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The asymptotic behavior of the R/S statistic for fractional Brownian motion

Wen Li[a,1], Cindy Yu[a], Alicia Carriquiry[a,*], Wolfgang Kliemann[b]

[a] Department of Statistics, Iowa State University, Ames, IA 50011, USA
[b] Department of Mathematics, Iowa State University, Ames, IA 50011, USA

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ABSTRACT

This paper provides a proof of the fact that asymptotically the R/S statistic and the self-similarity index of fractional Brownian motion agree in the expectation sense. In particular for fractional Gaussian noise time series, the R/S statistic is an estimator of the self-similarity index H. We also show that two other methods for estimating H yield consistent estimators.

1. Introduction: R/S statistic and Hurst exponent

The Hurst exponent, denoted by H, was proposed by Hurst (1951) to determine the design of an ideal water reservoir based upon observed discharges from a lake. To recall the original definition given by Hurst, let $\{X_k,\ k = 1, \dots, N\}$ be an observed time series and denote by $\bar{X}_n$ the average of X_k over n periods. For $k = 1, \dots, n$ one computes the running sum of the accumulated deviations from the mean as

$$Y_{k,n} = \sum_{u=1}^{k}(X_u - \bar{X}_n),$$

where $\bar{X}_n = n^{-1}\sum_u X_u$. The range over the time period n is defined as

$$R(n) = \max_{k=1,\dots,n}(Y_{k,n}) - \min_{k=1,\dots,n}(Y_{k,n}),$$

and the rescaled range is $\mathrm{R/S}(n) = R(n)/S(n)$, where $S(n)$ is the standard deviation of $X_k,\ k = 1, \dots, n$. This leads to the Hurst exponent of the observed time series on the time interval $k = 1, \dots, n$ as

$$H(n) = \frac{\log\{\mathrm{R/S}(n)\}}{\log(an)}, \tag{1}$$

where a is a constant. Hurst (1951) calculated R/S in many different datasets and concluded empirically that the value of 0.5 for a can describe well the linear relationship between $\log \mathrm{R/S}(n)$ and $\log n$.

* Corresponding author. Tel.: +1 515 450 3440; fax: +1 515 294 4040.
E-mail addresses: shirley@iastate.edu (W. Li), cindyyu@iastate.edu (C. Yu), alicia@iastate.edu (A. Carriquiry), kliemann@iastate.edu (W. Kliemann).

1 Present address: Singapore Clinical Research Institute, 31 Biopolis Way Nanos, Singapore, 138669, Singapore.

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In practice, to avoid using an arbitrary value of the unknown constant a, the Hurst exponent is estimated by averaging the rescaled range $\mathrm{R/S}(n)$ over several, non-overlapping periods of different length n. More precisely, one partitions the time interval $[1, N]$ into non-overlapping subintervals of length n for $n = N/2, N/4, \ldots$ and regresses $\log\{\mathrm{R/S}(n)\}$ on $\log n$ for several values of n. An estimate of the slope of this linear regression is taken as $\hat{H}_{\mathrm{R/S}}$. Hall et al. (2000) discuss the asymptotic distribution of $\hat{H}_{\mathrm{R/S}}$: for $3/4 < H < 1$ the asymptotic distribution of $\hat{H}_{\mathrm{R/S}}$ is the Rosenblatt distribution (Rosenblatt, 1961), while for $0 < H \leq 3/4$ one obtains the normal distribution.

Intuitively, the Hurst exponent measures the smoothness of a time series based on the asymptotic behavior of the rescaled range of the process. If $H = 1/2$, the behavior of the time series will be similar to that of a random walk; if $0 < H < 0.5$, the time series will be anti-persistent (i.e. exhibits short-term memory); if $0.5 < H < 1$, the time series will be persistent, usually characterized by long memory effects.

The Hurst exponent seems to have seen a resurgence in the literature over the past few years. Applications of H are found in the recent literature in areas as diverse as energy market analysis (Alvarez-Ramirez et al., 2008; Serletis and Rosenberg, 2007; Wang and Liu, 2010), neurosciences and other biological applications (Maxim et al., 2005; Valous et al., 2010), modeling Internet traffic (Kalagianis et al., 2004) and more traditional financial analyses and statistical theory (Alfarano and Lux, 2007; Carbone et al., 2004; Couillard and Davidson, 2004; Domino, in press; Grech and Pamula, 2008; Lillo and Farmer, 2004). (In some of these applications, a time-dependent version $H(t)$ of H is discussed.)

The R/S statistic (analysis) that defines the Hurst exponent is also a tool to determine long-range dependence. The statistic has several desirable properties relative to newer methods for detecting long-range dependence (e.g. analyzing autocorrelations, variance ratios, and spectral decompositions): Mandelbrot and Wallis (1969) show using Monte Carlo simulations that the R/S statistic is able to diagnose long-range dependence in highly non-Gaussian time series. Mandelbrot (1975) analyzes the distributional and almost sure convergence of the R/S statistic for certain stochastic processes, for which autocorrelations and variance ratios need not be well defined if the variances are infinite. Mandelbrot and Taqqu (1979) derive a robustness property of the R/S statistic. Finally, Mandelbrot (1972) argues that, unlike spectral analysis detecting periodic cycles, R/S analysis can detect non-periodic cycles with periods equal to or greater than the sample period (Lo and MacKinlay, 1999, pp. 147–166).

