On estimation of the extended Orey index for Gaussian processes

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Abstract

Orey suggested the definition of an index for Gaussian process with stationary increments which determines various properties of the sample paths of this process. We provide an extension of the definition of the Orey index towards a second order stochastic process which may not have stationary increments and estimate the Orey index towards a Gaussian process from discrete observations of its sample paths.

Keywords: Gaussian process, Hurst index, fractional Brownian motion, incremental variance function

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1 Introduction

Fractional Brownian motion (fBm) is a popular model in financial mathematics, economics and natural sciences. It is well known that the fBm B^H is the only continuous Gaussian process which is self-similar with stationary increments and that depends only on index 0 < H < 1. Moreover, a fBm with Hurst index H is Hölder up to order H.

For a real zero-mean Gaussian process with stationary increments, Orey suggested the following definition of the index.

Definition 1 (see [16], [13]) Let X be a real-valued zero-mean Gaussian stochastic process with stationary increments and continuous in mean-square sense. Let σ_X be the incremental variance of X given by $\sigma_X^2(h) = \mathbf{E}[X(t+h) - X(t)]^2$ for $t,h \geqslant 0$. Define

$$\widehat{\beta}_* := \inf \left\{ \beta > 0 \colon \lim_{h \downarrow 0} \frac{h^{\beta}}{\sigma_X(h)} = 0 \right\} = \limsup_{h \downarrow 0} \frac{\ln \sigma_X(h)}{\ln h}$$
 (1)

and

$$\widehat{\beta}^* := \sup \left\{ \beta > 0 \colon \lim_{h \downarrow 0} \frac{h^{\beta}}{\sigma_X(h)} = +\infty \right\} = \liminf_{h \downarrow 0} \frac{\ln \sigma_X(h)}{\ln h}. \tag{2}$$

If $\widehat{\beta}_* = \widehat{\beta}^*$ then X has the Orey index $\widehat{\beta}_* = \widehat{\beta}^* = \beta_X$.

If a Gaussian process with stationary increments has Orey index then almost all sample paths satisfy a Hölder condition of order γ for each $\gamma \in (0, \beta_X)$ (see Section 9.4 of Cramer and Leadbetter [5]). For fBm B^H with the Hurst index 0 < H < 1 the Orey index $\beta_{B^H} = H$. So we have a class of Gaussian processes with stationary increments depending on the Orey index β_X .

Recently there have been two extensions of fBm which preserve many properties of fBm, but have no stationary increments except for particular parameter values. One of them is a so called sub-fractional Brownian motion (sfBm) (see [1]) and another one is a bifractional Brownian motion (bifBm) (see [9], [17]). Thus it is very natural to extend the definition of

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the Orey index for Gaussian processes so that it would be possible to consider processes which may not have stationary increments and have the Orey index.

We will provide such extension of the Orey index. As will be shown later, processes sfBm and bifBm satisfy this extended definition of the Orey index and are Hölder up to the Orey index. Moreover, for fBm, sfBm, and bifBm, the Orey index coincides with their self-similarity parameter. Therefore it is enough to construct and consider the asymptotic behavior of an estimate of the Orey index instead of estimating parameters of each of the processes under consideration.

Many authors have already considered the asymptotic behavior of the first- and secondorder quadratic variations of Gaussian processes (see [2], [3], [8], [10], [11], [12], [14]). The conditions in those papers are expressed in terms of covariance of a Gaussian process and depend on some parameter $\gamma \in (0, 2)$. If a Gaussian process has the Orey index then conditions on a covariance function may be expressed in its terms. As it will be shown below, the Orey index can be obtained for some well-known Gaussian processes. Moreover, in order to consider stochastic differential equations (SDE) driven by processes with a bounded p-variation, we should know when the Riemann-Stieltjes (RS) integral is defined. For Gaussian processes the Orey index helps to obtain these conditions.

The purpose of this paper is to give an extension of the definition of the Orey index for the second order stochastic processes which may not have stationary increments and to estimate the Orey index for a Gaussian process from discrete observations of its sample paths.

Norvaiša [15] extends the definition of the Orey index for the second order stochastic processes which may not have stationary increments. He showed that sfBm and bifBm satisfy this extended definition of the Orey index. In this paper we give a different extension of the definition of the Orey index as it is more convenient for our purposes. Moreover, it is given in an explicit form.

The paper is organized in the following way. Section 2 contains the definition of the Orey index for a second order stochastic process. The conditions when the second order stochastic process has the Orey index are also given. Moreover, for some well-known Gaussian processes which do not have stationary increments the Orey index is obtained. Section 3 contains the results on an almost sure asymptotic behavior of the second-order quadratic variations of a Gaussian process. We also verify the obtained conditions for some well-known Gaussian processes.

2 Generalized Orey index for the second order stochastic processes

Let $X = \{X(t): t \in [0, T]\}$ be a second order stochastic process with the incremental variance function σ_X^2 defined on $[0, T]^2 := [0, T] \times [0, T]$ with values

$$\sigma_X^2(s,t) := \mathbf{E}[X(t) - X(s)]^2, \quad (s,t) \in [0,T]^2.$$

Denote by Ψ a class of continuous functions $\varphi \colon (0,T] \to [0,\infty)$ such that $\lim_{h\downarrow 0} \varphi(h) = 0$ and $\lim_{h\downarrow 0} [h\cdot L^3(h)] = 0$, where $L(h) = \varphi(h)/h \to \infty$, $h\downarrow 0$. For example, we can take $\varphi(h) = h\cdot |\ln h|^{\alpha}$ or $\varphi(h) = h^{1-\beta}$, where $\alpha > 0$, $0 < \beta < 1/3$. Set

$$\gamma_* := \inf \left\{ \gamma > 0 \colon \lim_{h \downarrow 0} \sup_{\varphi(h) \leqslant s \leqslant T - h} \frac{h^{\gamma}}{\sigma_X(s, s + h)} = 0 \right\}, \tag{3}$$

$$\widetilde{\gamma}_* := \inf \left\{ \gamma > 0 : \lim_{h \downarrow 0} \frac{h^{\gamma}}{\sigma_X(0, h)} = 0 \right\}$$
 (4)

and

$$\gamma^* := \sup \left\{ \gamma > 0 : \lim_{h \downarrow 0} \inf_{\varphi(h) \leqslant s \leqslant T - h} \frac{h^{\gamma}}{\sigma_X(s, s + h)} = +\infty \right\}, \tag{5}$$

$$\widetilde{\gamma}^* := \sup \left\{ \gamma > 0 : \lim_{h \downarrow 0} \frac{h^{\gamma}}{\sigma_X(0, h)} = +\infty \right\},$$
(6)

where $\varphi \in \Psi$. Note that $0 \leqslant \widetilde{\gamma}^* \leqslant \widetilde{\gamma}_* \leqslant +\infty$ and $0 \leqslant \gamma^* \leqslant \gamma_* \leqslant +\infty$.

We give the following extension of the Orey index.

Definition 2 Let $X = \{X(t): t \in [0,T]\}$ be a second order stochastic process with the incremental variance function σ_X^2 such that $\sup_{0 \le s \le T-h} \sigma_X(s,s+h) \to 0$ as $h \to 0$. If $\gamma_* = \widetilde{\gamma}_* = \gamma^* = \widetilde{\gamma}^*$ for any function $\varphi \in \Psi$, then we say that the process X has the Orey index $\gamma_X = \gamma_* = \widetilde{\gamma}_* = \gamma^* = \widetilde{\gamma}^*$.

Remark 3 If we consider a real-valued mean zero Gaussian stochastic process with stationary increments and continuous in mean square then the Orey indices in Definition 1 and Definition 2 coincide.

Let us introduce the notions

$$\widehat{\gamma}_* := \limsup_{h \downarrow 0} \sup_{\varphi(h) \leqslant s \leqslant T - h} \frac{\ln \sigma_X(s, s + h)}{\ln h} \quad \text{and} \quad \overline{\gamma}_* := \limsup_{h \downarrow 0} \frac{\ln \sigma_X(0, h)}{\ln h}, \tag{7}$$

$$\widehat{\gamma}^* := \liminf_{h \downarrow 0} \inf_{\varphi(h) \leqslant s \leqslant T - h} \frac{\ln \sigma_X(s, s + h)}{\ln h} \quad \text{and} \quad \overline{\gamma}^* := \liminf_{h \downarrow 0} \frac{\ln \sigma_X(0, h)}{\ln h}. \tag{8}$$

It follows from Remark 3 and (1), (2) that $\widetilde{\gamma}_* = \overline{\gamma}_*$ and $\widetilde{\gamma}^* = \overline{\gamma}^*$. Now we compare the values of $\widehat{\gamma}^*$ and $\widehat{\gamma}_*$ with γ^* and γ_* , respectively, for a second order stochastic process X.

Lemma 4 Let $X = \{X(t): t \in [0,T]\}$ be a second order stochastic process with the incremental variance function σ_X^2 such that

$$\sup_{\varphi(h)\leqslant s\leqslant T-h} \sigma_X(s,s+h) \longrightarrow 0 \quad as \ h\downarrow 0.$$
 (9)

If $0 < \widetilde{\gamma}^* \leqslant \widetilde{\gamma}_* < +\infty$, then $\widehat{\gamma}^* = \gamma^*$ and $\widehat{\gamma}_* = \gamma_*$.

Proof. The proof of the lemma follows the outlines of calculation of limits of the logarithmic ratios (see Annex A.4 in [18]). For completeness we give this proof in Appendix.

For $(s, t) \in [0, T]^2$, $s \neq t$, set

$$b(s,t) := \frac{\sigma_X(s,t)}{\kappa |t-s|^{\gamma}} - 1.$$

Assume that for some $\gamma \in (0,1)$ the second order stochastic process X satisfies the conditions:

- (C1) $\sigma_X(0,\delta) \simeq \delta^{\gamma}$, i.e., $\sigma_X(0,\delta)$ and δ^{γ} are of the same order as $\delta \downarrow 0$;
- (C2) there exists a constant $\kappa > 0$ such that

$$\Lambda(\delta) := \sup_{\varphi(\delta) \leqslant t \leqslant T - \delta} \sup_{0 < h \leqslant \delta} \left| b(t, t + h) \right| \longrightarrow 0 \qquad \text{as } \delta \downarrow 0$$

for every function $\varphi \in \Psi$.

For $(s,t) \in [0,T]^2$, $s \neq t$, set

$$c(s,t) := \frac{\sigma_X^2(s,t)}{\kappa^2 |t-s|^{2\gamma}} - 1. \tag{10}$$

It follows from (C1) and (C2) that for any $\varphi \in \Psi$

$$\sup_{0 \leqslant s \leqslant T-h} \sigma_X^2(s, s+h) \leqslant \sup_{0 \leqslant s \leqslant \varphi(h)} \sigma_X^2(s, s+h) + \sup_{\varphi(h) \leqslant s \leqslant T-h} \sigma^2(s, s+h)
\leqslant 4 \sup_{0 \leqslant \delta \leqslant \varphi(h)+h} \sigma_X^2(0, \delta) + \kappa^2 h^{2\gamma} \Big(\sup_{\varphi(h) \leqslant s \leqslant T-h} |c(s, s+h)| + 1 \Big)
\leqslant O\Big((\varphi(h))^{2\gamma} \Big) + \kappa^2 h^{2\gamma} \big[\Lambda^2(h) + 2\Lambda(h) + 1 \big] \longrightarrow 0 \text{ as } h \downarrow 0.$$
(11)

Thus the process X is continuous in quadratic mean for all $s \in [0, T - h]$.

Theorem 5 Assume that for some constant $\gamma \in (0,1)$ the second order stochastic process X satisfies conditions (C1) and (C2). Then it has the Orey index that equals γ .

Proof. Due to Lemma 4 it is sufficient to show that $\widehat{\gamma}_* = \widehat{\gamma}^* = \gamma$ and $\overline{\gamma}_* = \overline{\gamma}^* = \gamma$. We first note that condition (C1) implies $\overline{\gamma}_* = \overline{\gamma}^* = \gamma$. Since $\sigma_X(0,\delta) \simeq \delta^{\gamma}$, then there

exist constants $0 < m < M < \infty$ and δ_0 such that

$$m < \frac{\sigma_X(0,\delta)}{\delta^{\gamma}} < M$$
 for all $\delta < \delta_0$.

Thus

$$\gamma + \frac{\ln M}{\ln \delta} = \frac{\ln(M\delta^{\gamma})}{\ln \delta} < \frac{\ln \sigma_X(0,\delta)}{\ln \delta} < \frac{\ln(m\delta^{\gamma})}{\ln \delta} = \gamma + \frac{\ln m}{\ln \delta} \quad \text{as } \delta < 1,$$

and we obtain equality $\overline{\gamma}_* = \overline{\gamma}^* = \gamma$. It remains to prove that $\widehat{\gamma}^* = \widehat{\gamma}_*$. It follows from conditions (C1) and (C2) that there exists δ_0 such that for $\delta \leqslant \delta_0 < 1$ inequalities $\sigma_X(s,s+\delta) \leqslant 1/2$ for all $0 \leqslant s \leqslant T - \delta_0$ and $\Lambda(\delta) < 1/2$ hold. In what follows we suppose that inequalities hold for $\delta \leq \delta_0 < 1$.

