# Maximum Liklehood Estimation

**Mr Fugu Data Science**

# (◕‿◕✿)

[Youtube (https://www.youtube.com/channel/UCbni-TDI-Ub8VlGaP8HLTNw/)](https://www.youtube.com/channel/UCbni-TDI-Ub8VlGaP8HLTNw/) | [Github (https://github.com/MrFuguDataScience)](https://github.com/MrFuguDataScience)

# Purpose & Outcome:

- ddfdf

**`Maximum Likelihood Estimation`** : allows us to estimate our "*unknown*" parameters for a given probability distribution. Think of it like this: we are trying to find the likelihood "*probability*" of generating this dataset.

- The parameters are found where they maximize the likelihood of generating our model.
- We are trying to figure out what model best describes our data. Having domain knowledge can help, in these circustances.

`Desireable Effects` : (if you have a large sample size)

- As the sample size increases, we reach an unbiased variance estimator
- The ability to generate a normal distribution and confidence intervals can be used with large enough samples
- The ability to form hypothesis testing

`Drawbacks` :

- If you have small sample size there will be bias
- Can be a computational hog

```
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 Each probability distribution will have unique parameters. As the parameters
change, a different probability distribution is generated.
```

- The parameters for each model will fall into a family of distributions based on the parameters you have. The parameters here will be $(\mu, \sigma)$

For example you may have a `normal distribution` , but the mean $(\mu)$ and standard deviation $(\sigma)$ are not the same from one model to the other.

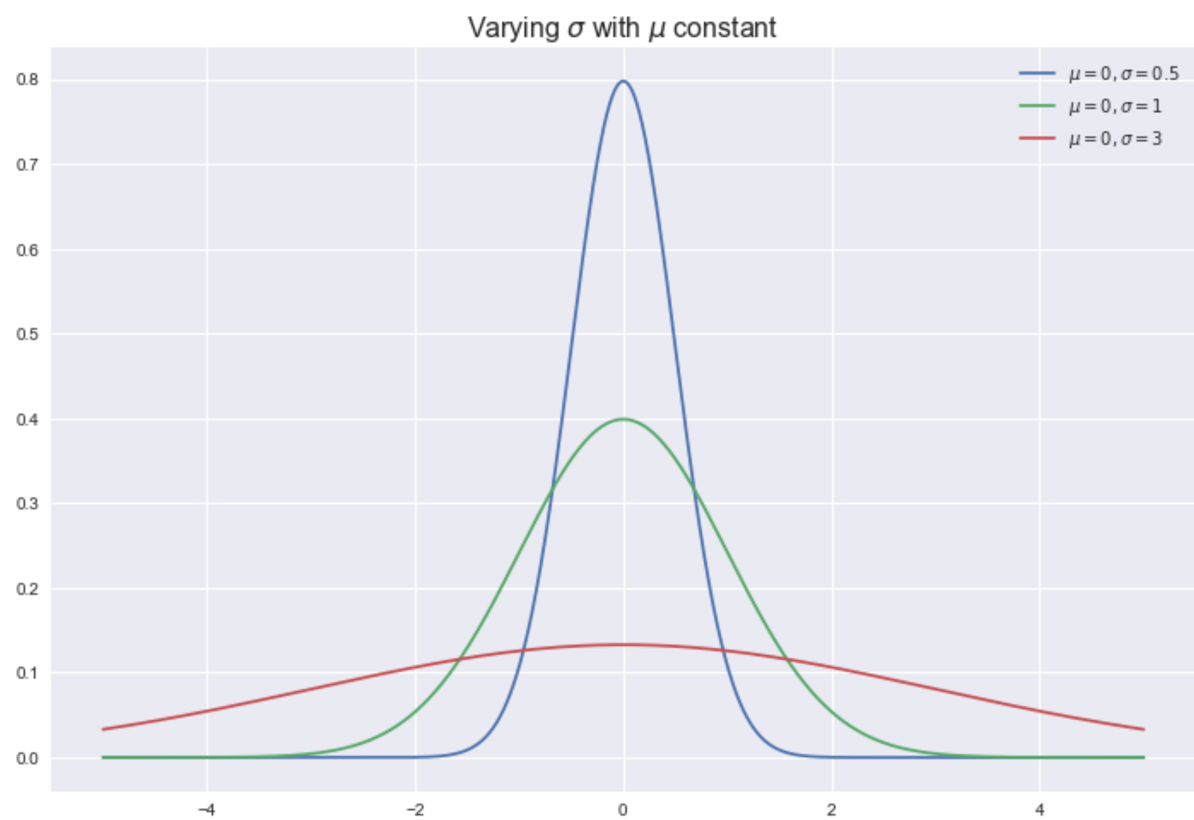
- Therefore, you may have a family of normal distributions with varying parameters.