Lo and MacKinlay (1999, pp. 158–161) argue that this classical rescaled range statistic may be sensitive to short-range dependence. They propose a modification of the standard deviation by introducing a (maximum) time lag that is chosen depending on the given dataset for the resulting short-term and long-term memory asymptotics. Several other weaknesses of the statistic have been identified in the literature (e.g., Alfarano and Lux, 2007; Giraitis et al., 2003; Mielniczuk and Wojdyllo, 2006). Overall, it is fair to say that statistics with better properties than R/S are available to quantify long-range dependence (LRD) (e.g., Giraitis et al., 2003; Mielniczuk and Wojdyllo, 2006).

This paper mainly addresses almost sure convergence and convergence in the first moment of the properly scaled R/S statistic of fractional Gaussian noise. These results have been mentioned in the literature, compare, e.g., Taqqu et al. (1995), but we have not found a published proof. Thus, we revisit the asymptotic behavior of the scaled R/S statistic and provide a formal proof of the results mentioned in Taqqu et al. (1995). Given that the R/S statistic is still often used to estimate H (see, e.g., Cajueiro and Tabak, 2005; Matteo et al., 2005; Norouzzadeha and Jafaria, 2005; Weng et al., 2006) the discussion in this manuscript can contribute to its understanding and as a consequence also to a more in-depth understanding of the properties of other estimators of H. The manuscript is organized as follows. In Section 2 we review some of the properties of the classical R/S statistic. The main results on the R/S statistic are discussed in Section 3. Finally, some brief conclusions are offered in Section 4.

2. The self-similarity index, fractional Brownian motion and fractional Gaussian noise

In 1975, Mandelbrot coined the word "fractal" to describe a self-similar structure in stochastic processes.

Definition 2.1. A real-valued stochastic process $Y = \{Y_t\}_{t\in\mathbb{R}}$ is self-similar with index $H > 0$ (H-ss) if, for any $a > 0$ and any $t \in \mathbb{R}$, $Y_{at} \stackrel{d}{=} a^H Y_t$, where $\stackrel{d}{=}$ denotes equality of the distributions. Self-similarity for processes $\{Y_t\}_{t\geq 0}$ and $\{Y_t\}_{t>0}$ is defined in the same way as for $Y = \{Y_t\}_{t\in\mathbb{R}}$.

The self-similarity index describes invariance under time and space scaling of a process and therefore the index H is also called the scaling exponent. It can be used to detect memory structures in stochastic processes. Note that an H-ss process $\{Y_t\}$ with the first and second moments cannot be stationary, unless it is degenerate, compare Beran (1994), Section 2.3. (A process $\{Y_t\}$ is degenerate if $Y_t \equiv 0$.) While there are many different self-similar processes, the interest in time series analysis is usually on those processes that have stationary increments. Recall that a real-valued process $\{Y_t\}_{t\in\mathbb{R}}$ is said to have (strictly) stationary increments if all finite-dimensional distributions are shift invariant, i.e. for all $h \in \mathbb{R}$ and all finite number of time points $t_1, \ldots, t_k$ it holds that $\mathcal{D}(Y_{t_1+h} - Y_{t_1+h-1}, \ldots, Y_{t_k+h} - Y_{t_k+h-1}) = \mathcal{D}(Y_{t_1} - Y_{t_1-1}, \ldots, Y_{t_k} - Y_{t_k-1})$, where $\mathcal{D}(\cdot)$ denotes the distribution of a random variable.

Definition 2.2. A process $\{Y_t\}_{t\in\mathbb{R}}$ is called H-sssi if it is self-similar with index H and has strictly stationary increments.

Self-similar processes with stationary increments are of great interest in applications in time series analysis. Here, we list some properties of H-sssi processes $\{Y_t\}_{t\in\mathbb{R}}$ with finite first and second moments that will be useful in later arguments in this manuscript. Corresponding results hold for processes defined on the time set $\{t \geq 0\}$, compare, e.g., Taqqu (2003). The underlying probability space is denoted by $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$.

1. If $H \neq 1$, then $\mathbb{E}(Y_t) = 0$ for all $t \in \mathbb{R}$.
2. $\mathbb{E}(Y_t^2) = |t|^{2H}\sigma^2$. If $\mathbb{E}(Y_1^2) = \sigma^2 = 1$, the process $\{Y_t\}_{t\in\mathbb{R}}$ is called as standard.
3. The covariance function $\gamma_Y(s,t) = \mathbb{E}[\{Y_s - \mathbb{E}(Y_s)\}\{Y_t - \mathbb{E}(Y_t)\}]$ for $s, t \in \mathbb{R}$, is equal to $\sigma^2(|s|^{2H} + |t|^{2H} - |s-t|^{2H})/2$. Note that for $0 < H \leq 1$ the function $\gamma_Y(s,t)$ is non-negative definite.

Next, we turn to some properties of the increments of an H-sssi process $\{Y_t\}_{t\in\mathbb{R}}$. Let $k \in \mathbb{Z}$ and denote by $X_k := Y_k - Y_{k-1}$ the increment process $\{X_k\}_{k\in\mathbb{Z}}$.

4. The autocovariance function of $\{X_k\}_{k\in\mathbb{Z}}$ is given by
$$\gamma_X(h) = \frac{\sigma^2}{2}(|h+1|^{2H} - 2|h|^{2H} + |h-1|^{2H}).$$
5. If $H \neq 1/2$, then $\gamma_X(h) \sim \sigma^2 H(2H-1)|h|^{2H-2}$ as $h \to \infty$.

