Volatility is rough, isn't it?

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August 2022


2005
2010
2015
2020
Is it rough?

What does it mean to be rough?

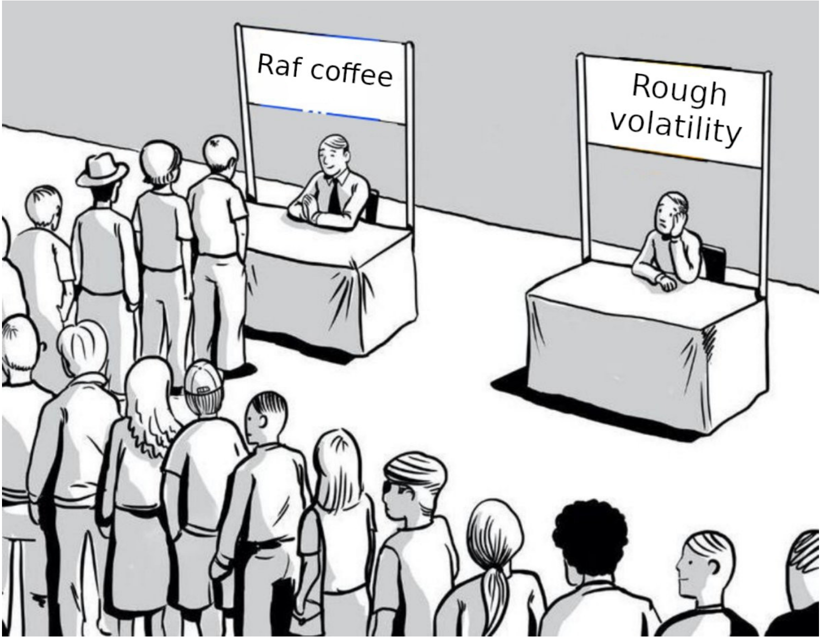
Definition

Fractional Brownian motion (fBM) Hurst parameter $H \in (0, 1)$ *is a process with the following properties:*

1. $W_0^H = 0,\ \mathbb{E}\left[W_t^H\right] = 0, \quad t \geq 0.$
2. $\mathrm{Law}(W_{t+s}^H - W_s^H) = \mathrm{Law}(W_t^H)$ — *stationary increments.*
3. W^H *is Gaussian and* $\mathbb{E}(W_t^H)^2 = |t|^{2H}, \quad 0 < H \leq 1.$
4. W^H *has continues trajectories.*

Definition

Rough volatility model is a model based on fBM with $H < \frac{1}{2}$.


Raf coffee
Rough volatility

Why volatility may be rough?

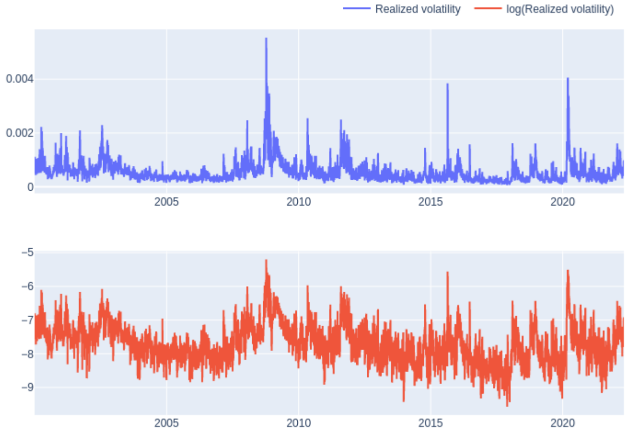


Fig. 1: S&P 500 realized volatility.

Why volatility may be rough?

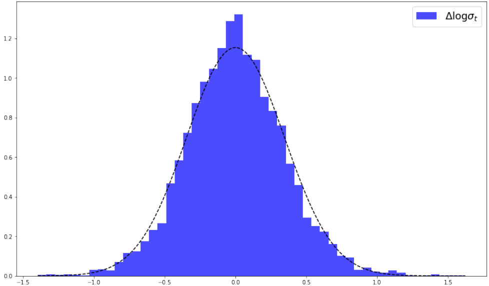
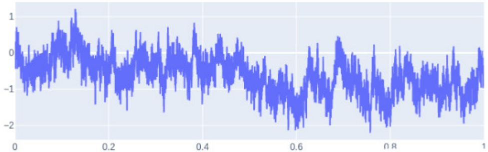
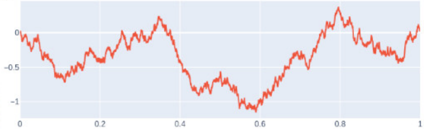


Fig. 2: S&P 500 log-volatility increments.

