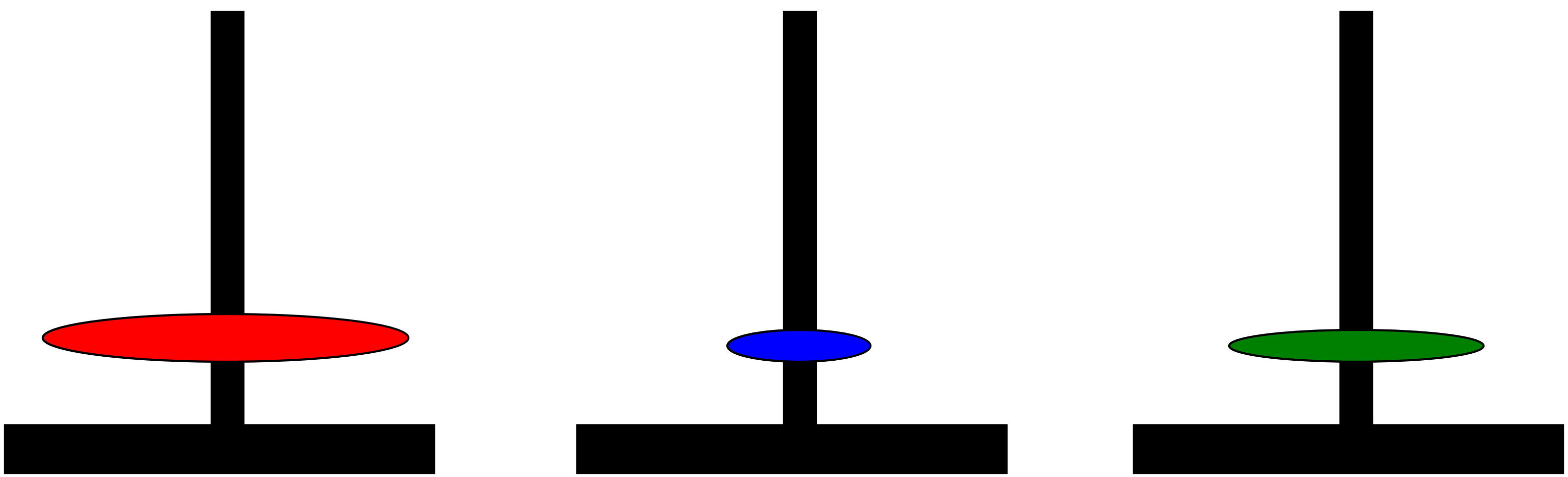


Tower of Hanoi



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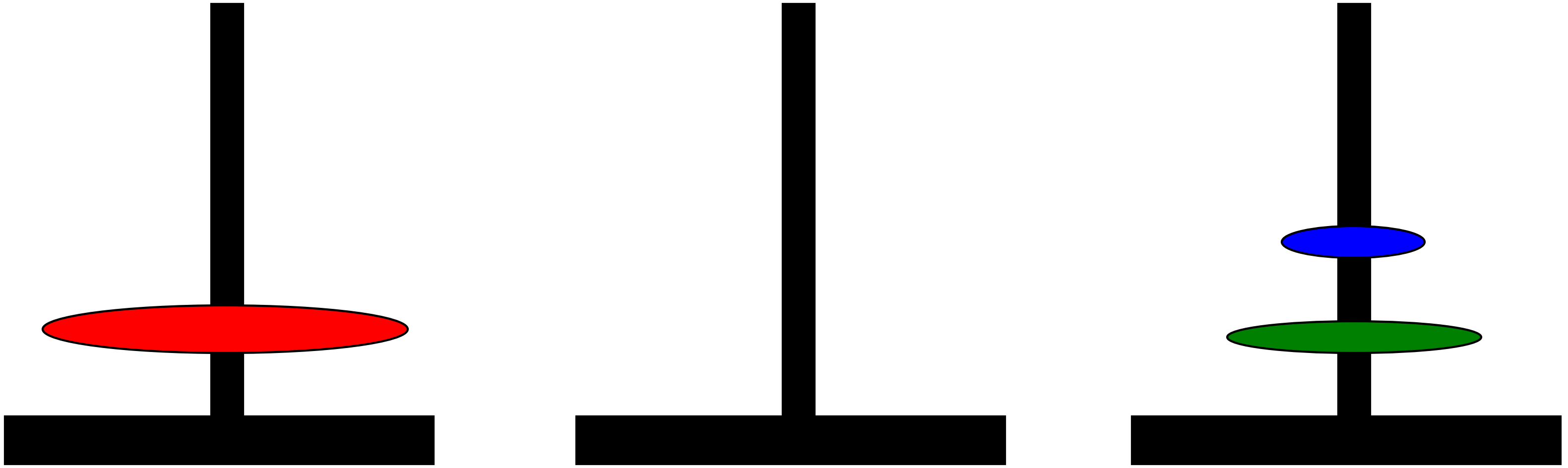

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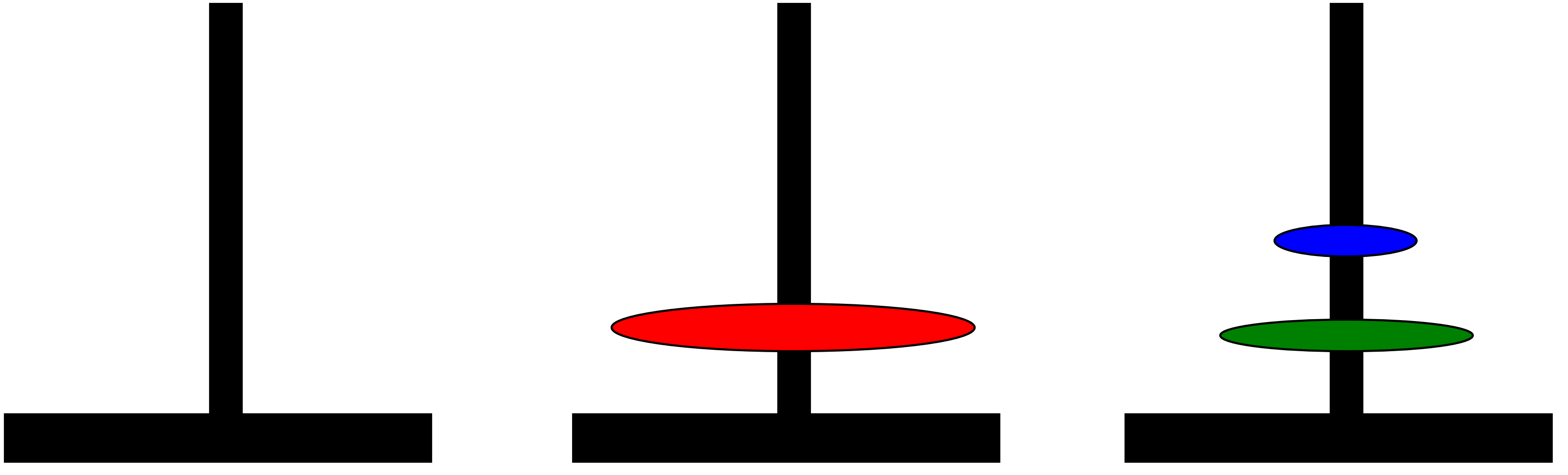

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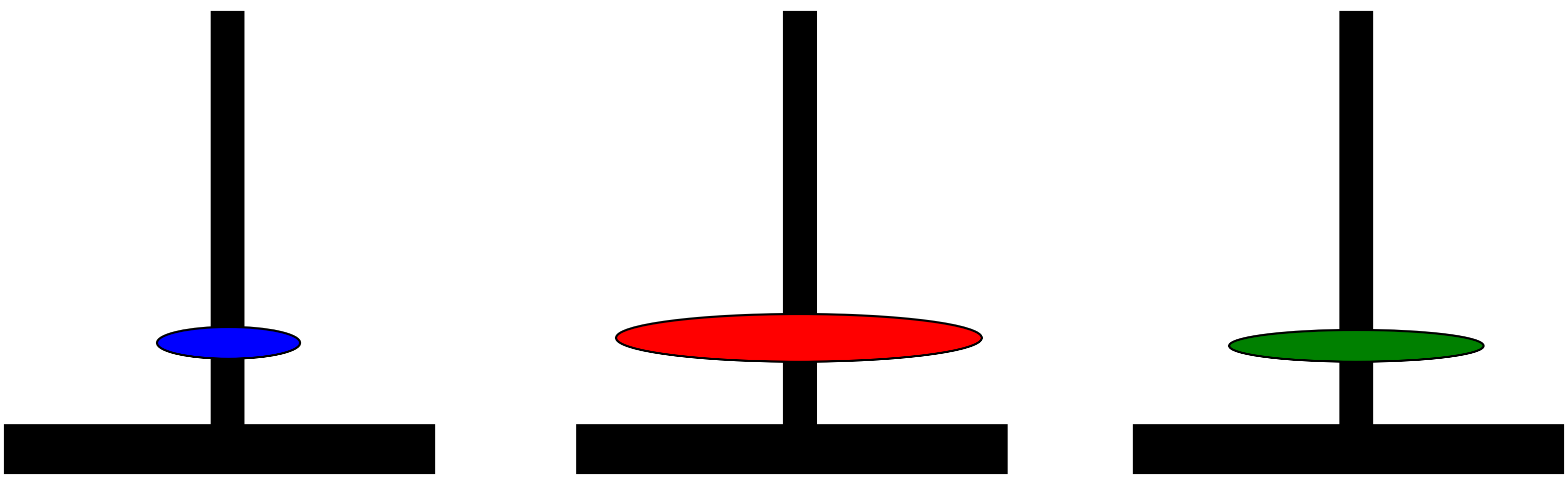