Note that, according to property 5., the covariance for the stationary increment sequence of a self-similar process with index H follows a power law structure approximately. In addition, this implies that for $0 < H < 1/2$ the autocovariance of $\{X_k\}_{k\in\mathbb{Z}}$ is negative and $\sum_{h=1}^{\infty}|\gamma_X(h)| < \infty$, so the increment sequence $\{X_k\}_{k\in\mathbb{Z}}$ of an H-sssi process $\{Y_t\}$ is mean-reverting and anti-persistent (i.e. has a short-term memory structure). If $1/2 < H < 1$, the covariance of $\{X_k\}_{k\in\mathbb{Z}}$ is positive and $\sum_{h=1}^{\infty}|\gamma_X(h)| = \infty$, hence in this case the sequence $\{X_k\}_{k\in\mathbb{Z}}$ is positively correlated and has a long-term memory structure.

The standard example of a self-similar process with stationary increments is fractional Brownian motion: any Gaussian H-sssi process $\{B_H(t)\}_{t\in\mathbb{R}}$ with $0 < H < 1$ is called a fractional Brownian motion (fBm). Its (stationary) increment process $\{X_k\}_{k\in\mathbb{Z}} := \{B_H(k) - B_H(k-1)\}_{k\in\mathbb{Z}}$ is called fractional Gaussian noise (fGn). When the self-similarity index H of the process is fixed at $H = 0.5$, fractional Brownian motion and fractional Gaussian noise become Brownian motion and Gaussian (white) noise, respectively. We refer the reader to Taqqu (2003) and the references therein for details on fractional Brownian motion.

3. Hurst exponent and self-similarity index for fractional Brownian motion

In 1995, Taqqu et al. mention that for fractional Gaussian noise (or fractional ARIMA) processes, one has the following asymptotic result: $\mathbb{E}\{\mathrm{R/S}(n)\} \sim C_H n^H$, as $n \to \infty$, where $\mathrm{R/S}(n)$ is the rescaled range (R/S) statistic (1), H is the self-similarity index defined in Definition 2.1, and C_H is a positive, finite constant not dependent on n. In this section we provide a proof for this and related results for fractional Gaussian noise.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space on which a continuous time fractional Brownian motion $\{B_H(t)\}_{t\geq 0}$ is defined with $B_H(0) = 0$ a.s. For a given $h > 0$, the increment process (fractional Gaussian noise) is defined as $X_0 = 0$, $X_k := B_H(kh) - B_H\{(k-1)h\}$ for $k \in \mathbb{N}$, with the variance $(\sigma h^H)^2$, where $\sigma^2 = \mathrm{Var}\{B_H(1)\}$. Note that $\sum_{i=1}^n X_i = B_H(nh)$, and we set $S^2(n) = \frac{1}{n}\sum_{i=1}^n X_i^2 - (\frac{1}{n}\sum_{i=1}^n X_i)^2$. Letting $Y_{k,n} = B_H(kh) - \frac{k}{n}B_H(nh)$, the R/S statistic of $\{B_H(kh)\}_{k\in\{1,\ldots,n\}}$ according to (1) reads

$$\mathrm{R/S}(n) = \frac{1}{S(n)}\left[\max_{k\in\{1,\ldots,n\}}\left\{B_H(kh) - \frac{k}{n}B_H(nh)\right\} - \min_{k\in\{1,\ldots,n\}}\left\{B_H(kh) - \frac{k}{n}B_H(nh)\right\}\right]. \tag{2}$$

By the definition of self-similar processes and by property 1. in Section 2, we have that

- $B_H(h) \stackrel{d}{=} (nh)^H B_H\left(\frac{1}{n}\right), \ldots, B_H(nh) \stackrel{d}{=} (nh)^H B_H(1)$,
- $\mathbb{E}\{B_H(h)\} = (nh)^H \mathbb{E}\left\{B_H\left(\frac{1}{n}\right)\right\} = 0, \ldots, \mathbb{E}\{B_H(nh)\} = (nh)^H\mathbb{E}\{B_H(1)\} = 0$, and
- by property 3. in Section 2 for any $k, m \in \mathbb{N}$ it holds that

$$\begin{aligned}\mathrm{Cov}\{B_H(kh), B_H(mh)\} &= \mathbb{E}\{B_H(kh)B_H(mh)\}\\ &= \frac{\sigma^2(nh)^{2H}}{2}\left(\left|\frac{k}{n}\right|^{2H} + \left|\frac{m}{n}\right|^{2H} - \left|\frac{k}{n} - \frac{m}{n}\right|^{2H}\right)\\ &= \mathrm{Cov}\left\{(nh)^H B_H\left(\frac{k}{n}\right), (nh)^H B_H\left(\frac{m}{n}\right)\right\}.\end{aligned}$$

Therefore, we obtain for fractional Brownian motion

$$\{B_H(h), \ldots, B_H(nh)\} \stackrel{d}{=} (nh)^H\left\{B_H\left(\frac{1}{n}\right), \ldots, B_H(1)\right\}.$$

We define

$$\tilde{S}^2(n) = \frac{(nh)^{2H}}{n} \sum_{i=1}^{n} \left\{ B_H\left(\frac{i}{n}\right) - B_H\left(\frac{i-1}{n}\right) \right\}^2 - \left[\frac{(nh)^H}{n} \sum_{i=1}^{n} \left\{ B_H\left(\frac{i}{n}\right) - B_H\left(\frac{i-1}{n}\right) \right\} \right]^2 \tag{3}$$

and

$$\widetilde{\mathrm{R/S}}(n) = \frac{(nh)^H}{\tilde{S}(n)} \left[\max_{k\in\{1,\dots,n\}} \left\{ B_H\left(\frac{k}{n}\right) - \frac{k}{n} B_H(1) \right\} - \min_{k\in\{1,\dots,n\}} \left\{ B_H\left(\frac{k}{n}\right) - \frac{k}{n} B_H(1) \right\} \right], \tag{4}$$

which results in

$$\mathbb{E}\{\mathrm{R/S}(n)\} = \mathbb{E}\{\widetilde{\mathrm{R/S}}(n)\}. \tag{5}$$

The following theorem on the asymptotic behavior of the R/S statistics for fractional Gaussian noise is the main result of this section.

