Research group ’Stochastic Volatility Models, Rough volatility’

Hybrid scheme for simulation in rough volatility models

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Heston Model Definition[1]

Recall Heston Model where the asset price S follows the following dynamic:

$$dS_t = S_t \sqrt{V_t} dW_t$$
$$dV_t = \lambda (\theta - V_t) dt + \lambda \nu \sqrt{V_t} dB_t.$$

Here the parameters λ, θ, V_0 and ν are positive, and W_t and B_t are two Brownian motions with correlation coefficient ρ, that is $dW_t dB_t = \rho dt$.

[1] S. L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies, 6(2):327–343, 1993

Mandelbrot-van Ness representation

Let us recall that a fractional Brownian motion W^H with Hurst parameter $H \in (0,1)$ can be constructed through the Mandelbrot-van Ness representation:

$$W_t^H = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^{0} \left((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right) dW_s + \\ + \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$$

The rough dynamics of the fBM for $H < 1/2$ is caused primarily by the kernel $(t-s)^{H-\frac{1}{2}}$. In particular,

$$\int_0^t (t-s)^{H-\frac{1}{2}} dW_s$$

has Hölder regularity ($H - \varepsilon$) for any $\varepsilon > 0$.

Rough Heston model[2]

In order to allow for a rough behavior of the volatility in a Heston-type model, we can introduce the kernel $(t-s)^{\alpha-1}$ in a Heston-like stochastic volatility process as follows:

$$dS_t = S_t\sqrt{V_t}dW_t$$
$$V_t = V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda\left(\theta - V_s\right)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda\nu\sqrt{V_s}dB_s$$

The parameters λ, θ, V_0 and ν are positive and play the same role as in standard Heston model, and here also W_t and B_t are two Brownian motions with correlation ρ. The additional parameter α belongs to $(1/2, 1)$ and governs the smoothness of the volatility sample paths.

[2] El Euch, O. and Rosenbaum, M. (2019). The characteristic function of rough Heston models. Mathematical Finance, 29(1):3–38

Rough Bergomi stochastic volatility model[3]

Let $V_t \geq$ be an instantaneous spot variance process.
$\xi_t^u, u \geq t$ is the instantaneous forward variance for date u at time t; in particular, $\xi_t^t = V_t$ corresponds to the spot variance.
The rough Bergomi stochastic volatility model takes the form

$$dS_t = S_t \sqrt{V_t} dW_t$$
$$d\xi_t^u = \xi_t^u \eta \sqrt{2\alpha - 1} (u - t)^{\alpha - 1} dB_t$$

where $\alpha = H + \frac{1}{2} \in \left(\frac{1}{2}, 1\right)$, and $d\langle W, B\rangle_t = \rho dt$, and η is a positive number depending on H. Straightforward integration gives us the formula for V_t:

$$V_t := \xi_0^t \exp\left\{\eta\sqrt{2\alpha - 1}\int_0^t (t-s)^{\alpha-1} dB_s - \frac{\eta^2}{2} t^{2\alpha - 1}\right\}$$

[3] C. Bayer, P. K. Friz and J. Gatheral (2016). Pricing under rough volatility. Quant. Finance 16 (6), 887–904

Stochastic convolutions and Volterra equation

So we need to analyze a **Stochastic Volterra equation** (SVE):

$$X_t = X_0 + \int_0^t K(t-s)b\left(X_s\right)ds + \int_0^t K(t-s)\sigma\left(X_s\right)dW_s,$$

We will refer to the second integral as **stochastic convolution** and write the equation in a shorter form

$$X = X_0 + K * \left(b\left(X_t\right)dt + \sigma\left(X_t\right)dW_t\right),$$

where W_t is a Brownian motion, and $K(s)$ is a convolution **kernel**.

When the stochastic convolution is well-defined?

