Vega research groups

Rough Bergomi model, ATM skew, Muravlev's representation of fBM

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Forward variance swap

Payoff of the forward variance swap on $[t, T]$ at time T is

$$h(S) = \sum_{i=1}^{n} \left(\log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 - K, \quad t = t_0 < \ldots < t_n = T,$$

i.e. it is the forward on realized variance on $[t, T]$.
For continuous time price model (under risk neutral measure with $r = 0$)

$$dS_t = \sigma_t S_t \, dW_t, \quad d \log S_t = -\frac{1}{2} \sigma_t^2 \, dt + \sigma_t \, dW_t,$$

the payoff can be approximated by $\int_t^T \sigma_u^2 \, du - K$, which fair price is equal to

$$K^* = \mathbb{E}_t \int_t^T \sigma_u^2 \, du.$$

Forward variance swap

The forward variance swap price can be rewritten in terms of *forward variances*:

$$K^* = \int_t^T \xi_t(u)\,du, \quad \xi_t(u) = \mathbb{E}_t\,\sigma_u^2.$$

On the other side, the integral can be calculated from SDE for $\log S_t$, hence

$$K^* = \int_t^T \xi_t(u)\,du = -2\mathbb{E}_t \log\left(\frac{S_T}{S_t}\right),$$

so the payoff of variance swap is proportional to the payoff of the European option with logarithmic payoff function $g(S_T) = \log S_T$.

Calibration of the variance curve

The logarithmic option can be approximated by puts and calls with the use of Carr–Madan formula[1], that holds for any $k > 0$:

$$g(x) = g(k) + g'(k)(x-k) + \\ + \int_{K>k} g''(K)(x-K)^+ \, dK + \int_{0<K<k} g''(K)(K-x)^+ \, dK.$$

Calibration of the initial variance curve $\xi_t(\cdot)$ can be done by the algorithm:

- Approximate variance swap prices by Carr–Madan formula.
- Interpolate the curve with respect to T.
- Differentiate it to obtain $\xi_t(u)$.

[1] Carr, P. and Madan, D. (1998). Towards a theory of volatility trading. In: Robert A. Jarrow (ed), Volatility: New Estimation Techniques for Pricing Derivatives, pp. 417–427. London: RISK Publications.

Forward variance swap: remarks

- Along with variance swaps one can consider volatility swaps, the payoff of which is $\sqrt{\int_t^T \sigma_u^2\, du} - K$. In this case pricing problem is more intricate due to the nonlinearity of the square root function.
- One of the well known tradable volatility indices is VIX, the CBOE volatility index, defined as the square root of the fair price of a variance swap on the S&P index over a 30-days time interval:

$$\mathrm{VIX}_t = \sqrt{\mathbb{E}_t \int_t^{t+30\text{ days}} \sigma_u^2\, du}.$$

Futures and options on VIX were intoduced in 2004 and 2006 respectively.

Stochastic volatility model in forward variance curve form

Let S_t denote the stock price. The variance process can be written in terms of forward variances:

$$v_t = \sigma_t^2 = \xi_t(t), \quad \xi_t(u) = \mathbb{E}_t\left[v_u\right].$$

Under risk neutral measure the price and forward variance dynamics is described by the equations

$$\begin{cases} \frac{dS_t}{S_t} = \sqrt{\xi_t(t)}\, dB_t, \\ d\xi_t(u) = \varepsilon\lambda(t, u, \xi_t(u))\, dW_t \end{cases}$$

with correlated Brownian motions $dB_t dW_t = \rho dt$.

Bergomi–Guyon expansion

The Bergomi–Guyon[2] small noise expansion provides asymptotics for *ATM skew*

$$\psi(T) = \frac{\partial}{\partial k}\sigma_{BS}(k,T)\Big|_{k=0} \approx \sigma_{VS}\left[\frac{1}{2w^2}C^{x\xi} + \frac{1}{8w^3}(4wC^{\mu} + 3(C^{x\xi})^2)\right]$$

where $w = \int_0^T \xi_0(u)\,du,\ \sigma_{VS} = \sqrt{\frac{w}{T}}$ and

$$C^{x\xi} = \int_0^T dt \int_t^T du \frac{\mathbb{E}_t\left[d\log(S_t)d\xi_t(u)\right]}{dt},$$

$$C^{\mu} = \int_0^T dt \int_t^T du \frac{\mathbb{E}_t\left[d\log(S_t)d\xi_t(u)\right]}{dt}\frac{\delta C_t^{x\xi}}{\delta\xi_t(u)}.$$

[2] L. Bergomi and J. Guyon. Stochastic volatility's orderly smiles. Risk May, pages 60–66, 2012.