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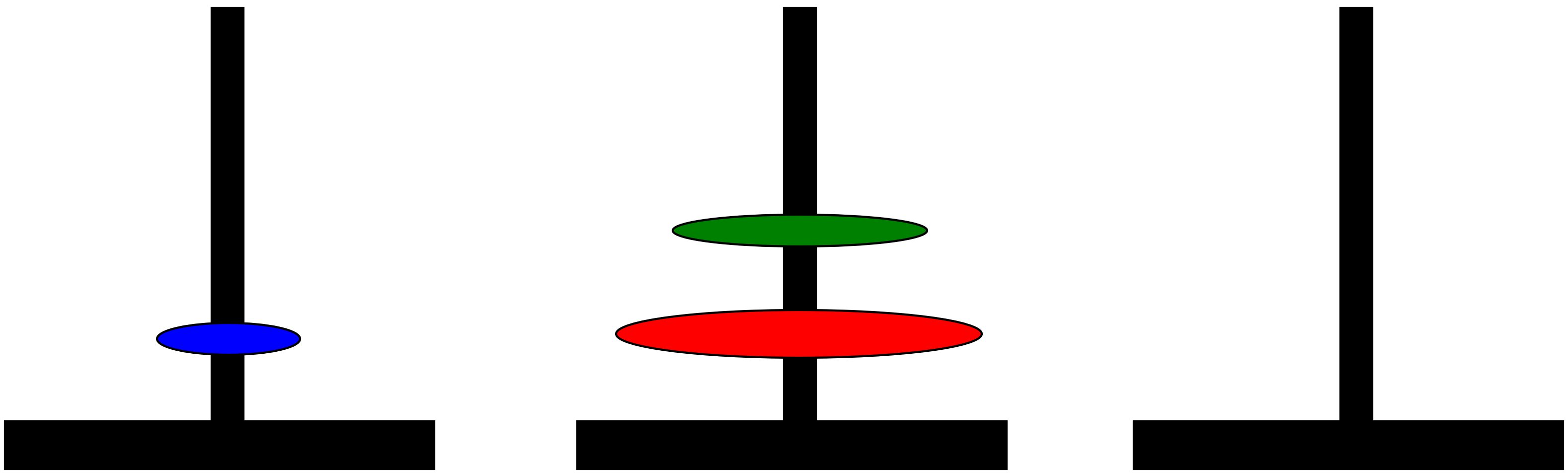
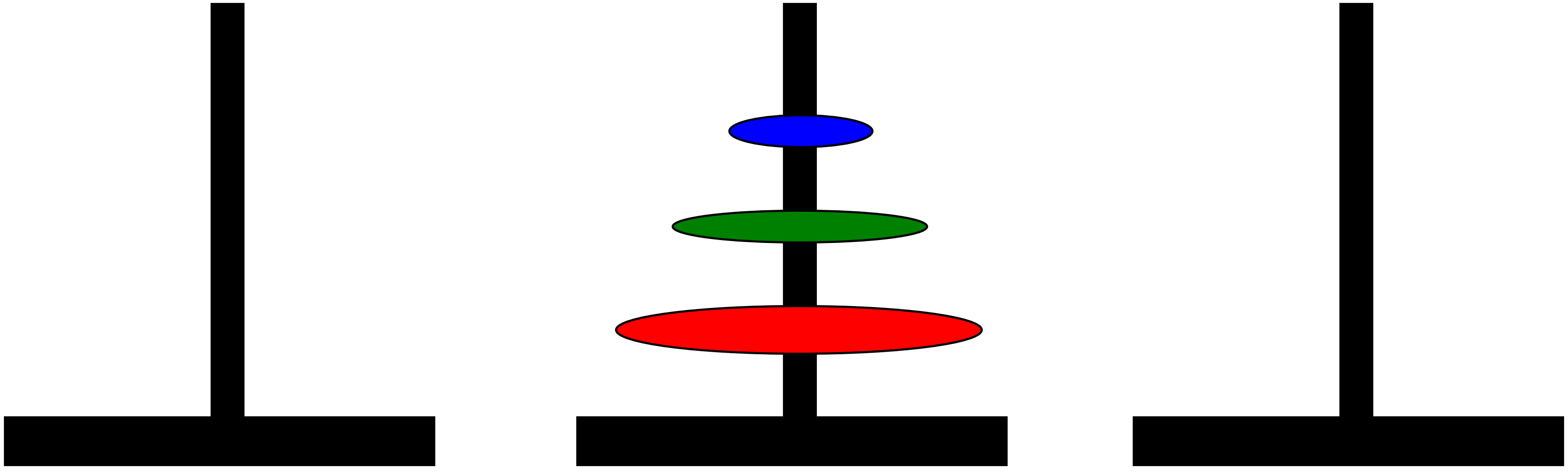


Tower of Hanoi



Towers of Hanoi

```
Hanoi(n, from, to, temp){
   if (n == 1)
      Move(from, to);
   else{
      Hanoi(n - 1, from, temp, to);
      Move(from, to);
      Hanoi(n - 1, temp, to, from);
   }
}
```

The recurrence relation for the running time of the method **hanoi** is:

$$T(1) = 1$$

$$T(n) = 2T(n-1) + 1 \qquad \text{if } n > 1$$

$$\mathbf{T(n) = \Theta(2^n)}$$

Guess and Prove

- Calculate T(n) for small n and look for a pattern.
- Guess the result and prove your guess correct using induction.

$T(n) = 2T(n - 1) + 1$

n	T(n)
1	1
2	3
3	7
4	15
5	31

$T(n) = 2^n - 1$

Iteration Method

Unwind recurrence, by repeatedly replacing T(n) by the r.h.s. of the recurrence until the base case is encountered.

$T(n) = 2T(n-1) + 1$

$= 2*[2*T(n-2)+1] + 1 = 2^2 * T(n-2) + 1+2$

$= 2^2 * [2*T(n-3)+1] + 1 + 2$

$= 2^3 * T(n-3) + 1+2 + 2^2$

Geometric Series

After k steps

$T(n) = 2^{k} * T(n\text{-}k) + 1+2 + 2^{2} + \ldots + 2^{n-k-1}$

$T(n) = 2^{n-1} * T(1) + 1+2 + 2^{2} + \ldots + 2^{n-2}$

$= 1 + 2 + \ldots + 2^{n-1} = \sum_{i=0}^{n-1} 2^{i}$

$\Theta(2^{n})$

If n=64 the 2^{64} seconds about 1.84×10^{19} seconds or 584+billion years

Forming Recurrence Relations

```
public void f (int n) {
    if (n > 0) {
        System.out.println(n);
        f(n-1);
    }else
        return;
}
```

The recurrence relation is:

$$T(0) = 1$$

$$T(n) = T(n-1) + b \qquad \text{if } n > 0$$

$$T(n) = \Theta(n)$$

Recurrences Solutions

- $T(n) = T(n-1) + n$ $\Theta(n^2)$
  - Recursive algorithm that loops through the input to eliminate one item
- $T(n) = T(n/2) + c$ $\Theta(\lg n)$
  - Recursive algorithm that halves the input in one step
- $T(n) = T(n/2) + n$ $\Theta(n)$
  - Recursive algorithm that halves the input but must examine every item in the input
- $T(n) = 2T(n/2) + 1$ $\Theta(n)$
  - Recursive algorithm that splits the input into 2 halves and does a constant amount of other work

Methods for Solving Recurrences

- Iteration method
- Substitution method
- Recursion tree method
- Master method
- Muster method

The Iteration Method

- Convert the recurrence into a summation and try to bound it using a known series
  - Iterate the recurrence until the initial condition is reached.
  - Use back-substitution to express the recurrence in terms of *n* and the initial (boundary) condition.

Iteration Method – Binary Search

$$\mathbf{T(n) = c + T(n/2)}$$

$$T(n) = c + T(n/2) \qquad T(n/2) = c + T(n/4)$$

$$= c + c + T(n/4) \qquad T(n/4) = c + T(n/8)$$

$$= c + c + c + T(n/8)$$

Stop when $n/2^i = 1 \quad \Rightarrow i = \lg n$

$$T(n) = \underbrace{c + c + \ldots + c +}_{\text{n times}} T(1)$$

$$= c\lg n + T(1)$$

$$= \Theta(\lg n)$$

Iteration - Mergesort

$$\mathbf{T(n) = n + 2T(n/2)}$$

$T(n) = n + 2T(n/2)$ $\qquad\qquad$ $T(n/2) = n/2 + 2T(n/4)$

$= n + 2(n/2 + 2T(n/4))$

$= n + n + 4T(n/4)$

$= n + n + 4(n/4 + 2T(n/8))$

$= n + n + n + 8T(n/8)$

… $= in + 2^iT(n/2^i)$ stop at $i = \lg n$

$= n\lg n + 2^{\lg n}T(1)$

$= n\lg n + nT(1)$

$= \Theta(n\lg n)$

Substitution Method

- Guess a solution

  $T(n) = O(g(n))$

  Induction goal: apply the definition of the asymptotic notation

  $T(n) \leq c\, g(n)$, for some $c > 0$ and $n \geq n_0$

- Induction hypothesis: $T(k) \leq c\, g(k)$ for all $k < n$

- Prove the induction goal
  - Use the **induction hypothesis** to find some values of the constants d and n_0 for which the **induction goal** holds

Substitution: $T(n) = T(n-1)+T(n-2)$

Guess: $T(n) = O(\phi^n)$

Induction goal: $T(n) \leq c\phi^n$, for some c and $n \geq n_0$

- Induction hypothesis: $T(k) \leq c\phi^k$ for $k < n$
- Proof of induction goal:

$$\begin{aligned} T(n) &= T(n-1) + T(n-2) \\ &\leq c\phi^{n-1} + c\phi^{n-2} \\ &\leq c\phi^{n-2}(\phi + 1) \\ &\leq c\phi^{n-2}(\phi^2) \end{aligned}$$

$$T(n) \leq c\,\phi^n$$

$$T(n) = O(\phi^n)$$

$$\boldsymbol{\Phi = \frac{1+\sqrt{5}}{2}}$$

$$\boldsymbol{\Phi^2 = \frac{3+\sqrt{5}}{2}}$$

$$\boldsymbol{\Phi + 1 = \Phi^2}$$

The Recursion-Tree method

Convert the recurrence into a tree:

- Each node represents the cost incurred at various levels of recursion
- Sum up the costs of all levels

Used to “guess” a solution for the recurrence

Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- Convert the recurrence into a tree:
  - Each node represents the cost incurred at various levels of recursion
  - Sum up the costs of all levels
- The recursion tree method is good for generating guesses for the substitution method.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- The recursion-tree method promotes intuition, however.