# Plot showing different normals and explain as family of functions

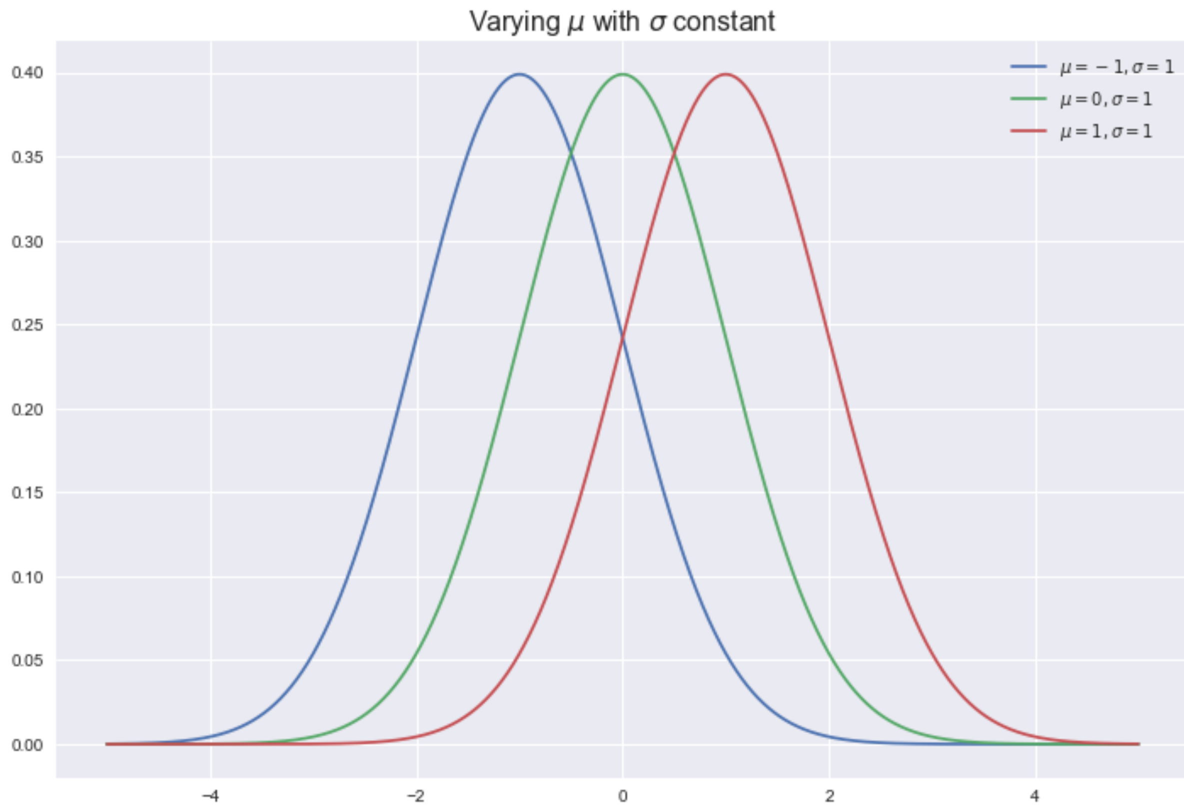
- These two plots reflect the parameters of a `Normal Distribution` you may want to estimate. If this were a different problem you would evaluate other parameters and which may not be $[\mu, \sigma]$

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline


plt.style.use('seaborn') # pretty matplotlib plots
plt.rcParams['figure.figsize'] = (12, 8)


x = np.linspace(-5, 5, 5000)
mu = [0,0,0]
sigma = [.5,1,3]

y_=[]
for i in range(len(sigma)):

    y = (1 / (np.sqrt(2 * np.pi * np.power(sigma[i], 2)))) * \
    (np.power(np.e, -(np.power((x - mu[i]), 2) / (2 * np.power(sigma[i],
2)))))

    y_.append(y)

labels=[('$\mu=0,\sigma=0.5$'),('$\mu=0,\sigma=1$'),('$\mu=0,\sigma=3$'
)]