We fix some function $\varphi \in \Psi$. Assume that $-1/2 < b(s_0, s_0 + \delta_0) \leq 0$ for some fixed $s_0 \in [\varphi(\delta_0), T - \delta_0]$. Furthermore, it is known that $-2x \leqslant \ln(1-x) \leqslant -x$ for $0 \leqslant x \leqslant 1/2$. From this inequality we get

$$\ln \sigma_X(s_0, s_0 + \delta_0) = \ln(\kappa \delta_0^{\gamma}) + \ln(1 + b(s_0, s_0 + \delta_0)) = \ln(\kappa \delta_0^{\gamma}) + \ln(1 - (-b(s_0, s_0 + \delta_0)))$$

$$\leq \ln(\kappa \delta_0^{\gamma}) + b(s_0, s_0 + \delta_0) \leq \ln(\kappa \delta_0^{\gamma}) + \Lambda(\delta_0)$$

and

$$\ln \sigma_X(s_0, s_0 + \delta_0) \ge \ln(\kappa \delta_0^{\gamma}) + 2b(s_0, s_0 + \delta_0) = \ln(\kappa \delta_0^{\gamma}) - 2|b(s_0, s_0 + \delta_0)|$$

$$\ge \ln(\kappa \delta_0^{\gamma}) - 2\Lambda(\delta_0).$$

It is known that $|\ln(1+x)| \le x$ for $x \ge 0$. Assume that $0 \le b(s_0, s_0 + \delta_0) < 1/2$ for some fixed $s_0 \in [\varphi(\delta_0), T - \delta_0]$, then

$$\ln \sigma_X(s_0, s_0 + \delta_0) = \ln(\kappa \delta_0^{\gamma}) + \ln(1 + b(s_0, s_0 + \delta_0)) \leqslant \ln(\kappa \delta_0^{\gamma}) + b(s_0, s_0 + \delta_0)$$

$$\leqslant \ln(\kappa \delta_0^{\gamma}) + \Lambda(\delta_0)$$

and

$$\ln \sigma_X(s_0, s_0 + \delta_0) = \ln(\kappa \delta_0^{\gamma}) + \ln(1 + b(s_0, s_0 + \delta_0)) \geqslant \ln(\kappa \delta_0^{\gamma}) - 2|b(s_0, s_0 + \delta_0)|$$

$$\geqslant \ln(\kappa \delta_0^{\gamma}) - 2\Lambda(\delta_0).$$

Thus for every $s \in [\varphi(\delta_0), T - \delta_0]$ we obtain

$$\ln(\kappa \delta_0^{\gamma}) - 2\Lambda(\delta_0) \leq \ln \sigma_X(s, s + \delta_0) \leq \ln(\kappa \delta_0^{\gamma}) + \Lambda(\delta_0).$$

Consequently,

$$\gamma + \frac{\ln \kappa}{\ln \delta_0} - \frac{\Lambda(\delta_0)}{|\ln \delta_0|} \leqslant \inf_{\varphi(\delta_0) \leqslant s \leqslant T - \delta_0} \frac{\ln \sigma_X(s, s + \delta_0)}{\ln \delta_0} \leqslant \sup_{\varphi(\delta_0) \leqslant s \leqslant T - \delta_0} \frac{\ln \sigma_X(s, s + \delta_0)}{\ln \delta_0}$$
$$\leqslant \gamma + \frac{\ln \kappa}{\ln \delta_0} + 2 \frac{\Lambda(\delta_0)}{|\ln \delta_0|}.$$

and both sides of the above inequality approach γ as $\delta_0 \to 0$. Thus $\widehat{\gamma}_* = \widehat{\gamma}^* = \gamma$.

2.1Subfractional Brownian motion

We shall prove that sfBm satisfies conditions (C1) and (C2).

Definition 6 ([1]) A sub-fractional Brownian motion with index $H, H \in (0,1)$, is a zero-mean Gaussian stochastic process $S^H = (S_t^H, t \ge 0)$ with covariance function

$$G_H(s,t) := s^{2H} + t^{2H} - \frac{1}{2} [(s+t)^{2H} + |s-t|^{2H}].$$

The incremental variance function of sfBm is of the following form

$$\sigma_{S^H}^2(s,t) = \mathbf{E}|S_t^H - S_s^H|^2 = |t - s|^{2H} + (s + t)^{2H} - 2^{2H-1}(t^{2H} + s^{2H}). \tag{12}$$

Since for any $0 \le s \le t \le T$ inequalities (see [1])

$$(t-s)^{2H} \le \sigma_{SH}^2(s,t) \le (2-2^{2H-1})(t-s)^{2H}, \quad \text{if} \quad 0 < H < 1/2,$$
 (13)

$$(2-2^{2H-1})(t-s)^{2H} \le \sigma_{SH}^2(s,t) \le (t-s)^{2H}, \quad \text{if} \quad 1/2 < H < 1$$
 (14)

hold, then condition (C1) is satisfied.

We get from (12) that

$$\sigma_{SH}^2(s, s+h) = h^{2H} + f_s(h),$$

where

$$f_s(h) := (2s+h)^{2H} - 2^{2H-1} [s^{2H} + (s+h)^{2H}].$$

Note that

$$f_s(0) = f_s'(0) = 0.$$

By Taylor formula we obtain

$$f_s(h) = f_s(0) + f_s'(0)h + \int_0^h f_s''(x)(h-x) dx = \int_0^h f_s''(x)(h-x) dx$$
$$= 2H(2H-1) \int_0^h \left[(2s+x)^{2H-2} - 2^{2H-1}(s+x)^{2H-2} \right] (h-x) dx.$$

From inequality

$$[2^{2H-1}(s+x)^{2H-2} - (2s+x)^{2H-2}] = \frac{1}{(s+x)^{2-2H}} \left[2^{2H-1} - \left(\frac{s+x}{2s+x}\right)^{2-2H} \right]$$

$$= \frac{1}{(s+x)^{2-2H}} \left[2^{2H-1} - \left(1 - \frac{s}{2s+x}\right)^{2-2H} \right] \leqslant \frac{1}{(s+x)^{2-2H}} \left[2^{2H-1} - 2^{-1} \right],$$

it follows that for s > 0

$$|f_s(h)| \le (2^{2H} - 1) \int_0^h \frac{h - x}{(s+x)^{2-2H}} dx \le \frac{1}{2} (2^{2H} - 1) s^{2H-2} h^2$$

and

$$\sup_{\varphi(\delta)\leqslant s\leqslant T-\delta} \sup_{0< h\leqslant \delta} \left| \frac{\sigma_{S^H}^2(s,s+h)}{h^{2H}} - 1 \right| = \sup_{\varphi(\delta)\leqslant s\leqslant T-\delta} \sup_{0< h\leqslant \delta} \frac{|f_s(h)|}{h^{2H}}$$

$$\leqslant \sup_{\varphi(\delta)\leqslant s\leqslant T-\delta} \frac{2^{2H-1}\delta^{2-2H}}{s^{2-2H}} \leqslant \frac{2^{2H-1}}{(L(\delta))^{2-2H}}$$

for every $\varphi \in \Psi$, where $L(h) = \varphi(h)/h$. So we get condition (C2) with $\kappa = 1$.

Remark 7 The function $\varphi(\delta)$ could not be replaced by δ or 0 in condition (C2). Indeed, e. g., H > 1/2. Then

$$\begin{split} \sup_{0 \leqslant s \leqslant T-\delta} \sup_{0 \leqslant h \leqslant \delta} |h^{-2H} f_s(h)| &\geqslant \sup_{\delta \leqslant s \leqslant T-\delta} \sup_{0 \leqslant h \leqslant \delta} |h^{-2H} f_s(h)| \\ &= 2H(2H-1) \sup_{\delta \leqslant s \leqslant T-\delta} \sup_{0 \leqslant h \leqslant \delta} \int_0^h \left[\frac{2^{2H-1}}{h^{2H} (s+x)^{2-2H}} - \frac{1}{h^{2H} (2s+x)^{2-2H}} \right] (h-x) \, dx \\ &\geqslant 2H(2H-1) \sup_{\delta \leqslant s \leqslant T-\delta} \sup_{0 \leqslant h \leqslant \delta} \int_0^h \frac{2^{2H-1}-1}{h^{2H} (2s+x)^{2-2H}} \, (h-x) \, dx \\ &\geqslant H(2H-1) \sup_{\delta \leqslant s \leqslant T-\delta} \sup_{0 \leqslant h \leqslant \delta} \frac{(2^{2H-1}-1)h^{2-2H}}{(2s+h)^{2-2H}} \\ &= H(2H-1)(2^{2H-1}-1) \sup_{\delta \leqslant s \leqslant T-\delta} \frac{\delta^{2-2H}}{(2s+\delta)^{2-2H}} \\ &= H(2H-1)(2^{2H-1}-1)3^{2H-2}. \end{split}$$

2.2 Bifractional Brownian motion

Definition 8 ([9]) A bifractional Brownian motion $B^{HK} = (B_t^{HK}, t \ge 0)$ with parameters $H \in (0, 1)$ and $K \in (0, 1]$ is a centered Gaussian process with covariance function

$$R_{HK}(t,s) = 2^{-K} ((t^{2H} + s^{2H})^K - |t - s|^{2HK}), \quad s, t \ge 0$$

The incremental variance function of bifBm is of the following form

$$\sigma_{B^{H,K}}^{2}(s,t) = \mathbf{E}|B_{t}^{H,K} - B_{s}^{H,K}|^{2} = 2^{1-K} \left[|t-s|^{2HK} - (t^{2H} + s^{2H})^{K} \right] + t^{2HK} + s^{2HK}.$$

Let $H \in (0,1)$ and $K \in (0,1]$. Then

$$2^{-K}|t-s|^{2HK} \leqslant \sigma_{B^{H,K}}^{2}(s,t) \leqslant 2^{1-K}|t-s|^{2HK}$$
 (15)

for all $s,t \in [0,\infty)$ (see [9]). Thus condition (C1) holds.

$$\sigma_{PH,K}^2(s,s+h) = 2^{1-K}(h^{2HK} - f_s(h))$$

with

$$f_s(h) := \left[s^{2H} + (s+h)^{2H}\right]^K - 2^{K-1}\left[s^{2HK} + (s+h)^{2HK}\right],$$

then $f_s(0) = f'_s(0) = 0$ and by Taylor formula we obtain

$$\frac{\sigma_{BHK}^2(s,s+h)}{2^{1-K}h^{2HK}} - 1 = -h^{-2HK} \int_0^h f_s''(x)(h-x) dx,$$

where

$$\begin{split} f_s''(x) = & 4K(K-1)H^2 \big[s^{2H} + (s+x)^{2H} \big]^{K-2} (s+x)^{2(2H-1)} \\ & + 2HK(2H-1) \big[s^{2H} + (s+x)^{2H} \big]^{K-1} (s+x)^{2H-2} \\ & - 2^K HK(2HK-1)(s+x)^{2HK-2}. \end{split}$$

Note that for $H \ge 1/2$

$$\frac{(s+x)^{2(2H-1)}}{\lceil s^{2H} + (s+x)^{2H} \rceil^{2-K}} = \left\lceil \frac{(s+x)^{2H}}{s^{2H} + (s+x)^{2H}} \right\rceil^{2-K} (s+x)^{2HK-2} \leqslant (s+x)^{2HK-2}.$$

Thus for s > 0

$$\sup_{0 \leqslant x \leqslant h} |f_s''(x)| \leqslant \frac{4}{s^{2-2HK}} \mathbf{1}_{\{H \geqslant 1/2\}} + \frac{4}{(2s^{2H})^{2-K} s^{2(1-2H)}} \mathbf{1}_{\{H < 1/2\}} + \frac{2}{(2s^{2H})^{1-K} s^{2-2H}} + \frac{2}{s^{2-2HK}} \leqslant \frac{8}{s^{2-2HK}}$$

and

$$\sup_{\varphi(\delta)\leqslant s\leqslant T-\delta}\sup_{0< h\leqslant \delta}\left|\frac{\sigma_{B_{HK}}^2(s,s+h)}{2^{1-K}h^{2H}}-1\right|\leqslant \sup_{\varphi(\delta)\leqslant s\leqslant T-\delta}\frac{8\delta^{2-2HK}}{s^{2-2HK}}\leqslant \frac{8}{(L(\delta))^{2-2HK}}$$

for every $\varphi \in \Psi$. So condition (C2) holds.

