Statistics and Probability Letters 79 (2009) 55–62

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

First order p-variations and Besov spaces

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ARTICLE INFO

Article history:
Received 12 September 2007
Received in revised form 25 February 2008
Accepted 21 July 2008
Available online 26 July 2008

MSC:
26A15
42C40
46F99
60G15
60G17

ABSTRACT

Based on the notion of first order dyadic p-variation, we give a new characterization of Besov spaces $\mathcal{B}^s_{p,q}([0,1])$ for $0 < s < 1$, $1 \le p, q \le +\infty$ and $s > 1/p$. We also give results in the case where $p < 1$. Hence we provide simple tools that enable us to derive new regularity properties for the trajectories of various continuous time stochastic processes.

1. Introduction

Besov spaces are a natural framework to study the smoothness of the sample paths of a continuous time random process, see Ciesielski et al. (1993). For $s > 0$ and $1 \le p, q \le +\infty$, the Besov spaces $\mathcal{B}^s_{p,q}([0,1])$ are usually defined in terms of modulus of continuity, see Appendix A.1 for definitions. Using the Schauder basis, Ciesielski et al. have proved that for $s > 1/p$, the usual Besov norm on $\mathcal{B}^s_{p,q}([0,1])$ is equivalent to a norm based on the *second order dyadic p-variation*, that is, for a real function f on $[0,1]$ and $0 < p < +\infty$, the quantity

$$\sum_{k=1}^{2^j} |-f(\{2k\}2^{-(j+1)}) + 2f(\{2k-1\}2^{-(j+1)}) - f(\{2k-2\}2^{-(j+1)})|^p,$$

see Appendix A.2 for details.[1] This result has been used to obtain regularity properties of the sample paths of some stochastic processes such as Brownian motion, fractional Brownian motion, Lévy stable processes and stochastic integrals, see Roynette (1993) and Ciesielski et al. (1993). The local times of the Brownian motion and of Lévy stable processes have been studied by Boufoussi and Roynette (1993) and Boufoussi and Ouknine (1999).

In this paper, we simplify the results of Ciesielski et al. and we extend them using either *first order dyadic p-variations* or *first order general p-variations*. For a real function f on $[0,1]$ and $0 < p < +\infty$, the first order dyadic p-variation $V_j^p(f)$ is defined by

$$V_j^p(f) = \sum_{k=1}^{2^j} |f(k2^{-j}) - f(\{k-1\}2^{-j})|^p$$

and the first order general p-variation $\nu_p(f)$ by

$$\nu_p(f) = \sup\left\{\sum_{k=0}^{m-1} |f(t_{k+1}) - f(t_k)|^p,\ 0 = t_0 < t_1 < \cdots < t_m = 1,\ m = 1, 2, \ldots\right\}.$$

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[1] The case $p = +\infty$ will be treated considering throughout the paper the following usual modification: for a sequence real $(a_j)_{j\ge 0}$, $(\sum_j |a_j|^p)^{1/p} = \sup_j |a_j|$ for $p = +\infty$.

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doi:10.1016/j.spl.2008.07.019

The behavior of these kinds of objects has been studied for many stochastic processes, see Section 3. Using these quantities, our objective is to be able to derive Besov smoothness properties for the sample paths of various continuous time processes such as Gaussian processes or Markov processes.

We give in Section 2 a new characterization of Besov spaces $\mathcal{B}^s_{p,q}([0,1])$ for $0 < s < 1$, $1 \le p, q \le \infty$, $s > 1/p$ and a useful result for the case $p < 1$. Section 3 contains new smoothness properties for the trajectories of various processes derived from our results. Some proofs are relegated to Section 4.

2. Characterization of Besov spaces by first order *p*-variations

We state in this section two theorems that we widely use in Section 3 to obtain Besov smoothness properties for the sample paths of stochastic processes. We refer to Appendix A.1 for the definition of the usual norm (or quasi-norm) $\|\cdot\|_{\mathcal{B}^s_{p,q}([0,1])}$ on $\mathcal{B}^s_{p,q}([0,1])$.

