The submission deadline for this exam is Sunday October 28 at 12:00 noon PDT. Note that you can save your entries as you work through the problems, and can change them at any time prior to submission, but once you submit your answers no further changes are possible. Because the peer grading process begins immediately after the submission deadline, the system cannot accept late submissions, so if you think there may be any possible delay in submitting on the Sunday morning, you should submit on the Saturday evening at the latest.

YOU ARE EXPECTED TO WORK ALONE ON THIS EXAM.

1. Say whether the following is true or false and support your answer by a proof.

$$(\exists m \in \mathcal{N})(\exists n \in \mathcal{N})(3m + 5n = 12)$$

2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

3. Say whether the following is true or false and support your answer by a proof: For any integer n, the number $n^2 + n + 1$ is odd.

4. Prove that every odd natural number is of one of the forms $4n + 1$ or $4n + 3$, where n is an integer.

5. Prove that for any integer n, at least one of the integers $n,\ n + 2,\ n + 4$ is divisible by 3.

6. A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

7. Prove that if the natural number n is not a perfect square, then $\sqrt{n}$ is irrational.

8. Prove that for any natural number n,

$$2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$$

9. Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number M, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

10. Given an infinite collection $A_n, n = 1, 2, \ldots$ of intervals of the real line, their *intersection* is defined to be

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid (\forall n)(x \in A_n)\}$$

Give examples of two families of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and (respectively):

(a) $\bigcap_{n=1}^{\infty} A_n = \emptyset$

(b) $\bigcap_{n=1}^{\infty} A_n$ consists of a single real number.