2.3 Ornstein-Uhlenbeck process

The fractional Ornstein-Uhlenbeck (fO-U) process of the first kind is the unique solution of the following stochastic differential equation

$$X_t = x_0 - \mu \int_0^t X_s \, ds + \theta B_t^H, \qquad t \leqslant T, \tag{16}$$

with $\mu, \theta > 0$, where B^H , 0 < H < 1, is a fBm. This equation has an explicit solution

$$X_t = x_0 e^{-\mu t} + \theta \int_0^t e^{-\mu(t-u)} dB_u^H,$$

where the integral exists as a Riemann-Stieltjes integral for all t > 0 (see, e.g., [4]).

First check condition (C1) for this process. From [4] we know that

$$\int_{0}^{t} e^{\mu u} dB_{u}^{H} = e^{\mu t} B_{t}^{H} - \mu \int_{0}^{t} e^{\mu u} B_{u}^{H} du.$$

Thus

$$\begin{split} X_t^2 = & \left(x_0 e^{-\mu t} + \theta \int_0^t e^{-\mu(t-u)} dB_u^H \right)^2 = \left(x_0 e^{-\mu t} + \theta B_t^H - \theta \mu e^{-\mu t} \int_0^t e^{\mu u} B_u^H du \right)^2 \\ \leqslant & 3x_0^2 + 3\theta^2 (B_t^H)^2 + 3\theta^2 e^{-2\mu t} \left(\int_0^t B_u^H de^{\mu u} \right)^2 \\ \leqslant & 3x_0^2 + 3\theta^2 (B_t^H)^2 + 3\theta^2 e^{-\mu t} \int_0^t (B_u^H)^2 de^{\mu u} \end{split}$$

and

$$\sup_{t \le T} \mathbf{E} X_t^2 \leqslant 3x_0^2 + 6\theta^2 T^2.$$

The incremental variance function of X has the following form

$$\sigma_X^2(t, t+h) = \mu^2 \mathbf{E} \left(\int_t^{t+h} X_s \, ds \right)^2 - 2\mu \theta \mathbf{E} \left([B^H(t+h) - B^H(t)] \int_t^{t+h} X_s \, ds \right) + \theta^2 \sigma_{BH}^2(t, t+h).$$

Cauchy-Schwarz inequality yields

$$\mathbf{E}\bigg(\int_{t}^{t+h} X_{s} \, ds\bigg)^{2} \leqslant h^{2} \sup_{t \leqslant s \leqslant t+h} \mathbf{E} X_{s}^{2}$$

and

$$\mathbf{E}\Big([B^{H}(t+h) - B^{H}(t)] \int_{t}^{t+h} X_{s} \, ds\Big) \leqslant \mathbf{E}^{1/2} [B^{H}(t+h) - B^{H}(t)]^{2} \Big(h \int_{t}^{t+h} \mathbf{E} X_{s}^{2} \, ds\Big)^{1/2}$$

$$\leqslant h^{H+1} \Big(\sup_{t \leqslant s \leqslant t+h} \mathbf{E} X_{s}^{2}\Big)^{1/2}.$$

Note that

$$\left|\sigma_X^2(0,h) - \theta^2 h^{2H}\right| = \left|\sigma_X^2(0,h) - \theta^2 \mathbf{E}(B_h^H)^2\right| \leqslant \mu^2 h^2 \sup_{t \leqslant h} \mathbf{E} X_t^2 + 2\mu\theta h^{1+H} \sqrt{\sup_{t \leqslant h} \mathbf{E} X_t^2} \,.$$

Condition (C1) follows from these calculations since $\sigma_X^2(0,h) \approx h^{2H}$ is equivalent to the requirement that $\sigma_X^2(0,h)/h^{2H}$ converges to some finite non-zero limit as $h \downarrow 0$.

Next, for every $\varphi \in \Psi$

$$\sup_{\varphi(\delta)\leqslant t\leqslant T-\delta}\sup_{0< h\leqslant \delta}\left|\frac{\sigma_X^2(t,t+h)}{\theta^2h^{2H}}-1\right|\leqslant \theta^{-2}\delta^{1-H}\Big[\delta^{1-H}\mu^2\sup_{t\leqslant T}\mathbf{E}X_t^2+2\mu\theta\Big(\sup_{t\leqslant T}\mathbf{E}X_t^2\Big)^{1/2}\Big]\longrightarrow 0$$

as $\delta \downarrow 0$. It follows from the inequality

$$\left|\frac{\sigma_X(t,t+h)}{\theta h^H} - 1\right| = \left|\frac{\sigma_X^2(t,t+h)}{\theta^2 h^{2H}} - 1\right| / \left|\frac{\sigma_X(t,t+h)}{\theta h^H} + 1\right| \leqslant \left|\frac{\sigma_X^2(t,t+h)}{\theta^2 h^{2H}} - 1\right|$$

that condition (C2) is satisfied.

2.4 Fractional Brownian bridge

The fractional Brownian bridge is defined in [0, T] as

$$X_t^H = B_t^H - \frac{t^{2H} + T^{2H} - |t - T|^{2H}}{2T^{2H}} B_T^H, \tag{17}$$

where B^H , 0 < H < 1, is a fBm on the interval [0, T] (see [7]).

Now we verify condition (C1). The incremental variance function of X^H has the following form

$$\sigma_{XH}^2(t, t+h) = h^{2H} - \frac{1}{4T^{2H}} f_t^2(h),$$

where

$$f_t^2(h) := [(t+h)^{2H} - t^{2H} - |t+h-T|^{2H} + |t-T|^{2H}]^2.$$

It is easy to verify that

$$\frac{f_0^2(h)}{h^{2H}} = \frac{[h^{2H} - |T - h|^{2H} + T^{2H}]^2}{h^{2H}} \longrightarrow 0 \qquad \text{as } h \downarrow 0.$$

Thus

$$\sigma_{XH}^2(t, t+h) \simeq h^{2H}$$
.

So condition (C1) is satisfied.

Assume that H < 1/2. Since

$$\left|(t+h)^{2H}-t^{2H}\right|\leqslant h^{2H}\quad\text{and}\quad \left|(T-t-h)^{2H}-(T-t)^{2H}\right|\leqslant h^{2H},$$

then for every $\varphi \in \Psi$

$$\sup_{\varphi(\delta)\leqslant t\leqslant T-\delta}\sup_{0< h\leqslant \delta}\left|\frac{\sigma_{X^H}^2(t,t+h)}{h^{2H}}-1\right|=\frac{1}{4T^{2H}}\sup_{\varphi(\delta)\leqslant t\leqslant T-\delta}\sup_{0< h\leqslant \delta}\frac{f_t^2(h)}{h^{2H}}\leqslant T^{-2H}\delta^{2H}.$$

Assume that $H \ge 1/2$. Then $f_t(0) = 0$ and we obtain from Taylor formula that

$$\frac{\sigma_{XH}^2(t,t+h)}{h^{2H}} - 1 = -\frac{1}{4T^{2H}h^{2H}} \left(\int_0^h f_t'(x) \, dx \right)^2,$$

where

$$f'_t(x) = 2H[(t+x)^{2H-1} - (T-t-x)^{2H-1}].$$

Thus for every $\varphi \in \Psi$ and $H \geqslant 1/2$ we get

$$\sup_{\varphi(\delta)\leqslant t\leqslant T-\delta}\sup_{0< h\leqslant \delta}\left|\frac{\sigma_{X^H}^2(t,t+h)}{h^{2H}}-1\right|\leqslant \frac{\delta^{2-2H}}{4T^{2H}}\cdot 4H^2T^{4H-2}=H^2T^{2H-2}\delta^{2-2H}.$$

3 The convergence of the second order quadratic variation of process X along arbitrary partition

Let $\pi_n = \{0 = t_0^n < t_1^n < \dots < t_{N_n}^n = T\}$, T > 0, be a sequence of partitions of the interval [0, T], where (N_n) is an increasing sequence of natural numbers. Define

$$m_n = \max_{1 \leqslant k \leqslant N_n} \Delta_k^n t, \qquad p_n = \min_{1 \leqslant k \leqslant N_n} \Delta_k^n t, \qquad \Delta_k^n t = t_k^n - t_{k-1}^n.$$

Definition 9 A sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ is regular if $m_n = p_n = TN_n^{-1}$ for all $n\in\mathbb{N}$ or, equivalently, $t_k^n = \frac{kT}{N_n}$ for all $n\in\mathbb{N}$ and all $k\in\{0,\ldots,N_n\}$.

Usually in practice observations of the process are available at discrete regular time intervals. However, it may happen that the part of observations is lost, resulting in observations at arbitrary time intervals. Therefore we define the second order quadratic variations of Gaussian processes along arbitrary partitions.

Definition 10 The second order quadratic variations of Gaussian processes X with Orey index γ along the partitions $(\pi_n)_{n\in\mathbb{N}}$ is defined by

$$V_{\pi_n}^{(2)}(X,2) = 2 \sum_{k=1}^{N_n-1} \frac{\Delta_{k+1}^n t (\Delta_{ir,k}^{(2)n} X)^2}{(\Delta_k^n t)^{\gamma+1/2} (\Delta_{k+1}^n t)^{\gamma+1/2} [\Delta_k^n t + \Delta_{k+1}^n t]} \,,$$

where

$$\Delta_{ir,k}^{(2)n}X_k^n = \Delta_k^n t X(t_{k+1}^n) + \Delta_{k+1}^n t X(t_{k-1}^n) - (\Delta_k^n t + \Delta_{k+1}^n t) X(t_k^n).$$

If the sequence $(\pi_n)_{n\in\mathbb{N}}$ is regular then one has

$$V_{N_n}^{(2)}(X,2) = (T^{-1}N_n)^{2\gamma - 1} \sum_{k=1}^{N_n - 1} \left(\Delta_{n,k}^{(2)}X\right)^2, \qquad \Delta_{n,k}^{(2)}X = X(t_{k+1}^n) - 2X(t_k^n) + X(t_{k-1}^n).$$

To study the almost sure convergence of the second order quadratic variation of X we need additional assumptions on the sequence $(\pi_n)_{n\in\mathbb{N}}$.

Definition 11 (see [2]) Let $(\ell_k)_{k\geqslant 1}$ be a sequence of real numbers in the interval $(0,\infty)$. We say that $(\pi_n)_{n\in\mathbb{N}}$ is a sequence of partitions with asymptotic ratios $(\ell_k)_{k\geqslant 1}$ if it satisfies the following assumptions:

- 1. There exists $c \ge 1$ such that $m_n \le cp_n$ for all n.
- 2. $\lim_{n\to\infty} \max_{1\leqslant k\leqslant N_n-1} \left| \frac{\Delta_k^n t}{\Delta_{k+1}^n t} \ell_k \right| = 0.$

The set $\mathcal{L} = \{\ell_1; \ell_2; \dots; \dot{\ell}_k; \dots\}$ will be called the range of the asymptotic ratios of the sequence $(\pi_n)_{n \in \mathbb{N}}$.

It is clear that if the sequence $(\pi_n)_{n\in\mathbb{N}}$ is regular, then it is a sequence with asymptotic ratios $\ell_k=1$ for all $k\geqslant 1$.

Definition 12 (see [2]) The function $g:(0,\infty)\to\mathbb{R}$ is invariant on \mathcal{L} if for all $\ell,\hat{\ell}\in\mathcal{L}$, $g(\ell)=g(\hat{\ell})$.

Definition 13 (see [6]) A function $f : [a, b] \to \mathbb{R}$ is called regulated provided there is a sequence $(f_n)_{n\geqslant 1}$ of step functions which converges uniformly to f.

Proposition 14 Let $X = \{X(t) : t \in [0,T]\}$, T > 0, be a zero mean second order process satisfying conditions (C1) and (C2). Let $(\pi_n)_{n \in \mathbb{N}}$ be a sequence of partitions with asymptotic ratios $(\ell_k)_{k \geqslant 1}$ and range of the asymptotic ratios \mathcal{L} . If the function

$$g(\lambda) = \frac{1 + \lambda^{2\gamma - 1} - (1 + \lambda)^{2\gamma - 1}}{\lambda^{\gamma - 1/2}}$$

is invariant on \mathcal{L} or if the sequence of step functions $\ell_n(t)$, i.e. $\ell_n(t) = \ell_k$ on (t_k^n, t_{k+1}^n) , $0 \le k \le N_n - 1$ and $\ell_0 = \ell_1$, converges to regulated function $\ell(t)$ on the interval [0, T], then

$$\mathbf{E}V_{\pi_n}^{(2)}(X,2) \longrightarrow 2\kappa^2 \int_0^T g(\ell(t)) dt$$
 as $n \to \infty$.