Theorem 3.1. *The $\widetilde{\mathrm{R/S}}(n)$ statistic of a fractional Brownian motion process $\{B_H(n)\}_{n\in\mathbb{N}}$ satisfies*

(i) *as $n \to \infty$,*

$$\frac{1}{n^H}\widetilde{\mathrm{R/S}}(n) \xrightarrow{\text{w.p.1}} \frac{1}{\sigma}[\max_{0\le s\le 1}\{B_H(s) - sB_H(1)\} - \min_{0\le s\le 1}\{B_H(s) - sB_H(1)\}],$$

(ii) *as $n \to \infty$,*

$$\frac{1}{n^H}\mathbb{E}\{\widetilde{\mathrm{R/S}}(n)\} \longrightarrow \frac{1}{\sigma}\mathbb{E}[\max_{0\le s\le 1}\{B_H(s) - sB_H(1)\} - \min_{0\le s\le 1}\{B_H(s) - sB_H(1)\}],$$

where $\sigma^2 = \mathrm{Var}\{B_H(1)\}$.

Proof. It is sufficient to prove the maximum case. Let $x_0 \in [0, 1]$ with $f_0 := f(x_0) = \max_{0\le x\le 1} f(x)$. Take $\varepsilon > 0$ arbitrary, then there exist $\delta > 0$ with $f_0 - f(y) < \varepsilon$ for all $y \in [0, 1]$ with $|x_0 - y| < \delta$. Let $N_0 \in \mathbb{N}$ with $\frac{1}{N_0} < \delta$, then for all $n \ge N_0$ there exists $m_n \in \mathbb{N}$ with $\left|x_0 - \frac{m_n}{n}\right| < \delta$ and hence $f_0 - f\left(\frac{m_n}{n}\right) < \varepsilon$. Therefore,

$$\max_{x\in\{1,\dots,n\}} f\left(\frac{x}{n}\right) > f_0 - \varepsilon \quad \text{for all } n \ge N_0. \quad \square$$

Proof of Theorem 3.1. (i) First we show, using the ergodic theorem, that $\frac{\tilde{S}(n)}{\sigma h^H} \to 1$ with probability 1 as $n \to \infty$ where $\tilde{S}(n)$ is defined in (3). Recall that a sufficient condition for a stationary Gaussian time series $\{X_k\}_{k\in\mathbb{Z}}$ to be ergodic is that $\mathrm{Cov}(X_k, X_{k+h}) \to 0$, as $h \to \infty$ (compare, e.g. Sinai, 1976, page 111). properties 4. and 5. from Section 2 then show that fractional Gaussian noise is ergodic. Applying the ergodic theorem to $X_i^* := (nh)^H \left\{B_H\left(\frac{i}{n}\right) - B_H\left(\frac{i-1}{n}\right)\right\}$ and to $(X_i^*)^2$, for $i = 1, \dots, n$, yields

$$\frac{1}{n}\sum_{i=1}^{n} X_i^* = \frac{(nh)^H B_H(1)}{n} \xrightarrow{\text{w.p.1}} 0, \qquad \frac{1}{n}\sum_{i=1}^{n}(X_i^*)^2 \xrightarrow{\text{w.p.1}} \sigma^2 h^{2H}, \quad n \to \infty.$$

Therefore, $\tilde{S}^2(n) \to \sigma^2 h^{2H}$, i.e. $\frac{1}{\tilde{S}(n)} \to \frac{1}{\sigma h^H}$ with probability 1 as $n \to \infty$.

This means that there exists a measurable set $B_0 \subset \Omega$ with $\mathrm{pr}(B_0) = 1$, such that for all $\omega \in B_0$ we have $\lim_{n\to\infty} \frac{1}{\tilde{S}(n)} = \frac{1}{\sigma h^H}$. Define $A(s) = B_H(s) - sB_H(1)$, then the continuity of the paths of $B_H(t)$ implies

$$\max_{k\in\{1,\dots,n\}} A\left(\frac{k}{n}\right) \xrightarrow{\text{w.p.1}} \max_{0\le s\le 1} A(s), \quad \text{as } n \to \infty.$$

This means that there exists a measurable set $B_1 \subset \Omega$ with $\mathrm{pr}(B_1) = 1$, such that for all $\omega \in B_1$ it holds that $\lim_{n\to\infty} \max_{k\in\{1,\dots,n\}} A\left(\frac{k}{n}\right) = \max_{0\le s\le 1} A(s)$. And similarly there exists a measurable set $B_2 \subset \Omega$ with $\mathrm{pr}(B_2) = 1$, such that for all $\omega \in B_2$ we have $\lim_{n\to\infty} \min_{k\in\{1,\dots,n\}} A\left(\frac{k}{n}\right) = \min_{0\le s\le 1} A(s)$. Note that $\mathrm{pr}(B_0 \cap B_1 \cap B_2) = 1$, and hence we obtain

$$\frac{\max_{k\in\{1,\dots,n\}} A\left(\frac{k}{n}\right) - \min_{k\in\{1,\dots,n\}} A\left(\frac{k}{n}\right)}{\tilde{S}(n)} \xrightarrow{\text{w.p.1}} \frac{\max_{0\le s\le 1} A(s) - \min_{0\le s\le 1} A(s)}{\sigma h^H}, \quad \text{as } n \to \infty.$$

In other words,

$$\frac{1}{n^H}\widetilde{\mathrm{R/S}}(n) \xrightarrow{\text{w.p.1}} \frac{\max_{0\le s\le 1} A(s) - \min_{0\le s\le 1} A(s)}{\sigma}, \quad \text{as } n \to \infty.$$