Bergomi model

n-factor Bergomi model[3] is a special case of model in variance forward curve form with

$$\xi_t(u) = \xi_0(u)\mathcal{E}\left(\sum_{i=1}^{n} \eta_i \int_0^t e^{-\kappa_i(u-s)}\, dW_s^{(i)}\right),$$

or, in differential form,

$$\frac{d\xi_t(u)}{\xi_t(u)} = \sum_{i=1}^{n} \eta_i e^{-\kappa_i(u-t)}\, dW_t^{(i)}.$$

The parameters of the model are $\{\eta_i\}, \{\kappa_i\}, \{\rho_{B,W^{(i)}}\}, \{\rho_{W^{(i)},W^{(j)}}\}$.

[3] L. Bergomi. Smile dynamics II. Risk October, pages 67–73, 2005.

Bergomi model

- For an appropriate fit on volatility surface one needs at least two factors, i.e. the model with 7 parameters.
- ATM skew in Bergomi model can be shown to have the following asymptotic behavior for $\tau \ll 1$:

$$\psi(\tau) \sim \sum_{i=1}^{n} \frac{\eta_i}{\kappa_i \tau} \left(1 - \frac{1 - e^{-\kappa_i \tau}}{\kappa_i \tau}\right),$$

  i.e. it does not explode at $\tau = 0$.
- Forward variance processes can be considered as stochastic convolutions with exponential kernels.

Rough Bergomi model

The goal of *rough Bergomi (rBergomi) model*[4] is to reproduce the exploding power law behavior of ATM skew for small maturities. It can be achieved by changing the convolution kernel from the exponential to the power law:

$$\xi_t(u) = \xi_0(u)\mathcal{E}\left(\eta \int_0^t \frac{dW_s}{(u-s)^\gamma}\, ds\right).$$

If $\xi_0(u)$ is constant, the variance process $v_t = v_0 \exp\left\{\eta V_t - \frac{\eta^2}{2}\mathbb{E}V_t^2\right\}$, where $V_t = \int_0^t \frac{dW_s}{(u-s)^\gamma}\, ds$ is a Volterra convolution process.

It can be shown that the ATM skew follows the power law for small values of τ:

$$\psi(\tau) \sim \frac{1}{\tau^\alpha}.$$

[4] Bayer, C., Friz, P., Gatheral, J.: Pricing under rough volatility, Quantitative Finance, 16(6):887–904, 2016.

Bergomi vs rBergomi

- rBergomi model is not Markovian, the variance process is not a semimartingale.
- According to the authors of the article, exponential kernels in Bergomi model approximate **more realistic** power-law kernels. Hence, the Bergomi model is just a Markovian **engineering approximation** to its rough alternative.
- Empirical analysis shows that for small maturities

$$\psi(\tau) \sim \frac{1}{\tau^{\alpha}}$$

for some $\alpha \in \left(0, \frac{1}{2}\right)$.

Validation of previously implemented method

- The idea of an exact approach is to find a joint distribution for Wiener and Volterra processes via Cholesky decomposition. If $s < t$:

$$\mathbb{E}[V_t V_s] = s^{2H} g\left(\frac{t}{s}\right) \quad \mathbb{E}[V_t W_s] = \rho \frac{\sqrt{2H}}{H + 1/2} \left\{ t^{H+1/2} - (t - \min(t, s))^{H+1/2} \right\}$$

  Where $g(x) = 2H \int_0^1 \frac{ds}{(1-s)^{1/2-H}(x-s)^{1/2-H}}$

- For validation of previous approach we implemented the exact method and applied Kolmogorov-Smirnov test at last moment.

As a result, we are sure that the hypothesis of equality of distributions should not be rejected.