2.1. Results

Theorem 1. *Let $0 < s < 1, s > 1/p$, $1 \le p, q \le \infty$. The usual norm on $\mathcal{B}^s_{p,q}([0,1])$ is equivalent to the norm defined by*

$$\|f\| = \max\left\{|f(0)|, \left(\sum_{j\ge 0} 2^{jq(s-1/p)}\{V^p_j(f)\}^{q/p}\right)^{1/q}\right\}.$$

The fact that the usual Besov norm is less than a constant times this new norm is a simple consequence of the results of Ciesielski et al. (1993), see Section 4.[2] The main contribution of Theorem 1 is to show the inverse inequality. Hence it gives a new tool, simpler than the second order dyadic p-variations used by Ciesielski et al., for proving that a trajectory does not belong to a Besov space. See Sections 2.2 and 3 for a discussion and some examples where the use of the first order dyadic p-variation is convenient whereas the use of the second order dyadic p-variation appears more difficult.

Theorem 2. *Let $f : [0,1] \to \mathbb{R}$ be a Borel function and $0 < p < \infty$. If $\nu_p(f) < +\infty$, then f belongs to $\mathcal{B}^{1/p}_{p,\infty}([0,1])$.*

Theorem 2 follows from Theorem 3 in Section 4. Theorem 3 is an analogue of Theorem 1 for distributional Besov spaces $\tilde{\mathcal{B}}^s_{p,q}$, $s \in \mathbb{R}$, $0 < p, q \le \infty$, as defined in Triebel (1983, 2006). Thus our contribution is to show that in some sense, the link between first order p-variations and Besov spaces established in Theorem 1 remains true in the distribution context. In particular, this framework enables us to treat the case $p < 1$. The use of Theorem 3 being mainly Theorem 2, its statement and proof are relegated within the proof of Theorem 2.

2.2. Discussion

• Theorem 1 simply shows that, to derive some Besov regularity of a trajectory, it is often enough to consider first order dyadic p-variation. A striking example is provided by the Brownian motion. The Besov regularity of the Brownian motion has been studied in particular by Roynette (1993). Let $(B_t)_{t\ge 0}$ be a Brownian motion. Using the Lévy construction of the Brownian motion, Roynette has proved the following proposition, where $a.s.$ means almost surely.

Proposition 1 (*Roynette, 1993*).

- *If $0 < s < 1/2$, for $1 \le p < \infty$ and $1 \le q \le \infty$, a.s., $(t \to B_t) \in \mathcal{B}^s_{p,q}([0,1])$.*
- *For $1 \le p < \infty$, a.s., $(t \to B_t) \in \mathcal{B}^{1/2}_{p,\infty}([0,1])$. Moreover, for $p > 2$, there exists a constant $c_p > 0$ such that a.s., $\|B\|_{\mathcal{B}^{1/2}_{p,\infty}([0,1])} > c_p$.*
- *For $2 < p < \infty$ and $1 \le q < \infty$, a.s., $(t \to B_t) \notin \mathcal{B}^{1/2}_{p,q}([0,1])$.*
- *If $s > 1/2$, for $2 < p < \infty$ and $1 \le q \le \infty$, a.s., $(t \to B_t) \notin \mathcal{B}^s_{p,q}([0,1])$.*

Thanks to Theorem 1, we have a very simple proof of the results of Roynette. Indeed, using L^2 convergence, Markov inequality and the Borel–Cantelli lemma, we classically have that for $1 \le p < \infty$, a.s.,

$$2^{j(p/2-1)}\sum_{k=0}^{2^j-1}|B_{(k+1)2^{-j}} - B_{k2^{-j}}|^p \to \mathbb{E}[|\eta|^p],$$

where η denotes a standard Gaussian variable. Hence, by Theorem 1, the positivity of $\mathbb{E}[|\eta|^p]$ and obvious embeddings, we have the result. We extend this result to more general Gaussian processes in Proposition 2.

[2] Note also that we directly get from Ciesielski et al. (1993) that for $0 < s < 1 + 1/p$, $1 \le p, q \le \infty$ and f càdlàg, $\|f\|_{\mathcal{B}^s_{p,q}([0,1])} \le c \max\left\{|f(0)|, \left(\sum_{j\ge 0} 2^{jq(s-1/p)}\{V^p_j(f)\}^{q/p}\right)^{1/q}\right\}$.

• A corollary of Theorem 1 is the following.

Corollary 1. *Let $\{X_t, t \in [0,1]\}$ be a continuous process such that for $p > 1$, $1/p \leq s < 1$ and Y an almost surely finite and positive random variable,*

$$2^{j(ps-1)} V_j^p(X) \to Y \quad \text{in probability}.$$

For any $\varepsilon > 0$, a.s. $(t \to X_t) \in \mathcal{B}^{s-\varepsilon}_{p,\infty}([0,1])$ and a.s. $(t \to X_t) \notin \mathcal{B}^{s+\varepsilon}_{p,\infty}([0,1])$.