(ii) Using Part (i), we only need to show that $\frac{1}{n^H h^H}\widetilde{\mathrm{R/S}}(n)$ is uniformly integrable, i.e., it is enough to show that $\frac{1}{n^H h^H}$ $\mathbb{E}\{\widetilde{\mathrm{R/S}}(n)\}^{1+\eta} < \infty$, for a some $\eta > 0$. By Hölder's inequality,

$$\frac{1}{n^H h^H}\mathbb{E}\{\widetilde{\mathrm{R/S}}(n)\}^{1+\eta} \le (\mathbb{E}[1/\{\widetilde{S}^2(n)\}])^{\frac{1+\eta}{2}} \times [\mathbb{E}\{\widetilde{R}(n)\}^{\frac{2(1+\eta)}{1-\eta}}]^{\frac{1-\eta}{2}} < \infty.$$

Let $\eta = 1/3$, and then it is sufficient to prove, for n large, that $\mathbb{E}[\{\widetilde{S}^2(n)\}^{-1}] < \infty$ and $\mathbb{E}\{\widetilde{R}^4(n)\} < \infty$, where $\widetilde{S}$ was defined in (3) and $\widetilde{R} = \max_{k\in\{1,\dots,n\}} A\left(\frac{k}{n}\right) - \min_{k\in\{1,\dots,n\}} A\left(\frac{k}{n}\right)$.

Let $X_i^{**} := B_H\left(\frac{i}{n}\right) - B_H\left(\frac{i-1}{n}\right)$, for $i = 1, \dots, n$, then $X_{**}{}^{(n)} := (X_1^{**}, \dots, X_n^{**})^T \sim N_n\left(0^{(n)}, \Sigma\right)$ with Σ is positive definite, where $0^{(n)}$ denotes an n-vector of 0's. By Imhoff (1961) it holds for $A = \mathbb{I}_n - \frac{1}{n}1^{(n)}1^{(n)T}$, where $\mathbb{I}_n$ is the identity matrix that

$$Q = \frac{1}{n}X_{**}{}^{(n)T}AX_{**}{}^{(n)} = \frac{1}{n}\sum_{r=1}^{m}\lambda_r\chi^2_{h_r}, \tag{6}$$

where the λ_r are the distinct non-zero roots of $A\Sigma$ (which are all positive in this case, see Lemma 3.3), the h_r are their respective orders of multiplicity, m is the number of these distinct non-zero roots, and then $\sum_{r=1}^m h_r = (n-1)$ (see Lemma 3.3). In (6) the $\chi^2_{h_r}$ are independent central χ^2-variables with h_r degrees of freedom. The following claim is needed in the proof.

- (C.1) $\int_0^\infty P(X > y)\mathrm{d}y = \mathbb{E}X$ for any positive variable X.

  Defining $\lambda_{(1)} := \min\{\lambda_1, \dots, \lambda_m\}$, we have

$$\begin{aligned}
\mathbb{E}[1/\{\widetilde{S}^2(n)\}] &= \mathbb{E}\{(nh)^{2H}Q\}^{-1} \overset{\text{(C.1)}}{=} n(nh)^{-2H}\int_0^\infty \mathrm{pr}\left(\frac{1}{nQ} > y\right)\mathrm{d}y \\
&= n(nh)^{-2H}\int_0^\infty \mathrm{pr}\left(nQ < \frac{1}{y}\right)\mathrm{d}y = n(nh)^{-2H}\int_0^\infty \mathrm{pr}\left(\sum_{r=1}^m \lambda_r\chi^2_{h_r} < \frac{1}{y}\right)\mathrm{d}y \\
&\le n(nh)^{-2H}\int_0^\infty \mathrm{pr}\left(\lambda_{(1)}\sum_{r=1}^m \chi^2_{h_r} < \frac{1}{y}\right)\mathrm{d}y = n\,(nh)^{-2H}\int_0^\infty \mathrm{pr}\left(\frac{1}{\lambda_{(1)}\chi^2_{n-1}} > y\right)\mathrm{d}y \\
&\overset{\text{(C.1)}}{=} \frac{n}{(nh)^{2H}\lambda_{(1)}}\mathbb{E}\left(\frac{1}{\chi^2_{n-1}}\right) = \frac{n}{(n-1)(nh)^{2H}\lambda_{(1)}} < \infty,
\end{aligned}$$

which proves the statement for $\mathbb{E}[\{\widetilde{S}^2(n)\}^{-1}]$.

To see the corresponding result for $\mathbb{E}\{\widetilde{R}^4(n)\}$, let $V_1 = \sigma^{-2}\max_{0\le s\le 1} B_H(s)$ with the cumulative probability function $F_{V_1}(v_1)$, $V_2 = \sigma^{-2}\min_{0\le s\le 1} B_H(s)$, and denote by $F_Z(z)$ the cumulative probability function of a random variable with $N(0, 1)$ distribution. By Adler (1990), Theorem 5.5 and Corollary 5.6, we then have $\lim_{x\to\infty}\frac{1-F_{V_1}(x)}{1-F_Z(x)} = 1$, which means that for all $\epsilon > 0$ there exists $x_0 \in \mathbb{R}^+$ such that for all $x \ge x_0$ it holds that $(1-\epsilon)\{1-F_Z(x)\} < 1-F_{V_1}(x) < (1+\epsilon)\{1-F_Z(x)\}$. Therefore