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

$$T(n)$$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

n^2

$T(n/4)$ $T(n/2)$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

n^2

$(n/4)^2$ $(n/2)^2$

$T(n/16)$ $T(n/8)$ $T(n/8)$ $T(n/4)$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

n^2

$(n/4)^2$ $(n/2)^2$

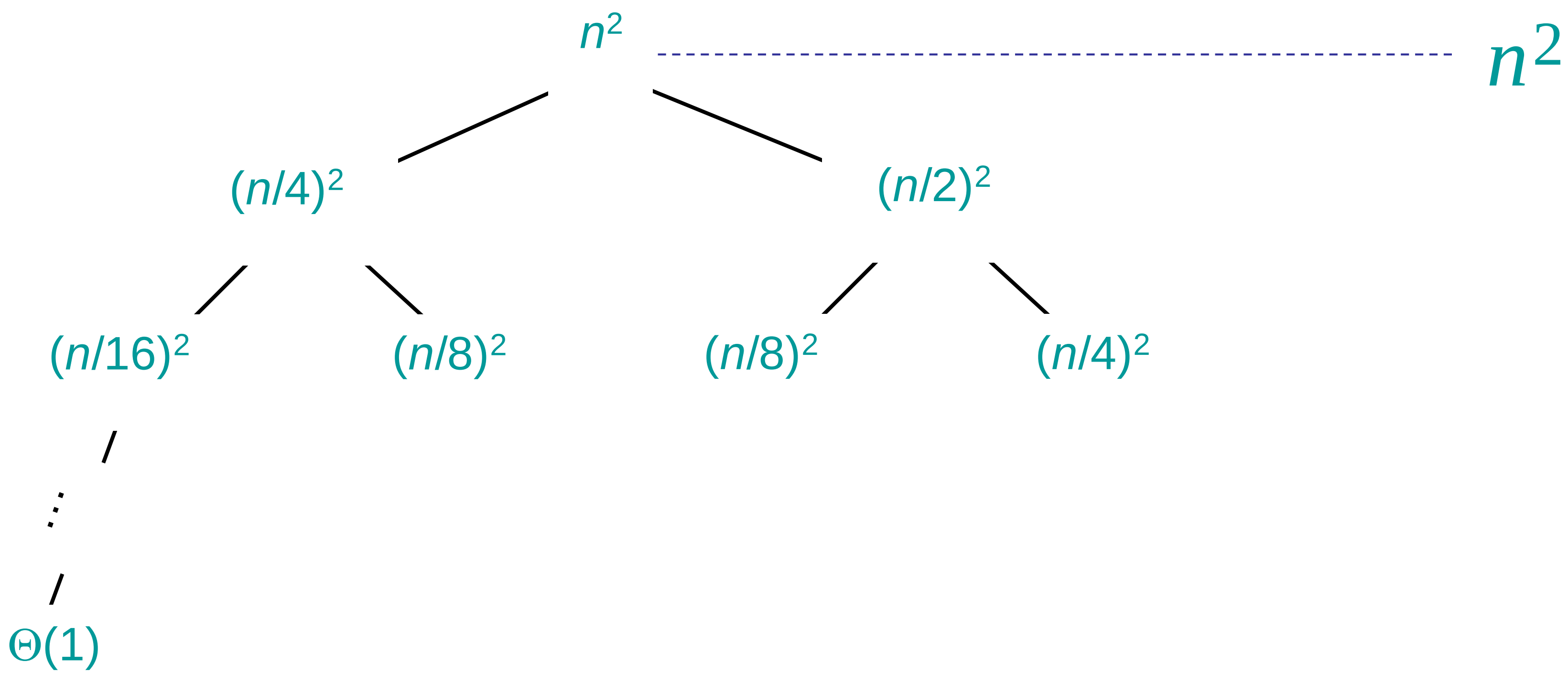
$(n/16)^2$ $(n/8)^2$ $(n/8)^2$ $(n/4)^2$

$\vdots$

$\Theta(1)$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



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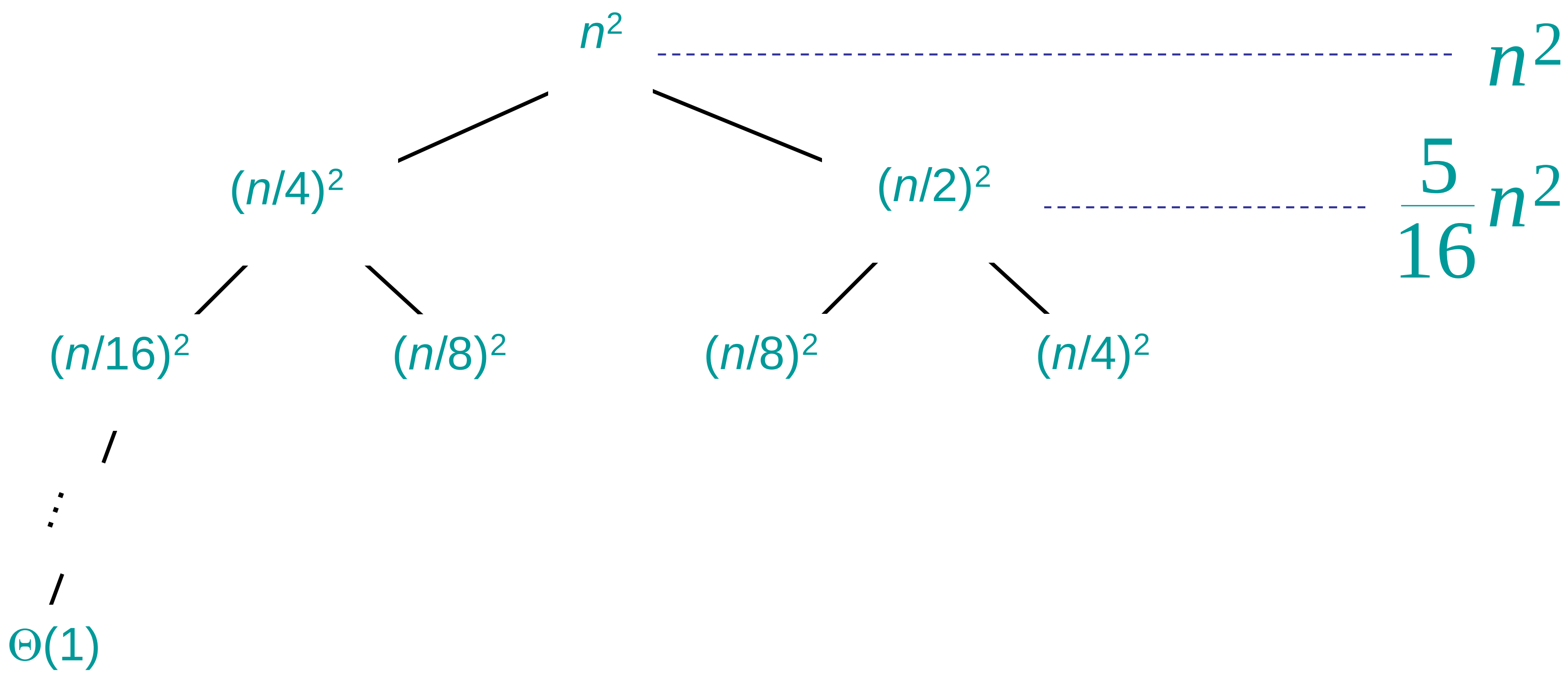

Example of recursion tree

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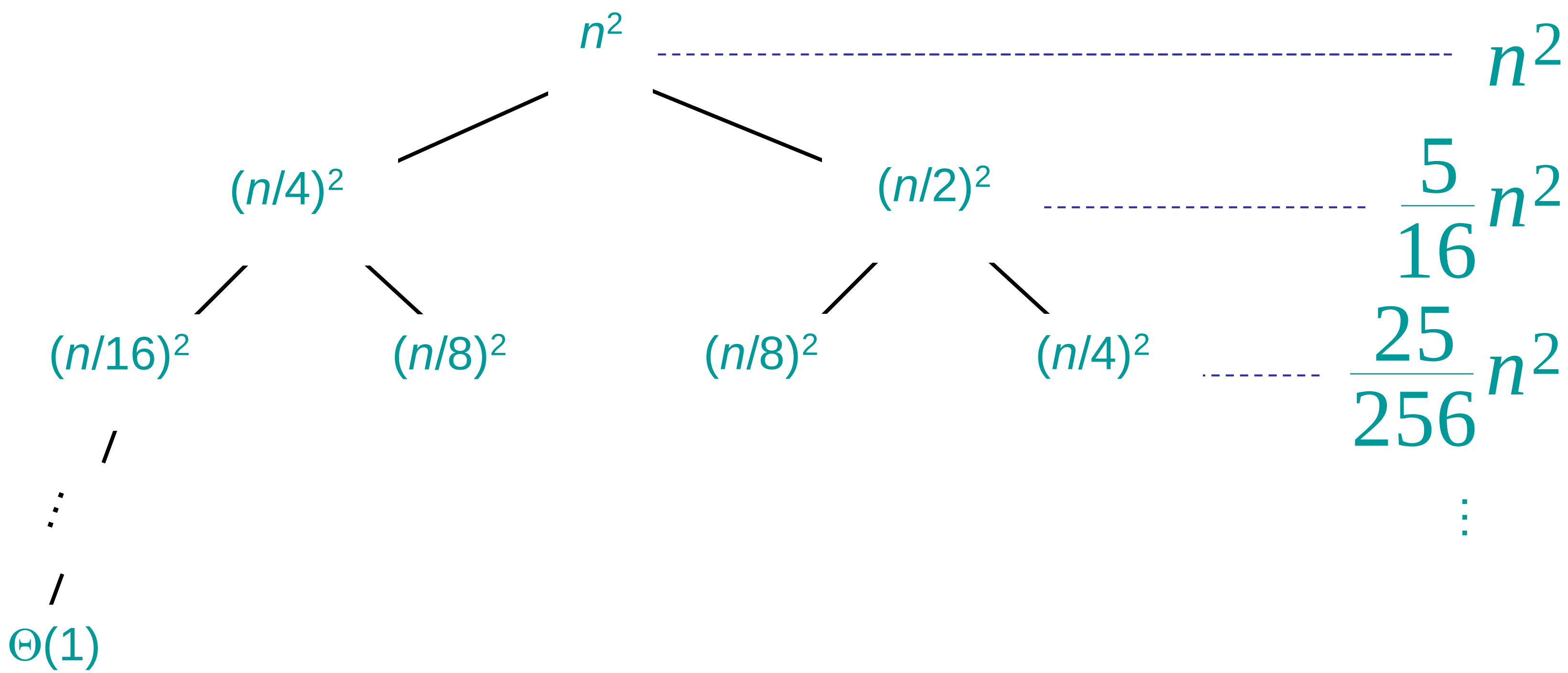
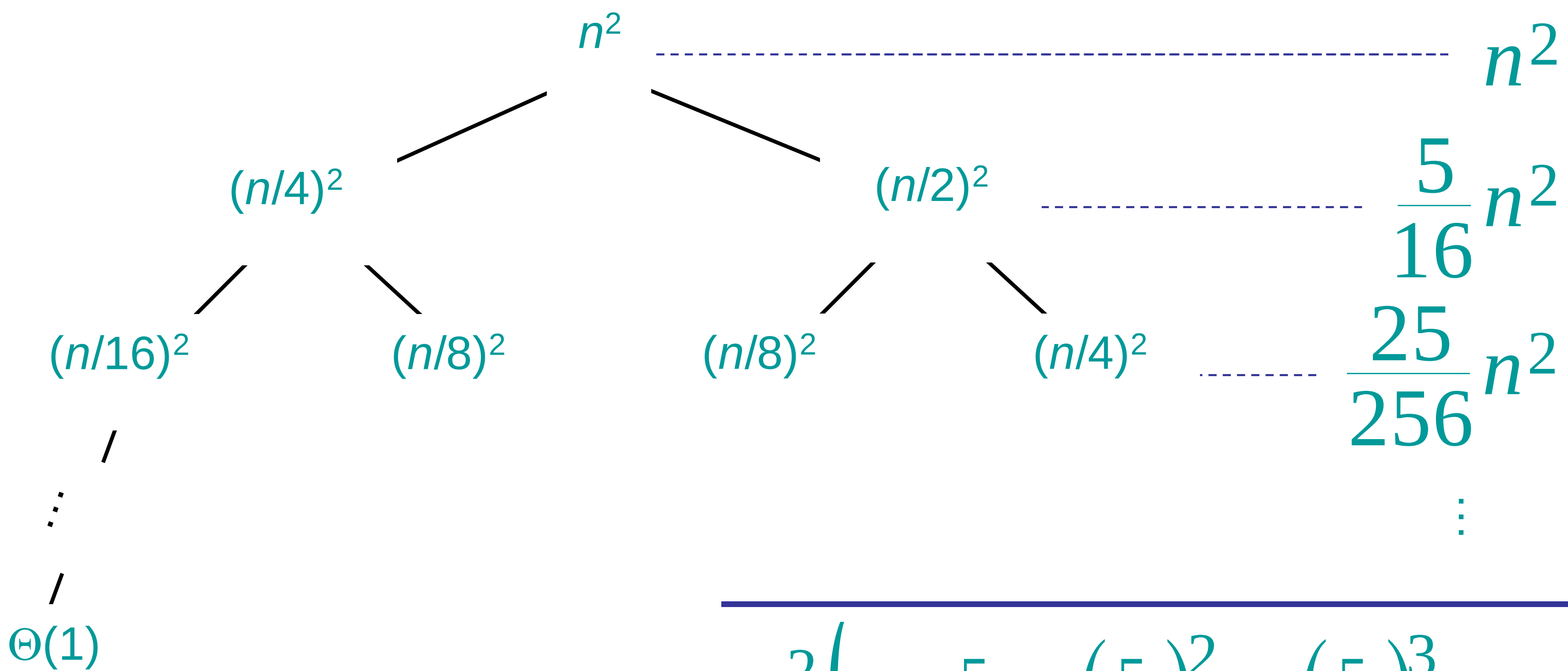


Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



$$\text{Total} \le n^2\left(1+\frac{5}{16}+\left(\frac{5}{16}\right)^2+\left(\frac{5}{16}\right)^3+\cdots\right)$$

geometric series

$$= \mathrm{O}(n^2)$$

Geometric series

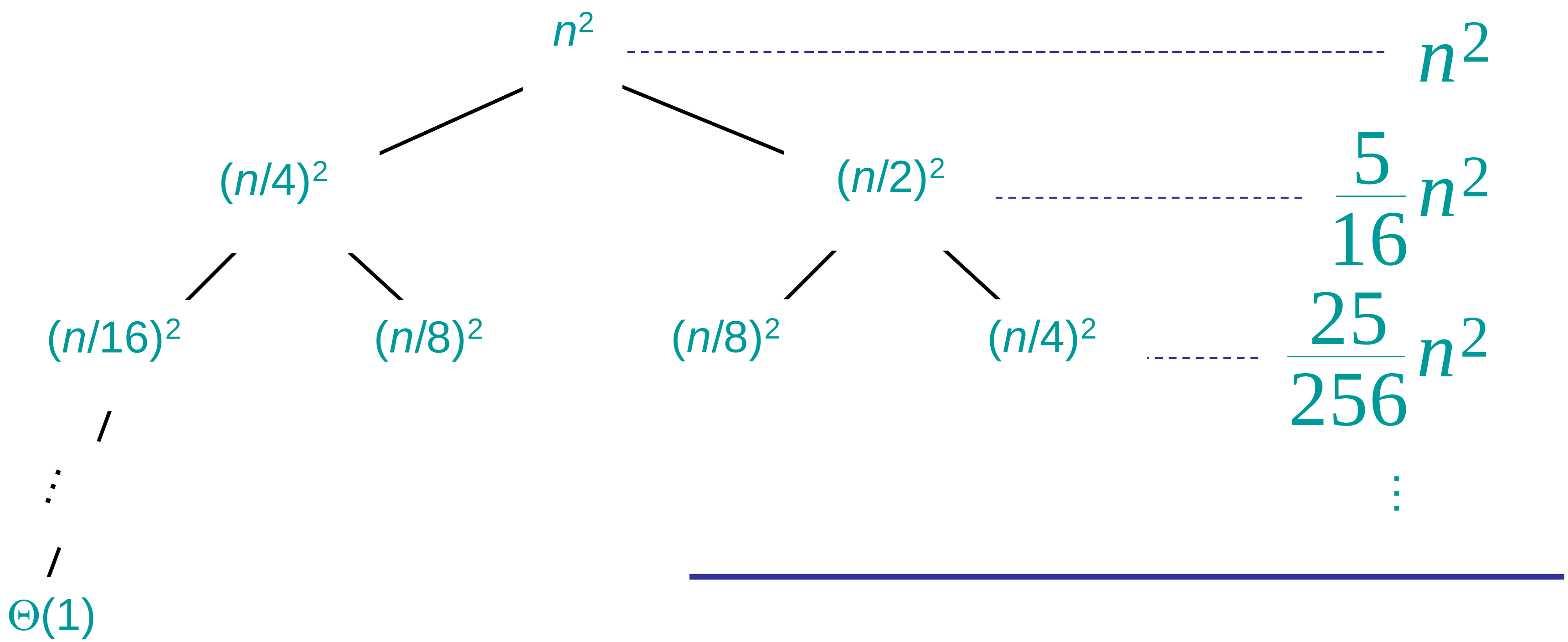
$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$ for $x \neq 1$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x}$$ for $|x| < 1$

$$n^2\left(1 + \tfrac{5}{16} + \left(\tfrac{5}{16}\right)^2 + \left(\tfrac{5}{16}\right)^3 + \cdots\right) = n^2\left(\frac{1}{1 - \tfrac{5}{16}}\right) = \tfrac{16}{11} n^2$$

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

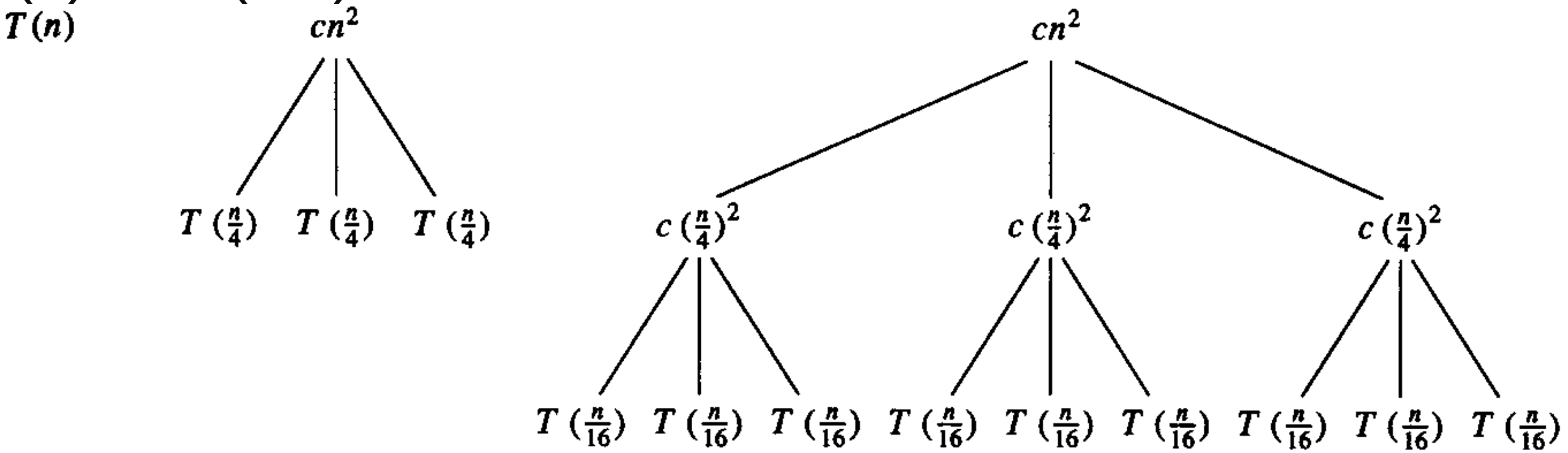


Total $\geq n^2 \quad = \Omega(n^2)$

Therefore $T(n) = \Theta(n^2)$

Recursion tree Example 2

$T(n) = 3T(n/4) + cn^2$



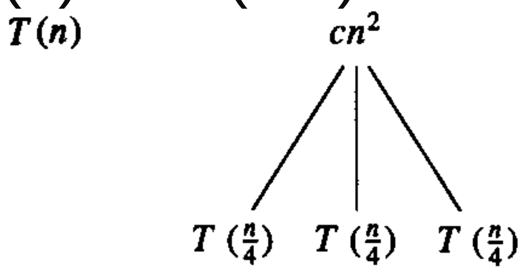
- Subproblem size at level i is: $n/4^i$
- Subproblem size hits 1 when $1 = n/4^i \Rightarrow i = \log_4 n$
- Cost of a node at level i = $c(n/4^i)^2$
- Number of nodes at level i = $3^i \Rightarrow$ last level has $3^{\log_4 n} = n^{\log_4 3}$ nodes
- Total cost:

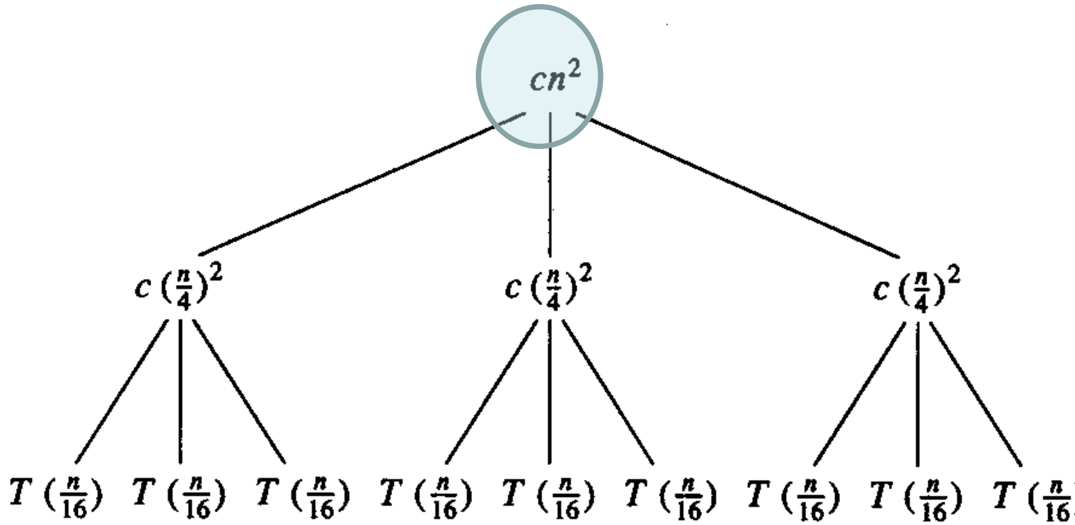
$$T(n) = \sum_{i=0}^{\log_4 n-1}\left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right) \le \sum_{i=0}^{\infty}\left(\frac{3}{16}\right)^i cn^2 + \Theta\left(n^{\log_4 3}\right) = \frac{1}{1-\frac{3}{16}}cn^2 + \Theta\left(n^{\log_4 3}\right) = O(n^2)$$