This result can be applied with $s = 1/2$ to various Ito semi-martingales using the results on the first order p-variations of these processes given in Jacod (2006).

• Our results can probably be of first interest in multifractal analysis. In this field, first order p-variations (called structure functions) appeared naturally. Moreover, the need for deriving the Besov regularity of a signal has become crucial since the Frisch–Parisi conjecture (Frisch and Paris, 1985), see Eyink (1995) and Bacry et al. (2008).

• The tools we provide are particularly relevant in finance, where the first order p-variations are common objects. In particular, the celebrated "realized volatility" corresponds to the case $p = 2$. An example of the use of our results is the analysis of market microstructure through the lenses of Besov spaces, see Rosenbaum (2007).

3. New regularity properties for stochastic processes

In this section we present new regularity properties for various processes. Our goal is to give an overview of the kinds of results one can easily obtain thanks to Theorems 1 and 2. From now on, we write $\mathcal{B}^s_{p,q}$ for $\mathcal{B}^s_{p,q}([0,1])$.

3.1. Gaussian processes

We first give a very simple characterization for some Gaussian processes for belonging to the Besov spaces $\mathcal{B}^s_{p,q}$.

Proposition 2. *Let $\{X_t, t \in [0,1]\}$ be a zero mean Gaussian process with stationary increments. Let $\sigma(h) = (\mathbb{E}[(X_{t+h} - X_t)^2])^{1/2}$. Assume that for some $0 < r < 1$ and $0 < \alpha < \infty$,*

$$\lim_{h \to 0} \sigma(h)/h^r = \alpha.$$

Then, we have the following result:

- *If $0 < s < r$, for $1 \leq p < \infty$ and $1 \leq q \leq \infty$, a.s., $(t \to X_t) \in \mathcal{B}^s_{p,q}$.*
- *For $1 \leq p < \infty$, a.s. $(t \to X_t) \in \mathcal{B}^r_{p,\infty}$. Moreover, for $p > 1/r$, there exists a constant $c_p > 0$ such that a.s., $\|X\|_{\mathcal{B}^r_{p,\infty}} > c_p$.*
- *For $2 \leq p < \infty$, $rp > 1$ and $1 \leq q < \infty$, a.s., $(t \to X_t) \notin \mathcal{B}^r_{p,q}$.*
- *If $s > r$, for $2 \leq p < \infty$, $rp > 1$ and $1 \leq q \leq \infty$, a.s., $(t \to X_t) \notin \mathcal{B}^s_{p,q}$.*

This proposition can for example be applied to fractional Brownian motion with Hurst parameter H in $(0, 1)$. Indeed, we have in this case $\sigma(h) = h^H$.

We now present results for Baxter and Gladyshev processes, see Baxter (1956) and Gladyshev (1961) for details and examples. Recall the following definitions.

Definition 1 (*Baxter Process*). A real-valued Gaussian process $\{X_t, t \in [0,1]\}$ is called a Baxter process if:

- $m(t) = \mathbb{E}[X_t]$ has a bounded first derivative,
- $r(s,t) = \mathbb{E}[X_t X_s] - m(s)m(t)$ is continuous on $[0,1]^2$ and has uniformly bounded second derivatives on $[0,1]^2 - \Delta$, $\Delta = \{(t,t), t \in [0,1]\}$.

Definition 2 (*Gladyshev Process*). A real-valued Gaussian process $\{X_t, t \in [0,1]\}$ is called a Gladyshev process if:

- $m(t)$ has a bounded first derivative,
- $r(s,t)$ is continuous on $[0,1]^2$, has second derivatives on $[0,1]^2 - \Delta$ and $|\partial^2 r(s,t)/\partial s \partial t| \leq L/|t-s|^\gamma$ for some constants $L > 0$ and $0 < \gamma < 2$,
- the expression $|r(t,t) - 2r(t,t-h) + r(t-h,t-h)|/h^{2-\gamma}$ converges uniformly on $[0,1]$ to some function $f(t)$ as $h \to 0$.

We have the following propositions.

