1. Prove or disprove the claim that there are integers m, n such that $m^2 + mn + n^2$ is a perfect square.
2. Prove or disprove the claim that for any positive integer m there is a positive integer n such that $mn + 1$ is a perfect square.
3. Prove that there is a quadratic $f(n) = n^2 + bn + c$, with positive integers coefficients b, c, such that $f(n)$ is composite (i.e, not prime) for all positive integers n, or else prove that the statement is false.
4. Prove that if every even natural number greater than 2 is a sum of two primes (the Goldbach Conjecture), then every odd natural number greater than 5 is a sum of three primes.
5. Use the method of induction to prove that the sum of the first n odd numbers is equal to n^2.
6. The notation
$$\sum_{i=1}^{n} a_i \quad (\text{or} \quad \textstyle\sum_{i=1}^{n} a_i)$$
is a common abbreviation for the sum
$$a_1 + a_2 + a_3 + \ldots + a_n$$
For instance,
$$\sum_{r=1}^{n} r^2$$
denotes the sum
$$1^2 + 2^2 + 3^2 + \ldots + n^2$$
Prove by induction that:
$$\forall n \in \mathcal{N} : \sum_{r=1}^{n} r^2 = \tfrac{1}{6} n(n+1)(2n+1)$$

OPTIONAL PROBLEMS

1. In the lecture we used induction to prove the general theorem
$$1 + 2 + \ldots + n = \tfrac{1}{2} n(n+1)$$
There is an alternative proof that does not use induction, famous because Gauss used the key idea to solve a "busywork" arithmetic problem his teacher gave to the class when he was a small child at school. The teacher asked the class to calculate the sum of the first hundred natural numbers. Gauss noted that if
$$1 + 2 + \ldots + 100 = N$$
then you can reverse the order of the addition and the answer will be the same:
$$100 + 99 + \ldots + 1 = N.$$
So by adding those two equations, you get
$$101 + 101 + \ldots + 101 = 2N$$

and since there are 100 terms in the sum, this can be written as

$$100 \cdot 101 = 2N$$

and hence

$$N = \tfrac{1}{2}(100 \cdot 101) = 5,050.$$

Generalize Gauss's idea to prove the theorem without recourse to the method of induction.

2. Prove that for any finite collection of points in the plane, not all collinear, there is a triangle having three of the points as its vertices, which contains none of the other points in its interior.

3. Prove the following by induction:

   (a) $4^n - 1$ is divisible by 3.

   (b) $(n+1)! > 2^{n+3}$ for all $n \geq 5$.

   (c) $\forall n \in \mathcal{N} : \displaystyle\sum_{r=1}^{n} 2^r = 2^{n+1} - 2$

   (d) $\forall n \in \mathcal{N} : \displaystyle\sum_{r=1}^{n} r.r! = (n+1)! - 1$