Proof. Rewrite the expectation of each increment from the second order variation in the following way

$$\begin{split} \mathbf{E}(\Delta_{ir,k}^{(2)n}X)^2 = & (\Delta_k^n t)^2 \sigma_X^2(t_k^n, t_{k+1}^n) + (\Delta_{k+1}^n t)^2 \sigma_X^2(t_{k-1}^n, t_k^n) \\ & + \Delta_k^n t \cdot \Delta_{k+1}^n t \big[\sigma_X^2(t_k^n, t_{k+1}^n) - \sigma_X^2(t_{k-1}^n, t_{k+1}^n) + \sigma_X^2(t_{k-1}^n, t_k^n) \big] \\ = & [\Delta_k^n t + \Delta_{k+1}^n t] \big[\Delta_k^n t \cdot \sigma_X^2(t_k^n, t_k^n + \Delta_{k+1}^n t) + \Delta_{k+1}^n t \cdot \sigma_X^2(t_{k-1}^n, t_{k-1}^n + \Delta_k^n t) \big] \\ & - \Delta_k^n t \cdot \Delta_{k+1}^n t \cdot \sigma_X^2(t_{k-1}^n, t_{k-1}^n + \Delta_k^n t + \Delta_{k+1}^n t) \\ = & I_k^{(1)} - I_k^{(2)} + I_k^{(3)}, \end{split}$$

where

$$\begin{split} I_k^{(1)} &:= & [\Delta_k^n t + \Delta_{k+1}^n t] \big\{ \Delta_k^n t \left[\sigma_X^2(t_k^n, t_{k+1}^n) - \kappa^2 (\Delta_{k+1}^n t)^{2\gamma} \right] \\ &\quad + \Delta_{k+1}^n t \left[\sigma_X^2(t_{k-1}^n, t_k^n) - \kappa^2 (\Delta_k^n t)^{2\gamma} \right] \big\}, \\ I_k^{(2)} &:= & \Delta_k^n t \cdot \Delta_{k+1}^n t \left[\sigma_X^2(t_{k-1}^n, t_{k+1}^n) - \kappa^2 (\Delta_k^n t + \Delta_{k+1}^n t)^{2\gamma} \right], \\ I_k^{(3)} &:= & \kappa^2 \left[\Delta_k^n t + \Delta_{k+1}^n t \right] \Delta_k^n t \cdot \Delta_{k+1}^n t \big\{ (\Delta_{k+1}^n t)^{2\gamma-1} + (\Delta_k^n t)^{2\gamma-1} - (\Delta_k^n t + \Delta_{k+1}^n t)^{2\gamma-1} \big\}. \end{split}$$

Set

$$\mu_k^n = [\Delta_k^n t + \Delta_{k+1}^n t] (\Delta_{k+1}^n t)^{\gamma + 1/2} (\Delta_k^n t)^{\gamma + 1/2} \quad \text{and} \quad \ell_k^n = \frac{\Delta_k^n t}{\Delta_{k+1}^n t}.$$

Then

$$\begin{split} I_k^{(1)} &= \kappa^2 \big[\Delta_k^n t + \Delta_{k+1}^n t \big] \Delta_k^n t \cdot \Delta_{k+1}^n t \big[\big(\Delta_{k+1}^n t \big)^{2\gamma - 1} c(t_k^n, t_{k+1}^n) + \big(\Delta_k^n t \big)^{2\gamma - 1} c(t_{k-1}^n, t_k^n) \big] \\ &= \kappa^2 \mu_k^n \big[(\ell_k^n)^{1/2 - \gamma} c(t_k^n, t_{k+1}^n) + (\ell_k^n)^{\gamma - 1/2} c(t_{k-1}^n, t_k^n) \big], \\ I_k^{(2)} &= \kappa^2 \mu_k^n (\Delta_k^n t)^{1/2 - \gamma} (\Delta_{k+1}^n t)^{1/2 - \gamma} (\Delta_k^n t + \Delta_{k+1}^n t)^{2\gamma - 1} c(t_{k-1}^n, t_{k+1}^n) \\ &= \kappa^2 \mu_k^n (\ell_k^n)^{1/2 - \gamma} (1 + \ell_k^n)^{2\gamma - 1} c(t_{k-1}^n, t_{k+1}^n) \end{split}$$

and

$$I_k^{(3)} = \kappa^2 \mu_k^n \left((\ell_k^n)^{1/2 - \gamma} + (\ell_k^n)^{\gamma - 1/2} - (\ell_k^n)^{1/2 - \gamma} (1 + \ell_k^n)^{2\gamma - 1} \right) = \kappa^2 \mu_k^n g(\ell_k^n),$$

where the function c(s,t) is defined in (10). Further, we note that

$$\mathbf{E}V_{\pi_n}^{(2)}(X,2) = 2\sum_{k=1}^{\tau_n+1} \frac{\Delta_{k+1}^n t \cdot \mathbf{E}(\Delta_{ir,k}^{(2)n} X)^2}{\mu_k^n} + 2\sum_{k=\tau_n+2}^{N_n-1} \frac{\Delta_{k+1}^n t \cdot \mathbf{E}(\Delta_{ir,k}^{(2)n} X)^2}{\mu_k^n}$$

$$= 2\sum_{k=1}^{\tau_n+1} \frac{\Delta_{k+1}^n t \cdot \mathbf{E}(\Delta_{ir,k}^{(2)n} X)^2}{\mu_k^n} + 2\kappa^2 \sum_{k=\tau_n+2}^{N_n-1} \Delta_{k+1}^n t \cdot J_k$$

$$+ 2\kappa^2 \sum_{k=\tau_n+2}^{N_n-1} \Delta_{k+1}^n t \cdot g(\ell_k^n), \tag{18}$$

where $\tau_n = [\varphi(m_n)N_n]$, [a] is an integer part of a real number a,

$$J_k = (\ell_k^n)^{1/2 - \gamma} \left[c(t_k^n, t_{k+1}^n) + (\ell_k^n)^{2\gamma - 1} c(t_{k-1}^n, t_k^n) - (1 + \ell_k^n)^{2\gamma - 1} c(t_{k-1}^n, t_{k+1}^n) \right].$$

Now we estimate the first term in the right-hand side of (18). Note that

$$\tau_n \leqslant \frac{\varphi(m_n)}{n_n} T \leqslant cL(m_n)T, \qquad 2p_n^{2\gamma+2} \leqslant \mu_k^n \leqslant 2m_n^{2\gamma+2}, \tag{19}$$

$$\sum_{k=1}^{\tau_n+1} \Delta t_{k+1}^n \leqslant \varphi(m_n) \frac{m_n}{p_n} T + m_n \leqslant cT\varphi(m_n) + m_n \leqslant c(T+1)\varphi(m_n) \quad \text{if } L(m_n) > 1. \quad (20)$$

We get from (C1), (C2) and inequalities (19), (20) that

$$2\sum_{k=1}^{\tau_{n}+1} \frac{\Delta_{k+1}^{n} t \cdot \mathbf{E}(\Delta_{ir,k}^{(2)n} X)^{2}}{\mu_{k}^{n}}$$

$$\leq \frac{4c^{3} (T+1)\varphi(m_{n})}{p_{n}^{2\gamma}} \max_{1 \leq k \leq \tau_{n}+2} \sigma_{X}^{2}(t_{k-1}^{n}, t_{k}^{n}) \leq \frac{16c^{3} (T+1)\varphi(m_{n})}{p_{n}^{2\gamma}} \sup_{1 \leq k \leq \tau_{n}+2} \sigma_{X}^{2}(0, t_{k}^{n})$$

$$= \frac{16c^{3} (T+1)\varphi(m_{n})}{p_{n}^{2\gamma}} O(L^{2\gamma}(m_{n})m_{n}^{2\gamma}) \leq 16c^{3} (T+1)\varphi(m_{n}) O(L^{2\gamma}(m_{n}))$$

as $m_n \downarrow 0$. We obtain from the properties of function φ that the right hand side of the above inequality tends to zero as $m_n \downarrow 0$.

Next, since $[\varphi(m_n)N_n] + 1 \geqslant \varphi(m_n)$, we get that the second term of equality (18) can be estimated as

$$\begin{split} &\sum_{k=\tau_{n}+2}^{N_{n}-1} \Delta_{k+1}^{n} t \cdot J_{k} \\ &\leqslant \max_{\tau_{n}+1 \leqslant k \leqslant N_{n}-1} \left| c(t_{k}^{n}, t_{k+1}^{n}) \right| \sum_{k=\tau_{n}+2}^{N_{n}-1} \Delta_{k+1}^{n} t \left[(\ell_{k}^{n})^{1/2-\gamma} + (\ell_{k}^{n})^{\gamma-1/2} \right] \\ &+ \max_{\tau_{n}+2 \leqslant k \leqslant N_{n}-1} \left| c(t_{k-1}^{n}, t_{k+1}^{n}) \right| \sum_{k=\tau_{n}+2}^{N_{n}-1} \Delta_{k+1}^{n} t \cdot (\ell_{k}^{n})^{1/2-\gamma} (1 + \ell_{k}^{n})^{2\gamma-1} \\ &\leqslant T \sup_{\varphi(m_{n}) \leqslant t \leqslant T-m_{n}} \sup_{0 < h \leqslant m_{n}} \left| c(t, t+h) \right| \max_{1 \leqslant k \leqslant N_{n}} \left[(\ell_{k}^{n})^{1/2-\gamma} + (\ell_{k}^{n})^{\gamma-1/2} \right] \\ &+ T \sup_{\varphi(m_{n}) \leqslant t \leqslant T-2m_{n}} \sup_{0 < h \leqslant m_{n}} \left| c(t, t+2h) \right| \max_{1 \leqslant k \leqslant N_{n}} \left[(\ell_{k}^{n})^{1/2-\gamma} (1 + \ell_{k}^{n})^{2\gamma-1} \right] \\ &\leqslant T \left[\Lambda^{2}(m_{n}) + 2\Lambda(m_{n}) \right] \max_{1 \leqslant k \leqslant N_{n}} \left[(\ell_{k}^{n})^{1/2-\gamma} + (\ell_{k}^{n})^{\gamma-1/2} \right] \\ &+ T \left[\Lambda^{2}(2m_{n}) + 2\Lambda(2m_{n}) \right] \max_{1 \leqslant k \leqslant N_{n}} \left[(\ell_{k}^{n})^{1/2-\gamma} (1 + \ell_{k}^{n})^{2\gamma-1} \right] \\ &\leqslant 2Tc \left[\Lambda^{2}(m_{n}) + 2\Lambda(m_{n}) \right] + T(1+c)c \left[\Lambda^{2}(2m_{n}) + 2\Lambda(2m_{n}) \right]. \end{split}$$

Thus the second term of equality (18) tends to zero as $n \to \infty$.

It still remains to investigate asymptotic behavior of the third term of equality (18). If the function q is invariant on \mathcal{L} , then

$$2\kappa^{2} \sum_{k=\tau_{n}+2}^{N_{n}-1} \Delta_{k+1}^{n} t \cdot g(\ell_{k}^{n}) = 2\kappa^{2} \sum_{k=\tau_{n}+2}^{N_{n}-1} \Delta_{k+1}^{n} t \cdot [g(\ell_{k}^{n}) - g(\ell_{k})] + 2\kappa^{2} T g(\ell_{1}) - 2\kappa^{2} g(\ell_{1}) \sum_{k=0}^{\tau_{n}+1} \Delta_{k+1}^{n} t.$$

Assumption 1 of Definition 11 implies that $(\ell_k)_{k\geqslant 1}\subset [c^{-1},c]$. Since the derivative of the function g is bounded on $[c^{-1},c]$ by $2c^{3/2}$, then

$$|q(\ell_k^n) - q(\ell_k)| \le 2c^{3/2}|\ell_k^n - \ell_k|$$
.

Thus

$$2\kappa^2 \sum_{k=\tau_n+2}^{N_n-1} \Delta_{k+1}^n t \cdot g(\ell_k^n) \longrightarrow 2\kappa^2 g(\ell_1) T \quad \text{as } n \to \infty$$

by assumption 2 of Definition 11 and the inequality (20).