for i in range(len(y_)):
    plt.plot(x,y_[i],label=labels[i])
    plt.title('Varying $\sigma$ with $\mu$ constant', size=16)
    plt.legend();
```



In [2]:
```
x = np.linspace(-5, 5, 5000)
mu_1=[-1,0,1]
sigma_1=[1,1,1]
y_1=[]
for i in range(len(sigma)):

    y_01= (1 / (np.sqrt(2 * np.pi * np.power(sigma_1[i], 2)))) * \
    (np.power(np.e, -(np.power((x - mu_1[i]), 2) / (2 * np.power(sigma_1
[i], 2)))))
    y_1.append(y_01)

labels_1=[('$\mu=-1,\sigma=1$'),('$\mu=0,\sigma=1$'),('$\mu=1,\sigma=1$'
)]
for i in range(len(y_)):
    plt.plot(x,y_1[i],label=labels_1[i])
    plt.legend();
    plt.title('Varying $\mu$ with $\sigma$ constant',size=16)
```



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In the real world we are given some data that we observed, and need to find some model of interest that will best describe our data using a `PDF` that will most likely represent our data.

- Realistically, we do not always have the option of transforming our data (*log,reciprocal,square root, etc*) to save the day.

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**Instead we have**: $L(\theta; x_i) = f(x; \theta)$

$L(\theta; x_i)$ `: This means that we have the likelihood of some true parameter theta given the observed data X`

`well what is the right side then?` : it is our `PDF` and there is a distinction

$f(x; \theta)$ : is a function of data given a particular set of parameters

I am using the semicolon **to denote parameters with respect to X instead of confusing conditional probability at this point, which would have the vertical bars**

# Math of MLE:

*Nasty Truth*:

- **First**, there doesn't have to be a unique value of the `MLE`
  - **Second**, there may not be a value that exists for the `MLE`

`Hint: because you need to think about concavity` , this is a problem if we are dealing with machine learning algorithms and a bit topic.

`-----------------`

**Super Fun Happy Times**, wait what?

# ʕ•ᴥ•ʔ

`MaTh TiMe Y'aLL`:

$L(\theta; x_i) = f(x; \theta)$

$f(x; \theta) = \prod f(X|\theta) =$

$f(x_1, x_2, x_3 \ldots x_n|\theta) =$