Proposition 3. *Let $\{X_t, t \in [0,1]\}$ be a Baxter process. Let $D^+(t)$ and $D^-(t)$ denote respectively the left and right derivatives of $s \to r(s,t)$ at point t and $f(t) = D^-(t) - D^+(t)$. Then, a.s., $(t \to X_t) \in \mathcal{B}^{1/2}_{2,\infty}$. Moreover, if $\int_0^1 f(t)\mathrm{d}t \neq 0$, for all $\varepsilon > 0$, a.s., $(t \to X_t) \notin \mathcal{B}^{1/2+\varepsilon}_{2,\infty}$.*

Proof. Directly from Theorem 1 together with Theorem 1 in Baxter (1956). □

Proposition 4. *Let $\{X_t, t \in [0,1]\}$ be a Gladyshev process of index $\gamma \in (0,2)$.*

- *A.s., $(t \to X_t)$ belongs to $\mathcal{B}^{\gamma/2}_{2,\infty}$.*

– *If* $\int_0^1 f(t)\mathrm{d}t \neq 0$, $\gamma > 1$ *and* $1 \leq q < \infty$, *a.s.,* $(t \to X_t) \notin \mathcal{B}^{\gamma/2}_{2,q}$.
– *If* $\int_0^1 f(t)\mathrm{d}t \neq 0$, $\gamma \geq 1$ *and* $\varepsilon > 0$, *a.s.,* $(t \to X_t) \notin \mathcal{B}^{\gamma/2+\varepsilon}_{2,\infty}$.

Proof. Directly from Theorem 1 together with Theorem 1 in Gladyshev (1961). □

3.2. Markov processes

Proposition 5. *Let* $\{X_t, t \in [0, 1]\}$ *be a strong Markov process. Denote the transition probability function of* X_t *by* $P_{s,t}(x, \mathrm{d}y)$. *For any* $h \in [0, 1]$ *and* $a > 0$, *define*

$$\alpha(h, a) = \sup\{P_{s,t}(x, \{y : |x - y| \geq a\}), 0 \leq s \leq t \leq (s + h) \wedge 1\}.$$

Let $\beta \geq 1$, $\gamma > 0$. *If there exist constants* $a_0 > 0$ *and* $K > 0$ *such that, for all* $h \in [0, 1]$ *and* $a \in (0, a_0]$,

$$\alpha(h, a) \leq Kh^{\beta} a^{-\gamma}.$$

then, for all $\gamma/\beta < p < \infty$, *a.s.,* $(t \to X_t)$ *belongs to* $\mathcal{B}^{1/p}_{p,\infty}$.

Proof. Directly from Theorem 2 together with Theorem 1.3 in Manstavicius (2004). □

Remark. It is proved in Manstavicius (2004) that for symmetric stable Lévy processes with index $\alpha \in (0, 2]$, the condition $p > \gamma/\beta$ applies with $\gamma/\beta = \alpha$.

3.3. Functionals of Lévy processes

The Besov regularity of Lévy processes has been studied by Ciesielski et al. (1993), in the case of stable processes of index β, $1 < \beta < 2$, $p \geq 1$, and by Schilling (2000) in the general case.[3] We give here other results related to Lévy processes.

Proposition 6. *Let* $\{X_t, t \in [0, 1]\}$ *be a real-valued stable process of index* β, $1 < \beta \leq 2$. *For* $0 < \gamma < (\beta - 1)/2$, *we define* H_t, *its fluctuating continuous additive functional of order* γ *by*

$$H_t = \frac{1}{\Gamma(-\gamma)} \int_0^{\infty} y^{-1-\gamma} (L_t^{-y} - L_t^0)\mathrm{d}y,$$

where $\{L(t, x), t \geq 0\}$ *is the local time at* x *for* X. *Let* $p_0 = (\beta - 1)/(\beta - 1 - \gamma)$. *Then for all* $p_0 < p < \infty$, *a.s.,* $(t \to H_t)$ *belongs to* $\mathcal{B}^{1/p}_{p,\infty}$ *and for* $\varepsilon > 0$, *a.s.,* $(t \to H_t)$ *does not belong to* $\mathcal{B}^{1/p_0+\varepsilon}_{p_0,\infty}$.

Proof. Directly from Theorem 2 together with Theorem 4.3 in Fitzsimmons and Getoor (1992). □

Proposition 7. *Let* $f(x) = x^{\delta}$ *with* $\delta \in (\{3 - e\}/\{e - 1\}, 1)$. *Let* $\{X_t, t \in [0, 1]\}$ *be a symmetric real-valued* α*-stable Lévy process with* $\alpha \in (0, \delta)$. *We define* $Y_t = X_{f(t)}$ *for* $t \in [0, 1]$. *For any* $\alpha/\delta < p < \infty$, *a.s.,* $(t \to Y_t)$ *belongs to* $\mathcal{B}^{1/p}_{p,\infty}$.