$\Rightarrow T(n) = O(n^2)$

Recursion tree Example 2

$T(n) = 3T(n/4) + cn^2$





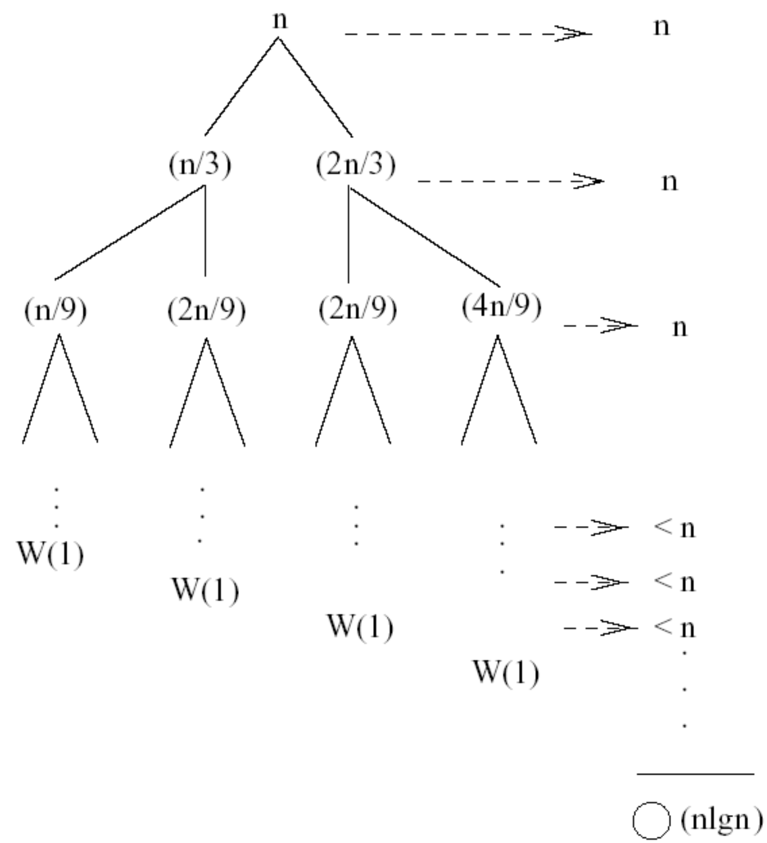
$T(n) = O(n^2)$

$T(n) = \Omega(n^2)$

$T(n) = \Theta(n^2)$

Recursion Tree – Example 3

$T(n) = T(n/3) + T(2n/3) + n$



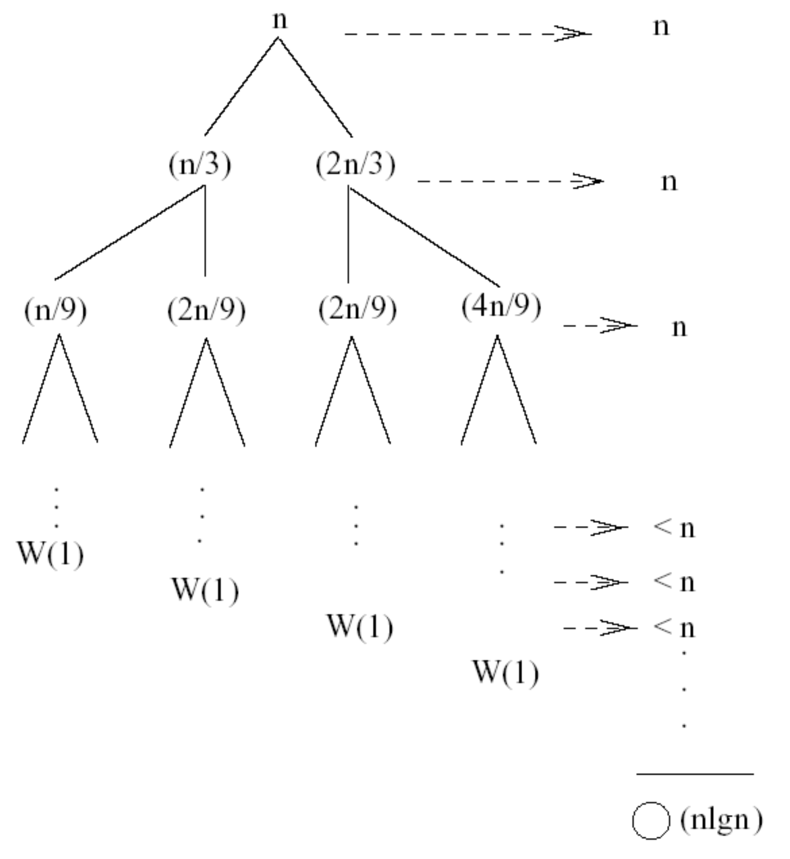
- The longest path from the root to a leaf is: $n \to (2/3)n \to (2/3)^2\, n \to \ldots \to 1$
- Subproblem size hits 1 when $1 = (2/3)^i n \Leftrightarrow i=\log_{3/2}n$
- cost of the problem at level i = n
- Total cost:

$$T(n) < n + n + \ldots = n(\log_{3/2} n) = n\frac{\lg n}{\lg \frac{3}{2}} = O(n \lg n)$$

$\Rightarrow T(n) = O(n\lg n)$

Recursion Tree – Example 3

$T(n) = T(n/3) + T(2n/3) + n$



$T(n) = \Omega(n)$

$T(n) = O(n\lg n)$

The Master Method

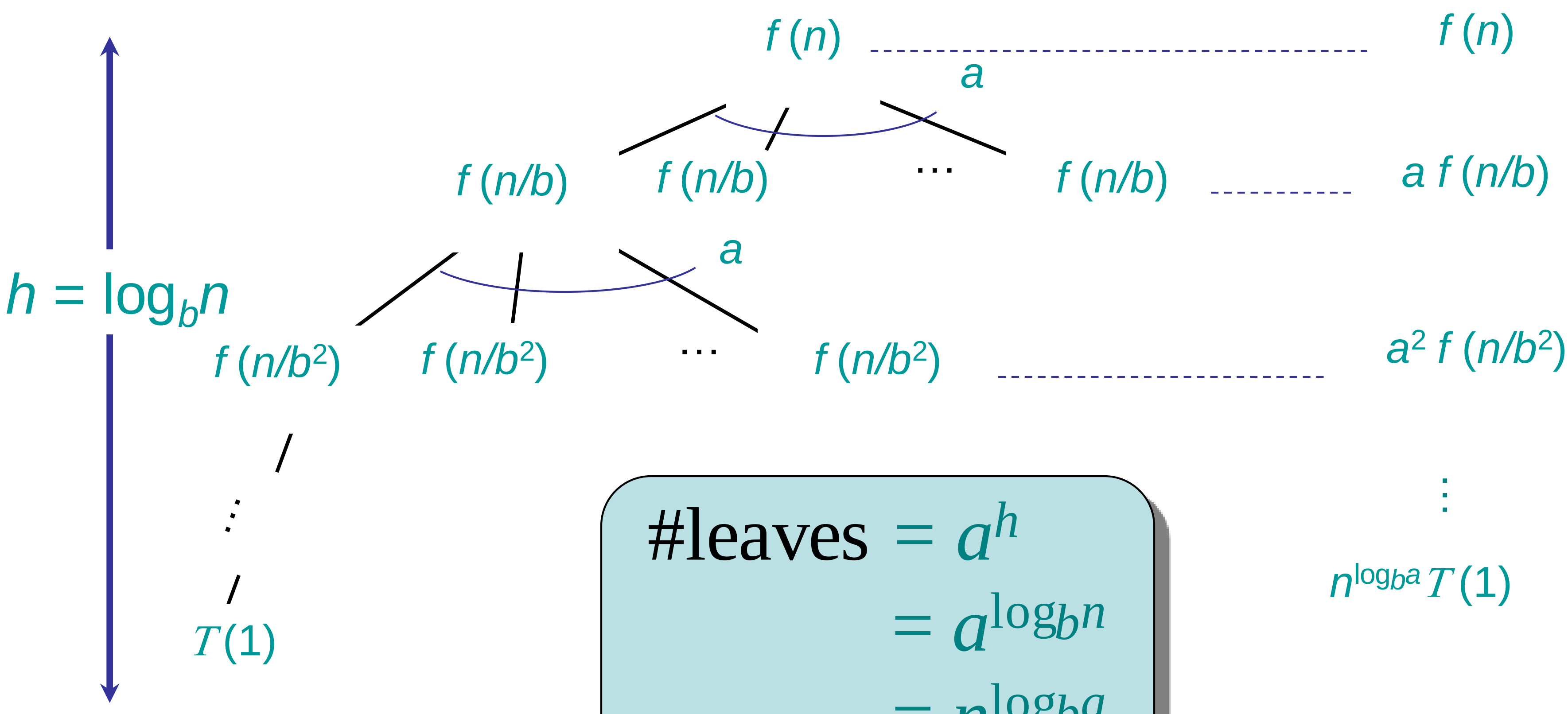
The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f\,(n)\ ,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

Idea of Master Method

Recursion tree:

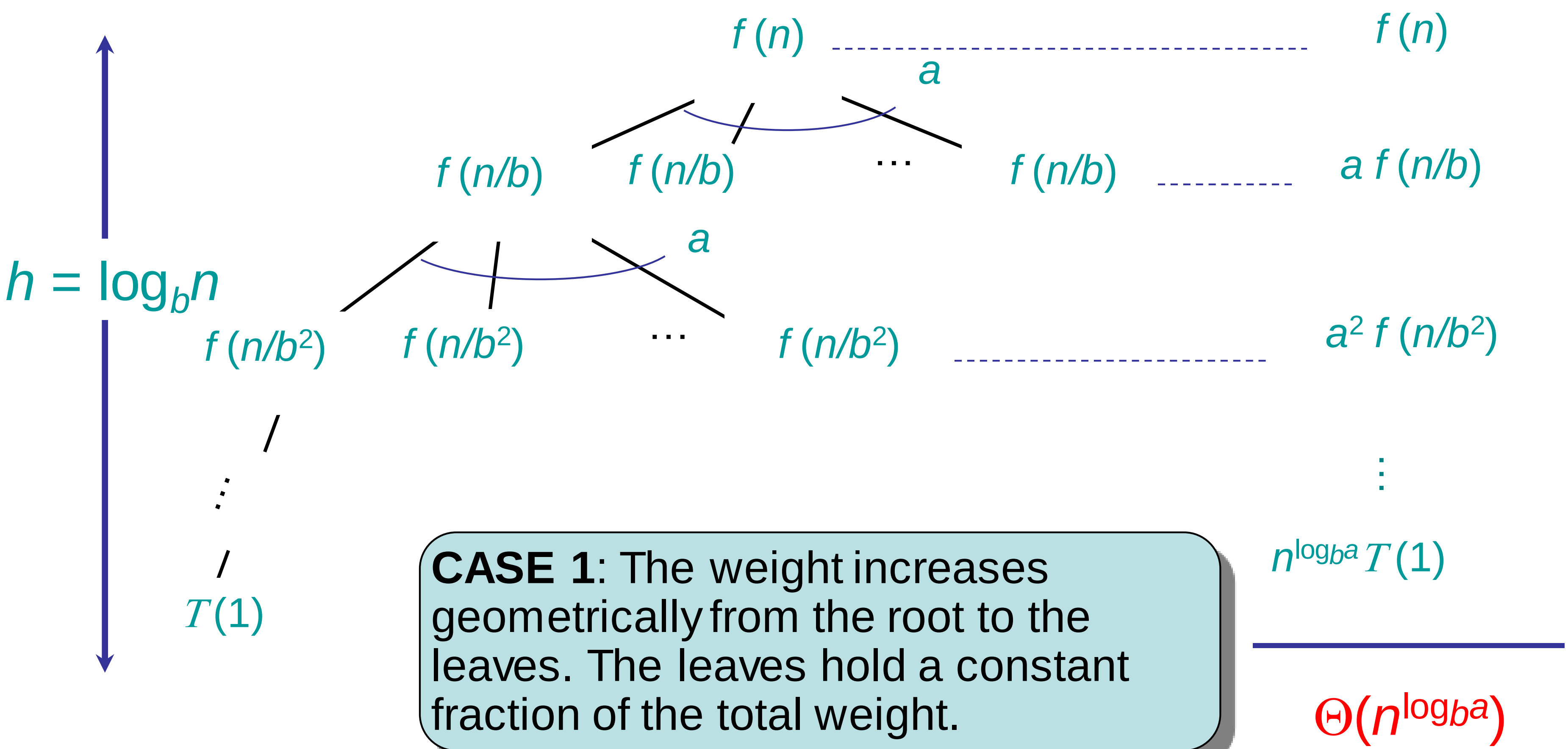


$$\begin{aligned}\text{\#leaves} &= a^h \\ &= a^{\log_b n} \\ &= n^{\log_b a}\end{aligned}$$

Idea of Master Method

$$f(n) = O(n^{\log_b a - \varepsilon})$$

Recursion tree:



CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.

   $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.

Case 1

Ex. $T(n) = 4T(n/2) + n$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n$.

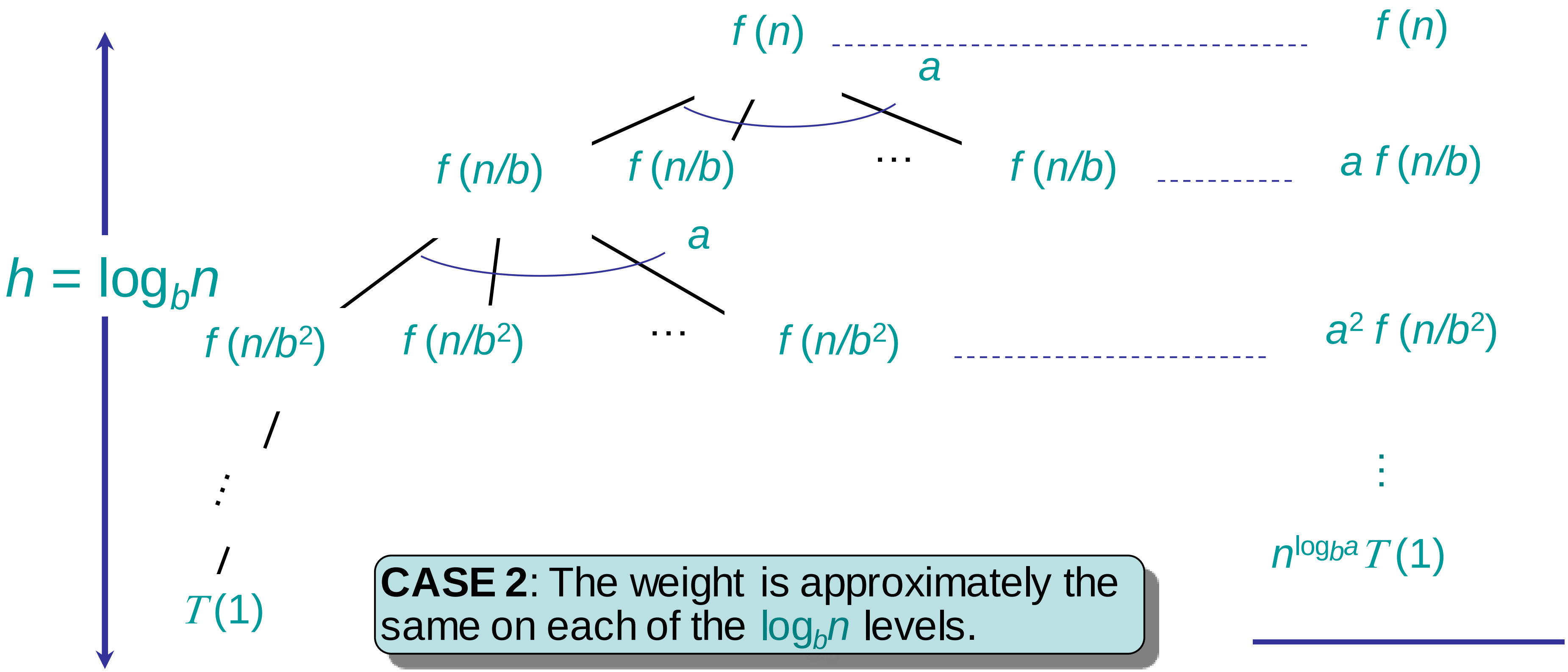
CASE 1: $f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^2)$.

Idea of Master Method

$f(n) = \Theta(n^{\log_b a})$

Recursion tree:



CASE 2: The weight is approximately the same on each of the $\log_b n$ levels.

$\Theta(n^{\log_b a} \lg n)$

Case 2

Ex. $T(n) = 4T(n/2) + n^2$

$a = 4,\ b = 2 \Rightarrow n^{\log_b a} = n^2;\ f(n) = n^2.$

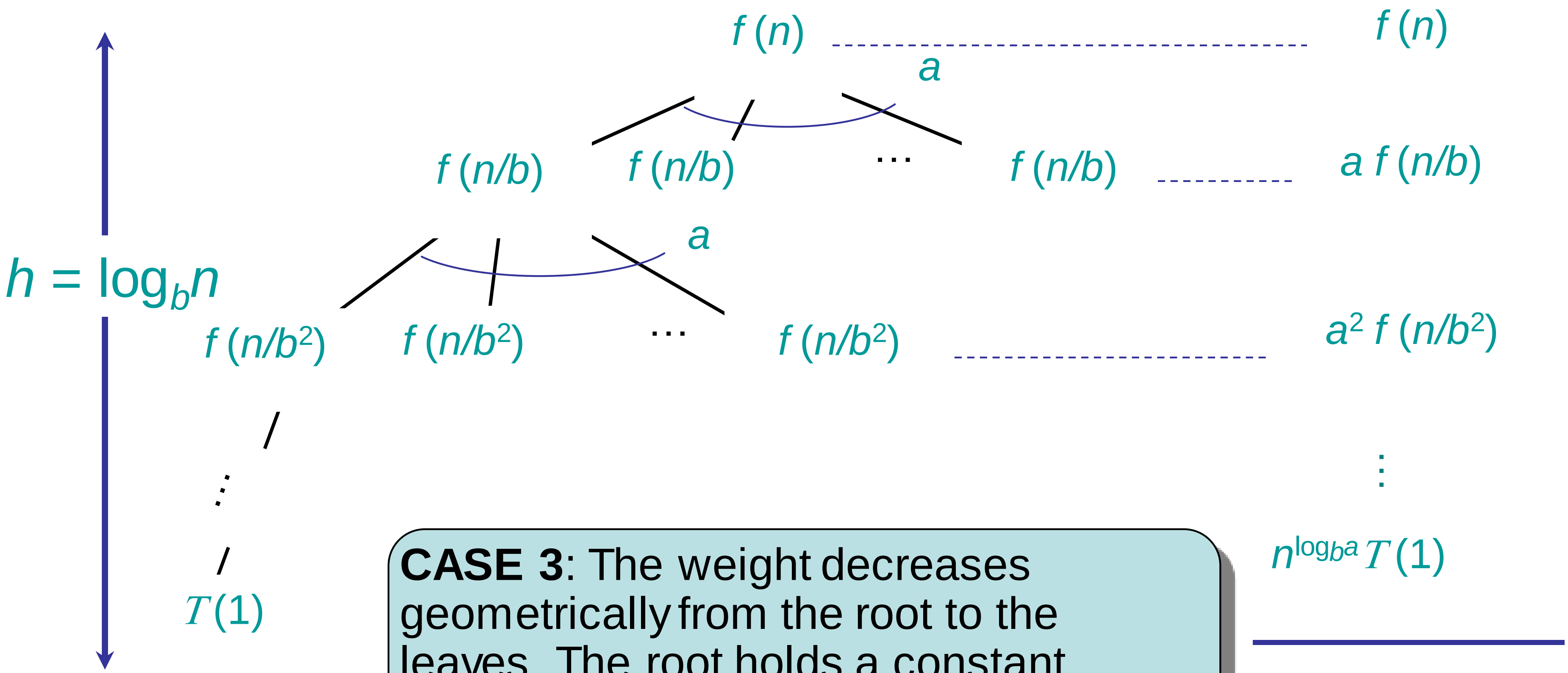
CASE 2: $f(n) = \Theta(n^2)$

$\therefore\ T(n) = \Theta(n^2 \lg n).$

Idea of master theorem

$$f(n) = \Omega(n^{\log_b a + \varepsilon})$$

Recursion tree:



CASE 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

$\Theta(f(n))$

Case 3

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^3$.

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ ***and***

$4(cn/2)^3 \leq cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore$ $T(n) = \Theta(n^3)$.

No Cases

Ex. $T(n) = 4T(n/2) + n^2/\lg n$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^2/\lg n$.

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^{\varepsilon} = \omega(\lg n)$.

Master Method

"Formula" for solving recurrences of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where, $a \geq 1$, $b > 1$, and $f(n) > 0$

case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$, then: $T(n) = \Theta(n^{\log_b a})$

case 2: if $f(n) = \Theta(n^{\log_b a})$, then: $T(n) = \Theta(n^{\log_b a} \lg n)$

case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$, and if

$af(n/b) \leq cf(n)$ for some $c < 1$ and all sufficiently large n, then:

$$T(n) = \Theta(f(n))$$

regularity

Master Method – Binary Search

$T(n) = T(n/2) + c$ $\quad$ $a = 1,\ b = 2,\ \log_2 1 = 0$

compare $n^{\log_2 1} = n^0 = 1$ with $f(n) = c$

Case 2: if $f(n) = \Theta(n^{\log_b a})$, then: $T(n) = \Theta(n^{\log_b a} \lg n)$

$f(n) = \Theta(1) \Rightarrow$ case 2

$\Rightarrow T(n) = \Theta(\lg n)$

Master Method – Example 1

$T(n) = 2T(n/2) + n^2$ $\qquad$ $a = 2$, $b = 2$, $\log_2 2 = 1$

compare n with $f(n) = n^2$

case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$

$\Rightarrow f(n) = \Omega(n^{1+\varepsilon})$ case 3 $\Rightarrow$ verify regularity cond.