But does ATM skew follow a power law?[5]

In the article for 4 models of the form

$$\frac{\xi_t(u)}{\xi_t(u)} = K(u-t)\,dW_t, \quad dB_t dW_t = \rho dt,$$

the quality of ATM curve fitness was compared for SPX, SX5E and DAX indices over the date range from 2020 Jan 1 to 2021 Dec 31. The considered models are

- 2-factor Bergomi, $K_{2fB}(\tau) = c_1 e^{-\kappa_1 \tau} + c_2 e^{-\kappa_2 \tau}$;
- Rough Bergomi, $K_{rB}(\tau) = \frac{w}{\tau^\alpha}$;
- Time shifted power law, $K_{TSPL}(\tau) = \frac{w}{(\tau+\delta)^\alpha}$;
- Power law of the function $I(x) = \frac{1-e^{-x}}{x}$, $K_{PLI}(\tau) = wI(\kappa\tau)^\alpha$.

[5] Guyon, Julien and El Amrani, Mehdi, Does the Term-Structure of Equity At-the-Money Skew Really Follow a Power Law? (July 27, 2022).

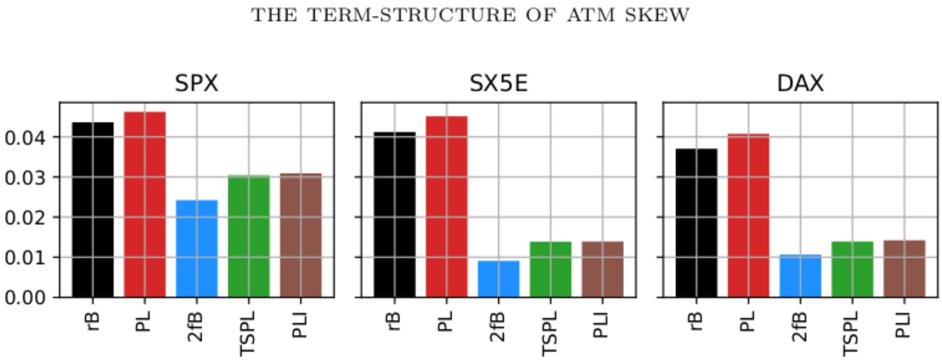
Models calibration process

- For T smaller than 3 years interpolate smiles using cubic spline.
- Calculate ATM skews at monthly maturities.
- Calculate initial forward variance $\xi_0(u) = \dfrac{d}{du}\left[u\mathrm{VS}(u)\right]$, where the variance swap rates are obtained via Carr-Madan approximation.
- Estimate ATM skew using Bergomi–Guyon representation. Integrals in the coefficients are computed with the use of Gaussian quadrature.
- Optimization of the model parameters using least-square minimization with L^2 regularization to fit market data.
- Fit the power-law curve (PL).
- Root Mean Squared Error (RMSE) is then used as a fit quality metric.

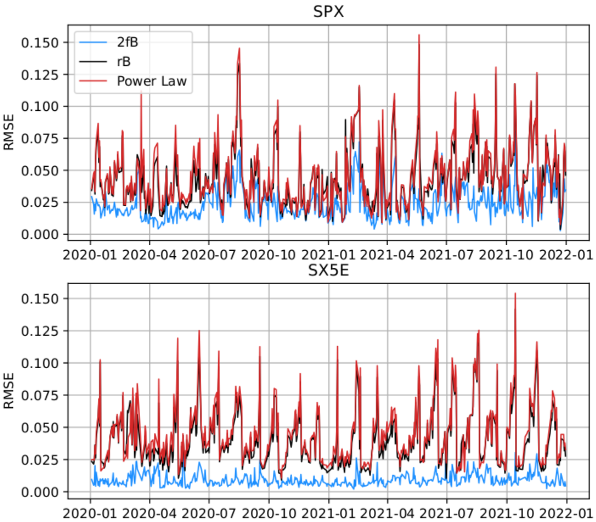
Comparison results

- 2-factor Bergomi model performed better that rough Bergomi and power law in more than 99% cases.
- A strong seasonality of error is observed for rough Bergomi and power law models, the error is getting larger when time to maturity is close to 0.
- Often, the power-law decay has not enough flexibility to accurately fit both the short and long ATM skews.
- The distribution of the extrapolated zero-maturity ATM skew in 2fB, TSPL and PLI models resembles the lognormal that peaks around 1.5.