$f(x_1|\theta) * f(x_1|\theta) * f(x_2|\theta) * \ldots f(x_n|\theta)$

But, now the trickery `¯\_(ツ)_/¯` :

We will decide to do the `Log` because, it will get rid of our multiplcations and simply everything. Also, since the log is strickly increasing it shouldn't effect our function.

- We will need to find the Max/Min, hince we will need the `first derivative` with respect to $\theta$ and equate to zero.

$log(L(\theta; x_i)) = log[f(\theta|x_1) * f(\theta|x_1) * f(\theta|x_2) * \ldots f(\theta|(x_n)]$

$= log(f(\theta|x_1)) + log(f(\theta|x_1)) + log(f(\theta|x_2)) + \ldots log(f(\theta|x_n))$

$= \sum_{i=1}^{n} log(f(\theta|x))$

**`Derivative Time`**

`First Derivative`: $\dfrac{\partial ln(L(\theta|x))}{\partial \theta_i} = 0$

`Second Derivative`: $\dfrac{\partial^2 ln(L(\theta|x))}{\partial \theta_i^2} < 0$

*At this point if you have a suspected distribution, you would replace with your distribution function*. There are times when:

`Real World` : Either the data are highly paramertized , non-convex (*are a few reasons*) and you will need to use optimazation techinues to attempt solving these problems with approximation methods.

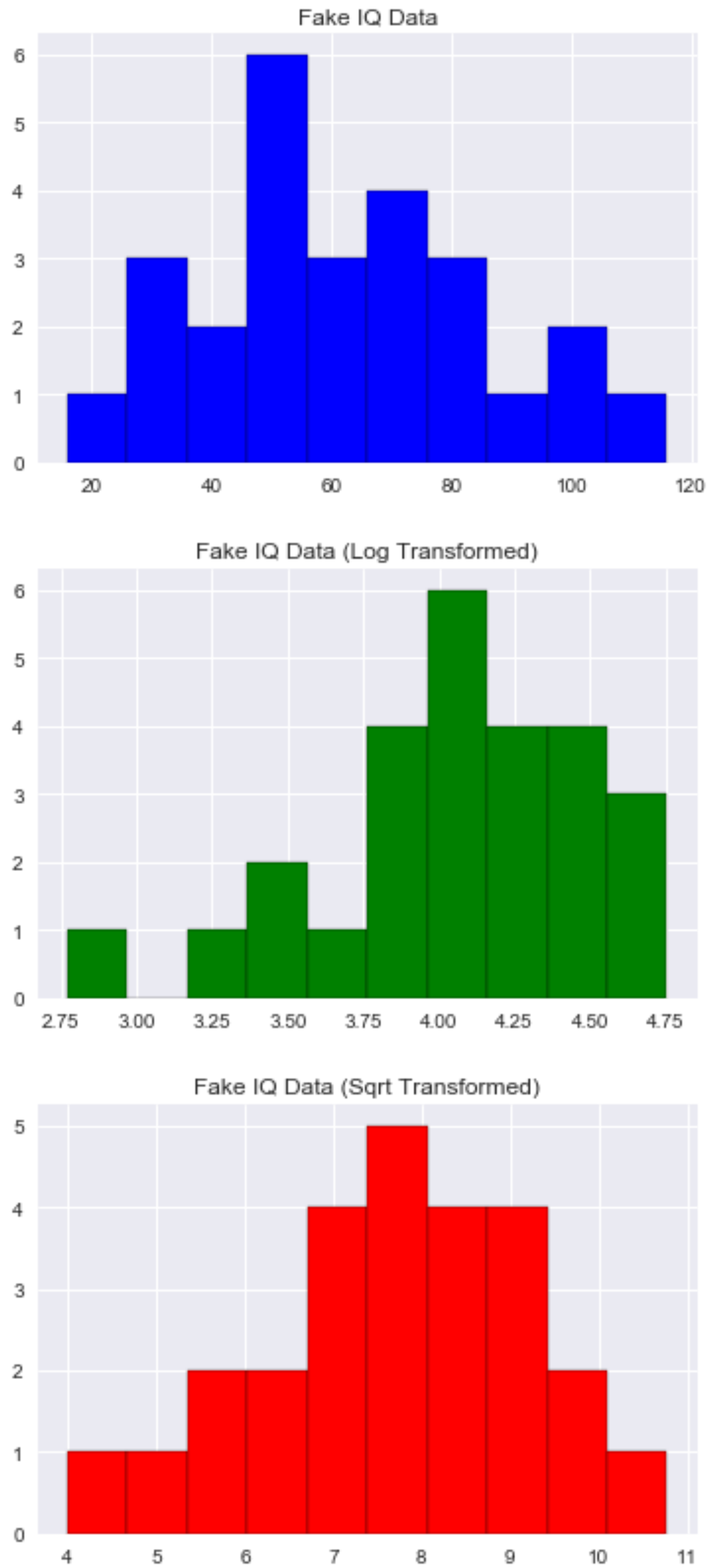
- `Examples` : *Newton’s method, Fisher scoring, various conjugate gradient-based approaches, steepest descent, Nelder-Mead type (simplex) approaches, BFGS* and a wide variety of other techniques (from: [https://www.analyticsvidhya.com/blog/2018/07/introductory-guide-maximum-likelihood-estimation-case-study-r/ (https://www.analyticsvidhya.com/blog/2018/07/introductory-guide-maximum-likelihood-estimation-case-study-r/)](https://www.analyticsvidhya.com/blog/2018/07/introductory-guide-maximum-likelihood-estimation-case-study-r/))

*a great resource*: [https://www.stat.tamu.edu/~suhasini/teaching613/chapter2.pdf (https://www.stat.tamu.edu/~suhasini/teaching613/chapter2.pdf)](https://www.stat.tamu.edu/~suhasini/teaching613/chapter2.pdf)

# Assume we have some `(*fake*)` data based on the preceived `IQ` for people who vote for a particular president

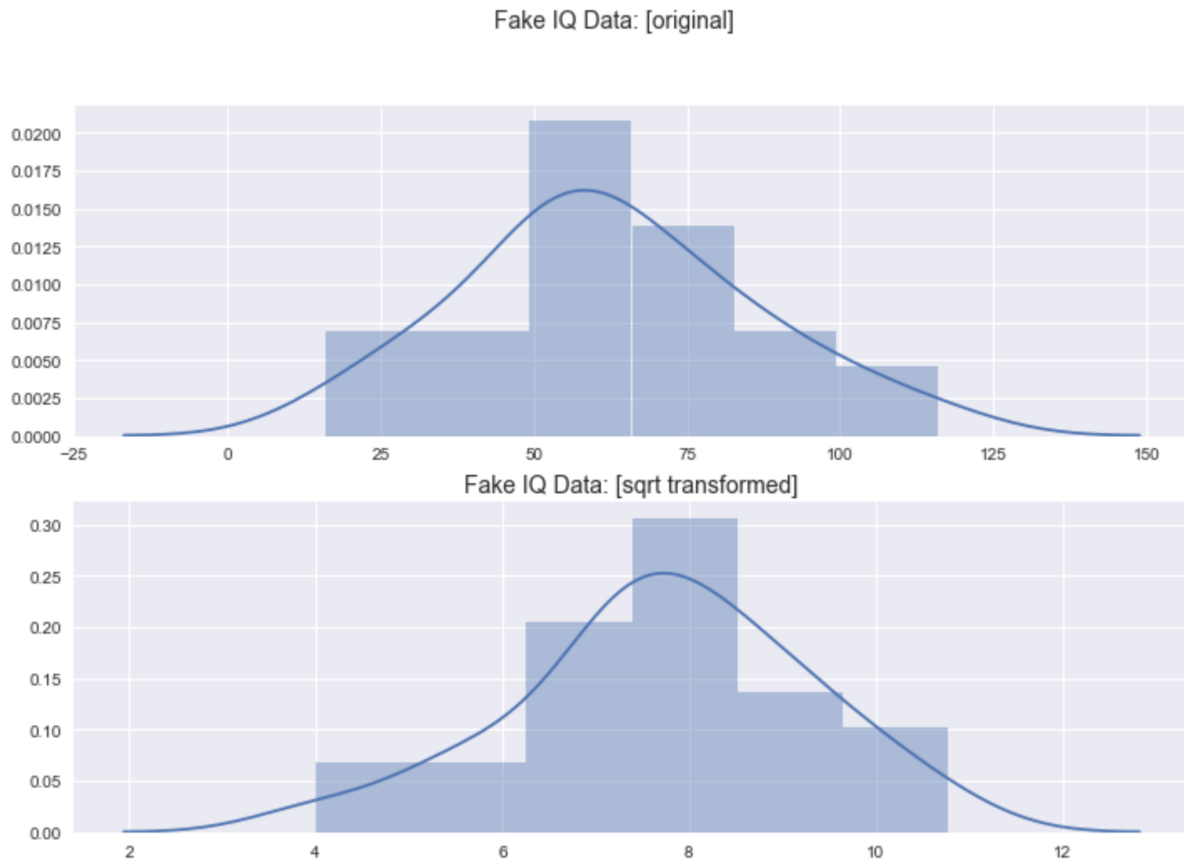
In [122]:
```
# Histogram with density plot

