CS 331: Artificial Intelligence
Probability I

Thanks to Andrew Moore for some course material

Dealing with Uncertainty

- We want to get to the point where we can reason with uncertainty
- This will require using probability e.g. probability that it will rain today is 0.99
- We will review the fundamentals of probability

Outline

1. Random variables
2. Probability

Random Variables

- The basic element of probability is the random variable
- Think of the random variable as an event with some degree of uncertainty as to whether that event occurs
- Random variables have a domain of values it can take on

Random Variables

3 types of random variables:

1. Boolean random variables
2. Discrete random variables
3. Continuous random variables

Random Variables

Example:

- *ProfLate* is a random variable for whether your prof will be late to class or not
- The domain of *ProfLate* is {*true*, *false*}
  - *ProfLate* = *true*: proposition that prof will be late to class
  - *ProfLate* = *false*: proposition that prof will not be late to class

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Random Variables

Example:

- *ProfLate* is a random variable for whether your prof will be late to class or not
- The domain of *ProfLate* is <*true*, *false*>
  - *ProfLate* = *true*: proposition that prof will be late to class
  - *ProfLate* = will not be l

You can assign some degree of belief to this proposition e.g. *P(ProfLate = true)* = 0.9

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Random Variables

Example:

- *ProfLate* is a random variable for whether your prof will be late to class or not
- The domain of *ProfLate* is <*true*, *false*>
  - *ProfLate* = *true*: proposition that prof will be late to class
  - *ProfLate* = *false*: proposition that prof will not be late to class

And to this one e.g. *P(ProfLate = false)* = 0.1

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Random Variables

- We will refer to random variables with capitalized names e.g. *X*, *Y*, *ProfLate*
- We will refer to names of values with lower case names e.g. *x*, *y*, *proflate*
- This means you may see a statement like *ProfLate* = *proflate*
  - This means the random variable *ProfLate* takes the value *proflate* (which can be *true* or *false*)
- Shorthand notation:

  ProfLate = *true* is the same as *proflate* and *ProfLate* = *false* is the same as ¬*proflate*

Boolean Random Variables

- Take the values *true* or *false*
- E.g. Let *A* be a Boolean random variable
  - $P(A = false) = 0.9$
  - $P(A = true) = 0.1$

Discrete Random Variables

Allowed to taken on a finite number of values e.g.

- *P(DrinkSize=small)* = 0.1
- *P(DrinkSize=medium)* = 0.2
- *P(DrinkSize=large)* = 0.7

Discrete Random Variables

Values of the domain must be:

- Mutually Exclusive i.e. $P(A = v_i \text{ AND } A = v_j) = 0$ if $i \neq j$

  This means, for instance, that you can't have a drink that is both *small* and *medium*

- Exhaustive i.e. $P(A = v_1 \text{ OR } A = v_2 \text{ OR } \ldots \text{ OR } A = v_k) = 1$

  This means that a drink can only be either *small*, *medium* or *large*. There isn't an *extra large*.

Discrete Random Variables

Values of the domain must be:

- Mutually Exclusive i.e. $P(A = v_i \text{ AND } A = v_j) = 0$ if $i \neq j$

  This means, for i ... drink that is both ...

  The AND here means intersection i.e. $(A = v_i) \cap (A = v_j)$

- Exhaustive i.e. $P(A = v_1 \text{ OR } A = v_2 \text{ OR } ... \text{ OR } A = v_k) = 1$

  This means that a dri ... ly be either *small*, *medium* or *larg* ...

  The OR here means union i.e. $(A = v_1) \cup (A = v_2) \cup ... \cup (A = v_k)$

Discrete Random Variables

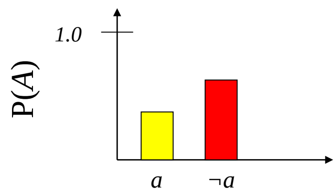
- Since we now have multi-valued discrete random variables we can't write *P(a)* or *P(¬a)* anymore
- We have to write $P(A = v_i)$ where v_i = a value in $\{v_1, v_2, \ldots, v_k\}$

Continuous Random Variables

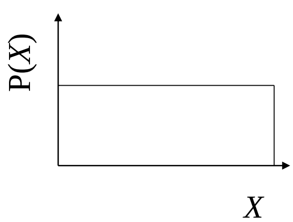
- Can take values from the real numbers
- E.g. They can take values from [0, 1]
- Note: We will primarily be dealing with discrete random variables
- (The next slide is just to provide a little bit of information about continuous random variables)