Proof. Directly from Theorem 2 together with Remark 1 in Manstavicius (2005). □

4. Proofs

4.1. Proof of Theorem 1

In this proof, c denotes a positive constant that may vary from line to line. Let $1 \leq p, q \leq \infty$ and $s > 1/p$. The fact that $\|f\|_{\mathcal{B}^s_{p,q}}$ is smaller than $c\|f\|$ is obvious because of Proposition 9 together with a convexity inequality. For the other inequality, we use the development of f in the Schauder basis, see Appendix A.2. Let $f \in \mathcal{B}^s_{p,q}([0, 1])$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $s > 1/p$.

$$V_n^p(f) = \sum_{r \leq 2^n} |f(\{r + 1\}2^{-n}) - f(r2^{-n})|^p.$$

Consider the Schauder basis (ϕ_{jk}) and the coefficients of the expansion of f defined as in Appendix A.2. We have

$$V_n^p(f) \leq 2^{p-1} \sum_{r \leq 2^n} \left| \sum_{j=0}^{n} \sum_{k=1}^{2^j} f_{jk}[\phi_{jk}(\{r + 1\}2^{-n}) - \phi_{jk}(r2^{-n})] \right|^p + 2^{p-1} f_1^p 2^{n(1-p)}.$$

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[3] Note that for $p < 1$, we can give simple proofs of some results of Schilling (2000) on Lévy processes. We just use Theorem 2 together with classical results on the p-variation of Lévy processes (for stable processes, Theorem 2 in Fristedt and Taylor (1973), for general Lévy processes, Theorem 2 in Monroe (1972)).

For given j, as $j \leq n$, there exists a unique integer $k = k_{j,r,n}$ such that $[\phi_{jk}(\{r+1\}2^{-n}) - \phi_{jk}(r2^{-n})] \neq 0$. Let

$$\widetilde{V}_n^p(f) = \sum_{r \leq 2^n} \left| \sum_{j=0}^{n} \sum_{k=1}^{2^j} f_{jk} [\phi_{jk}(\{r+1\}2^{-n}) - \phi_{jk}(r2^{-n})] \right|^p .$$

Since $|\phi_{jk}(\{r+1\}2^{-n}) - \phi_{jk}(r2^{-n})| \leq 2^{j/2-n}$, we have $\widetilde{V}_n^p(f) \leq M_n$ with

$$M_n = \sum_{r \leq 2^n} \left(\sum_{j=0}^{n} 2^{j/2-n} |f_{jk_{j,r,n}}| \right)^p .$$

Let $\varepsilon = p - \lfloor p \rfloor$, we have

$$M_n \leq 2^{-n\lfloor p \rfloor} \sum_{j_1,\ldots,j_{\lfloor p \rfloor}} 2^{(j_1+\cdots+j_{\lfloor p \rfloor})/2} \sum_{r \leq 2^n} |f_{j_1 k_{j_1,r,n}}| \cdots |f_{j_{\lfloor p \rfloor} k_{j_{\lfloor p \rfloor},r,n}}| \left\{ \sum_{j=0}^{n} 2^{j/2-n} |f_{jk_{j,r,n}}| \right\}^{\varepsilon} .$$

By the Hölder inequality, we obtain

$$M_n \leq 2^{-n\lfloor p \rfloor} \sum_{j_1,\ldots,j_{\lfloor p \rfloor}} 2^{(j_1+\cdots+j_{\lfloor p \rfloor})/2} M_n^{\varepsilon/p} \prod_{i=1}^{\lfloor p \rfloor} \left(\sum_{r \leq 2^n} |f_{j_i k_{j_i,r,n}}|^p \right)^{1/p} .$$

Thus,

$$M_n^{1-\varepsilon/p} \leq 2^{-n\lfloor p \rfloor} \sum_{j_1,\ldots,j_{\lfloor p \rfloor}} 2^{(j_1+\cdots+j_{\lfloor p \rfloor})/2} \prod_{i=1}^{\lfloor p \rfloor} \left(2^{n-j_i} \sum_k |f_{j_i k}|^p \right)^{1/p} \leq 2^{-n\lfloor p \rfloor} \left\{ \sum_{j=0}^{n} 2^{j/2} \left(2^{n-j} \sum_k |f_{jk}|^p \right)^{1/p} \right\}^{\lfloor p \rfloor} .$$