import seaborn as sns
import matplotlib.mlab as mlab
%matplotlib inline



fake_data=[30,40,62,61,85,80,16,55,26,53,51,70,116,
           66,51,101,99,88,71,55,51,35,62,71,81,44]


plt.hist(fake_data, color = 'blue', edgecolor = 'black')
plt.title('Fake IQ Data')
plt.subplots()

plt.hist(np.log(fake_data), color = 'green', edgecolor = 'black')
plt.title('Fake IQ Data (Log Transformed)')
plt.subplots()

plt.hist(np.sqrt(fake_data), color = 'red', edgecolor = 'black')
plt.title('Fake IQ Data (Sqrt Transformed)')
# plt.subplot()
# plt.hist(np.log(fake_data), color = 'green', edgecolor = 'black')
# plt.title('Fake IQ Data (Log Transformed)')
```

Out[122]: `Text(0.5, 1.0, 'Fake IQ Data (Sqrt Transformed)')`




# Lets look at this a little more clearly:

In [296]:
```
fig, axarr = plt.subplots(2, 1, figsize=(12, 8))


sns.distplot(fake_data,ax=axarr[0])
plt.suptitle('Fake IQ Data: [original]',size=14)

sns.distplot(np.sqrt(fake_data),ax=axarr[1])
plt.title('Fake IQ Data: [sqrt transformed]',size=14)