Assume that the sequence of step functions $\ell_n(t)$ converges uniformly to $\ell(t)$ on the interval [0,T]. Then $\ell_n(t), \ell(t) \in [c^{-1},c]$ and

$$|g(\ell_n(t)) - g(\ell(t))| \le 2c^{3/2} \sup_{0 \le t \le T} |\ell_n(t) - \ell(t)| \longrightarrow 0$$
 as $n \to \infty$,

i.e. the sequence $g(\ell_n(t))$ converges uniformly to $g(\ell(t))$ on [0,T] and $g(\ell(t))$ is regulated function. Thus

$$2\kappa^{2} \sum_{k=\tau_{n}+2}^{N_{n}-1} \Delta_{k+1}^{n} t \cdot g(\ell_{k}^{n}) = 2\kappa^{2} \sum_{k=\tau_{n}+2}^{N_{n}-1} \Delta_{k+1}^{n} t \cdot [g(\ell_{k}^{n}) - g(\ell_{k})] + 2\kappa^{2} \int_{0}^{T} g(\ell_{n}(t)) dt$$
$$-2\kappa^{2} \sum_{k=0}^{\tau_{n}+1} \Delta_{k+1}^{n} t \cdot g(\ell_{k}) \longrightarrow 2\kappa^{2} \int_{0}^{T} g(\ell(t)) dt$$

since regulated functions are Riemann integrable and

$$\sum_{k=\tau_n+2}^{N_n-1} \Delta_{k+1}^n t \cdot |g(\ell_k^n) - g(\ell_k)| \le 2c^{3/2} T \max_{1 \le k \le N_n - 1} |\ell_k^n - \ell_k|,$$

$$g(\ell_k) \le c^{1/2} (1+c) \quad \text{for all } k \ge 1.$$

Consequently, in both cases we obtain that

$$\mathbf{E}V_{\pi_n}^{(2)}(X,2) \longrightarrow 2\kappa^2 \int_0^T g(\ell(t)) dt$$
 as $n \to \infty$.

Corollary 15 Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of regular partitions of the interval [0,T], T>0, and let $X=\{X(t):t\in[0,T]\}$, T>0, be a zero mean second order process satisfying conditions (C1) and (C2). Then

$$\mathbf{E}V_{N_n}^{(2)}(X,2) \longrightarrow \kappa^2(4-2^{2\gamma})T$$
 as $n \to \infty$.

Proof. For regular subdivision we have that $\ell_k = 1$. Thus $g(\lambda) = 2 - 2^{2\gamma - 1}$ and the statement of the corollary follows immediately from Proposition 14.

Now we formulate a slightly more general version of Corollary 15.

Proposition 16 Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of regular partitions of the interval [0,T], T>0. Assume that condition (C1) is fulfilled for some constant $\gamma\in(0,1)$ and there exists a continuous bounded function $g_0:(0,T)\to\mathbb{R}$ such that

$$\lim_{h \to 0+} \sup_{\varphi(h) \le t \le T-h} \left| \frac{\mathbf{E} (X_{t+h} - 2X_t + X_{t-h})^2}{h^{2\gamma}} - g_0(t) \right| = 0.$$
 (21)

Then

$$\mathbf{E}V_{N_n}^{(2)}(X,2) \longrightarrow \int_0^T g_0(t) dt$$
 as $n \to \infty$.

Proof. Note that

$$\left| \mathbf{E} V_{N_{n}}^{(2)}(X,2) - \int_{0}^{T} g_{0}(t) dt \right| \\
\leq \left(\frac{T}{N_{n}} \right)^{1-2\gamma} \sum_{k=1}^{\tau_{n}} \mathbf{E} (\Delta_{n,k}^{(2)} X)^{2} + \frac{T}{N_{n}} \sum_{k=\tau_{n}+1}^{N_{n}-1} \left| \frac{\mathbf{E} (\Delta_{n,k}^{(2)} X)^{2}}{T^{2\gamma} N_{n}^{-2\gamma}} - g_{0} \left(\frac{kT}{N_{n}} \right) \right| \\
+ \left| \int_{0}^{T} g_{0}(t) dt - \frac{T}{N_{n}} \sum_{k=\tau_{n}+1}^{N_{n}-1} g_{0} \left(\frac{kT}{N_{n}} \right) \right|, \tag{22}$$

where $\tau_n = [\varphi(TN_n^{-1})N_n]$. We get from condition (C1) that

$$\max_{1\leqslant k\leqslant \tau_n+1}\sigma^2(t_{k-1}^n,t_k^n)\leqslant 2\sup_{1\leqslant k\leqslant \tau_n+1}\sigma^2(0,t_k^n)=O\big((TN_n^{-1}(\tau_n+1))^{2\gamma}\big)=O\big((\varphi(TN_n^{-1}))^{2\gamma}\big)$$

Thus

$$\left(\frac{T}{N_n}\right)^{1-2\gamma} \sum_{k=1}^{\tau_n} \mathbf{E}(\Delta_{n,k}^{(2)} X)^2 \leqslant 4T \left(\frac{T}{N_n}\right)^{-2\gamma} \varphi\left(\frac{T}{N_n}\right) \max_{1 \leqslant k \leqslant \tau_n + 1} \sigma^2(t_{k-1}^n, t_k^n)$$

$$= 4T \left(\frac{T}{N_n}\right)^{-2\gamma} \varphi\left(\frac{T}{N_n}\right) O\left(\left(\varphi\left(\frac{T}{N_n}\right)\right)^{2\gamma}\right)$$

and the first term in inequality (22) tends to zero as $n \to \infty$. Assumption (21) yields

 $\max_{\substack{\tau_n+1\leqslant k\leqslant N_n-1\\ \tau_n+1\leqslant k\leqslant N_n-1}} \left| \frac{\mathbf{E}(\Delta_{n,k}^{(2)}X)^2}{T^{2\gamma}N_n^{-2\gamma}} - g_0\left(\frac{kT}{N_n}\right) \right|$

$$\max_{n+1 \le k \le N_n - 1} \left| \frac{\frac{t}{T^{2\gamma} N_n^{-2\gamma}} - g_0\left(\frac{N^2}{N_n}\right)}{T^{2\gamma} N_n^{-2\gamma}} - g_0\left(\frac{N^2}{N_n}\right) \right|$$

$$\le \sup_{\varphi(TN_n^{-1}) \le t \le T - TN_n^{-1}} \left| \frac{\mathbf{E}\left(X_{t+TN_n^{-1}} - 2X_t + X_{t-TN_n^{-1}}\right)^2}{(TN_n^{-1})^{2\gamma}} - g_0(t) \right| \longrightarrow 0 \quad \text{as } n \to \infty.$$

The third term of the right hand side of (22) also converges towards 0 as $n \to \infty$ that is a consequence of classical results for Riemann sums and inequality

$$\left| \frac{T}{N_n} \sum_{k=1}^{\tau_n} \left| g_0\left(\frac{kT}{N_n}\right) \right| \leqslant \sup_{0 \leqslant t \leqslant T} \left| g_0(t) \right| \varphi\left(\frac{T}{N_n}\right).$$

Theorem 17 Assume that conditions of Proposition 14 are satisfied and the partition π_n is such that $p_n = o(\ln^{-1} n)$. Moreover assume that X is a Gaussian process with the Orey index γ and

$$\max_{1 \leqslant k \leqslant N_n - 1} \sum_{i=1}^{N_n - 1} |d_{jk}^{(2)n}| \leqslant C p_n^{2 + 2\gamma}, \tag{23}$$

for some constant C and every sequence of partitions (π_n) of the interval [0,T], where $d_{jk}^{(2)n} = \mathbf{E}(\Delta_{ir,j}^{(2)n} X \Delta_{ir,k}^{(2)n} X)$, $1 \leq j,k \leq n$. Then

$$V_{\pi_n}^{(2)}(X,2) \longrightarrow 2\kappa^2 \int_0^T g(\ell(t)) dt$$
 a.s. as $n \to \infty$.

Proof. The proof of the theorem follows the outlines of the proof of Theorem 4 in [2].

Corollary 18 Let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence of regular partitions of the interval [0,T], T>0. Assume that X is a Gaussian process satisfying conditions (C1) and (C2) and having the Orey index γ . Moreover, assume that

$$\max_{1 \le k \le N_n - 1} \sum_{i=1}^{N_n - 1} |d_{jk}^{(2)n}| \le C \left(\frac{T}{N_n}\right)^{2\gamma} \tag{24}$$

for some constant C, and every sequence of partitions (π_n) of the interval [0,T], where $d_{jk}^{(2)n} = \mathbf{E}(\Delta_{n,j}^{(2)} X \Delta_{n,k}^{(2)} X), 1 \leq j, k \leq N_n - 1$. Then

$$V_{N_n}^{(2)}(X,2) \longrightarrow \kappa^2(4-2^{2\gamma})T$$
 a.s. as $n \to \infty$.

Proof. For regular partition π_n condition (23) transforms to (24).

Theorem 19 Assume that conditions of Proposition 16 are satisfied. Moreover, assume that inequality (24) holds, then

$$V_{N_n}^{(2)}(X,2) \longrightarrow \int_0^T g_0(t) dt$$
 a.s. as $n \to \infty$.

Proof. The proof of the theorem evidently follows from Proposition 16 and arguments used to prove Theorem 17.

Remark 20 Generally speaking, the function $\varphi(h)$ could not be replaced by h in assumption (21). Indeed, let us consider sfBm X with $H \neq 1/2$. Observe that following equality

$$\mathbf{E}(X_{t+h} - 2X_t + X_{t-h})^2 = (4 - 2^{2H})h^{2H} - 2^{2H-1}(t+h)^{2H} - 3 \cdot 2^{2H}t^{2H} - 2^{2H-1}(t-h)^{2H} + 2(2t+h)^{2H} + 2(2t-h)^{2H}$$

holds. Set

$$\lambda_t(h) := \mathbf{E}(X_{t+h} - 2X_t + X_{t-h})^2 - (4 - 2^{2H})h^{2H}$$

and note that $\lambda_t(0) = \lambda_t'(0) = \lambda_t''(0) = \lambda_t^{(3)}(0) = 0$. The Taylor formula yields

$$\lambda_t(h) = \int_0^h \frac{(h-x)^3}{3!} \, \lambda_t^{(4)}(x) \, dx, \qquad \forall \ h \leqslant t \leqslant T - h,$$

where

$$\lambda_t^{(4)}(x) = C_H \left(2\left[(2t+x)^{2H-4} + (2t-x)^{2H-4} \right] - 2^{2H-1} \left[(t+x)^{2H-4} + (t-x)^{2H-4} \right] \right),$$

$$C_H = 2H(2H-1)(2H-2)(2H-3).$$

Note that

$$\sup_{h \leqslant t \leqslant T - h} \left| \frac{\mathbf{E} (X_{t+h} - 2X_t + X_{t-h})^2}{h^{2H}} - (4 - 2^{2H}) \right|$$

$$= \sup_{h \leqslant t \leqslant T - h} \left| \int_0^h \frac{(h - x)^3}{3! h^{2H}} \lambda_t^{(4)}(x) \, dx \right| \geqslant \left| \int_0^h \frac{(h - x)^3}{3! h^{2H}} \lambda_h^{(4)}(x) \, dx \right|.$$

After a change of variable $y = \frac{h-x}{ah+bx}$ with certain constants a and b, we obtain equality

$$h^{-2H} \int_0^h (h-x)^3 \lambda_h^{(4)}(x) dx = 2 \cdot 3^{2H} C_H \int_0^{1/2} y^3 (1+y)^{-2H-1} dy + 2C_H \int_0^{1/2} y^3 (1-y)^{-2H-1} dy - 2^{4H-1} C_H \int_0^1 y^3 (1+y)^{-2H-1} dy - 2^{2H-2} H^{-1} C_H.$$

All these integrals are finite and don't depend on h. Moreover,

$$\lim_{h \to 0+} \sup_{h \leqslant t \leqslant T-h} \left| \frac{\mathbf{E} (X_{t+h} - 2X_t + X_{t-h})^2}{h^{2H}} - (4 - 2^{2H}) \right| > 0.$$
 (25)

On the other hand, assumption (21) is satisfied for sfBm. From inequality

$$\sup_{\varphi(h) \leqslant t \leqslant T-h} \left| \frac{\mathbf{E} \left(X_{t+h} - 2X_t + X_{t-h} \right)^2}{h^{2H}} - (4 - 2^{2H}) \right| \\
\leqslant h^{-2H} \sup_{\varphi(h) \leqslant t \leqslant T-h} \sup_{0 \leqslant x \leqslant h} \left| \lambda_t^{(4)}(x) \right| \int_0^h (h-x)^3 dx \\
\leqslant |C_H| \cdot h^{4-2H} \sup_{\varphi(h) \leqslant t \leqslant T-h} \left(\frac{2}{(2t)^{4-2H}} + \frac{2}{(2t-h)^{4-2H}} + \frac{2^{2H-1}}{t^{4-2H}} + \frac{2^{2H-1}}{(t-h)^{4-2H}} \right) \\
\leqslant |C_H| \cdot h^{4-2H} \left(\frac{2}{(2\varphi(h))^{4-2H}} + \frac{2}{(2\varphi(h)-h)^{4-2H}} + \frac{2^{2H-1}}{\varphi(h)^{4-2H}} + \frac{2^{2H-1}}{(\varphi(h)-h)^{4-2H}} \right) \\
\leqslant |C_H| \cdot \left[\left(\frac{h}{\varphi(h)} \right)^{4-2H} + \frac{2}{(2L(h)-1)^{4-2H}} + 2^{2H-1} \left(\frac{h}{\varphi(h)} \right)^{4-2H} + \frac{2^{2H-1}}{(L(h)-1)^{4-2H}} \right]$$

we obtain the required assertion.