$a\, f(n/b) \leq c\, f(n)$

$\Leftrightarrow 2\, n^2/4 \leq c\, n^2 \Rightarrow c = ½$ is a solution $(c<1)$

$\Rightarrow T(n) = \Theta(n^2)$

Master Method – Example 2

$T(n) = 2T(n/2) + \sqrt{n}$ $\quad$ $a = 2, b = 2, \log_2 2 = 1$

compare n with $f(n) = n^{1/2}$

$\Rightarrow f(n) = O(n^{1-\varepsilon})$ $\quad$ case 1

$\Rightarrow T(n) = \Theta(n)$

Master Method - Example 3

$T(n) = 3T(n/4) + n\lg n$ $\quad$ $a = 3$, $b = 4$, $\log_4 3 = 0.793$

compare $n^{0.793}$ with $f(n) = n\lg n$

$f(n) = \Omega(n^{\log_4 3+\varepsilon})$ case 3

check regularity condition:

$$3 * (n/4)\lg(n/4) \le (3/4)n\lg n = c * f(n),\ c=3/4$$

$\Rightarrow T(n) = \Theta(n\lg n)$

Master Method: Merge-Sort

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$$T(n) = 2T\left(\frac{n}{2}\right) + kn$$

where, $a = 2$, $b = 2$, and $f(n) = nn$

$n^{\log_b a} = n^{\log_2 2} = n$

case 1: if $f(n) = O(n^{\log_b a - \varepsilon})$ for some $\varepsilon > 0$, then: $T(n) = \Theta(n^{\log_b a})$

case 2: if $f(n) = \Theta(n^{\log_b a})$, then: $T(n) = \Theta(n^{\log_b a} \lg n)$

case 3: if $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some $\varepsilon > 0$, and if

$$T(n) = \Theta(n \lg n)$$

Decrease and Conquer

Muster Theorem for *"decrease and conquer"* recurrences of the form

$$T(n) = a\,T(n-b) + f(n)$$

for some integer constants a, b > 0, d ≥ 0 .

If *f(n)* is $O(n^d)$ then

$$T(n) = \begin{cases} O\left(n^d\right), & if\ a < 1, \\ O\left(n^{d+1}\right), & if\ a = 1 \\ O\left(n^d a^{n/b}\right), & if\ a > 1. \end{cases}$$

Decrease and Conquer: Towers

$$T(n) = 2\,T(n\text{-}1) + 1$$

$$T(n) = a\,T(n\text{-}b) + f(n)$$

a = 2, b= 1, f(n) = 1 so d = 0.

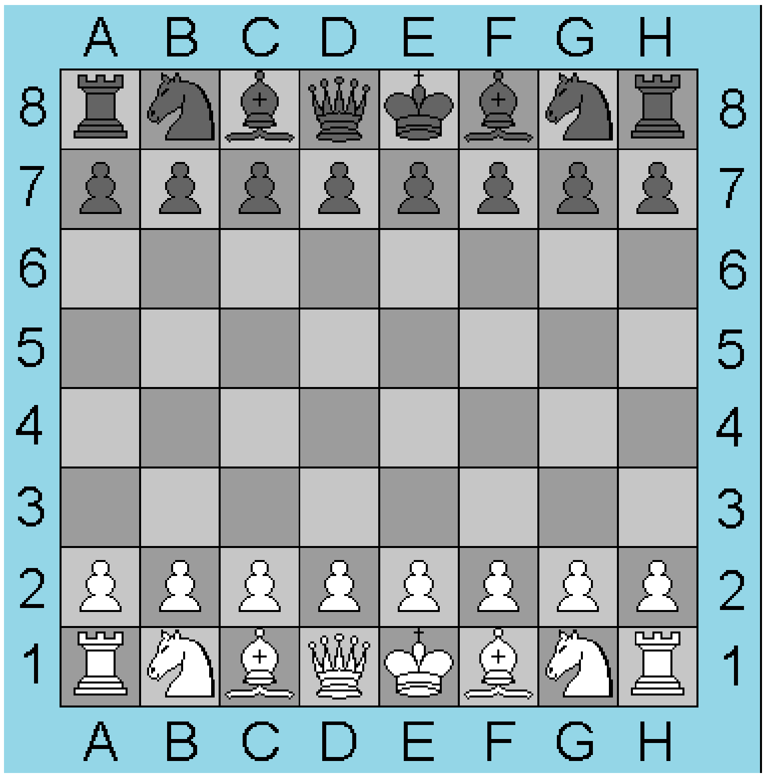
$$T(n) = \begin{cases} O(n^d), & if\ a < 1, \\ O(n^{d+1}), & if\ a = 1 \\ O(n^d a^{n/b}), & if\ a > 1. \end{cases}$$

T(n) is $O(2^n)$ even better *f(n)* is $\Theta(n^d)$ so we could conclude that T(n) is $\Theta(2^n)$.

More Applications

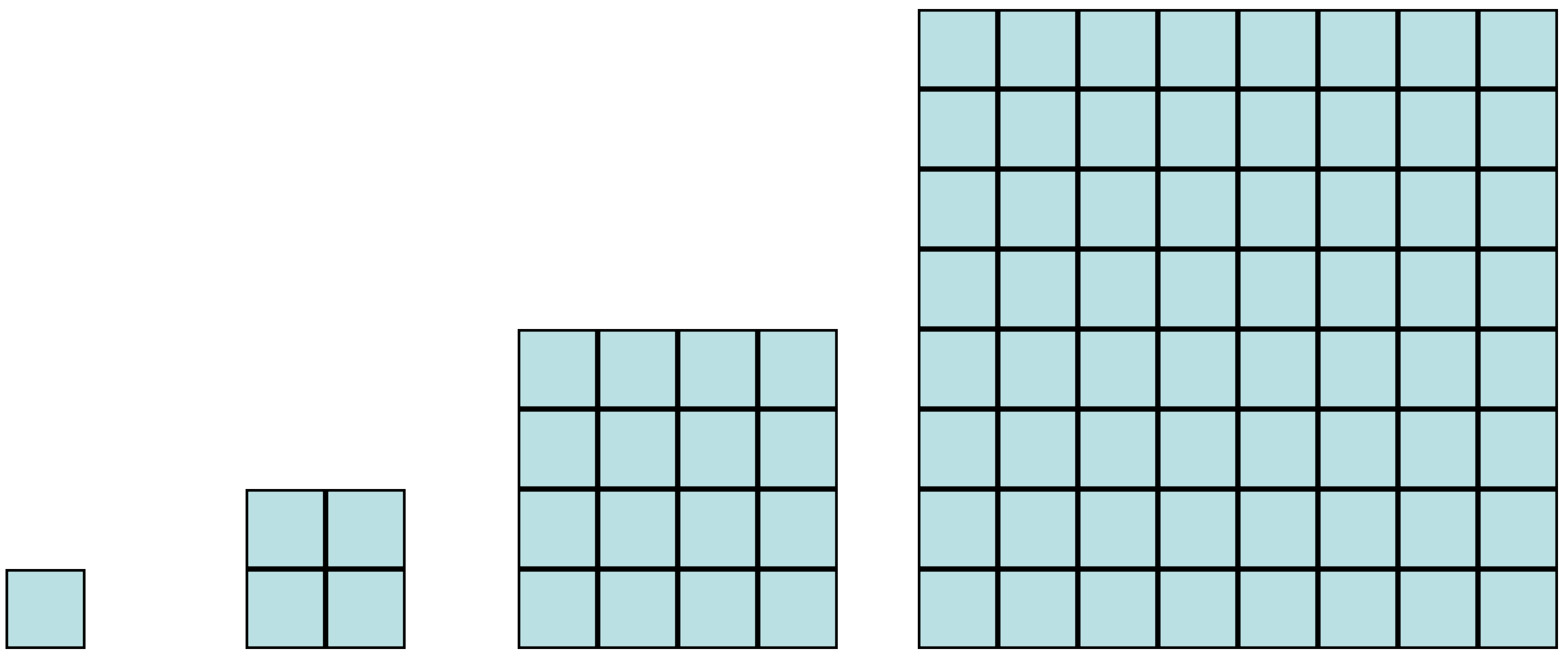
- Tiling
- Skylines

Tiling A Defective chessboard



Our Definition Of A chessboard

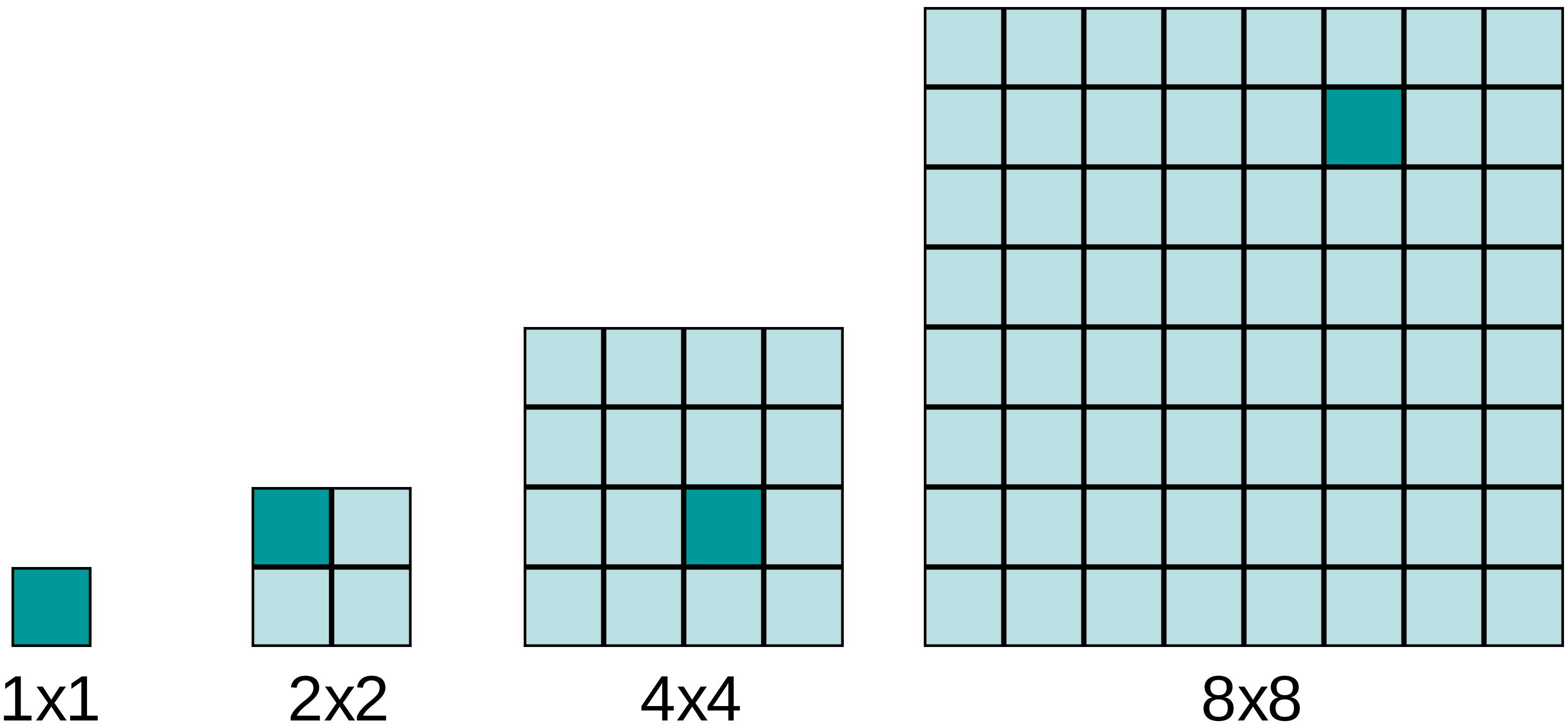
A chessboard is an n x n grid, where n is a power of 2.



