Representation of a fractional Brownian motion in terms of an infinite-dimensional Ornstein-Uhlenbeck process

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1. A fractional Brownian motion with Hurst parameter $H \in (0,1)$ is defined as a Gaussian process $B^H = (B_t^H)_{t\geqslant 0}$ starting from zero and having zero mean and covariance function

$$R(s,t) = \frac{1}{2}\bigl(t^{2H} + s^{2H} - |t-s|^{2H}\bigr), \qquad s,t \geqslant 0. \tag{1}$$

It is well known that, for $H \in (0,1/2) \cup (1/2,1)$, such a process B^H is neither a semimartingale nor a Markov process (see, for example, [1]). In spite of this, here it will be shown that B^H can be represented as a linear functional of an infinite-dimensional Markov process.

Let $\xi = (\xi_\beta)_{\beta>0}$ be a Gaussian process with zero mean and covariance function $R_\xi(\alpha,\beta) = (\alpha+\beta)^{-1}$, and let $B = (B_t)_{t\geqslant 0}$ be a standard Brownian motion independent of ξ. We construct from ξ and B the family of processes $\{Z^\beta\}_{\beta>0}$, where $Z^\beta = (Z_t^\beta)_{t\geqslant 0}$ is an Ornstein–Uhlenbeck process, which is a solution of the stochastic differential equation

$$dZ_t^\beta = -\beta Z_t^\beta\,dt + dB_t, \qquad Z_0^\beta = \xi_\beta.$$

Theorem 1. *Let $H \in (0,1/2)\cup(1/2,1)$ and let $\varepsilon > 0$ be arbitrary. Then the process $\overline{B}^{H,\varepsilon} = \bigl(\overline{B}_t^{H,\varepsilon}\bigr)_{t\geqslant 0}$ defined by*

$$\overline{B}_t^{H,\varepsilon} = c_H \int_0^\infty \beta^{-1/2-H}(Z_t^\beta - \xi_\beta - e^{-\beta\varepsilon_0}B_t)\,d\beta + \varepsilon B_t, \tag{2}$$

where

$$c_H = \frac{[\Gamma(2H+1)\sin(\pi H)]^{1/2}}{\mathrm{B}(1/2+H, 1/2-H)}, \qquad \varepsilon_0 = \biggl(\frac{\varepsilon}{c_H\Gamma(1/2-H)}\biggr)^{1/(H-1/2)},$$

is a fractional Brownian motion with Hurst parameter H.

Corollary 1. *The process $\overline{B}^H = \bigl(\overline{B}_t^H\bigr)_{t\geqslant 0}$ defined by*

$$\overline{B}_t^H = \begin{cases} c_H \displaystyle\int_0^\infty \beta^{-1/2-H}(Z_t^\beta - \xi_\beta)\,d\beta & \text{for } H \in \Bigl(0,\dfrac{1}{2}\Bigr), \\ c_H \displaystyle\int_0^\infty \beta^{-1/2-H}(Z_t^\beta - \xi_\beta - B_t)\,d\beta & \text{for } H \in \Bigl(\dfrac{1}{2},1\Bigr) \end{cases} \tag{3}$$

is also a fractional Brownian motion with Hurst parameter H.

The representations obtained here make B^H amenable to some methods of the theory of Markov processes. In particular, the general optimal stopping theory (see [2]) for the family of Markov processes $\{Z^\beta\}_{\beta>0}$ can be useful in obtaining inequalities involving B^H.

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Theorem 2. *Let B^H be a fractional Brownian motion with Hurst parameter H. Then*

$$-k_H(\mathbf{E}\tau)^H \leqslant \mathbf{E}B^H_\tau \leqslant k_H(\mathbf{E}\tau)^H \tag{4}$$

for all stopping times τ of the process B^H. Also,

$$k_H \leqslant c_H \frac{(2\pi)^{-H/2}}{H\sqrt{2}} \int_{\mathbb{R}} \Phi(u)\biggl[\int_{-\infty}^{u} \Phi^2(v)e^{v^2/2}\,dv\biggr]^{-H} du.$$

We note that the inequality (4) for $H \in (1/2, 1)$ follows from the results of [3]; however, for $H \in (0, 1/2)$ our bound for $\mathbf{E}B^H_\tau$ is a new result.