If $q = +\infty$, as $s < 1$, using Proposition 9 we easily obtain $M_n \leq c2^{n(1-sp)}$. The result follows using that $V_n^p(f) \leq M_n + 2^{p-1} f_1^p 2^{n(1-p)}$. If $1 \leq q < +\infty$, we have

$$2^{nq(s-1/p)} M_n^{q/p} \leq \left\{ \sum_{j=0}^{n} 2^{(j-n)(1-s)} 2^{j(-1/2+s-1/p)} \left(\sum_k |f_{jk}|^p \right)^{1/p} \right\}^q .$$

Let γ be such that $0 < \gamma < 1-s$ and q' be the conjugate exponent of q. Then, using the Hölder inequality and the fact that

$$\sum_{j=0}^{n} 2^{(j-n)q'(1-s-\gamma)} < c,$$

we get

$$2^{nq(s-1/p)} M_n^{q/p} \leq c \sum_{j=0}^{n} 2^{(j-n)q\gamma} 2^{qj(-1/2+s-1/p)} \left(\sum_k |f_{jk}|^p \right)^{q/p} . \tag{1}$$

Moreover,

$$V_n^p(f)^{q/p} \leq c(M_n^{q/p} + 2^{nq(1/p-1)}).$$

The result follows using Proposition 9 and remarking that the series in j in (1) is the nth term of the Cauchy product of two convergent series. The case $p = +\infty$ is treated the same way.

4.2. *Proof of* Theorem 2

We denote by $\mathcal{S}'$ the space of all tempered distributions, by $\mathcal{C}^r$ the set of all compactly supported r-times continuously differentiable real functions and by $\mathcal{W}^r$ the set of all couple (ψ_0, ψ) of Daubechies scaling function and mother wavelet, both in $\mathcal{C}^r$, see Daubechies (1988) for details. For $s \in \mathbb{R}$ and $0 < p, q \leq \infty$, let $\sigma_p = (1/p - 1)_+$ and $r(s,p) = \max(s, \sigma_p - s)$. Let $b_{p,q}^s$ be the set of all sequences $\lambda = \{\lambda_{jk} \in \mathbb{C}, j \in \mathbb{N}, k \in \mathbb{Z}\}$ such that

$$\|\lambda | b_{p,q}^s\| = \left(\sum_{j=0}^{\infty} 2^{jq(s-1/p)} \left\{ \sum_{k \in \mathbb{Z}} |\lambda_{jk}|^p \right\}^{q/p} \right)^{1/q} < \infty.$$

We now define the distributional Besov space $\tilde{\mathcal{B}}_{p,q}^s$, see Triebel (2006).

Definition 3. Let $(\psi_0, \psi) \in \mathcal{W}^r$, with $r > r(s, p)$. Then, $f \in \mathcal{S}'$ is an element of $\tilde{\mathcal{B}}^s_{p,q}$ if and only if it can be represented as

$$f = \sum_{\substack{j \in \mathbb{N},\\ k \in \mathbb{Z}}} \lambda_{jk} \psi_{jk}, \tag{2}$$

with $\|\lambda|b^s_{p,q}\| < \infty$, convergence in $\mathcal{S}'$ and $\psi_{jk}(x) = \psi(2^{j-1}x - k)$. Furthermore, if $f \in \tilde{\mathcal{B}}^s_{p,q}$, then representation (2) is unique with $\lambda_{jk} = 2^j \langle f, \psi_{jk} \rangle$. Equipped with $\|f\|_{\tilde{\mathcal{B}}^s_{p,q}} = \|\lambda(f)|b^s_{p,q}\|$, $\tilde{\mathcal{B}}^s_{p,q}$ is a quasi-Banach space.

Remark that for duality reasons, the notation $\langle f, \psi_{jk} \rangle$ makes sense as soon as $f \in \tilde{\mathcal{B}}^s_{p,q}$ and $\psi_{jk} \in \mathcal{C}^r$ with $r > -s + \sigma_p$. This definition coincides with the definition given in Appendix A.1 for $s > \sigma_p$. The next proposition will enable us to give an analogue of Theorem 1 for distributional Besov spaces.