3.1 Bifractional Brownian motion

We shall prove that the conditions of Theorem 17 are satisfied for bifBm. The bifBm satisfies conditions (C1) and (C2). So it suffices to verify the inequality (23).

Following the outlines of the proof of Theorem 4 of Begyn [2], we divide the study of the asymptotic properties of $d_{jk}^{(2)n}$ into three steps, according to the value of k-j.

If j = k then (15) yields

$$d_{kk}^{(2)n} \leq 2\left[(\Delta_k^n t)^2 \mathbf{E} (\Delta_{k+1}^n B^{HK})^2 + (\Delta_{k+1}^n t)^2 \mathbf{E} (\Delta_k^n B^{HK})^2 \right]$$

$$\leq 2^{2-K} \left[(\Delta_k^n t)^2 |t_{k+1} - t_k|^{2HK} + (\Delta_{n,k+1} t)^2 |t_k - t_{k-1}|^{2HK} \right]$$

$$\leq 2^{3-K} m_n^{2+2HK}.$$
(26)

By using the Cauchy-Schwarz inequality we get

$$\left| d_{jk}^{(2)n} \right| \leqslant \mathbf{E}^{1/2} \left| \left(\Delta_{ir,j}^{(2)n} B^{HK} \right) \right|^2 \cdot \mathbf{E}^{1/2} \left| \left(\Delta_{ir,k}^{(2)n} B^{HK} \right) \right|^2 \leqslant 2^{3-K} m_n^{2+2HK}$$
 (27)

for $1 \leqslant k - j \leqslant 2$ and

$$d_{j1}^{(2)n} \leq 2^{3-K} m_n^{2+2HK} \quad \text{for } 1 \leq j \leq N_n - 1,$$
 (28)

$$d_{1k}^{(2)n} \leq 2^{3-K} m_n^{2+2HK} \quad \text{for } 1 \leq k \leq N_n - 1.$$
 (29)

Now consider the case $|j-k| \ge 3$. By symmetry of $d_{jk}^{(2)n}$ one can take $j-k \ge 3$. Note that for $j \ne 1$ and $k \ne 1$ equality

$$d_{jk}^{(2)n} = \int_{t_j^n}^{t_{j+1}^n} du \int_{t_{j-1}^n}^{t_j^n} dv \int_v^u dw \int_{t_k^n}^{t_{k+1}^n} dx \int_{t_{k-1}^n}^{t_k^n} dy \int_y^x \frac{\partial^4 R_{HK}}{\partial s^2 \partial t^2}(w, z) dz$$

holds. The fourth order mixed partial derivative of the covariance function $R_{HK}(s,t)$ is of the following form

$$\begin{split} \frac{\partial^4 R_{HK}}{\partial s^2 \partial t^2}(s,t) &= -\frac{2HK(2H-1)(2HK-2)(2HK-3)}{2^K |s-t|^{2(2-KH)}} \\ &+ \frac{K(K-1)(K-2)(K-3)(2H)^4}{2^K} \left(st\right)^{4H-2} \left(s^{2H} + t^{2H}\right)^{K-4} \\ &+ \frac{K(K-1)(2H)^2(2H-1)}{2^K} \left[(K-2)(2H) + (2H-1) \right] \left(st\right)^{2H-2} \left(s^{2H} + t^{2H}\right)^{K-2} \end{split}$$

for each s,t>0 such that $s\neq t$. Since $2s^Ht^H\leqslant s^{2H}+t^{2H}$ and $K-2<0,\,K-4<0$ it follows that

$$(st)^{2H-2} (s^{2H} + t^{2H})^{K-2} \le 2^{K-2} (st)^{KH-2} (st)^{4H-2} (s^{2H} + t^{2H})^{K-4} \le 2^{K-4} (st)^{KH-2}.$$

Thus

$$\left| \frac{\partial^4 R^{HK}}{\partial s^2 \partial t^2}(s,t) \right| \leqslant \frac{C_1}{|s-t|^{2(2-KH)}} + \frac{C_2}{(st)^{2-KH}}$$

and

$$\begin{split} |d_{jk}^{(2)n}| &\leqslant \int_{t_{j}^{n}}^{t_{j+1}^{n}} du \int_{t_{j-1}^{n}}^{t_{j}^{n}} dv \int_{v}^{u} dw \int_{t_{k}^{n}}^{t_{k+1}^{n}} dx \int_{t_{k-1}^{n}}^{t_{k}^{n}} dy \int_{y}^{x} \frac{C_{1}}{|w-z|^{2(2-KH)}} dz \\ &+ \int_{t_{j}^{n}}^{t_{j+1}^{n}} du \int_{t_{j-1}^{n}}^{t_{j}^{n}} dv \int_{v}^{u} dw \int_{t_{k}^{n}}^{t_{k+1}^{n}} dx \int_{t_{k-1}^{n}}^{t_{k}^{n}} dy \int_{y}^{x} \frac{C_{2}}{(wz)^{2-KH}} dz \\ =: I_{jk}^{n,1} + I_{jk}^{n,2}, \end{split} \tag{30}$$

where constants C_1 and C_2 depends on H and K. Inequality

$$|w-z| \ge t_{j-1}^n - t_{k+1}^n = \sum_{i=k+2}^{j-1} \Delta_{n,i} t \ge (j-k-2)p_n$$

on the integration set imply

$$I_{jk}^{n,1} \leqslant \frac{4C_1 m_n^6}{(j-k-2)^{2(2-HK)} p_n^{2(2-HK)}} \leqslant \frac{4C_1 c^6 p_n^{2+2HK}}{(j-k-2)^{2(2-HK)}},$$
(31)

where c is a constant defined in Definition 11. Moreover,

$$\sum_{j-k\geqslant 3}^{n-1} \frac{1}{(j-k-2)^{2(2-HK)}} \leqslant \sum_{j=1}^{\infty} \frac{1}{j^{2(2-KH)}} < \infty.$$
 (32)

Now we estimate $I_{jk}^{n,2}$. By modifying the computations above we similarly find that

$$I_{jk}^{n,2} \leqslant \frac{4C_2 m_n^6}{(t_{j-1}t_{k-1})^{2-KH}} = \frac{4C_2 m_n^6}{(t_{k-1} \sum_{i=k}^{j-1} \Delta_i t + t_{k-1}^2)^{2-KH}}$$

$$\leqslant \frac{4C_2 m_n^6}{p_n^{2-KH} ((t_{j-1} - t_{k-1}) + t_{k-1})^{2-KH}} \leqslant \frac{4C_2 c^6 p_n^{4+KH}}{(t_{j-1} - t_{k-1})^{2-KH}}$$

$$\leqslant 4C_2 c^6 \frac{p_n^{2+2KH}}{(i-k)^{2-KH}}.$$

$$(33)$$

Note that

$$\sum_{j=k\geqslant 3}^{N_n-1} \frac{1}{(j-k)^{2-KH}} \leqslant \sum_{j=1}^{\infty} \frac{1}{j^{2-KH}} < \infty.$$
 (34)

The inequality (23) follows from inequalities (30)-(34).

Subfractional Brownian motion

We recall that conditions (C1) and (C2) are satisfied for sfBm. So the statement of Theorem 17 is satisfied if inequality (23) holds. To prove inequality (23), we apply similar arguments as for bifBm.

If j = k or $1 \le k - j \le 2$ then (13) and (14) yields

$$d_{jk}^{(2)n} \leqslant 8m_n^{2+2H}$$

The same inequality holds for $d_{j1}^{(2)n}$, $1 \leq j \leq N_n - 1$ and $d_{1k}^{(2)n}$, $1 \leq k \leq N_n - 1$. The fourth order mixed partial derivative of the covariance function $G_H(s,t)$ is of the following form

$$\frac{\partial^4 G_H}{\partial s^2 \partial t^2}(s,t) = -H(2H-1)(2H-2)(2H-3) \left[\frac{1}{|s-t|^{2(2-H)}} + \frac{1}{(s+t)^{2(2-H)}} \right].$$

for each s, t > 0 such that $s \neq t$. Note that $(s+t)^{2(2-H)} \geqslant |s-t|^{2(2-H)}$ if $s \neq t$.

$$\left|\frac{\partial^4 G_H}{\partial s^2 \partial t^2}(s,t)\right| \leqslant \frac{2H(2H-1)(2H-2)(2H-3)}{|s-t|^{2(2-H)}}$$

and

$$|d_{jk}^{(2)n}| \leqslant \frac{4C_H m_n^6}{(j-k-2)^{2(2-H)} p_n^{2(2-H)}} \leqslant \frac{4C_H c^6 p_n^{2+2H}}{(j-k-2)^{2(2-H)}}$$

for $j-k \ge 3$, $2 \le k \le N_n-1$, where $C_H = 2H(2H-1)(2H-2)(2H-3)$, c is a constant defined in Definition 11. So, we have

$$\max_{2 \leqslant k \leqslant N_n - 1} \sum_{j - k \geqslant 3} d_{jk}^{(2)n} \leqslant 4C_H c^6 p_n^{2+2H} \max_{2 \leqslant k \leqslant N_n - 1} \sum_{j - k \geqslant 3} \frac{1}{(j - k - 2)^{2(2-H)}}
\leqslant 4C_H c^6 p_n^{2+2H} \sum_{j=1}^{\infty} \frac{1}{j^{2(2-H)}} \leqslant C p_n^{2+2H}$$
(35)

for some constant C. This proves (23).

3.3 Ornstein-Uhlenbeck process

Before to proof the inequality (23) we give an auxiliary lemma. In its formulation we shall use the notion O_r . Let (a_n) be a sequence of real numbers. The notion of symbol $Y_n = O_r(a_n)$, $a_n \downarrow 0$, means that there exists a.s. finite r.v. ς with the property $|Y_n| \leqslant \varsigma \cdot a_n$.

Lemma 21 Let X be the solution of equation (16). Then

$$\left| V_{\pi_n}^{(2)}(X,2) - \theta^2 V_{\pi_n}^{(2)}(B^H,2) \right| = O_r(m_n^{1-2\varepsilon})$$

for every $0 < \varepsilon < 1/2 \wedge H$

Proof. It is evident that

$$\Delta_{ir,k}^{(2)n} X = -\mu \left(\Delta_k^n t \int_{t_k^n}^{t_{k+1}^n} X_s \, ds - \Delta_{k+1}^n t \int_{t_{k-1}^n}^{t_k^n} X_s \, ds \right) + \theta \Delta_{ir,k}^{(2)n} B^H.$$

For simplicity, we denote $X_k = X(t_k^n)$ and $B_k^H = B^H(t_k^n)$. After simple calculations we get the estimate

$$\begin{split} \sup_{t_k^n \leqslant s \leqslant t_{k+1}^n} |X_s - X_k| \leqslant & \mu(\Delta_{k+1}^n t) \sup_{t \leqslant T} |X_t| + \theta \sup_{t_k^n \leqslant s \leqslant t_{k+1}^n} |B_s^H - B_k^H| \\ \leqslant & \mu \min_{t \leqslant T} |X_t| + \theta G_T^{H,H-\varepsilon} m_n^{H-\varepsilon}, \end{split}$$

where

$$G_T^{H,H-\varepsilon} := \sup_{\substack{s \neq t \\ s,t \leqslant T}} \frac{|B_t^H - B_s^H|}{|t-s|^{H-\varepsilon}} < \infty \qquad \text{for every } 0 < \varepsilon < H. \tag{36}$$