Conclusion

Gaussian process is an adequate model for log-volatility.





H = 0.15 H = 0.5

Why volatility may be rough? m-estimator[1]

fBM property:

$$\mathbb{E}|W^H_{t+\Delta} - W^H_t|^q = K_q \Delta^{qH}.$$

Sample estimate for q-th moment

$$m(q, \Delta) = \frac{1}{N} \sum_{k=1}^{N} \left| \log(\sigma_{k\Delta}) - \log(\sigma_{(k-1)\Delta}) \right|^q \approx K_q \Delta^{qH}$$

and double linear regression

$$\log m(q, \Delta) = b + \zeta_q \log \Delta, \quad \zeta_q = \hat{H} q.$$

give us *m-estimator* $\hat{H}$.

[1] Gatheral, J., Jaisson, T., & Rosenbaum, M. (2014). Volatility is rough. Available at SSRN.

Why volatility may be rough?

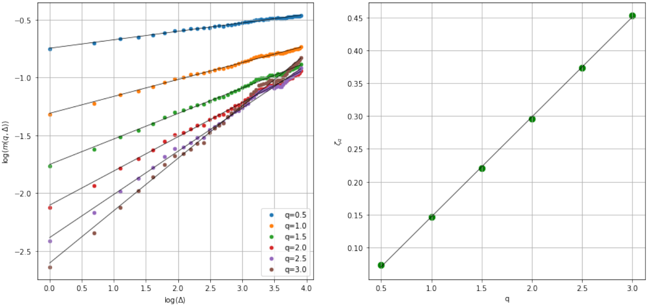


Fig. 3: m-estimator for market data. $\hat{H} = 0.15$

Hypothesis (gfBM)

Market volatility is a geometric fractional Brownian motion

$$\sigma_t = \exp\{\nu W_t^H\}, \quad H \approx 0.15, \quad \nu > 0.$$

Realized volatility

We assume the following price dynamics

$$dS_t = \sigma_t S_t \, dB_t, \quad S_0 = s_0.$$

Problem

Volatility σ_t is **not directly observable** in the market.

Daily *realized volatility* given m price process observations a day is an estimate

$$\hat{\sigma}_t = \sqrt{\sum_{j=1}^{m} \left| \log S_{t+\frac{j}{m}} - \log S_{t+\frac{j-1}{m}} \right|^2} \approx \sigma_t, \quad t = 1, \dots, N.$$

We also introduce the *microstructure noise*

$$\varepsilon_t = \log \sigma_t - \log \hat{\sigma}_t, \quad t = 1, \dots N.$$

Microstructure noise

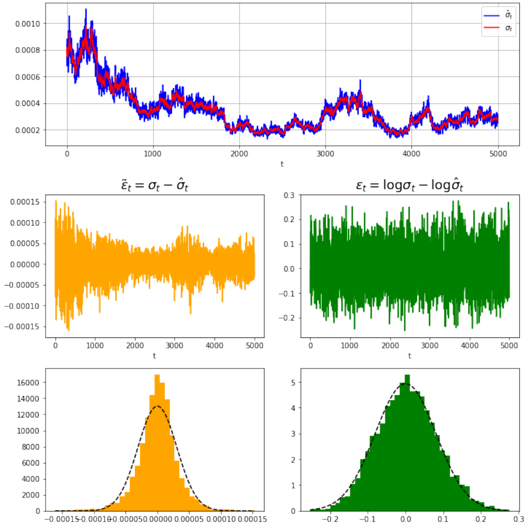


Fig. 4: Microstructure noise for gBM instantaneous volatility
$\sigma_t = \exp\{\nu W_t\}$ with $\nu = 0.02,\ m = 78,\ \sigma_0 = e^{-7.2}$.