$$\begin{aligned}
0 &\le \mathbb{E}_{V_1}(V_1^4) \overset{\text{(C.1)}}{=} \int_0^\infty \mathrm{pr}\left(V_1^4 > y\right)\mathrm{d}y = \int_0^{x_0^4} \mathrm{pr}\left(V_1^4 > y\right)\mathrm{d}y + \int_{x_0^4}^\infty \mathrm{pr}\left(V_1^4 > y\right)\mathrm{d}y \\
&\le x_0^4 + \int_{x_0^4}^\infty \mathrm{pr}\left(V_1 > y^{\frac{1}{4}}\right)\mathrm{d}y + \int_{x_0^4}^\infty \mathrm{pr}\left(V_1 < -y^{\frac{1}{4}}\right)\mathrm{d}y \\
&\le x_0^4 + (1+\epsilon)\int_{x_0^4}^\infty \mathrm{pr}\left(Z > y^{\frac{1}{4}}\right)\mathrm{d}y + \int_{x_0^4}^\infty \mathrm{pr}\left\{(B_H(0)/\sigma^2) < -y^{\frac{1}{4}}\right\}\mathrm{d}y \\
&\le x_0^4 + (1+\epsilon)\int_{x_0^4}^\infty \mathrm{pr}\left(Z > y^{\frac{1}{4}}\right)\mathrm{d}y + \int_{x_0^4}^\infty \mathrm{pr}\left\{-(B_H(0)/\sigma^2) > y^{\frac{1}{4}}\right\}\mathrm{d}y \\
&= x_0^4 + (2+\epsilon)\int_{x_0^4}^\infty \mathrm{pr}\left(Z > y^{\frac{1}{4}}\right)\mathrm{d}y = x_0^4 + (2+\epsilon)\int_{x_0^4}^\infty \mathrm{pr}\left(Z^4 > y\right)\mathrm{d}y \\
&\le x_0^4 + (2+\epsilon)\int_0^\infty \mathrm{pr}\left(Z^4 > y\right)\mathrm{d}y \overset{\text{(C.1)}}{=} x_0^4 + (2+\epsilon)\,\mathbb{E}(Z^4) < \infty.
\end{aligned}$$

Thus, we have $0 \le \mathbb{E}\{\max_{0\le s\le 1} B_H(s)\}^4 = \sigma^2\mathbb{E}(V_1^4) < \infty$. To prove $\mathbb{E}\{\widetilde{R}^4(n)\} < \infty$, we need the following four claims:

- (C.2) For any $a, b \in \mathbb{R}$, $(a \pm b)^4 \le 8(a^4 + b^4)$.
- (C.3) $\max_{0\le s\le 1} B_H(s) \overset{d}{=} -\min_{0\le s\le 1} B_H(s)$, i.e. $V_1 \overset{d}{=} -V_2$.

- (C.4) $0 \leq \left|\max_{0\leq s\leq 1}\{B_H(s) - sB_H(1)\}\right| \leq \max_{0\leq s\leq 1} B_H(s) + |B_H(1)|$.
- (C.5) $0 \leq \left|\min_{0\leq s\leq 1}\{B_H(s) - sB_H(1)\}\right| \leq -\min_{0\leq s\leq 1} B_H(s) + |B_H(1)|$.

Using the same notation that was defined in (i), we see that

$$\begin{aligned}
\mathbb{E}\left\{\widetilde{R}^4(n)\right\} &= \mathbb{E}\left\{\max_{k\in\{1,\dots,n\}} A\left(\frac{k}{n}\right) - \min_{k\in\{1,\dots,n\}} A\left(\frac{k}{n}\right)\right\}^4 \\
&\leq \mathbb{E}\{\max_{0\leq s\leq 1} A(s) - \min_{0\leq s\leq 1} A(s)\}^4 \overset{(C.2)}{\leq} 8\mathbb{E}\{\max_{0\leq s\leq 1} A(s)\}^4 + 8\mathbb{E}\{\min_{0\leq s\leq 1} A(s)\}^4 \\
&= 8\mathbb{E}[\max_{0\leq s\leq 1}\{B_H(s) - sB_H(1)\}]^4 + 8\mathbb{E}[\min_{0\leq s\leq 1}\{B_H(s) - sB_H(1)\}]^4, \text{ by (C.4) and (C.5)} \\
&\leq 8\mathbb{E}\left|\max_{0\leq s\leq 1} B_H(s) + |B_H(1)|\right|^4 + 8\mathbb{E}\left|-\min_{0\leq s\leq 1} B_H(s) + |B_H(1)|\right|^4, \text{ then by (C.2)} \\
&\leq 64\mathbb{E}\{\max_{0\leq s\leq 1} B_H(s)\}^4 + 64\mathbb{E}\{B_H(1)\}^4 + 64\mathbb{E}\{-\min_{0\leq s\leq 1} B_H(s)\}^4 + 64\mathbb{E}\{B_H(1)\}^4, \text{ then by (C.3)} \\
&= 128\mathbb{E}\{\max_{0\leq s\leq 1} B_H(s)\}^4 + 128\mathbb{E}\{B_H(1)\}^4 < \infty. \quad \square
\end{aligned}$$

Corollary 3.2. *The* R/S(n) *statistic of a fractional Brownian motion process* $\{B_H(n)\}_{n\in\mathbb{N}}$ *satisfies, as* $n \to \infty$,

$$\frac{1}{n^H}\mathbb{E}\{\mathrm{R/S}(n)\} \to \frac{1}{\sigma}\mathbb{E}[\max_{0\leq s\leq 1}\{B_H(s) - sB_H(1)\} - \min_{0\leq s\leq 1}\{B_H(s) - sB_H(1)\}].$$