Thus

$$\left(\Delta_{k}^{n} t \int_{t_{k}^{n}}^{t_{k+1}^{n}} (X_{s} - X_{k}) ds - \Delta_{k+1}^{n} t \int_{t_{k-1}^{n}}^{t_{k}^{n}} (X_{s} - X_{k}) ds\right)^{2}$$

$$\leq 2m_{n}^{3} \int_{t_{k}^{n}}^{t_{k+1}^{n}} (X_{s} - X_{k})^{2} ds + 2m_{n}^{3} \int_{t_{k-1}^{n}}^{t_{k}^{n}} (X_{s} - X_{k})^{2} ds$$

$$\leq 2m_{n}^{4} \left(\sup_{t_{k}^{n} \leq s \leq t_{k+1}^{n}} (X_{s} - X_{k})^{2} + \sup_{t_{k-1}^{n} \leq s \leq t_{k}^{n}} (X_{k} - X_{s})^{2}\right)$$

$$\leq 8m_{n}^{4+2H-2\varepsilon} \left(\mu^{2} m_{n}^{2-2H+2\varepsilon} \sup_{t \leq T} X_{t}^{2} + \theta^{2} (G_{T}^{H,H-\varepsilon})^{2}\right)$$

and

$$\begin{split} &\left| \left(\Delta_k^n t \int_{t_k^n}^{t_{k+1}^n} X_s \, ds - \Delta_{k+1}^n t \int_{t_{k-1}^n}^{t_k^n} X_s \, ds \right) \Delta_{ir,k}^{(2)n} B^H \right| \\ &= \left| \left(\Delta_k^n t \int_{t_k^n}^{t_{k+1}^n} (X_s - X_k) \, ds - \Delta_{n,k+1} t \int_{t_{k-1}^n}^{t_k^n} (X_s - X_k) \, ds \right) \Delta_{ir,k}^{(2)n} B^H \right| \\ &\leqslant 2 m_n^{2+H-\varepsilon} \left(\mu \, m_n^{1-H+\varepsilon} \sup_{t \leqslant T} |X_t| + \theta G_T^{H,H-\varepsilon} \right) \cdot 2 m_n G_T^{H,H-\varepsilon} m_n^{H-\varepsilon} \\ &= 4 m_n^{3+2H-2\varepsilon} \left(\mu \, m_n^{1-H+\varepsilon} \sup_{t \leqslant T} |X_t| + \theta G_T^{H,H-\varepsilon} \right) \cdot G_T^{H,H-\varepsilon}. \end{split}$$

We get from the obtained inequalities and definition of $V_{\pi_n}^{(2)}(\cdot,2)$ that

$$\begin{aligned}
|V_{\pi_n}^{(2)}(X,2) - \theta^2 V_{\pi_n}^{(2)}(B^H,2)| &\leq 8c^{2+2H} m_n^{2-2\varepsilon} \left(\mu^2 m_n^{2-2H+2\varepsilon} \sup_{t \leq T} X_t^2 + 2\theta^2 (G_T^{H,H-\varepsilon})^2 \right) T \\
&+ 4c^{2+2H} m_n^{1-2\varepsilon} \left(\mu m_n^{1-H+\varepsilon} \sup_{t \leq T} |X_t| + \theta G_T^{H,H-\varepsilon} \right) \cdot G_T^{H,H-\varepsilon} T \\
&= O_r(m_n^{1-2\varepsilon}).
\end{aligned}$$

As in previous cases it is enough to verify condition (23) of Theorem 17 for fBm B^H . The following inequality

$$\left|\frac{\partial^4 F_H}{\partial s^2 \partial t^2}(s,t)\right| \leqslant \frac{H|(2H-1)(2H-2)(2H-3)|}{|s-t|^{2(2-H)}}.$$

holds for the covariance function $F_H(s,t)$ of B^H . Applying similar arguments as for sfBm we obtain

$$\max_{1 \leqslant k \leqslant N_n - 1} \sum_{i=1}^{N_n - 1} |d_{jk}^{(2)n}| \leqslant C p_n^{2 + 2H}.$$

From Lemma 21 and inequality above we get the statement of Theorem 17.

3.4 Fractional Brownian bridge

For brevity, we rewrite the fractional Brownian bridge X_t^H given by (17) as follows:

$$X_t^H = B_t^H - g(t, T),$$

where

$$g(t,T) = \frac{t^{2H} + T^{2H} - |t - T|^{2H}}{2T^{2H}} B_T^H$$

It is evident that

$$\Delta_{ir,k}^{(2)n} X^H = \Delta_{ir,k}^{(2)n} B^H - \Delta_{ir,k}^{(2)n} g(\cdot, T),$$

where

$$\begin{split} \Delta_{ir,k}^{(2)n}g(\cdot,T) = & \Delta_k t \, \frac{(t_{k+1}^{2H} - t_k^{2H}) - (|T - t_{k+1}|^{2H} - |T - t_k|^{2H})}{2T^{2H}} \, B_T^H \\ & - \Delta_{k+1} t \, \frac{(t_k^{2H} - t_{k-1}^{2H}) - (|T - t_k|^{2H} - |T - t_{k-1}|^{2H})}{2T^{2H}} \, B_T^H. \end{split}$$

Since

$$|\Delta_{ir,k}^{(2)n}g(\cdot,T)| \leqslant \frac{4m_n^{1+2H}}{2T^{2H}} 2^{(2H-1)\vee 0} |B_T^H| \leqslant \frac{4m_n^{1+2H}}{T^{2H}} |B_T^H|,$$

then

$$V_{\pi_n}^{(2)}(g(\cdot,T),2) \leqslant 2T \frac{16m_n^{2+4H}}{T^{4H}2p_n^{2+2H}} |B_T^H| = 16T^{1-4H}c^{2+4H}p_n^{2H} |B_T^H|.$$

We get from the obtained inequalities and definition of $V_{\pi_n}^{(2)}(\cdot,2)$ that

$$\begin{split} \left| V_{\pi_n}^{(2)}(X^H, 2) - V_{\pi_n}^{(2)}(B^H, 2) \right| &= \left| V_{\pi_n}^{(2)}(g(\cdot, T), 2) - 4 \sum_{k=1}^{N_n - 1} \frac{\Delta_{k+1}^n t \Delta_{ir,k}^{(2)n} B^H \Delta_{ir,k}^{(2)n} g(\cdot, T)}{(\Delta_k^n t)^{H+1/2} (\Delta_{k+1}^n t)^{H+1/2} [\Delta_k^n t + \Delta_{k+1}^n t]} \right| \\ &\leq V_{\pi_n}^{(2)}(g(\cdot, T), 2) + 2 \frac{4m_n^{1+2H} G_T^{H,H-\varepsilon} m_n^{1+H-\varepsilon}}{T^{2H} p_n^{2+2H}} \left| B_T^H \right| \\ &\leq V_{\pi_n}^{(2)}(g(\cdot, T), 2) + 8c^{2+2H} \frac{G_T^{H,H-\varepsilon} m_n^{H-\varepsilon}}{T^{2H}} \left| B_T^H \right| = O_r(m_n^{H-\varepsilon}) \end{split}$$

for $0 < \varepsilon < H$, where $G_T^{H,H-\varepsilon}$ is defined in (36). By using similar arguments as in previous subsection we get the statement of Theorem 17.

4 On the estimation of Orey index for arbitrary partition

Let $(\pi_n)_{n\geqslant 1}$ be a sequence of partitions of [0,T] such that $0=t_0^n < t_1^n < \cdots < t_{m(n)}^n = T$ for all $n\geqslant 1$. Assume that we have two sequences of partitions $(\pi_{i(n)})_{n\geqslant 1}$ and $(\pi_{j(n)})_{n\geqslant 1}$ of [0,T] such that $\pi_{i(n)}\subset\pi_{j(n)}\subseteq\pi_n$, $i(n)< j(n)\leqslant m(n)$, for all $n\in\mathbb{N}$, where $\pi_{i(n)}=\{0=t_0^n< t_{i(1)}^n< t_{i(2)}^n< \cdots < t_{i(n)}^n = T\}$ and $\pi_{j(n)}=\{0=t_0^n< t_{j(1)}^n< t_{j(2)}^n< \cdots < t_{j(n)}^n = T\}$. Set

$$\Delta_{i(k)}^n t = t_{i(k)}^n - t_{i(k-1)}^n, \qquad m_{i(n)} = \max_{1 \le k \le i(n)} \Delta_{i(k)}^n t, \qquad p_{i(n)} = \min_{1 \le k \le i(n)} \Delta_{i(k)}^n t.$$

Moreover, assume that $p_{j(n)} \neq m_{i(n)}$ and $m_{i(n)} \leqslant cp_{i(n)}$, for all i(n), $n \geqslant 1$, $c \geqslant 1$. Note that $p_{j(n)} \leqslant p_{i(n)}$.

Let X be a Gaussian process with Orey index $\gamma \in (0,1)$. Set

$$V_{\pi_{i(n)}}^{(2)}(X,2) = 2 \sum_{k=1}^{i(n)-1} \frac{\Delta_{i(k+1)}^n t(\Delta_{ir,k}^{(2)n}X)^2}{(\Delta_{i(k)}^n t)^{\gamma+1/2} (\Delta_{i(k+1)}^n t)^{\gamma+1/2} [\Delta_{i(k)}^n t + \Delta_{i(k+1)}^n t]} \,,$$

where

$$\Delta_{ir,i(k)}^{(2)n}X = \Delta_{i(k)}^{n}t \cdot X(t_{i(k+1)}^{n}) + \Delta_{i(k+1)}^{n}t \cdot X(t_{i(k-1)}^{n}) - (\Delta_{i(k)}^{n}t + \Delta_{i(k+1)}^{n}t)X(t_{i(k)}^{n}).$$

Denote

$$V_{i(n)}^{(2)}(X,2) = \sum_{k=1}^{i(n)-1} (\Delta_{ir,k}^{(2)n} X)^2 \quad \text{and} \quad \mu_k^n = (\Delta t_{i(k)}^n)^{\gamma+1/2} (\Delta_{i(k)}^n t)^{\gamma+1/2} [\Delta_{i(k)}^n t + \Delta_{i(k+1)}^n t].$$

Define

$$\widehat{\gamma}_n = -\frac{1}{2} + \frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \ln \frac{V_{j(n)}^{(2)}(X,2)}{V_{i(n)}^{(2)}(X,2)}.$$

Theorem 22 Assume that conditions of Proposition 14 are satisfied for two sequences of partitions $(\pi_{i(n)})_{n\geqslant 1}$ and $(\pi_{j(n)})_{n\geqslant 1}$ of [0,T] with the properties mentioned above. Then

$$V_{\pi_{k(n)}}^{(2)}(X,2) \longrightarrow 2\kappa^2 \int_0^T g(\ell(t)) dt \quad a.s. \quad as \ n \to \infty$$
 (37)

for k(n) = i(n) and for k(n) = j(n). If sequences of partitions $\{\pi_{i(n)}\}$ and $\{\pi_{j(n)}\}$, i(n) < j(n), are regular or such that $p_{j(n)}/p_{i(n)} \to 0$ as $n \to \infty$, then

$$\widehat{\gamma}_n \xrightarrow{\mathbf{a.s.}} \gamma.$$

Proof. Proposition 14 yields the limit (37). It is evident that

$$\frac{1}{2m_n^{2\gamma+1}} \leqslant \frac{\Delta t_{i(k)}^n}{\mu_k^n} \leqslant \frac{1}{2p_n^{2\gamma+1}}$$

and

$$\left(\frac{p_{i(n)}}{m_{j(n)}}\right)^{2\gamma+1} \frac{V_{j(n)}^{(2)}(X,2)}{V_{i(n)}^{(2)}(X,2)} \leqslant \frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)} \leqslant \left(\frac{m_{i(n)}}{p_{j(n)}}\right)^{2\gamma+1} \frac{V_{j(n)}^{(2)}(X,2)}{V_{i(n)}^{(2)}(X,2)}.$$

Next, since $\ln(p_{i(n)}/m_{i(n)}) \leq 0$ and

$$\frac{m_{i(n)}^{2\gamma+1}V_{j(n)}^{(2)}(X,2)}{p_{j(n)}^{2\gamma+1}V_{i(n)}^{(2)}(X,2)}\Big/\frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\geqslant 1,$$

we have

$$\begin{split} \widehat{\gamma}_{n} &= -\frac{1}{2} + \frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \left((2\gamma + 1)\ln(p_{j(n)}/m_{i(n)}) + \ln \frac{m_{i(n)}^{2\gamma + 1}V_{j(n)}^{(2)}(X, 2)}{p_{j(n)}^{2\gamma + 1}V_{i(n)}^{(2)}(X, 2)} \right) \\ &= \gamma + \frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \ln \frac{m_{i(n)}^{2\gamma + 1}V_{j(n)}^{(2)}(X, 2)}{p_{j(n)}^{2\gamma + 1}V_{i(n)}^{(2)}(X, 2)} \\ &= \gamma + \frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \ln \frac{V_{\pi_{j(n)}}^{(2)}(X, 2)}{V_{\pi_{i(n)}}^{(2)}(X, 2)} \\ &+ \frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \ln \left(\frac{m_{i(n)}^{2\gamma + 1}V_{j(n)}^{(2)}(X, 2)}{p_{j(n)}^{2\gamma + 1}V_{i(n)}^{(2)}(X, 2)} \middle/ \frac{V_{\pi_{j(n)}}^{(2)}(X, 2)}{V_{\pi_{i(n)}}^{(2)}(X, 2)} \right) \\ &\leqslant \gamma + \frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \ln \frac{V_{\pi_{j(n)}}^{(2)}(X, 2)}{V_{\pi_{j(n)}}^{(2)}(X, 2)} . \end{split}$$