Microstructure noise

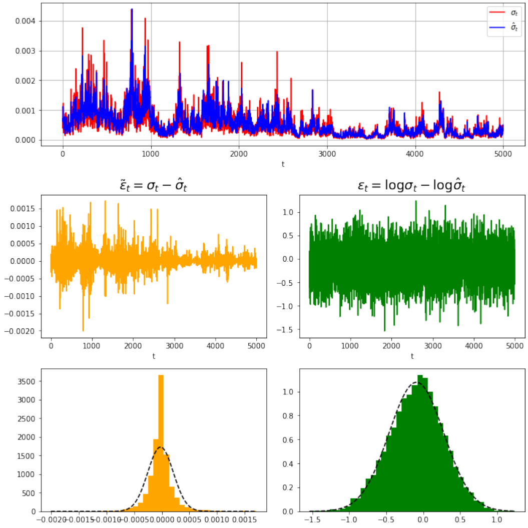


Fig. 5: Microstructure noise for gfBM instantaneous volatility $\sigma_t = \exp\{\nu W_t^H\}$ with $H = 0.1,\ \nu = 0.5,\ m = 78,\ \sigma_0 = e^{-7.2}$.

Microstructure noise

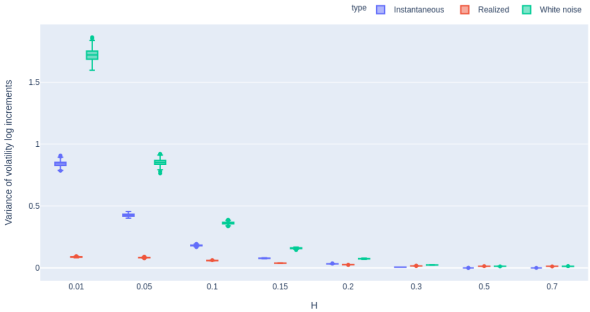


Fig. 6: Sample variances of $\Delta \log \sigma_t$ for instantaneous, realized and corrupted by white noise volatility processes.

Microstructure noise $\neq$ white noise

For rough trajectories the effect of microstructure noise is opposite to the one of white noise.

The notion of roughness

But what is *roughness*?

There exist many definitions of roughness, but we can divide them into two categories:

- Definitions based on the properties of continuous paths, such as fractional dimension, variation, Hölder regularity e.t.c. We will call it *roughness in a wide sense*. An important advantage of such definitions is model independence.
- However, one can use properties of fBM, such as self-similarity, implicitly assuming that it takes place for all rough processes. Such notion of roughness we will call *roughness in a narrow sense*.

OUOU process

Double Ornstein–Uhlenbeck process (OUOU) is a solution of

$$\begin{cases} dY_t = -\lambda_Y (Y_t - \mu)\, dt + \sigma_Y\, dB_t, \\ dX_t = -\lambda_X (X_t - Y_t)\, dt + \sigma_X\, dW_t. \end{cases}$$

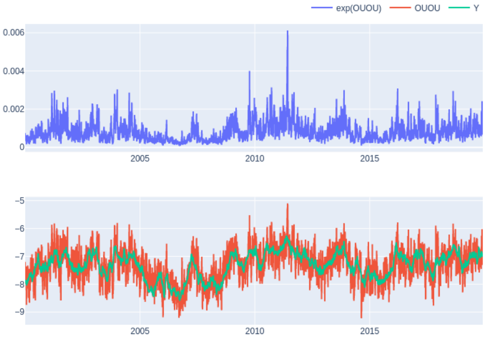


Fig. 7: OUOU volatility process.


fBM
Finally, a worthy opponent
fBM
Our battle will be legendary

Roughness estimators

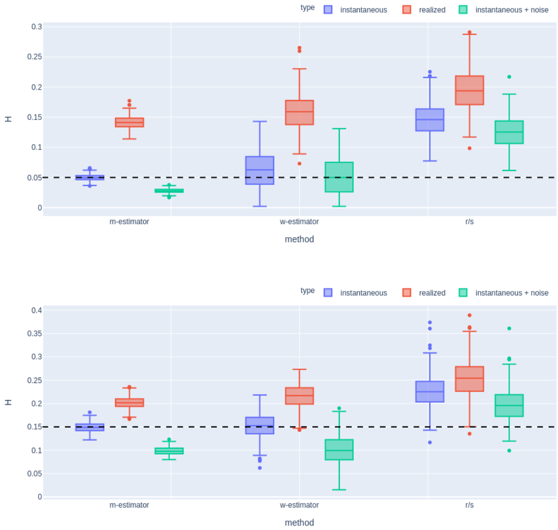


Fig. 8: fgBM with $H = 0.05$ and $H = 0.15$.