1x1 2x2 4x4 8x8

A Defective chessboard

A defective chessboard is a chessboard that has one unavailable (defective) position.



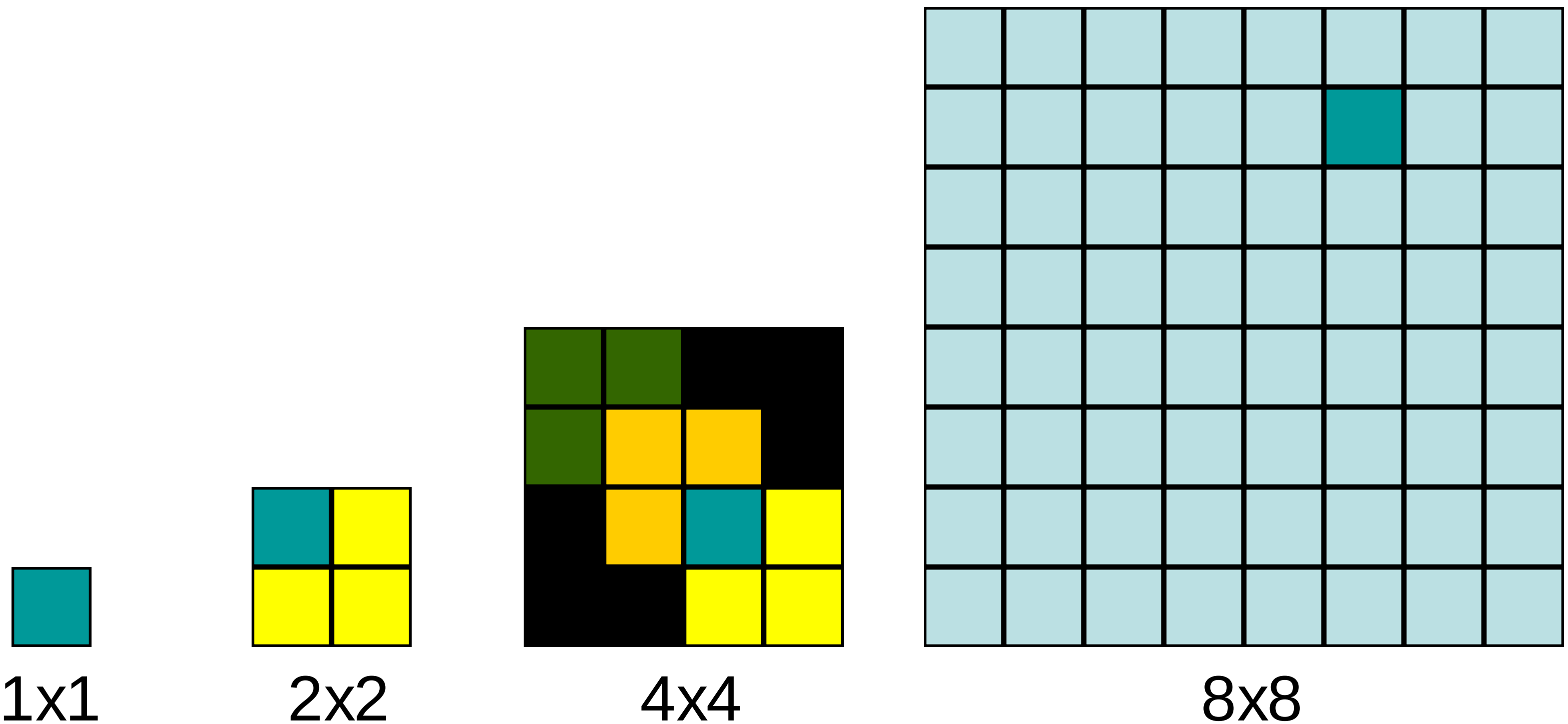
A Triomino

A triomino is an L shaped object that can cover three squares of a chessboard.

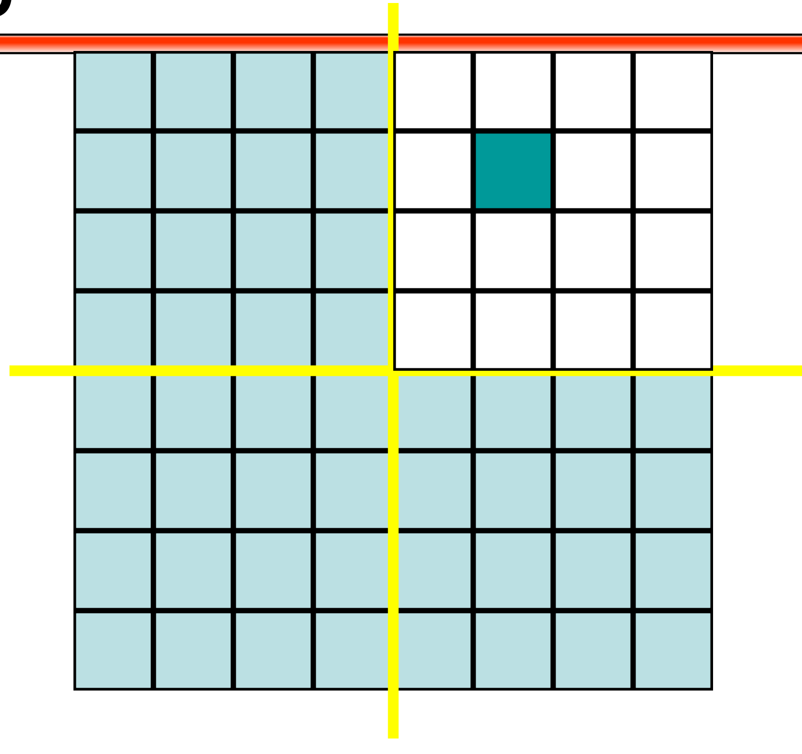
A triomino has four orientations.

Tiling A Defective chessboard

Place triominoes on an 2^n x 2^n defective chessboard so that all nondefective positions are covered.



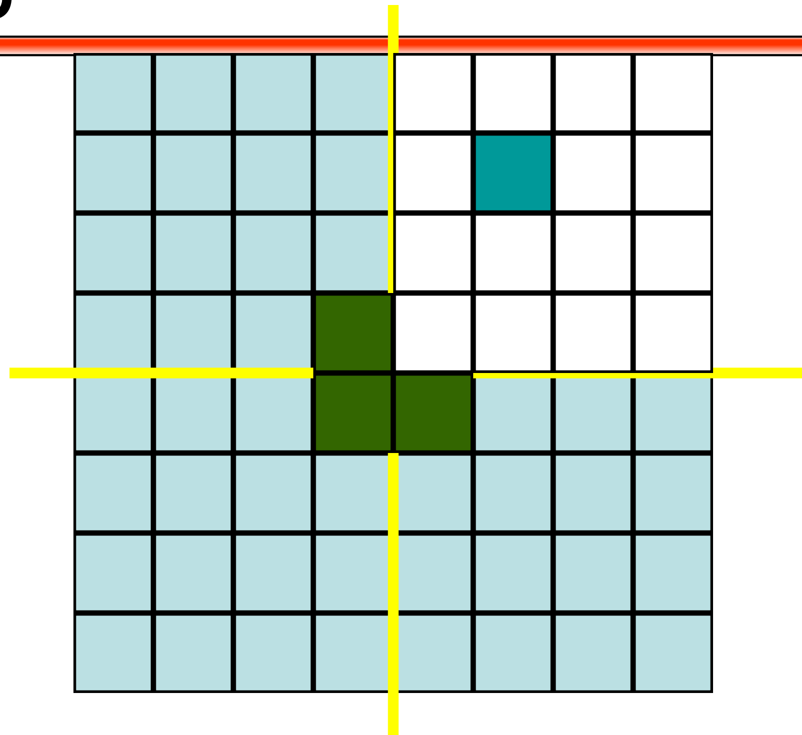
Tiling A Defective chessboard



Divide into four smaller chessboards. 4 x 4

One of these is a defective 4 x 4 chessboard.

Tiling A Defective chessboard



Mane the other three 4 x 4 chessboards defective by placing a triomino at their common corner.

Recursively tile the four defective 4 x 4 chessboards.

Tiling: Algorithm

INPUT: *n* – the board size (2^nx2^n board), *L* – location of the hole.
OUTPUT: tiling of the board

```
Tile(n, L)
   if n = 1 then
      Trivial case
      Tile with one tromino
      return
   Divide the board into four equal-sized boards
   Place one tromino at the center to cut out 3 additional
    holes (orientation based on where existing hole, L, is)
   Let L1, L2, L3, L4 denote the positions of the 4 holes
   Tile(n-1, L1)
   Tile(n-1, L2)
   Tile(n-1, L3)
   Tile(n-1, L4)
```

Recurrence

Let $T(n)$ be the time taken to tile a 2^n x 2^n defective chessboard.

$T(1) = 1,$

$T(n) = 4T(n-1) + c$, when $n > 0$.

Substitution Method

$$
\begin{aligned}
T(n) &= 4T(n-1) + c \\
&= 4[4T(n-2) + c] + c \\
&= 4^2\ T(n-2) + 4c + c \\
&= 4^2[4T(n-3) + c] + 4c + c \\
&= 4^3\ T(n-3) + 4^2c + 4c + c \\
&= \ldots \\
&= 4^{n-1}\ T(1) + 4^{n-2}c + \ldots + 4^2c + 4c + c \\
&= 4^n * 1 + 4^{n-1}c + 4^{n-2}c + \ldots + 4^2c + 4c + c \\
&= \Theta(4^n)
\end{aligned}
$$