Proposition 8 *(See Kerkyacharian and Picard (2004)). Let $(\psi_0, \psi) \in \mathcal{W}^r$ for given $r \geq 1$. Then there exists a function θ in $\mathcal{C}^r$, whose support is included in the support of ψ, such that $\psi(x) = \theta(x) - \theta(x - 1/2)$.*

For $f \in \tilde{\mathcal{B}}^s_{p,q}$ and g a compactly supported function in $\mathcal{C}^r$ with $r > s - \sigma_p$, we classically set

$$\langle f_a, g \rangle = \langle f, g_{-a} \rangle, \qquad \langle f^\lambda, g \rangle = |\lambda| \langle f, g^{1/\lambda} \rangle \quad \text{and} \quad \langle f^\lambda_a, g \rangle = \langle (f^\lambda)_a, g \rangle,$$

with $g_a(x) = g(x - a)$ and $g^\lambda(x) = g(x/\lambda)$. Let $(\psi_0, \psi) \in \mathcal{W}^r$ and θ be the function associated to ψ by Proposition 8. For $j \in \mathbb{N}$, we define $\tilde{V}^p_{j,\psi}(f)$ by

$$\tilde{V}^p_{0,\psi}(f) = \sum_k |\langle f, \psi_{0k} \rangle|^p \quad \text{and} \quad \tilde{V}^p_{j+1,\psi}(f) = 2^{-p} \sum_k |\langle f^{2^j}_{-k} - f^{2^j}_{-(k+1/2)}, \theta \rangle|^p.$$

We now deduce the following proposition.[4]

Theorem 3. *Let f be a tempered distribution such that $f \in \tilde{\mathcal{B}}^s_{p,q}$ for given $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. Let $(\psi_0, \psi) \in \mathcal{W}^r$, with $r > r(s, p)$. Then*

$$\|f\|_{\tilde{\mathcal{B}}^s_{p,q}} = \left(\sum_{j \geq 0} 2^{jq(s-1/p)} \{\tilde{V}^p_{j,\psi}(f)\}^{q/p} \right)^{1/q}.$$

From Theorem 3, obvious computations lead to Theorem 2.

4.3. *Proof of* Proposition 2

We write η for a standard Gaussian variable. Let $2 \leq p < \infty$. We first prove the following almost sure equality which slightly extends a result from Marcus and Rosen (1992):

$$\lim_{j \to \infty} 2^{j(ps-1)} V^p_j(X) = \alpha^p \mathbb{E}[|\eta|^p].$$

Then, we obtain the proposition applying Theorem 1. We adapt here the method used by Marcus and Rosen (1992). Consider our process $\{X_t, t \in [0, 1]\}$. We will write here V^p_j for $V^p_j(X)$. Let q be the conjugate exponent of p. Let S^j denote the set of all sequences of cardinal 2^j of the unit ball of l_q and U^j be a countable dense subset in S^j. For $(u^j = \{u^j_k, k = 1, \ldots, 2^j\}) \in U^j$, we set

$$H(u^j) = 2^{j(s-1/p)} \sum_{k=1}^{2^j} u^j_k (X_{(k+1)/2^j} - X_{k/2^j}).$$

Let

$$Z_j = 2^{j(s-1/p)} \{V^p_j\}^{1/p} - \mathbb{E}[2^{j(s-1/p)} \{V^p_j\}^{1/p}].$$

Remarking that

$$\sup_{u_j \in U_j} H(u^j) = 2^{j(s-1/p)} \left(\sum_{k=1}^{2^j} |X_{(k+1)/2^j} - X_{k/2^j}|^p \right)^{1/p},$$

we can apply the Borel inequality and we obtain that for $t > 0$

$$\mathbb{P}[|Z_j| > t] \leq 2e^{-t^2/(2v_j^2)},$$

[4] Note that Proposition 3 can be easily extended to distributional Besov spaces with functional smoothness using the results of Almeida (2005).

where $\nu_j^2 = \sup_{u^j \in U^j} \mathbb{E}[H(u^j)^2]$. As $p \geq 2$, by the Jensen inequality, we get

$$\nu_j^2 \leq 2^{2j(s-1/p)} \left(\mathbb{E}\left[\sum_{k=0}^{2^j-1} |X_{(k+1)/2^j} - X_{k/2^j}|^p \right] \right)^{2/p} \leq 2^{2js} (\mathbb{E}[|\eta|^p])^{2/p} \sigma^2(2^{-j}) \leq c,$$

with c a constant finite value. Hence for $t > 0$

$$\mathbb{P}[|Z_j| > t] \leq 2\mathrm{e}^{-t^2/(2c)}.$$

Let $M_j = \mathbb{E}[2^{j(s-1/p)}\{V_j^p\}^{1/p}]$, we have

$$M_j \leq c 2^{j(s-1/p)} \mathbb{E}[\{V_j^p\}^{1/p}] \leq c 2^{js} (\mathbb{E}[|\eta|^p])^{1/p} \sigma(2^{-j}) \leq c (\mathbb{E}[|\eta|^p])^{1/p}.$$