Proof. The proof follows from $\mathbb{E}\{\mathrm{R/S}(n)\} = \mathbb{E}\{\widetilde{\mathrm{R/S}}(n)\}$ (compare (5)), and the theorem above. □

Lemma 3.3. *Under the assumptions of Theorem 3.1 the matrix* $A\Sigma$ *of* (6) *has* n *non-negative real eigenvalues, with* $(n-1)$ *positive ones.*

Proof. The matrix A is a (symmetric) idempotent real matrix with $\mathrm{rank}(A) = n-1$, and so there exists an orthogonal matrix P, such that $P'P = \mathbb{I}_n$ and

$$P'AP = D_A = \begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since P is an orthogonal matrix, the eigenvalues of $A\Sigma$ are same as the eigenvalues of

$$P'A\Sigma P = P'APP'\Sigma P = D_AP'\Sigma P = \begin{pmatrix} M_{11} & M_{12} \\ 0 & 0 \end{pmatrix}, \quad \text{for } P'\Sigma P := \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

Therefore, if λ is the eigenvalue of $D_AP'\Sigma P$, then

$$0 = \det\left(\lambda\mathbb{I}_n - D_AP'\Sigma P\right) = \det\begin{pmatrix} \lambda\mathbb{I}_{n-1} - M_{11} & -M_{12} \\ 0 & \lambda \end{pmatrix} = \lambda\det\left(\lambda\mathbb{I}_{n-1} - M_{11}\right).$$

Note that M_{11} is also positive definite, since it is an $n-1$ dimensional matrix on the diagonal of the positive definite matrix $P'\Sigma P$. Therefore, the eigenvalues of $D_AP'\Sigma P$ (or $A\Sigma$) are 0 (with the multiplicity 1), or they are the eigenvalues of the matrix M_{11}, and therefore there are n non-negative real eigenvalues with $(n-1)$ positive ones. □

Remark 3.4. 1. Note that $C_H = \frac{1}{\sigma}\mathbb{E}[\max_{0\leq s\leq 1}\{B_H(s) - sB_H(1)\} - \min_{0\leq s\leq 1}\{B_H(s) - sB_H(1)\}]$ is always positive.

2. For $H = 1/2$, i.e. for regular Brownian motion $B(t)$, $t \in [0, 1]$, the difference $B(t) - tB(1)$ is the Brownian bridge on the unit interval and $\frac{1}{\sigma}\{B(t) - tB(1)\}$ is a process with variance 1. Similarly, for $H \in (0, 1)$, and $t \in [0, 1]$, $B_H(t) - tB_H(1)$ may be called "fractional Brownian bridge" on the unit interval, but this is not the most natural definition despite the fact that it is "tied down", compare Jonas (unpublished), Chapter 3.3 for a discussion. Our results in Theorem 3.1 are consistent with the long-term memory asymptotics of the R/S statistic described in Section 1.
3. Corollary 3.2 implies that for fractional Gaussian noise time series, the R/S statistic is an estimator of the H-ss index.
4. For a discussion of convergence in distribution (and in the weak sense) of more general processes, see Mandelbrot (1975) (Lemma 5 and Theorem 5 on pp. 276) and Mandelbrot and Taqqu (1979, Section 3 on pp. 78–83).

In addition to the R/S statistic for characterizing LRD, other statistics with better statistical properties have been proposed; two such statistics are the KPSS statistic of Kwiatkowski et al. (1992) and the R/V statistic of Giraitis et al. (2003). Giraitis et al. (2003) found that the simple transformation of the asymptotic distribution of V/S is connected to the limiting distribution of the standard Kolmogorov statistic or its fractional extension, and the V/S statistic is also more sensitive to shifts in variance, thus has higher power than both modified R/S and KPSS test statistics. Giraitis et al. (2003) considered a modified version of the R/S statistic, denoted by Q_N, in their paper. By adopting very general assumptions, they showed the convergence of Q_N in distribution and studied the tests based on the asymptotic distribution of the Q_N. Note that under the assumptions made by Giraitis et al. (2003), the R/S in (2) can be shown to converge in distribution. To show convergence in expectation (which is the theoretical basis for using R/S to estimate H), the uniform integrability condition needs to hold. In the proof for part (ii) of Theorem 3.1, we showed that for the fBM processes, the uniform integrability condition is satisfied.

Other estimators for the self-similarity index of fractional Brownian motion have been proposed. Two of the approaches discussed in the literature are the aggregated variance method (AVM) and an approach based on the absolute values of the aggregated time series (AVA) (Taqqu et al., 1995). In the Appendix, we show that the AVM estimator converges to the true H with probability 1, which implies the probability convergence and thus also consistency. We also show that similar results hold for the AVA estimator.

4. Discussion

The R/S statistic that defines the Hurst exponent is often used to determine the long-range dependence in a time series. In this paper, we show that asymptotically, the R/S statistic and the self-similarity index of fractional Brownian motion agree in the expectation sense. This result has been mentioned in the literature but – to the best of our knowledge – without formal proof; see, e.g., Taqqu et al. (1995). As preliminaries, we review some properties of the classical R/S statistic and provide a listing of relevant results in Section 2.

In addition to the R/S statistic, many other estimators of the self-similarity index in fractional Brownian motion have been proposed in the literature. Two of those are based on the aggregated variance of the process. We show that for a fixed number of blocks (with fixed block size), the Aggregated Variance Method (AVM) and the Absolute Value of the Aggregated Series (AVA) method result in estimators of H that converge to H with probability 1. Therefore, both estimators are consistent. To prove these results we have fixed both the number of blocks and the block size. It is possible to show that both results still hold if block size and number are not fixed, as long as they are bounded. Our proof does not extend in a natural way to the situation where either block size or block number become unbounded. From a practical viewpoint, this does not constitute a limitation of these approaches if we are interested in the statistics of observed data series.