Lemma 1

If $K \in L^2(0,t)$ and $\sup_{s \le t} |\varphi_s| < \infty$ a.s. Then the convolution $\int_0^t K(t-s)\varphi_s \, dW_s$ is well-defined.

Proof.

The stochastic integral is well-defined since

$$\int_0^t K^2(t-s)\varphi_s^2 ds \leqslant \sup_{s \le t} |\varphi_s|^2 \cdot \int_0^t K^2(t-s) ds < \infty.$$

$\square$

When the stochastic convolution is well-defined?

Lemma 2

If φ is locally bounded, $H \in (0, 1]$, then

$$X_t := \int_0^t (t-s)^{H-1/2} \varphi_s dW_s, \quad t \geqslant 0$$

has a version which is $(H-\varepsilon)$-Hölder continuous for any $\varepsilon > 0$.

Remark Similar result holds for $\int_0^t (t-s)^{H-\frac{1}{2}} \varphi_s ds$.

Remark $X_t := \int_0^t (t-s)^{H-1/2} \varphi_s dW_s$ is not a semimartingale in general. So, there is no Itô formula, and standard stochastic analysis is inapplicable.

Existence of Stochastic Volterra equation solution

Given: $\left(\Omega, F, (F_t)_{t\geqslant 0}, \mathbb{P}\right)$, $(W_t)_{t\geqslant 0}, b = b(x), \quad \sigma = \sigma(x), \quad K = K(t), \quad x_0 \in \mathbb{R}$
Strong solution if a continuous adapted X that satisfies

$$X_t = X_0 + \int_0^t K(t-s) b\left(X_s\right) ds + \int_0^t K(t-s)\sigma\left(X_s\right) dW_s, t \geqslant 0.$$

Theorem 3

Let b and σ be Lipschitz, $K(t) = t^{H-\frac{1}{2}}$, $H \in (0,1)$. Then the SVE has a unique strong solution.
Moreover the solution X is $(H-\varepsilon)$-Hölder continuous for any $\varepsilon > 0$.

Remark Solution X is not Markov in general. Hence, backward/forward Kolmogorov equations cannot be used.

Idea of the proof

The proof is based on Picard iteration. Let T be the time horizon. Define a Banach space: $\mathbb{H}_p =$ {adopted X with $\sup_{t\leq T}\mathbb{E}|X_t|^p < \infty$}, with a norm

$$\|X\|_p = \sup_{t\leq T}\mathbb{E}[e^{-\lambda t}|X_t|^p]^{\frac{1}{p}}, \quad \lambda > 0.$$

Define a mapping $\mathbb{A} : \mathbb{H}_p \to \mathbb{H}_p$ by

$$(\mathbb{A}X)_t = X_0 + \int_0^t K(t-s)(b\left(X_s\right)ds + \sigma\left(X_s\right)dW_s)$$

It can be shown that

$$\|\mathbb{A}X - \mathbb{A}Y\|_p \leqslant c \cdot \left(\int_0^\top e^{-2\lambda t}K(t)^2 dt\right)^{1/2} \|X - Y\|_p$$

Where $c = c(p, t, b, \sigma)$ doesn't depend on λ. Then choose λ large enough, that $\mathbb{A}$ is a contraction. Then apply Banach fixed point theorem.

Solution of SVE in Rough Heston model

In variance process

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda \left(\theta - V_s\right) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda \nu \sqrt{V_s} dB_s$$

the dB_S coefficient $\sqrt{x}$ is not Lipschitz, so the theorem can not be applied.

1. Make a regularization and replace $\sqrt{x}$ by $\sqrt{\varepsilon + (x-\varepsilon)^+}$.
2. Provided $\varepsilon \to 0$ one can show that the limit will be a weak solution.

Non-negative solutions

Definition 4

K is completely monotone, if for all k: $(-1)^k \frac{d^k K}{dt^k} \geq 0$.

