This assignment is fairly long, but it deals with a number of (related) crucial notions that pervade modern mathematics. It's impossible to progress in advanced mathematics without mastering these ideas. You probably won't be able to get it all done before the next lecture, but keep coming back to it until you have at least tried all the questions. The concepts and methods covered by this assignment are the key to mathematical thinking, and work on this one assignment will yield wide-ranging benefits not just in mathematics but in many others parts of your life.

1. Build a truth table to prove the claim I made earlier that $\phi \Leftrightarrow \psi$ is true if ϕ and ψ are both true or both false, and $\phi \Leftrightarrow \psi$ is false if exactly one of ϕ, ψ is true and the other false. (To constitute a proof, your table should have columns that show how the entries for $\phi \Leftrightarrow \psi$ are derived, one operator at a time.)

2. Build a truth table to show that

$$(\phi \Rightarrow \psi) \Leftrightarrow (\neg\phi \vee \psi)$$

   is true for all truth values of ϕ and ψ. A statement whose truth values are all T is called a *logical validity*, or sometimes a *tautology*.

3. Build a truth table to show that

$$(\phi \not\Leftrightarrow \psi) \Leftrightarrow (\phi \wedge \neg\psi)$$

   is a tautology.

4. The ancient Greeks formulated a basic rule of reasoning for proving mathematical statements. Called *modus ponens*, it says that if you know ϕ and you know $\phi \Rightarrow \psi$, then you can conclude ψ.

   (a) Construct a truth table for the logical statement

$$[\phi \wedge (\phi \Rightarrow \psi)] \Rightarrow \psi$$

   (b) Explain how the truth table you obtain demonstrates that *modus ponens* is a valid rule of inference.

5. [This question has a long set-up. The question itself is the very last sentence. TAKE YOUR TIME.]

   One way to prove that

$$\neg(\phi \wedge \psi) \quad \text{and} \quad (\neg\phi) \vee (\neg\psi)$$

   are equivalent is to show they have the same truth table:

| | | | * | | | * |
ϕ	ψ	$\phi \wedge \psi$	$\neg(\phi \wedge \psi)$	$\neg\phi$	$\neg\psi$	$(\neg\phi) \vee (\neg\psi)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

   Since the two columns marked * are identical, we know that the two expressions are equivalent.

   Thus, negation has the affect that it changes $\vee$ into $\wedge$ and changes $\wedge$ into $\vee$. An alternative approach way to prove this is to argue directly with the meaning of the first statement:

   1. $\phi \wedge \psi$ means both ϕ and ψ are true.
   2. Thus $\neg(\phi \wedge \psi)$ means it is not the case that both ϕ and ψ are true.
   3. If they are not both true, then at least one of ϕ, ψ must be false.

4. This is clearly the same as saying that at least one of $\neg\phi$ and $\neg\psi$ is true. (By the definition of negation).

5. By the meaning of *or*, this can be expressed as $(\neg\phi) \vee (\neg\psi)$.

Provide an analogous logical argument to show that $\neg(\phi \vee \psi)$ and $(\neg\phi) \wedge (\neg\psi)$ are equivalent.

6. By a *denial* of a statement ϕ we mean any statement equivalent to $\neg\phi$. Give a useful (and hence natural sounding) denial of each of the following statements.

   (a) 34,159 is a prime number.

   (b) Roses are red and violets are blue.

   (c) If there are no hamburgers, I'll have a hot-dog.

   (d) Fred will go but he will not play.

   (e) The number x is either negative or greater than 10.

   (f) We will win the first game or the second.

7. Show that $\phi \Leftrightarrow \psi$ is equivalent to $(\neg\phi) \Leftrightarrow (\neg\psi)$.

8. Construct truth tables to illustrate the following:

   (a) $\phi \Leftrightarrow \psi$

   (b) $\phi \Rightarrow (\psi \vee \theta)$

9. Use truth tables to prove that the following are equivalent:

   (a) $\neg(\phi \Rightarrow \psi)$ and $\phi \wedge (\neg\psi)$

   (b) $\phi \Rightarrow (\psi \wedge \theta)$ and $(\phi \Rightarrow \psi) \wedge (\phi \Rightarrow \theta)$

   (c) $(\phi \vee \psi) \Rightarrow \theta$ and $(\phi \Rightarrow \theta) \wedge (\psi \Rightarrow \theta)$

10. Verify the equivalences in (b) and (c) in the previous question by means of a logical argument. (So, in the case of (b), for example, you must show that assuming ϕ and deducing $\psi \wedge \theta$ is the same as both deducing ψ from ϕ and θ from ϕ.)