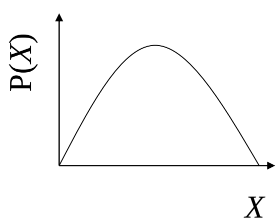
Probability Density Functions

Discrete random variables have probability distributions:



Continuous random variables have probability density functions e.g:





Probabilities

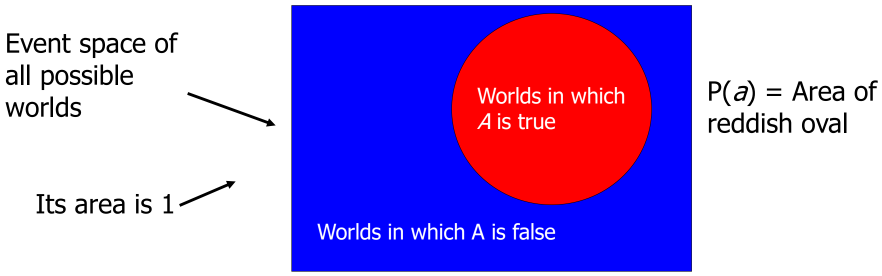
- We will write *P(A=true)* as "the fraction of possible worlds in which *A* is true"
- We can debate the philosophical implications of this for the next 4 hours
- But we won't

Probabilities

- We will sometimes talk about the probabilities of all possible values of a random variable
- Instead of writing
  - *P(A=false)* = 0.25
  - *P(A=true)* = 0.75
- We will write ***P***(*A*) = (0.25, 0.75)

Note the boldface!

Visualizing A



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The Axioms of Probability

- $0 \leq P(a) \leq 1$
- P(*true*) = 1
- P(*false*) = 0
- P(*a* OR *b*) = P(*a*) + P(*b*) - P(*a* AND *b*)

The logical OR is equivalent to set union $\cup$.

The logical AND is equivalent to set intersection ($\cap$). Sometimes, I'll write it as *P(a, b)*

These axioms are often called Kolmogorov's axioms in honor of the Russian mathematician Andrei Kolmogorov

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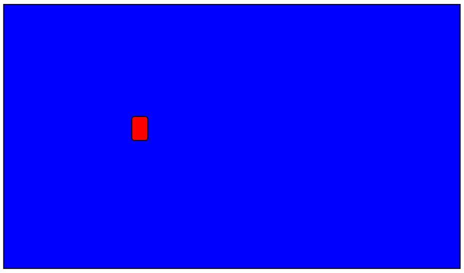
Interpreting the axioms

- $0 \leq P(a) \leq = 1$
- $P(true) = 1$
- $P(false) = 0$
- $P(a \text{ OR } b) = P(a) + P(b) - P(a, b)$



The area of P(*a)* can't get any smaller than 0

And a zero area would mean that there is no world in which *a* is not false

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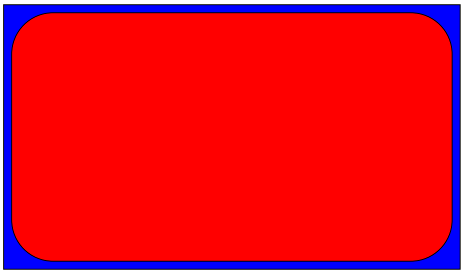
Interpreting the axioms

- $0 \leq P(a) \leq 1$
- $P(true) = 1$
- $P(false) = 0$
- $P(a \text{ OR } b) = P(a) + P(b) - P(a, b)$



The area of P(*a)* can't get any bigger than 1

And an area of 1 would mean all worlds will have *a* is true

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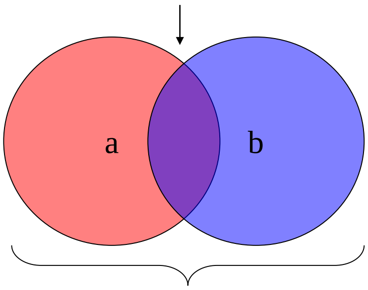
Interpreting the axioms

- $0 \le P(a) \le 1$
- $P(true) = 1$
- $P(false) = 0$
- $P(a \text{ OR } b) = P(a) + P(b) - P(a, b)$

P(*a*, *b*) [The purple area]



P(*a* OR *b*) [the area of both circles]