Theorem 5

Assume K is completely monotone and $\exists \varepsilon > 0$ such that $b(x) \geq 0$, $\sigma(x) = 0$ for every $x \leq \varepsilon$. If $X_0 \geq 0$ then $X_t \geq 0$ for all $t \geq 0$.

Simulation of stochastic convolutions

The idea of the algorithm

The trickiest part here is the simulation of

$$Y_t = Y_0 + \int_0^t (t-s)^{\alpha-1}\sigma(s, Y_s)\, dW_s, \quad t \in [0, T].$$

The first naïve idea is to use plain Euler's discretization with step $\frac{1}{n}$:

$$Y_{i/n} \approx Y_0 + \sum_{k=1}^{i} \sigma\left(\frac{i-k}{n}, Y_{\frac{i-k}{n}}\right) \int_{\frac{i-k}{n}}^{\frac{i-k+1}{n}} \left(\frac{i}{n} - s\right)^{\alpha-1} dW_s$$

and approximate the integral by $\left(\frac{i}{n} - \frac{i-k}{n}\right)^{\alpha-1} (W_{\frac{i-k+1}{n}} - W_{\frac{i-k}{n}})$. However, due to the explosive behavior of the kernel this approximation fails to reproduce the rough structure of the process.

Simulation of stochastic convolutions

The idea of the algorithm

Hence, for small values of k the stochastic integrals should be simulated exactly, i.e. for a fixed parameter $\kappa \geq 0$

$$Y_{i/n} \approx Y_0 + \sum_{k=1}^{\min(\kappa,i)} \sigma\left(\frac{i-k}{n}, Y_{\frac{i-k}{n}}\right) \int_{\frac{i-k}{n}}^{\frac{i-k+1}{n}} \left(\frac{i}{n} - s\right)^{\alpha-1} dW_s + \\ + \sum_{k=\kappa+1}^{i} \sigma\left(\frac{i-k}{n}, Y_{\frac{i-k}{n}}\right) \left(\frac{b_k^*}{n}\right)^{\alpha-1} \left(W_{\frac{i-k+1}{n}} - W_{\frac{i-k}{n}}\right).$$

Here $\frac{b_k^*}{n}$ is an optimal value of $\left(\frac{i}{n} - s\right)$ for the integral approximation to be the most accurate.

Hybrid scheme

Algorithm [4]

For processes of the form: $Y(t) = \int_0^t (t-s)^{\alpha-1}\sigma(s)dW(s), t \geq 0$, we define the hybrid scheme to discretize $Y(t)$, for any $t \geq 0$, as

$$Y_n(t) := \check{Y}_n(t) + \hat{Y}_n(t),$$

where

$$\check{Y}_n(t) := \sum_{k=1}^{\min\{\lfloor nt \rfloor, \kappa\}} \sigma\left(t - \frac{k}{n}\right) \int_{t-\frac{k}{n}}^{t-\frac{k}{n}+\frac{1}{n}} (t-s)^{\alpha-1} dW(s),$$

$$\hat{Y}_n(t) := \sum_{k=\kappa+1}^{\lfloor nt \rfloor} \left(\frac{b_k}{n}\right)^{\alpha-1} \sigma\left(t - \frac{k}{n}\right) \left(W_{t-\frac{k}{n}+\frac{1}{n}} - W_{t-\frac{k}{n}}\right).$$

[4] M. Bennedsen, A. Lunde, and M. S. Pakkanen. Hybrid scheme for Brownian semistationary processes. arXiv preprint arXiv:1507.03004, 2015