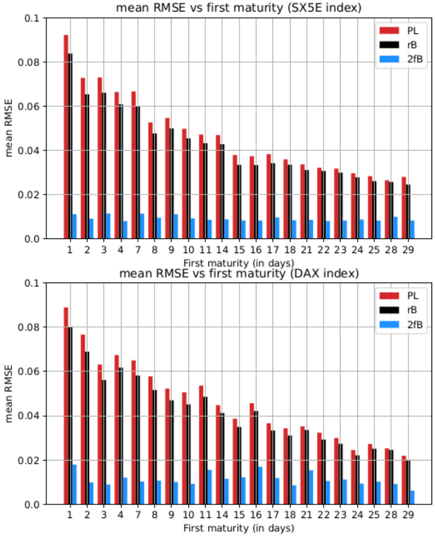
ATM skew fit: figures



ATM skew fit: figures



ATM skew fit: figures



ATM skew fit: figures

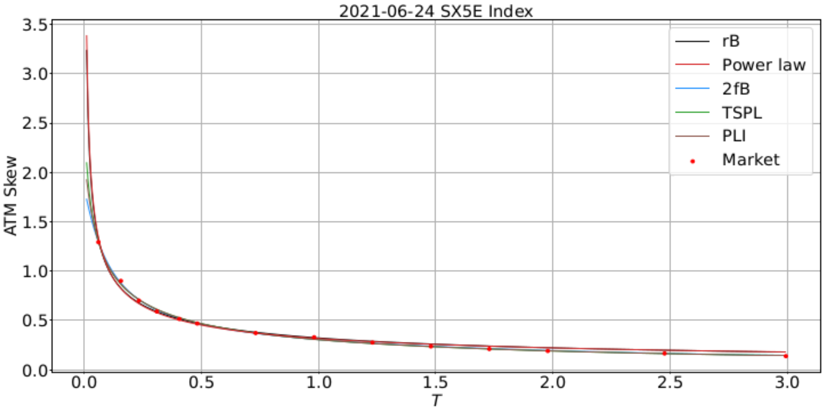


Figure: Successful fit for all models.

ATM skew fit: figures

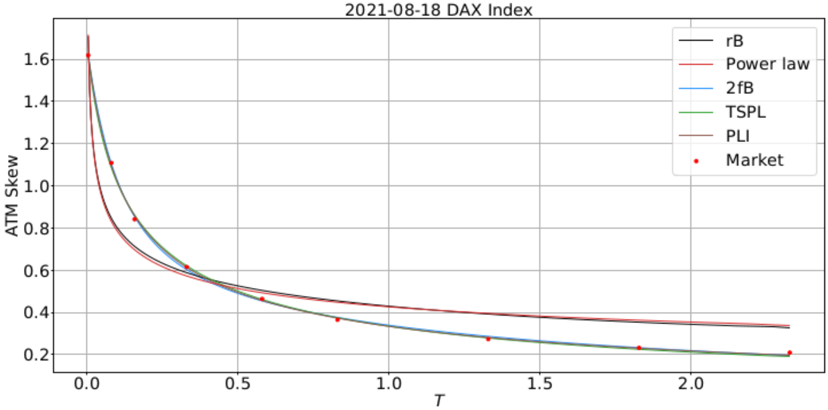


Figure: rB and PL fails to fit the curve for the first maturity date T_1 close to 0.

Muravlev's representation[6]

Let $\xi = (\xi_\beta)_{\beta>0}$ be Gaussian process with zero mean and auto-correlation function $R_\xi(\alpha, \beta) = (\alpha + \beta)^{-1}$, and B_t is an independent standard Brownian motion.
Let $\{Z_\beta\}_{\beta>0}$ be the family of processes, where $Z^\beta = (Z_t^\beta)_{t\geq 0}$ is an Ornstein-Uhlenbeck process, i.e. a solution to the SDE:

$$dZ_t^\beta = -\beta Z_t^\beta dt + dB_t, \quad Z_0^\beta = \xi_\beta.$$

[6] A. A. Muravlev, "Representation of a fractional Brownian motion in terms of an infinite-dimensional Ornstein–Uhlenbeck process", Russian Math. Surveys, 66:2 (2011), 439–441

Muravlev's representation[7]

Theorem 1

Process $\overline{B}^H = (\overline{B}^H_t)_{t\geq 0}$ *with* $H \in (0, \frac{1}{2})$

$$\overline{B}^H_t = c_H \int_0^\infty \beta^{-1/2-H} \left(Z^\beta_t - \xi_\beta\right) d\beta, \text{ where}$$

$$c_H = \frac{[\Gamma(2H+1)\sin(\pi H)]^{1/2}}{\mathrm{B}(1/2+H, 1/2-H)},$$

is a fractional Brownian motion with Hurst parameter H.