plt.show()
```



# transform data and looks close? What now?

- `Well, lets consider that maybe we think these data reflect a normal distribution.`
  - We could do a comparison after finding the parameters and see how it matches up.
  - We also, may end up in a situation where we need to reconsider our distribution as well. (*which is*) very likely in the real world.

**`For simplicity we are assuming Normal distribution`** :

$$f(x_1, x_2, x_3 \dots x_n; \mu, \sigma) =$$

**`Replace with distribution:`**

$$P(X; \mu, \sigma) = \frac{1}{\sigma\sqrt(2\pi)} exp(-\frac{(x-\mu)^2}{2\sigma^2}) =$$

$$\prod \frac{1}{\sigma\sqrt(2\pi)} exp(-\frac{(x-\mu)^2}{2\sigma^2}) =$$

**`Expand:`**

$$\frac{1}{\sigma\sqrt(2\pi)} exp(-\frac{(x_1-\mu)^2}{2\sigma^2}) * \frac{1}{\sigma\sqrt(2\pi)} exp(-\frac{(x_2-\mu)^2}{2\sigma^2}) * \frac{1}{\sigma\sqrt(2\pi)} exp(-\frac{(x_3-\mu)^2}{2\sigma^2}) * \dots \frac{1}{\sigma\sqrt(2\pi)} exp(-\frac{(x_n-\mu)^2}{2\sigma^2})$$

**`Now take the Log:`**

$$ln(L(\theta; X)) = ln(\frac{1}{\sigma\sqrt(2\pi)} exp(-\frac{(x_1-\mu)^2}{2\sigma^2})) + ln(\frac{1}{\sigma\sqrt(2\pi)} exp(-\frac{(x_2-\mu)^2}{2\sigma^2})) + ln(\frac{1}{\sigma\sqrt(2\pi)} exp(-\frac{(x_3-\mu)^2}{2\sigma^2})) + \dots ln(\frac{1}{\sigma\sqrt(2\pi)} exp(-$$

**`Simplify`** :

$ln(L(\theta; X)) = ln(\frac{1}{\sigma\sqrt(2\pi)}) + ln(exp(-\frac{(x_1-\mu)^2}{2\sigma^2})$, this is done for each x_i so I will simplify:

$$= \frac{-1}{2} ln(2\sigma\pi) - \frac{(x_1-\mu)^2}{2\sigma^2}$$

$= \frac{-1}{2} ln(2\pi) - \frac{1}{2} ln(\sigma) - \frac{(x_1-\mu)^2}{2\sigma^2}$ , Now repeat for a X=[x_1,x_2...x_n]

To speed this up and understand that we have: (n) observations (data points)

$= \frac{-n}{2} ln(2\pi) - n * ln(\sigma) - \frac{1}{2\sigma^2} \sum_{i=1}^{n}(x_i - \mu)^2$ *`This is what we will take the derivative of`*

**`Now the Derivatives`** :

The first derivative will be the slope of the log-likelihood curve:

$$\frac{\partial L}{\partial \mu} = 0 - 0 - \frac{1}{2\sigma^2} \sum_{i=1}^{n}(x_i - \mu)^2 =$$

$\frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - \mu =$

$\frac{1}{\sigma^2}\sum_{i=1}^{n} x_i - n\mu$

$\hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n}$ =sample mean

```
---------Now Standard Deviation-----------
```

short cut:

$= \frac{-n}{2} ln(2\pi) - n * ln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$

$\frac{\partial L}{\partial \sigma} = 0 - \frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \mu)^2 =$, set equal to zero

$-\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \mu)^2 = 0$

**Manipulate:**

$\frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \mu)^2 = \frac{n}{\sigma}$

$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2 = \frac{n}{\sigma}}{n}}$, biased estimator