2. To show that the processes $\overline{B}^{H,\varepsilon}$ and $\overline{B}^H$ are fractional Brownian motions, it suffices to verify that their correlation functions agree with (1). We explain how (2) and (3) were derived. For an arbitrary $\varepsilon > 0$ the Mandelbrot–van Ness representation [4] can be written in the form

$$\begin{aligned} B^H_t &= c_H \int_{-\infty}^{0} \biggl[\int_0^\infty (e^{-\beta(t-s)} - e^{\beta s})\beta^{-1/2-H}\,d\beta\biggr]\,dB_s \\ &\quad + c_H \int_0^t \biggl[\int_0^\infty (e^{-\beta(t-s)} - e^{-\beta\varepsilon_0})\beta^{-1/2-H}\,d\beta\biggr]\,dB_s + \varepsilon B_t. \end{aligned} \tag{5}$$

Applying Fubini's theorem for stochastic integrals (see, for example, [5]) to change the order of integration, we easily find exactly (2).

As $\varepsilon \downarrow 0$, the representation (2) becomes (3). For $H \in (0, 1/2)$ we can instead immediately put $\varepsilon = 0$ in (5).

3. To prove (4) we shall invoke (2). Let $(\mathscr{F}_t)_{t\geqslant 0}$ and $(\mathscr{F}^B_t)_{t\geqslant 0}$ be the natural filtrations of the processes $\overline{B}^{H,\varepsilon}$ and B, and let $\mathfrak{M}$ and $\mathfrak{M}^B$ be the sets of stopping times τ with respect to these filtrations for which $\mathbf{E}\tau < \infty$. Clearly, $\mathscr{F}_t = \sigma(\xi) \vee \mathscr{F}^B_t$. Similarly to [6], we have

$$W^\beta_*(c, z) = \sup_{\tau \in \mathfrak{M}^B} \mathbf{E}(Z^\beta_\tau - c\tau \mid \xi_\beta = z) = \begin{cases} z_* - 2c\displaystyle\int_z^{z^*} e^{\beta x^2} \int_{-\infty}^{x} e^{-\beta t^2}\,dt\,dx & \text{for } z < z_*, \\ z & \text{for } z \geqslant z_*, \end{cases}$$

where z_* is the unique solution of the equation $2ce^{\beta z^2}\displaystyle\int_{-\infty}^{z} e^{-\beta t^2}\,dt = 1$. Hence, for any $\tau \in \mathfrak{M}$,

$$\mathbf{E}(Z^\beta_\tau - c\tau) = \mathbf{E}\mathbf{E}(Z^\beta_\tau - c\tau \mid \xi) \leqslant \mathbf{E}W^\beta_*(c, \xi_\beta), \qquad \mathbf{E}Z^\beta_\tau \leqslant \inf_{c>0}\bigl[\mathbf{E}W^\beta_*(c, \xi_\beta) + c\mathbf{E}\tau\bigr].$$

Performing the necessary computations, we obtain

$$\mathbf{E}Z^\beta_\tau \leqslant \frac{1}{\sqrt{2\beta}} \int_{-\infty}^{A} \Phi(\alpha)\,d\alpha, \tag{6}$$

where A is the unique solution of the equation $\sqrt{2\pi}\displaystyle\int_{-\infty}^{A} \Phi^2(\gamma)e^{\gamma^2/2}\,d\gamma = \beta\,\mathbf{E}\tau$. Since $\mathbf{E}B_\tau = 0$ for any $\tau \in \mathfrak{M}$, to prove (4) it suffices to integrate both sides of (6) with respect to $\beta^{-1/2-H}\,d\beta$ and apply Fubini's theorem.

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