[7] A. A. Muravlev, "Representation of a fractional Brownian motion in terms of an infinite-dimensional Ornstein–Uhlenbeck process", Russian Math. Surveys, 66:2 (2011), 439–441

Idea of the proof

It is sufficient to check that the auto-correlation function of $\overline{B}^H$ is equal to the one of fBM.
Let us describe from where we have such representation. Recall Mandelbrot-van Ness representation of fBM with $H \in (0, \frac{1}{2})$:

$$B_t^H = c_H \int_{-\infty}^{0} \left[\int_0^{\infty} \left(e^{-\beta(t-s)} - e^{\beta s} \right) \beta^{-1/2-H} d\beta \right] dB_s \\ + c_H \int_0^t \left[\int_0^{\infty} e^{-\beta(t-s)} \beta^{-1/2-H} d\beta \right] dB_s$$

Idea of the proof

We use the stochastic Fubini theorem to interchange the order of integration.

$$B_t^H = c_H \int_0^\infty \beta^{-1/2-H} \left(e^{-\beta t} - 1\right) \left[\int_{-\infty}^0 e^{\beta s} dB_s\right] d\beta$$
$$+ c_H \int_0^\infty \beta^{-1/2-H} \left[\int_0^t e^{-\beta(t-s)} dB_s\right] d\beta =$$
$$= c_H \int_0^\infty \beta^{-1/2-H} \left[\left(-1 + e^{-\beta t}\right) \int_{-\infty}^0 e^{\beta s} dB_s + \int_0^t e^{-\beta(t-s)} dB_s\right] d\beta$$

Recall the solution to the Ornstein-Uhlenbeck SDE:

$$Z_t^\beta = \xi_\beta e^{-\beta t} + \int_0^t e^{-\beta(t-s)} dB_s$$

Idea of the proof

Take $X_\beta = \int_0^\infty e^{-\beta s} dB_s$ and calculate the correlation

$$\mathbb{E} X_\alpha X_\beta = \int_0^\infty e^{-(\alpha+\beta)s} ds = \frac{1}{\alpha+\beta}$$

Thus it is exactly process ξ_β.
And we get the wanted representation

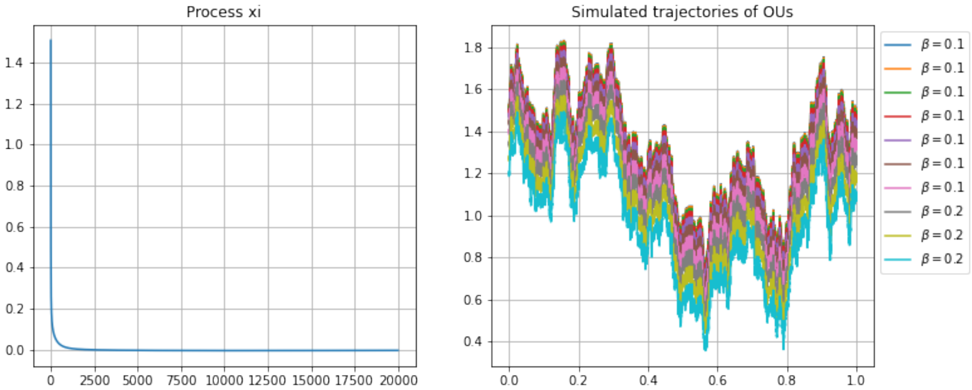
$$B_t^H = c_H \int_0^\infty \beta^{-1/2-H} \left[\left(-1 + e^{-\beta t} \right) \xi_\beta + \int_0^t e^{-\beta(t-s)} dB_s \right] d\beta =$$
$$= c_H \int_0^\infty \beta^{-1/2-H} \left[-\xi_\beta + Z_t^\beta \right] d\beta$$

Simulation of fBM via Muravlev's representation

We simulate this process as follows:

- Take uniform/non-uniform grid β_j and simulate values of ξ_{β_j} from the multivariate normal distribution,
- Simulate a trajectory of the Brownian motion at points $t_i = i\Delta t$, $i = 0, \ldots, n$,
- Simulate O-U processes with the above ξ_{β_j} as initial condition using Euler's scheme,
- Approximate the integral via Simpson's method.

Ornstein-Uhlenbeck trajectories



Analysis of simulation accuracy across different time grids