In the same way we get

$$\widehat{\gamma}_{n} = -\frac{1}{2} + \frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \left((2\gamma + 1) \ln(m_{j(n)}/p_{i(n)}) + \ln \frac{p_{i(n)}^{2\gamma+1}V_{j(n)}^{2\gamma}(X,2)}{m_{j(n)}^{2\gamma+1}V_{i(n)}^{2\gamma}(X,2)} \right) \\
= -\frac{1}{2} + \left(\gamma + \frac{1}{2} \right) \frac{\ln(m_{j(n)}/p_{i(n)})}{\ln(p_{j(n)}/m_{i(n)})} + \frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \ln \frac{p_{i(n)}^{2\gamma+1}V_{j(n)}^{2\gamma}(X,2)}{m_{j(n)}^{2\gamma+1}V_{i(n)}^{2\gamma}(X,2)} \\
= \gamma + \left(\gamma + \frac{1}{2} \right) \frac{\ln(m_{j(n)}/p_{i(n)}) - \ln(p_{j(n)}/m_{i(n)})}{\ln(p_{j(n)}/m_{i(n)})} + \frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \ln \frac{V_{\pi_{j(n)}}^{2\gamma}(X,2)}{V_{\pi_{i(n)}}^{2\gamma}(X,2)} \\
+ \frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \ln \left(\frac{p_{i(n)}^{2\gamma+1}V_{j(n)}^{2\gamma}(X,2)}{m_{j(n)}^{2\gamma+1}V_{i(n)}^{2\gamma}(X,2)} / \frac{V_{\pi_{j(n)}}^{2\gamma}(X,2)}{V_{\pi_{i(n)}}^{2\gamma}(X,2)} \right) \\
\geqslant \gamma + \left(\gamma + \frac{1}{2} \right) \frac{\ln(m_{j(n)}/p_{i(n)}) - \ln(p_{j(n)}/m_{i(n)})}{\ln(p_{j(n)}/m_{i(n)})} + \frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \ln \frac{V_{\pi_{j(n)}}^{2\gamma}(X,2)}{V_{\pi_{i(n)}}^{2\gamma}(X,2)}, \tag{38}$$

since

$$\frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \ln\left(\frac{p_{i(n)}^{2\gamma+1}V_{j(n)}^{(2)}(X,2)}{m_{j(n)}^{2\gamma+1}V_{i(n)}^{(2)}(X,2)} \middle/ \frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\right) \geqslant 0$$

and

$$\left(\gamma + \frac{1}{2}\right) \frac{\ln(m_{j(n)}/p_{i(n)}) - \ln(p_{j(n)}/m_{i(n)})}{\ln(p_{j(n)}/m_{i(n)})} \leqslant 0.$$

If sequences of partitions $\{\pi_{i(n)}\}$ and $\{\pi_{j(n)}\}$, i(n) < j(n), are regular then the second term in the inequality (38) is equal to 0 and

$$|\widehat{\gamma}_n - \gamma| \le \frac{1}{2\ln(m_{i(n)}/p_{j(n)})} \left| \ln \frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{j(n)}}^{(2)}(X,2)} \right|.$$

Under conditions of the theorem in the regular case of partitions the statement of the theorem holds. For arbitrary partitions we obtain inequalities

$$\left|\widehat{\gamma}_n - \gamma - \frac{1}{2\ln(p_{j(n)}/m_{i(n)})} \ln \frac{V_{\pi_{j(n)}}^{(2)}(X,2)}{V_{\pi_{i(n)}}^{(2)}(X,2)}\right| \leq \left(\gamma + \frac{1}{2}\right) \frac{\ln(m_{j(n)}/p_{j(n)}) + \ln(m_{i(n)}/p_{i(n)})}{\ln(m_{i(n)}/p_{j(n)})}$$

and

$$|\widehat{\gamma}_{n} - \gamma| \leq \left| \widehat{\gamma}_{n} - \gamma - \frac{1}{2 \ln(p_{j(n)}/m_{i(n)})} \ln \frac{V_{\pi_{j(n)}}^{(2)}(X, 2)}{V_{\pi_{i(n)}}^{(2)}(X, 2)} + \frac{1}{2 \ln(p_{j(n)}/m_{i(n)})} \ln \frac{V_{\pi_{j(n)}}^{(2)}(X, 2)}{V_{\pi_{i(n)}}^{(2)}(X, 2)} \right|$$

$$\leq \frac{3}{2} \frac{\ln(m_{j(n)}/p_{j(n)}) + \ln(m_{i(n)}/p_{i(n)})}{\ln(m_{i(n)}/p_{j(n)})} + \frac{1}{2 \ln(m_{i(n)}/p_{j(n)})} \left| \ln \frac{V_{\pi_{j(n)}}^{(2)}(X, 2)}{V_{\pi_{j(n)}}^{(2)}(X, 2)} \right|.$$

For arbitrary partitions $\{\pi_{i(n)}\}$ and $\{\pi_{j(n)}\}$, i(n) < j(n), the second term in above inequality goes to 0 as $\ln(p_{i(n)}/p_{j(n)}) \to \infty$, $n \to \infty$. Thus the statement of the theorem holds.

5 Appendix

5.1 Proof of Lemma 4

Assume, without lost of generality, that 0 < h < 1. We first prove that $\widehat{\gamma}_* \leqslant \gamma_*$, where

$$\widehat{\gamma}_* := \limsup_{h\downarrow 0} \sup_{\varphi(h) \leqslant s \leqslant T-h} \frac{\ln \sigma_X(s,s+h)}{\ln h} \,, \quad \gamma_* := \inf\bigg\{\gamma > 0 \colon \lim_{h\downarrow 0} \sup_{\varphi(h) \leqslant s \leqslant T-h} \frac{h^\gamma}{\sigma_X(s,s+h)} = 0\bigg\}.$$

Let $\gamma > \gamma_*$. It suffices to show that $\gamma \geqslant \widehat{\gamma}_*$. By definition of the infimum, there exists a real number α such that $\gamma > \alpha > \gamma_*$, and

$$\sup_{\varphi(h) \leqslant s \leqslant T-h} \frac{h^{\alpha}}{\sigma_X(s,s+h)} \longrightarrow 0 \quad \text{as } h \downarrow 0.$$

But

$$\sup_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^{\gamma}}{\sigma_X(s,s+h)} = h^{\gamma-\alpha} \sup_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^{\alpha}}{\sigma_X(s,s+h)} \longrightarrow 0 \quad \text{as } h\downarrow 0$$
 (39)

as the product of two functions tending to 0. Under the statement

$$\sup_{\varphi(h) \leqslant s \leqslant T-h} \sigma_X(s, s+h) \longrightarrow 0 \quad \text{as } h \downarrow 0$$
 (40)

and relation (39) there exists an h_0 such that for all $h \leq h_0 < 1$

$$\sup_{\varphi(h)\leqslant s\leqslant T-h}\frac{h^{\gamma}}{\sigma_X(s,s+h)}=\frac{h^{\gamma}}{\inf_{\varphi(h)\leqslant s\leqslant T-h}\sigma_X(s,s+h)}<1\quad\text{and}\quad \sup_{0\leqslant s\leqslant T-h}\sigma_X(s,s+h)<1.$$

Moreover,

$$h^{\gamma} < \inf_{\varphi(h) \leqslant s \leqslant T-h} \sigma_X(s, s+h)$$

for all $h \leq h_0 < 1$. So

$$\ln h^{\gamma} < \ln \left(\inf_{\varphi(h) \leqslant s \leqslant T - h} \sigma_X(s, s + h) \right) \leqslant \ln \left(\sup_{\varphi(h) \leqslant s \leqslant T - h} \sigma_X(s, s + h) \right)$$

and

$$\gamma > \frac{\ln\left(\sup_{\varphi(h)\leqslant s\leqslant T-h} \sigma_X(s,s+h)\right)}{\ln h} = \sup_{\varphi(h)\leqslant s\leqslant T-h} \frac{\ln \sigma_X(s,s+h)}{\ln h}$$
$$\geqslant \limsup_{h\downarrow 0} \sup_{\varphi(h)\leqslant s\leqslant T-h} \frac{\ln \sigma_X(s,s+h)}{\ln h} = \widehat{\gamma}_* .$$

Thus $\widehat{\gamma}_* \leqslant \gamma_*$.

Next we prove that $\widehat{\gamma}_* \ge \gamma_*$. Let $\gamma > \alpha > \widehat{\gamma}_*$. It suffices to show that $\gamma \ge \gamma_*$. Under the condition $\alpha > \widehat{\gamma}_*$ and statement (40) there exists h_0 such that for $h \le h_0 < 1$

$$\inf_{\varphi(h) \leqslant s \leqslant T-h} \frac{\ln \sigma_X(s,s+h)}{\ln h} < \alpha, \qquad \sup_{0 \leqslant s \leqslant T-h} \sigma_X(s,s+h) < 1.$$

This implies the inequality

$$\ln\left(\inf_{\varphi(h)\leq s\leq T-h}\sigma_X(s,s+h)\right) > \ln h^{\alpha}$$

and

$$\inf_{\varphi(h)\leqslant s\leqslant T-h}\sigma_X(s,s+h)>h^{\alpha}.$$

Thus

$$\sup_{\varphi(h)\leqslant s\leqslant T-h}\frac{h^{\alpha}}{\sigma_X(s,s+h)}<1.$$

So

$$\sup_{\varphi(h)\leqslant s\leqslant T-h}\frac{h^{\gamma}}{\sigma_X(s,s+h)}< h^{\gamma-\alpha}\longrightarrow 0\quad \text{as }h\to 0.$$

Therefore $\gamma \geqslant \gamma_*$.

Now we prove that $\widehat{\gamma}^* = \gamma^*$, where

$$\widehat{\gamma}^* := \liminf_{h\downarrow 0} \inf_{\varphi(h)\leqslant s\leqslant T-h} \frac{\ln \sigma_X(s,s+h)}{\ln h}, \quad \gamma^* := \sup\bigg\{\gamma > 0 \colon \lim_{h\downarrow 0} \inf_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^\gamma}{\sigma_X(s,s+h)} = +\infty\bigg\}.$$

We first prove $\hat{\gamma}^* \ge \gamma^*$. By definition of supremum, there exists a real number γ such that $\gamma^* > \gamma$, and

$$\lim_{h \downarrow 0} \inf_{\varphi(h) \leqslant s \leqslant T - h} \frac{h^{\gamma}}{\sigma_X(s, s + h)} = +\infty$$
(41)

It suffices to show that $\widehat{\gamma}^* \geqslant \gamma$. Under the condition $\gamma^* > \gamma$ and statements (40)-(41) there exists h_0 such that for $h \leqslant h_0 < 1$

$$\inf_{\varphi(h)\leqslant s\leqslant T-h} \frac{h^{\gamma}}{\sigma_X(s,s+h)} > 1, \qquad \sup_{0\leqslant s\leqslant T-h} \sigma_X(s,s+h) < 1.$$

Moreover,

$$h^{\gamma} > \sup_{\varphi(h) \leqslant s \leqslant T-h} \sigma_X(s,s+h) \geqslant \inf_{\varphi(h) \leqslant s \leqslant T-h} \sigma_X(s,s+h)$$

and

$$\gamma \ln h > \ln \inf_{\varphi(h) \le s \le T - h} \sigma_X(s, s + h), \qquad \inf_{\varphi(h) \le s \le T - h} \frac{\ln \sigma_X(s, s + h)}{\ln h} > \gamma.$$

So $\widehat{\gamma}^* > \gamma$

We show that $\gamma^* \geqslant \widehat{\gamma}^*$. Assume that $\widehat{\gamma}^* > \alpha > \gamma$. It sufficient to show that $\gamma^* > \gamma$. Under the condition $\widehat{\gamma}^* > \alpha$ and statement (40) there exists h_0 such that for $h \leqslant h_0 < 1$

$$\inf_{\varphi(h)\leqslant s\leqslant T-h}\frac{\ln\sigma_X(s,s+h)}{\ln h}>\alpha,\qquad \sup_{0\leqslant s\leqslant T-h}\sigma_X(s,s+h)<1.$$

Moreover,

$$\sup_{\varphi(h) \leq s \leq T-h} \frac{\ln \sigma_X(s, s+h)}{\ln h} > \alpha$$

and

$$\ln \Big(\sup_{\varphi(h) \leqslant s \leqslant T-h} \sigma_X(s, s+h) \Big) < \ln h^{\alpha}.$$

Thus

$$\sup_{\varphi(h) \leqslant s \leqslant T-h} \sigma_X(s,s+h) < h^{\alpha} \quad \text{and} \quad \inf_{\varphi(h) \leqslant s \leqslant T-h} \frac{h^{\alpha}}{\sigma_X(s,s+h)} > 1.$$

Then

$$\inf_{\varphi(h)\leqslant s\leqslant T-h}\frac{h^{\gamma}}{\sigma_X(s,s+h)}>h^{\gamma-\alpha}\to\infty$$

and $\gamma^* > \gamma$.

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