These Axioms are Not to be Trifled With

- There have been attempts to do different methodologies for uncertainty
  - Fuzzy Logic
  - Three-valued logic
  - Dempster-Shafer
  - Non-monotonic reasoning

- But the axioms of probability are the only system with this property:

If you gamble using them you can't be unfairly exploited by an opponent using some other system [de Finetti 1931]

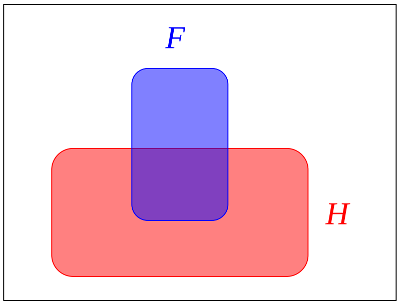
Prior Probability

- We can consider *P(A)* as the unconditional or prior probability
  - E.g. *P(ProfLate = true)* = 1.0
- It is the probability of event *A* in the absence of any other information
- If we get new information that affects *A*, we can reason with the conditional probability of *A* given the new information.

Conditional Probability

- P(*A* | *B*) = Fraction of worlds in which *B* is true that also have *A* true
- Read this as: "Probability of *A* conditioned on *B*"
- Prior probability P(*A*) is a special case of the conditional probability P(*A* |) conditioned on no evidence

Conditional Probability Example



H = "Have a headache"
F = "Coming down with Flu"

P(*H*) = 1/10
P(*F*) = 1/40
P(*H* | *F*) = 1/2

"Headaches are rare and flu is rarer, but if you're coming down with 'flu there's a 50-50 chance you'll have a headache."

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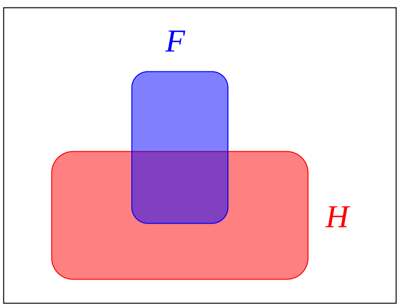
Conditional Probability



H = "Have a headache"
F = "Coming down with Flu"

P(*H*) = 1/10
P(*F*) = 1/40
P(*H* | *F*) = 1/2

P(*H*|*F*) = Fraction of flu-inflicted worlds in which you have a headache

$$= \frac{\text{\# worlds with flu and headache}}{\text{\# worlds with flu}}$$

$$= \frac{\text{Area of "H and F" region}}{\text{Area of "F" region}}$$

$$= \frac{\text{P(H, F)}}{\text{P(F)}}$$

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Definition of Conditional Probability

$$P(A \mid B) = \frac{P(A,B)}{P(B)}$$

Corollary: The Chain Rule (aka The Product Rule)

$$P(A,B) = P(A \mid B)P(B)$$

Important Note

$$P(A \mid B) + P(\neg A \mid B) = 1$$

But:

$$P(A \mid B) + P(A \mid \neg B) \text{ does not always} = 1$$

The Joint Probability Distribution

- P(*A*, *B*) is called the joint probability distribution of *A* and *B*
- It captures the probabilities of all combinations of the values of a set of random variables

The Joint Probability Distribution

- For example, if *A* and *B* are Boolean random variables, then P(*A*,*B*) could be specified as:

P(*A*=*false*, *B*=*false*)	0.25
P(*A*=*false*, *B*=*true*)	0.25
P(*A*=*true*, *B*=*false*)	0.25
P(*A*=*true*, *B*=*true*)	0.25

The Joint Probability Distribution

- Now suppose we have the random variables:
  - *Drink* = {*coke*, *sprite*}
  - *Size* = {*small*, *medium*, *large*}
- The joint probability distribution for P(*Drink*,*Size*) could look like:

P(*Drink*=*coke*, *Size*=*small*)	0.1
P(*Drink*=*coke*, *Size*=*medium*)	0.1
P(*Drink*=*coke*, *Size*=*large*)	0.3
P(*Drink*=*sprite*, *Size*=*small*)	0.1
P(*Drink*=*sprite*, *Size*=*medium*)	0.2
P(*Drink*=*sprite*, *Size*=*large*)	0.2

Full Joint Probability Distribution

- Suppose you have the complete set of random variables used to describe the world
- A joint probability distribution that covers this complete set is called the full joint probability distribution
- Is a complete specification of one's uncertainty about the world in question
- Very powerful: Can be used to answer any probabilistic query