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Appendix. Aggregated Variance Method—AVM

Consider an (observed) time series $\{Y_k, k \geq 0\}$ with increment series $\{X_k, k \geq 1\}$. The AVM approach considers the aggregated series $X^{(m)}(k) = \frac{1}{m}\sum_{i=(k-1)m+1}^{km} X_i,\ k = 1, 2, 3, \ldots$, for successive values of m, and uses the sample variance of $X^{(m)}(k)$ for sample size N

$$\widehat{\mathrm{Var}}(X^{(m)}) = \frac{1}{m^{-1}N}\sum_{k=1}^{m^{-1}N}\left\{X^{(m)}(k)\right\}^2 - \left\{\frac{1}{m^{-1}N}\sum_{k=1}^{m^{-1}N}X^{(m)}(k)\right\}^2,$$

as an estimator of $\mathrm{Var}(X^{(m)})$. Since for fractional Gaussian noise, $\mathrm{Var}(X^{(m)}) = \sigma^2 m^\beta$ for $\beta := 2H-2 < 0$ as $m \to \infty$, the slope of the straight line $-\log\{\widehat{\mathrm{Var}}(X^{(m)})\}$ versus $\log m$ will (approximately) be $2H-2$, i.e. an estimator of H is $\hat{H}_{\mathrm{AVM}} = \hat{\beta}/2 + 1$, where $\hat{\beta}$ is the estimated slope of the regression line.

Proposition. *Let $\{X_k, k \geq 1\}$ be a fractional Gaussian noise sequence. We fix a set of block sizes $\{m_i\}_{i=1}^M$. Denote by $\hat{H}_{\mathrm{AVM}}(N)$ the estimator $\hat{H}_{\mathrm{AVM}}$ obtained as above for the sequence $\{X_k, 1 \leq k \leq N\}$. Then $\hat{H}_{\mathrm{AVM}}(N)$ converges to H with probability 1 as $N \to \infty$.*

Proof. Given a set of block sizes $\{m_i\}_{i=1}^M$, define $y_{m_i} := \log\left\{\mathrm{Var}(X^{(m_i)})\right\}$ and $\hat{y}_{m_i} := \log\{\widehat{\mathrm{Var}}(X^{(m_i)})\}$, and let $\bar{y}_M$ and $\bar{\hat{y}}_M$ be their respective sample means. By the ergodic theorem and the continuous mapping theorem, for the set of block sizes $\{m_i\}_{i=1}^M$, we have $\left|y_{m_i} - \hat{y}_{m_i}\right| \overset{\text{w.p.1}}{\to} 0$ and $\left|\bar{y}_M - \bar{\hat{y}}_M\right| \overset{\text{w.p.1}}{\to} 0$ as $\frac{N}{m_i} \to \infty$ for any $i \in \{1, \ldots, M\}$, i.e. as $N \to \infty$.

Furthermore, defining

$$\beta := \frac{\sum_{i=1}^{M}(y_{m_i} - \bar{y}_{N_m})(\log m_i - \overline{\log m_i})}{\sum_{i=1}^{M}(\log m_i - \overline{\log m_i})^2} \quad \text{and} \quad \hat{\beta}_M := \frac{\sum_{i=1}^{M}(\hat{y}_{m_i} - \bar{\hat{y}}_{N_m})(\log m_i - \overline{\log m_i})}{\sum_{i=1}^{M}(\log m_i - \overline{\log m_i})^2},$$

we obtain as $N \to \infty$,

$$\begin{aligned} |\hat{\beta}_M - \beta| &= \left| \sum_{i=1}^{M} \left\{ \frac{\hat{y}_{m_i} - \bar{\hat{y}}_M - y_{m_i} + \bar{y}_M}{\sqrt{\sum_{i=1}^{M}(\log m_i - \overline{\log m_i})^2}} \right\} \left\{ \frac{\log m_i - \overline{\log m_i}}{\sqrt{\sum_{i=1}^{M}(\log m_i - \overline{\log m_i})^2}} \right\} \right| \\ &\le \left[\sum_{i=1}^{M} \left\{ \frac{\hat{y}_{m_i} - \bar{\hat{y}}_M - y_{m_i} + \bar{y}_M}{\sqrt{\sum_{i=1}^{M}(\log m_i - \overline{\log m_i})^2}} \right\}^2 \right]^{\frac{1}{2}} \left[\sum_{i=1}^{M} \left\{ \frac{\log m_i - \overline{\log m_i}}{\sqrt{\sum_{i=1}^{M}(\log m_i - \overline{\log m_i})^2}} \right\}^2 \right]^{\frac{1}{2}} \\ &= \left\{ \frac{\sum_{i=1}^{M}(\hat{y}_{m_i} - \bar{\hat{y}}_M - y_{m_i} + \bar{y}_M)^2}{\sum_{i=1}^{M}(\log m_i - \overline{\log m_i})^2} \right\}^{\frac{1}{2}} \le \left[\frac{2\sum_{i=1}^{M}\{(\hat{y}_{m_i} - y_{m_i})^2 + (\bar{y}_M - \bar{\hat{y}}_M)^2\}}{\sum_{i=1}^{M}(\log m_i - \overline{\log m_i})^2} \right]^{\frac{1}{2}} \xrightarrow{\text{w.p.1}} 0. \end{aligned}$$

Therefore, as $N \to \infty$, $\hat{H}_{\text{AVM}} = \hat{\beta}_{N_m}/2 + 1 \xrightarrow{\text{w.p.1}} H$, for $H = \beta_{N_m}/2 + 1$. $\square$

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