```
side note: the (n-1) will be unbiased look into proof, second webpage below
```

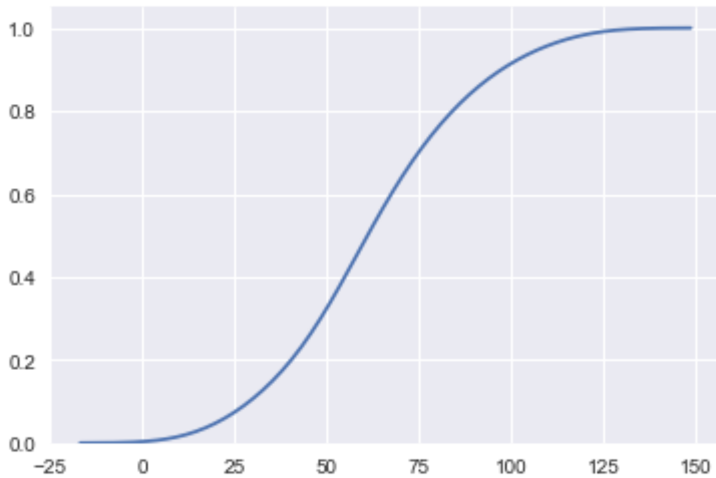
https://machinelearningmastery.com/probability-density-estimation/ (https://machinelearningmastery.com/probability-density-estimation/)

https://daijiang.name/en/2014/10/08/mle-normal-distribution/ (https://daijiang.name/en/2014/10/08/mle-normal-distribution/)

# Check the CDF: so we can get an idea of if these data will fit a Normal Distr

In [511]:
```
kwargs = {'cumulative': True}
# sns.distplot(fake_data, hist_kws=kwargs, kde_kws=kwargs)
sns.kdeplot(fake_data,cumulative=True)
```

Out[511]: <matplotlib.axes._subplots.AxesSubplot at 0x1a424b5208>

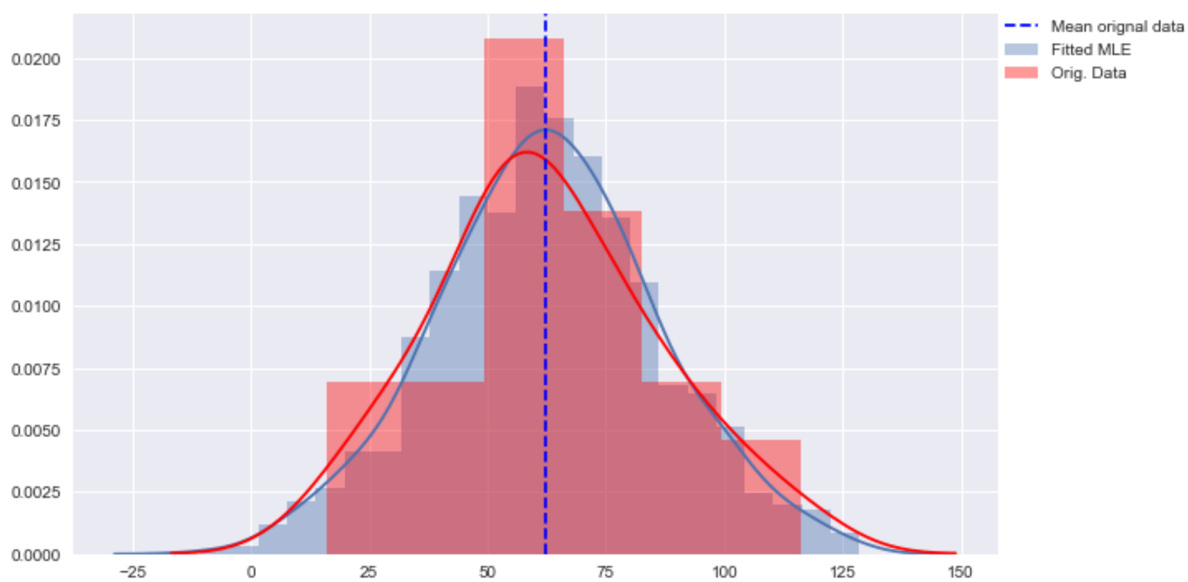



# We can directly compute the $\mu$ & $\sigma$ from the data, but this doesn't mean that we are actually representing the population the data came from.

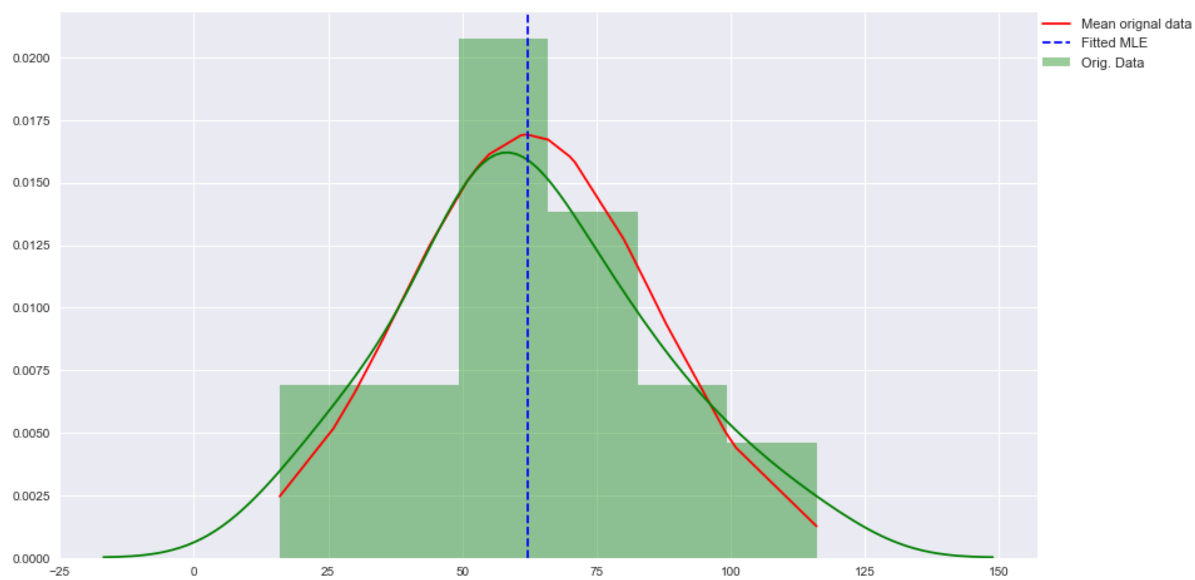
- Hence, the use of `MLE` .

In [452]:
```
print(np.mean(fake_data),np.std(fake_data))
fig_dims = (10, 6)
fig, ax = plt.subplots(figsize=fig_dims)
sns.distplot(np.random.normal(np.mean(fake_data),np.std(fake_data),1000
))
# plt.legend('MLE')
sns.distplot(fake_data,color='red')
plt.axvline(np.mean(fake_data),color='b', linestyle='--')
plt.legend(['Mean orignal data','Fitted MLE','Orig. Data'],bbox_to_ancho
r=(1, 1),
           loc=2, borderaxespad=0.) #,fontsize=25
plt.show()