Hybrid scheme

Algorithm

The observations $Y_n(\frac{i}{n})$, for $i = 0, 1, ..., \lfloor Tn \rfloor$, given by the hybrid scheme can be computed via

$$Y_n\left(\frac{i}{n}\right) = \sum_{k=1}^{\min\{i,\kappa\}} \sigma_{i-k}^n W_{i-k,k}^n + \sum_{k=\kappa+1}^{i} \left(\frac{b_k^*}{n}\right)^{\alpha-1} \sigma_{i-k}^n W_{i-k}^n,$$

where

$$b_k^* = \left(\frac{k^\alpha - (k-1)^\alpha}{\alpha}\right)^{1/(\alpha-1)}.$$

Hybrid scheme

Algorithm

The notation used in the scheme:

$$W_{i,j}^n := \int_{\frac{i}{n}}^{\frac{i+1}{n}} \left(\frac{i+j}{n} - s\right)^{\alpha-1} dW(s), \quad i = 0, \ldots, \lfloor nT \rfloor - 1, \quad j = 1, \ldots, \kappa,$$

$$W_i^n := \int_{\frac{i}{n}}^{\frac{i+1}{n}} dW(s), \qquad i = 0, \ldots, \lfloor nT \rfloor - 1,$$

$$\sigma_i^n := \sigma\left(\frac{i}{n}\right), \qquad i = 0, \ldots, \lfloor nT \rfloor - 1,$$

Hybrid scheme

Algorithm

The covariance matrix of Gaussian vector

$$W_i^n = \left(W_i^n, W_{i,1}^n, \dots, W_{i,\kappa}^n\right)$$

can be calculated explicitly:

$$\Sigma_{1,1} = \frac{1}{n}, \Sigma_{i,1} = \Sigma_{1,i} = \frac{i^\alpha - (i-1)^\alpha}{\alpha n^\alpha}, \Sigma_{i,i} = \frac{i^{2\alpha-1} - (i-1)^{2\alpha-1}}{(2\alpha-1)n^{2\alpha-1}}$$

$$\Sigma_{i,k} = \frac{1}{\alpha n^{2\alpha-1}} \left(a_{ik} - b_{ik}\right),$$

where

$$a_{ik} = i^\alpha k^{\alpha-1} \mathsf{F}\left(-\alpha-1, 1, \alpha+1, \frac{i}{k}\right)$$

$$b_{ik} = (i-1)^\alpha (k-1)^{\alpha-1} \mathsf{F}\left(-\alpha-1, 1, \alpha+1, \frac{i-1}{k-1}\right)$$

Hybrid scheme

Asymptotics of mean square error

Theorem 6

Suppose that the kernel function g is continuously differentiable on $(0, \infty)$, and that for some $\delta > 0$,

$$\mathbb{E}\left[|\sigma(s) - \sigma(0)|^2\right] = \mathcal{O}\left(s^{2\alpha-1+\delta}\right), \quad s \downarrow 0.$$

Then for all $t > 0$,

$$\mathbb{E}\left[|Y(t) - Y_n(t)|^2\right] \sim J(\alpha, \kappa, \mathbf{b})\mathbb{E}\left[\sigma(0)^2\right] n^{-(2\alpha-1)}, \quad n \to \infty,$$

where $J(\alpha, \kappa, \mathbf{b})$ doesn't depend on n.

Hybrid scheme

Complexity

Let's assume $N = \lfloor Tn \rfloor$. We care about $\sum_{k=\kappa+1}^{i} \left(\frac{b_k^*}{n}\right)^{\alpha-1} \sigma_{i-k}^n W_{i-k}^n$ term, since numerically it's the most difficult part.

Its straightforward computation gives us the complexity $O(N^2)$.

But in case of time-dependent $\sigma(s)$ we can calculate this term as convolution via FFT, which gives us

FFT modified algorithm complexity $O(N \log N)$.

To-Do

1. Numerically analyze the behavior of equity ATM skew for different rough stochastic volatility models and their alternatives (OUOU process).
2. Study and implement the infinite-dimensional Ornstein-Uhlenbeck representation of fBM proposed in (Muravlev, 2011).
3. Try to speed up the algorithm for Y-dependent coefficients and prove its correctness.
4. Empirically check whether the term-structure of equity ATM skew follow a power law (Amrani, Guyon, 2022).