11. Use truth tables to prove the equivalence of $\phi \Rightarrow \psi$ and $(\neg\psi) \Rightarrow (\neg\phi)$.

    $(\neg\psi) \Rightarrow (\neg\phi)$ is called the *contrapositive* of $\phi \Rightarrow \psi$. The logical equivalence of a conditional and its contrapositive means that one way to prove an implication it is to verify the contrapositive. This is a common form of proof in mathematics that we'll encounter later.

12. Write down the contrapositives of the following statements:

    (a) If two rectangles are congruent, they have the same area.

    (b) If a triangle with sides a, b, c (c largest) is right-angled, then $a^2 + b^2 = c^2$.

    (c) If $2^n - 1$ is prime, then n is prime.

    (d) If the Yuan rises, the Dollar will fall.

13. It is important not to confuse the contrapositive of a conditional $\phi \Rightarrow \psi$ with its *converse* $\psi \Rightarrow \phi$. Use truth tables to show that the contrapositive and the converse of $\phi \Rightarrow \psi$ are not equivalent.

14. Write down the converses of the four statements in question 12.

OPTIONAL PROBLEMS

1. Express the combinator

$$\phi \text{ unless } \psi$$

in terms of the standard logical combinators.

2. Show that for any two statements ϕ and ψ either $\phi \Rightarrow \psi$ or its converse is true (or both). This is another reminder that the conditional is not the same as implication.

3. Let $\dot{\vee}$ denote the 'exclusive or' that corresponds to the English expression "either one or the other but not both". Construct a truth table for this connective.

4. Express $\phi \dot{\vee} \psi$ it terms of the basic combinators $\wedge, \vee, \neg$.

5. Give, if possible, an example (one example in each case) of a true conditional sentence for which

   (a) the converse is true.
   (b) the converse is false.
   (c) the contrapositive is true.
   (d) the contrapositive is false.

6. *Mod-2 arithmetic* has just the two numbers 0 and 1 and follows the usual rules of arithmetic together with the additional rule $1 + 1 = 0$. (It is the arithmetic the takes place in a single bit location in a digital computer.) Complete the following table:

M	N	$M \times N$	$M + N$
1	1	?	?
1	0	?	?
0	1	?	?
0	0	?	?

7. In the table you obtained in the above exercise, interpret 1 as T and 0 as F and view M, N as denoting statements.

   (a) Which of the logical combinators $\wedge$, $\vee$ corresponds to $\times$?
   (b) Which logical combinator corresponds to $+$?
   (c) Does $\neg$ correspond to $-$ (minus)?

8. Repeat the above exercise, but interpret 0 as T and 1 as F. What conclusions can you draw?

9. The following puzzle was introduced by the psychologist Peter Wason in 1966, and is one of the most famous subject tests in the psychology of reasoning. Most people get it wrong. (So you have been warned!)

   Four cards are placed on the table in front of you. You are told (truthfully) that each has a letter printed on one side and a digit on the other, but of course you can only see one face of each. What you see is:

$$\text{B} \quad \text{E} \quad 4 \quad 7$$

   You are now told that the cards you are looking at were chosen to follow the rule "If there is a vowel on one side, then there is an odd number on the other side." What is the least number of cards you *have* to turn over to verify this rule, and which cards do you in fact *have* to turn over?

10. Let m, n denote any two natural numbers. Prove that mn is odd iff m and n are odd.

11. With reference to the previous question, is it true that mn is even iff m and n are even? Prove your answer.

12. You are in charge of a party where there are young people. Some are drinking alcohol, others soft drinks. Some are old enough to drink alcohol legally, others are under age. You are responsible for ensuring that the drinking laws are not broken, so you have asked each person to put his or her photo ID on the table. At one table are four young people. One person has a beer, another has a Coke, but their IDs happen to be face down so you cannot see their ages. You can, however, see the IDs of the other two people. One is under the drinking age, the other is above it. Unfortunately, you are not sure if they are drinking Seven-up or vodka and tonic. Which IDs and/or drinks do you need to check to make sure that no one is breaking the law?

13. Compare the logical structure of the previous question with Wason's problem (Optional Problem 9 above). Comment on your answers to those two questions. In particular, identify any logical rules you used in solving each problem, say which one was easier, and why you felt it was easier.