Roughness estimators

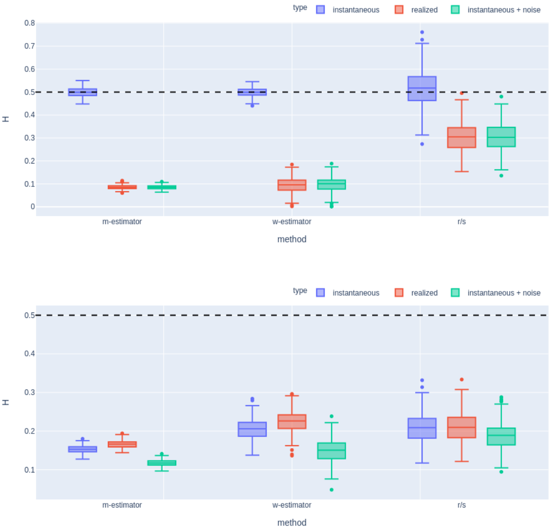


Fig. 9: fgBM with $H = 0.5$ and OUOU.

Market ACF

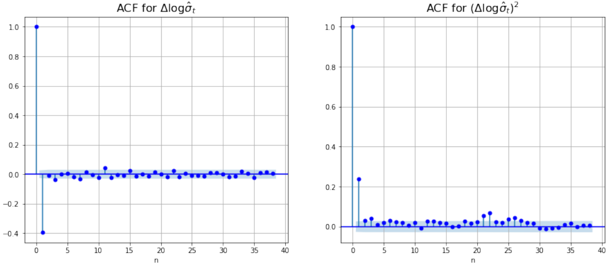


Fig. 10: ACF of $\Delta \log \hat{\sigma}_t$ and $(\Delta \log \hat{\sigma}_t)^2$ for market data.

What do we know about market volatility?

- The observed process is a realized volatility.
- The estimation of H is 0.152.
- Autocorrelation $\mathrm{Corr}((\Delta \log \hat{\sigma}_t)^2, (\Delta \log \hat{\sigma}_{t+1})^2) = 0.238$.
- The variance $\mathrm{Var}[\Delta \log \hat{\sigma}_t]$ is equal to 0.119.

Statistic

Given H we can estimate ν and sample the trajectory $\hat{\sigma}_t$. Then we can use m-estimator to obtain $\hat{H}$ and calculate the autocorrelation $\hat{\rho} = \mathrm{Corr}((\Delta \log \hat{\sigma}_t)^2, (\Delta \log \hat{\sigma}_{t+1})^2)$.

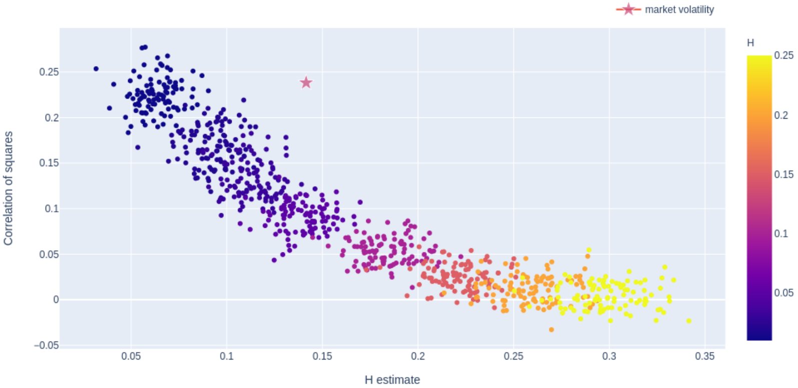


Fig. 11: The distribution of the statistic $(\hat{H}, \hat{\rho})$

Conclusions

1. Microstructure noise $\neq$ Gaussian white noise for rough gfBM ($H < 0.2$).
2. To claim that volatility is rough one should use notion of roughness in a wide sense and take in account microstructure noise.
3. The considered estimators fail to estimate roughness correctly for several reasons: microstructure noise, low sampling frequency, narrow sense notion of roughness.
4. Instantaneous volatility is not gfBM, but this model reproduces well some market volatility properties.

References

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