Choose some convergent subsequence $\{M_{j_i}\}_i$ of $\{M_j\}_j$ with limit $\bar{M}$. Then, using the Borel–Cantelli lemma, we get that a.s.

$$\lim_{i\to\infty} 2^{j_i(s-1/p)} \{V_{j_i}^p\}^{1/p} = \bar{M}.$$

Since $\mathbb{E}[2^{j(ps-1)} V_j^p] \leq c$, we can deduce that

$$\lim_{i\to\infty} \mathbb{E}[2^{j_i(ps-1)} V_{j_i}^p] = \bar{M}^p.$$

Now, it is clear that

$$\lim_{j\to\infty} \mathbb{E}[2^{j(ps-1)} V_j^p] = \alpha^p \mathbb{E}[|\eta|^p].$$

Hence, $\bar{M}^p = \alpha^p \mathbb{E}[|\eta|^p]$ and the bounded sequence (M_j) has a unique limit point $\bar{M}$.

Appendix

A.1. Besov spaces

For the definition of Besov spaces, we refer to Cohen (1999) and Triebel (1983). Let Δ_h^n be the operator defined by $\Delta_h^1 f(x) = f(x+h) - f(x)$ and $\Delta_h^n f(x) = \Delta_h^1(\Delta_h^{n-1}) f(x)$. The nth order L^p modulus of smoothness of f on $[0, 1]$ is

$$\omega_n(f, t)_p = \sup_{|h|\leq t} \|\Delta_h^n f\|_{L^p(\Omega_{h,n})},$$

where $\Omega_{h,n} = \{x \in [0, 1]; x + kh \in [0, 1], k = 0, \dots, n\}$. For $p, q \geq 1$, $s > 0$, the Besov space $\mathcal{B}_{p,q}^s([0, 1])$ consists of those functions $f \in L^p[0, 1]$ such that

$$\{2^{sj} \omega_n(f, 2^{-j})_p\}_{j\geq 0} \in l^q,$$

where $n \in \mathbb{N}$ such that $s < n$. It is a Banach space when equipped with the norm

$$\|f\|_{\mathcal{B}_{p,q}^s([0,1])} = \|f\|_{L^p} + \|\{2^{sj}\omega_n(f, 2^{-j})_p\}_{j\geq 0}\|_{l^q}.$$

For p or q less than 1 and $s > \max\{1/p - 1, 0\}$, the Besov space $\mathcal{B}_{p,q}^s([0, 1])$ can be defined the same way but is only a quasi-Banach space.

A.2. Besov spaces and Schauder basis

For $j \geq 0$, $k = 1, \dots, 2^j$, let $\chi_{jk} = 2^{j/2}(1_{[(k-1)/2^j,(2k-1)/2^{j+1}]} - 1_{[(2k-1)/2^{j+1},k/2^j]})$, where 1 is the indicator function. The Schauder basis is defined as follows:

$$\phi_0(t) = 1_{[0,1]}, \qquad \phi_1(t) = t 1_{[0,1]}, \qquad \phi_{jk}(t) = \int_0^t \chi_{jk}(s)\mathrm{d}s, \quad s \in [0, 1].$$

Any continuous function $f : [0, 1] \to \mathbb{R}$ can be expressed, for all $t \in [0, 1]$, as

$$f(t) = f_0\phi_0(t) + f_1\phi_1(t) + \sum_{j\geq 0} \sum_{k=1}^{2^j} f_{jk}\phi_{jk}(t),$$

where $f_0 = f(0)$, $f_1 = f(1) - f(0)$ and

$$f_{jk} = 2^{j/2}[-f(\{2k\}2^{-(j+1)}) + 2f(\{2k-1\}2^{-(j+1)}) - f(\{2k-2\}2^{-(j+1)})],$$

with convergence in uniform norm.

Proposition 9 (Ciesielski et al., 1993). *Let $0 < s < 1$, $1 \le p, q \le \infty$. If $s > 1/p$, then $\mathcal{B}^s_{p,q}([0,1])$ is a space of real continuous functions on [0, 1], isomorphic to a space of real sequences, with the following equivalence between the norms:*

$$\|f\|_{\mathcal{B}^s_{p,q}([0,1])} \sim \max\left\{|f_0|,\, |f_1|,\, \left(\sum_{j\geq 0} 2^{-jq(1/2-s+1/p)} \left(\sum_{k=1}^{2^j} |f_{jk}|^p\right)^{q/p}\right)^{1/q}\right\}.$$

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