# sigma=np.std(fake_data)
# mu=np.mean(fake_data)

# vals=[]
# for x in sorted(fake_data):
# #     norm_d=1/(sigma*np.sqrt(2*np.pi))*np.exp(-(x-mu)^2/2*sigma^2)
#     norm_d=1/(sigma*np.sqrt(2*np.pi))*np.exp(-np.power(x - mu, 2.) /
 (2 * np.power(sigma, 2.)))
#     vals.append(norm_d)
# sorted(fake_data)
```

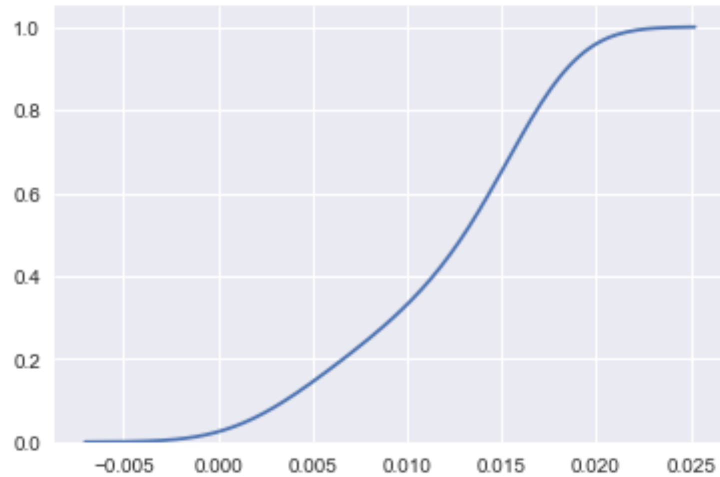
```
62.30769230769231 23.57601328398981
```



In [508]:
```
import numpy as np
from scipy.stats import norm
import matplotlib.pyplot as plt
import scipy.stats as stats
fig_dims = (14, 8)
fig, ax = plt.subplots(figsize=fig_dims)
meany=np.mean(fake_data)
sd_=np.std(fake_data)


pdf = stats.norm.pdf(sorted(fake_data), meany, sd_)
plt.plot(sorted(fake_data), pdf,color='red') # including h here is cruci
al

sns.distplot(fake_data,color='green')
plt.axvline(np.mean(fake_data),color='b', linestyle='--')

plt.legend(['Mean orignal data','Fitted MLE','Orig. Data'],bbox_to_ancho
r=(1, 1),
              loc=2, borderaxespad=0.,fontsize=11)
plt.show()
```



In [513]:
```
sns.kdeplot(pdf,cumulative=True)
plt.show()
```




# Last Thoughts:

- You can't always, assume your data will fit an exact model
- Approximation methods are usually a good call on real data
- Do not blindly, believe everything you hear!

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# Citations & Help:


https://www.itl.nist.gov/div898/handbook/apr/section4/apr412.htm (https://www.itl.nist.gov/div898/handbook/apr/section4/apr412.htm)

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